Optimization with set relations: Conjugate Duality

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Abstract

The aim of this paper is to develop a conjugate duality theory for convex set–valued maps. The basic idea is to understand a convex set–valued map as a function with values in the space of closed convex subsets of $\mathbb{R}^p$. The usual inclusion of sets provides a natural ordering relation in this space. Infimum and supremum with respect to this ordering relation can be expressed with aid of union and intersection. Our main result is a strong duality assertion formulated along the lines of classical duality theorems for extended real–valued convex functions.

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1 Introduction

Set–valued optimization problems have been investigated by many authors, see Jahn [8] and the references therein. Furthermore, set–valued maps naturally occur in vector optimization. Tanino [18] gave a nice summary of the theory of conjugate duality in vector optimization using set–valued maps. In Hamel et al. [6] a set–valued approach is used to solve the problem of the duality gap in linear vector optimization in the case $b = 0$.

Optimization with set relations provides a quite different approach to set–valued optimization. The main idea is to understand the set–valued objective map $f : X \rightrightarrows Y$ as a function $f : X \to \hat{\mathcal{P}}(Y)$ into the space $\hat{\mathcal{P}}(Y)$ of all subsets of Y. This space is provided with an appropriate ordering relation. Kuroiwa [11] used such ordering relations in order to formulate corresponding optimization problems. In Jahn [8], these relations are called *KN partial orderings* because the basic idea can be already found in Nishnianidze [16]. Jahn [8] designated the KN partial ordering approach to be promising in set–valued optimization.

In this paper we will investigate convex problems in this context. It turns out that the KN partial ordering can be replaced by the usual inclusion of sets in order to describe all the relevant assertions in the paper's framework. This is due to the fact that the KN partial orderings are not antisymmetric, in general. In order to obtain antisymmetry it is necessary to switch to equivalence classes. Choosing appropriate representatives of these equivalence classes is the same thing as using the usual inclusion of sets instead of the KN partial orderings.

A more detailed discussion can be found in [14]. Infimum and supremum with respect to the relation "inclusion of sets" are expressed via union and intersection. We develop a set–valued conjugate duality theory, based on the relation "inclusion of sets", and we proceed completely analogous to the scalar theory of conjugate duality. This work is organized as follows.

In the next section we investigate the structure of the objective function's image space, namely the space of closed convex subsets of $\mathbb{R}^p$. We observe that this space is not a linear space. Moreover, it is not possible to embed this space into a linear space. However, as we will see in the third section, certain subsets of this space can be embedded. Essentially, a convex function only attains values in such a subset. This ensures the linearity of the image spaces which is usually needed in duality theory. Furthermore, it is necessary to observe whether infimum and supremum are changed while the embedding procedure. We observe that the infimum is not changed, but the supremum is so. At the first glance, this problem seems to be asymmetric in this sense. With aid of the concept of oriented sets, due to Rockafellar [1], the symmetry can be re–established. For the details we also refer to [14]. In Section 4 we develop the duality theory. Weak and strong duality assertions will be proved. The last section is devoted to some examples.

As for prerequisites, the reader is expected to be familiar with Rockafellar's "Convex Analysis" [1]. Up to a few exceptions, we frequently use the notation therein. The following notations are not in accordance with Rockafellar's book. The symbol $\oplus$ does not mean the direct sum, because it will get an other meaning. If A is a real $m \times n$ matrix, $\operatorname{rg} A := \{Ax \in \mathbb{R}^m |\ x \in \mathbb{R}^n\}$ denotes the range of A and A^T is the transposed matrix. Further we write $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n |\ \forall i \in \{1, ..., n\} : x_i \geq 0\}$, $\mathbb{R}^n_- := -\mathbb{R}^n_+$ and $\mathbb{R}_+ := \mathbb{R}^1_+$.

The main results of this paper were announced in [13].

2 The structure of the image space

Throughout the paper, Y stands for the Euclidian space $\mathbb{R}^p$, where p is a positive integer. The space of all nonempty closed convex subsets of Y is denoted by $\mathcal{F}(Y)$. For simplicity of notation, we write $\mathcal{F}$ instead of $\mathcal{F}(Y)$. In $\mathcal{F}$ we introduce an addition $\oplus :\ \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and a multiplication with nonnegative real numbers $\cdot :\ \mathbb{R}_+ \times \mathcal{F} \to \mathcal{F}$, defined by

$$\forall A, B \in \mathcal{F} :\ A \oplus B := \operatorname{cl}(A + B) = \operatorname{cl}\ \{a + b |\ a \in A, b \in B\}\,,$$

$$\forall A \in \mathcal{F},\ \alpha \geq 0 :\ \alpha\, A := \alpha\ \cdot\ A := \{\alpha\, a |\ a \in A\}\,.$$

Clearly, both operations are well–defined and for all $A, B, C \in \mathcal{F}$ and all $\alpha, \beta \in \mathbb{R}_+$ the following calculus rules hold true:

(S1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(S2) $\{0\} \oplus A = A$,
(S3) $A \oplus B = B \oplus A$,
(S4) $\alpha\,(\beta\, A) = (\alpha\beta)\, A$,
(S5) $1 \cdot A = A$,
(S6) $\alpha\,(A \oplus B) = \alpha\, A \oplus \alpha\, B$,
(S7) $0 \cdot A = \{0\}$,
(S8) $\alpha\, A \oplus \beta\, A = (\alpha + \beta)\, A$.

In Hamel [5], a set $(W, \oplus, \cdot)$ is called a *conlinear space* if the axioms (S1)–(S7) are satisfied. In this manner $\mathcal{F}$ is *conlinear space* where, additionally, the second distributive law (S8) holds true. For further concepts of this type refer [5], [10] and the references therein. It is easy to see that $\mathcal{F}$ is not a linear space, since the axiom of the existence of an inverse element is violated. Moreover, it is *not possible to embed $\mathcal{F}$ into a linear space.* Indeed, assuming there is an injective homomorphism j (an embedding) from $\mathcal{F}$ into a linear space L. Given a nonempty closed convex cone $K \subseteq Y$ such that $K \neq \{0\}$ we have $K \in \mathcal{F}$ and $K = K \oplus K$. Hence $j(K) = j(K) + j(K) \neq 0$. Then there must be an inverse element $l \in L$ of $j(K)$, i.e. $j(K) + l = 0$. It follows $0 = j(K) + l = j(K) + l + j(K) = j(K)$, a contradiction.

Although, $\mathcal{F}$ is not a linear space nor can it be embedded into a linear space, its structure is rich enough to define the concept of convexity and that of a cone. A subset $\mathcal{A} \subseteq \mathcal{F}$ is said to be *convex* if $A, B \in \mathcal{A}$ implies that $\lambda A + (1 - \lambda)B \in \mathcal{A}$ for all $\lambda \in [0, 1]$. A subset $\mathcal{A} \subseteq \mathcal{F}$ is said to be a *cone* if $A \in \mathcal{A}$ implies that $\alpha A \in \mathcal{A}$ for all $\alpha > 0$.

Rockafellar [1, Section 39] introduced the concept of *orientation* of convex sets in $\mathbb{R}^p$. A convex set A which is identified with its convex indicator function $\delta(\,\cdot\,|A)$ is said to be *supremum oriented* and a convex set A which is identified with the concave function $-\delta(\,\cdot\,|A)$ is called *infimum oriented*. This concept will play a crucial role in our theory. Thus we introduce the following notation: The space $\mathcal{F}^\star$ is defined to be the space of all nonempty closed convex subset of Y having supremum orientation. By $\mathcal{F}^\diamond$ we denote the space of all nonempty closed convex subset of Y having infimum orientation. If not stated otherwise, the orientation is not changed while manipulating sets. For instance, this means $\mathcal{F}^\star$ and $\mathcal{F}^\diamond$ are conlinear spaces, the recession cone of a supremum (infimum) oriented set is supremum (infimum) oriented, and so on.

Let $K \subseteq Y$ be a nonempty closed convex cone. The set $\mathcal{F}_K \subseteq \mathcal{F}$ is defined to be the set of all elements $A \in \mathcal{F}$ with $0^+A = K$. If these sets additionally have an orientation we write $\mathcal{F}_K^\star$ and $\mathcal{F}_K^\diamond$, respectively.

Proposition 2.1 *$\mathcal{F}_K$ is a convex cone in $\mathcal{F}$.*

Proof. Let $A, B \in \mathcal{F}_K$. It remains to show $0^+(A \oplus B) = K$. This is a consequence of [1, Corollary 9.1.1.] if we can verify the following condition: If $z_1 \in 0^+A$ and $z_2 \in 0^+B$ such that $z_1 + z_2 = 0$, then z_1 belongs to the lineality space of A and z_2 belongs to the lineality space of B. Indeed, we have $0^+A = 0^+B = K$ and the lineality spaces of A and B are equal, namely $0^+A \cap (-0^+A) = 0^+B \cap (-0^+B) = K \cap (-K)$. Hence the mentioned condition is satisfied. $\square$

Note that [1, Corollary 9.1.1.] also implies that in $\mathcal{F}_K \subseteq \mathcal{F}$ the addition $\oplus$ reduces to the usual Minkowski addition $+$, i.e. the closure operation is superfluous.

The space $\mathcal{F}$ is now equipped with one of the reflexive, transitive and antisymmetric relations $\supseteq$ and $\subseteq$. We establish standard relations, in dependence of the orientation of the members of the space. In fact, let the standard relation be $\supseteq$ in the space $\mathcal{F}^\star$ and $\subseteq$ in $\mathcal{F}^\diamond$. Both these standard relations should have the meaning of "less or equal". This identification makes it easier to distinguish between convex and concave functions with values in $\mathcal{F}$ (which will be defined below).

We observe the following relation between the conlinear structure and the ordering structure in $(\mathcal{F},\supseteq)$ and $(\mathcal{F},\subseteq)$.

$$\forall A,B,C,D\in\mathcal{F},\ \forall\alpha\in\mathbb{R}_+:\ A\supseteq B,\ C\supseteq D\ \Rightarrow \alpha\,(A\oplus C)\supseteq\alpha\,(B\oplus D).$$

In [5], a conlinear space equipped with a partial ordering and satisfying the latter condition is called a *partially ordered conlinear space*. This means our spaces $\mathcal{F}^\star$ and $\mathcal{F}^\diamond$ (with its standard relations) are partially ordered conlinear spaces.

We now repeat some concepts with respect to partially ordered sets, see for instance [19]. Moreover, we illustrate some crucial facts according to these concepts by simple examples. If $(W,\leq)$ is a partially ordered set, V is a subset of W and the point $w_0\in W$ satisfies $v\leq w_0$ for all $v\in V$, then w_0 is called an *upper bound* of V. The subset V is now said to be *bounded above*. The definitions of *bounded below* and *lower bound* are analogous. Note that the boundedness depends on the "basis set" W. For instance, letting $W=\mathbb{R}$, ordered by the usual ordering $\leq$, the open interval $V=(0,1)$ is bounded. If we take instead $W=(0,1)$, then the same set $V=(0,1)$ is not bounded. If $w_0\in W$ is an upper bound of V such that $w_0\leq\bar w$ for any other upper bound $\bar w\in W$ of V, then w_0 is called *least upper bound* or *supremum* of V and is denoted by $\sup V$. If V has a supremum then it is uniquely defined. This is an easy consequence of the antisymmetry of the relation $\leq$. The *greatest lower bound* or *infimum* is defined analogously and denoted by $\inf V$. Supremum and infimum of a set $V\subseteq W$ also depend on the "basis set" W as the following example shows. As above, let $W=\mathbb{R}$ and $V=(0,1)$. Then $\sup V=1$. If we have $W=\{r\in\mathbb{R}|\ r<1\vee r\geq 2\}$ instead, $\sup V=2$. In the case where $W=\{r\in\mathbb{R}|\ r<1\vee r>2\}$, the supremum of V does not exist. A partially ordered set W is said to be *order complete* if every subset of W has supremum and infimum. If W is order complete and $V=\emptyset$, then $\sup V=\inf W$ and $\inf V=\sup W$. The set W is called *Dedekind complete* if every nonempty subset of W which is bounded above (bounded below) has a supremum (infimum). Note that for Dedekind completeness an one–sided condition is already sufficient, this means W is Dedekind complete if and only if every nonempty subset of W which is bounded above has a supremum [19, Theorem 1.4]. An element $\bar w\in W$ is called the *largest* element of $(W,\leq)$ if $w\leq\bar w$ for all $w\in W$. The *smallest* element is defined analogously. If $(W,\leq)$ has a largest (smallest) element, then it is uniquely defined. The examples given above motivate the following theorem.

Theorem 2.2 *Let $(W,\leq)$ and $(W^\star,\leq^\star)$ be Dedekind complete partially ordered sets with $W^\star\subseteq W$ and such that $\leq$ and $\leq^\star$ coincide on $W^\star$. Denote the supremum (infimum) of a subset $V\subseteq W$ with respect to $(W,\leq)$ by $\sup V$ $(\inf V)$ and the supremum (infimum) of a subset $V\subseteq W^\star$ with respect to $(W^\star,\leq^\star)$ by $\sup^\star V$ $(\inf^\star V)$. Then it holds.*

(i) $\emptyset\neq V\subseteq W^\star$ *bounded above* $\quad\Rightarrow\quad \sup V\leq\sup^\star V$,

(ii) $\emptyset \neq V \subseteq W^\star$ *bounded below* $\quad \Rightarrow \quad \inf^\star V \leq \inf V$.

Under the additional assumption $\sup V \in W^\star$ *it even holds equality in (i). The same is true in (ii) if* $\inf V \in W^\star$.

Proof. (i) By hypothesis, the set $V \subseteq W^\star$ is bounded above (with respect to $(W^\star, \leq^\star)$). Hence $V \subseteq W$ is bounded above (with respect to $(W, \leq)$). Since V is nonempty and $(W, \leq)$ and $(W^\star, \leq^\star))$ are Dedekind complete, $\sup V$ and $\sup^\star V$ exist. Of course, $\sup^\star V \in W^\star$ is an upper bound of V with respect to $(W, \leq)$. By the definition of the supremum we get $\sup V \leq \sup^\star V$. (ii) Analogous. The second statement follows immediately from the definition of supremum and infimum. $\square$

Let us apply the concepts introduced above to the spaces $\mathcal{F}^\star$ and $\mathcal{F}^\diamond$.

Proposition 2.3 *The spaces* $\mathcal{F}^\star$ *and* $\mathcal{F}^\diamond$ *are Dedekind complete and the infimum and supremum can be expressed as follows:*

$$
\begin{array}{lllll}
\text{(i)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{F}^\star \text{ bounded above} & \Rightarrow & \sup \mathcal{A} & = \bigcap_{A \in \mathcal{A}} A, \\
\text{(ii)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{F}^\star & \Rightarrow & \inf \mathcal{A} & = \operatorname{cl} \operatorname{conv} \bigcup_{A \in \mathcal{A}} A, \\
\text{(iii)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{F}^\diamond & \Rightarrow & \sup \mathcal{A} & = \operatorname{cl} \operatorname{conv} \bigcup_{A \in \mathcal{A}} A, \\
\text{(iv)} & \emptyset \neq \mathcal{A} \subseteq \mathcal{F}^\diamond \text{ bounded below} & \Rightarrow & \inf \mathcal{A} & = \bigcap_{A \in \mathcal{A}} A.
\end{array}
$$

Proof. (i) Set $S := \bigcap_{A \in \mathcal{A}} A$. Let $\bar{A}$ be an upper bound of $\mathcal{A}$, i.e. $A \supseteq \bar{A}$ for all $A \in \mathcal{A}$. Hence $S \neq \emptyset$. Of course, S is convex and closed. Thus, S belongs to $\mathcal{F}^\star$. For all $A \in \mathcal{A}$ we have $A \supseteq S$, i.e. S is an upper bound of $\mathcal{A}$. Let $\bar{S} \in \mathcal{F}^\star$ be another upper bound of $\mathcal{A}$, i.e. for all $A \in \mathcal{A}$ it holds $A \supseteq \bar{S}$, then it follows $\bigcap_{A \in \mathcal{A}} A \supseteq \bar{S}$, i.e. $S \supseteq \bar{S}$.

(ii) Set $I := \operatorname{cl} \operatorname{conv} \bigcup_{A \in \mathcal{A}} A$. Of course, $I \in \mathcal{F}^\star$. For all $A \in \mathcal{A}$ we have $I \supseteq A$, i.e. I is a lower bound of $\mathcal{A}$. Let $\bar{I} \in \mathcal{F}^\star$ be another lower bound of $\mathcal{A}$, i.e. for all $A \in \mathcal{A}$ it holds $\bar{I} \supseteq A$, then it follows $\bar{I} \supseteq \bigcup_{A \in \mathcal{A}} A$. Since $\bar{I}$ is closed and convex, we obtain $\bar{I} \supseteq I$. The same reasoning applies to (iii) and (iv). $\square$

In many situations, it is convenient to extend a Dedekind complete partially ordered set by a smallest or largest element in order to obtain an order complete set. In $\mathcal{F}^\star$ there already exists a smallest element, namely $Y \in \mathcal{F}^\star$. Therefore, we extent the space $\mathcal{F}^\star$ only by a largest element. Intuitively, we denote this element by $\emptyset$. Then we have $A \supseteq \emptyset$ for all $A \in \mathcal{F}$. This new element is also provided with an orientation, in this case with supremum orientation. Addition and multiplication with this new element are defined as:

$$\forall A \in \mathcal{F} \cup \{\emptyset\}: \; A \oplus \emptyset = \emptyset \oplus A = \emptyset, \qquad \forall \alpha > 0: \; \alpha \cdot \emptyset = \emptyset, \qquad 0 \cdot \emptyset = \{0\}.$$

The resulting order complete conlinear space is denoted by $\hat{\mathcal{F}}^\star$. In the same way the space $\mathcal{F}^\diamond$ is extended by a smallest (infimum oriented) element $\emptyset$ and the resulting order complete conlinear space is denoted by $\hat{\mathcal{F}}^\diamond$.

In every order complete partially ordered conlinear space $(W, \oplus, \cdot, \leq)$ it is evident that

$$\inf(\mathcal{A}+\mathcal{B}) \geq \inf \mathcal{A} \oplus \inf B \qquad \text{and} \qquad \sup(\mathcal{A}+\mathcal{B}) \leq \sup \mathcal{A} \oplus \sup B, \tag{1}$$

where $\mathcal{A}+\mathcal{B} := \{A \oplus B |\ A \in \mathcal{A}, B \in \mathcal{B}\}$. In general, it does not hold equality in (1).

Example 2.4 Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $\mathbb{B}(\bar{x}, r) := \left\{x \in \mathbb{R}^2 |\ \|x-\bar{x}\| \leq r\right\}$, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}^\star$, $\mathcal{A} := \left\{\{(0,1)^T\}+K, \{(1,0)^T\}+K\right\}$, $\mathcal{B} := \left\{\mathbb{B}\big((0,0)^T,1\big)+K\right\}$. Then, $\sup \mathcal{A} \oplus \sup \mathcal{B} = \big((\{(0,1)^T\}+K) \cap (\{(1,0)^T\}+K)\big) + \big(\mathbb{B}\big((0,0)^T,1\big)+K\big) = \mathbb{B}\big((1,1)^T,1\big)+K$. However, $\sup(\mathcal{A}+\mathcal{B}) = \sup\left\{\mathbb{B}\big((0,1)^T,1\big)+K,\ \mathbb{B}\big((1,0)^T,1\big)+K\right\} = \big(\mathbb{B}\big((0,1)^T,1\big)+K\big) \cap \big(\mathbb{B}\big((1,0)^T,1\big)+K\big) = K \supsetneq \mathbb{B}\big((1,1)^T,1\big)+K$. Hence $\sup(\mathcal{A}+\mathcal{B}) \neq \sup \mathcal{A} \oplus \sup \mathcal{B}$.

However, for the infimum in $\mathcal{F}^\star$ and supremum in $\mathcal{F}^\diamond$, (1) holds even with equality.

Proposition 2.5 *For nonempty sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}^\star$ and $\mathcal{C}, \mathcal{D} \subseteq \mathcal{F}^\diamond$ it holds*

$$\inf(\mathcal{A}+\mathcal{B}) = \inf \mathcal{A} \oplus \inf \mathcal{B} \qquad \textit{and} \qquad \sup(\mathcal{C}+\mathcal{D}) = \sup \mathcal{C} \oplus \sup \mathcal{D}.$$

Proof. For nonempty subsets $A, B \subseteq Y$ it holds $\operatorname{conv} A + \operatorname{conv} B = \operatorname{conv}(A+B)$, see for instance [9]. Furthermore, it is easy to check that $\operatorname{cl}(\operatorname{cl} A + \operatorname{cl} B) = \operatorname{cl}(A+B)$. Hence, we conclude $\operatorname{cl}(\operatorname{cl}\operatorname{conv} A + \operatorname{cl}\operatorname{conv} B) = \operatorname{cl}\operatorname{conv}(A+B)$. This yields

$$\begin{aligned}
\inf(\mathcal{A}+\mathcal{B}) &\overset{\text{Prop. 2.3}}{=} \operatorname{cl}\operatorname{conv} \bigcup_{C \in \mathcal{A}+\mathcal{B}} C = \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A},\, B \in \mathcal{B}} (A \oplus B) \\
&\leq \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A},\, B \in \mathcal{B}} (A+B) = \operatorname{cl}\operatorname{conv} \left(\bigcup_{A \in \mathcal{A}} A + \bigcup_{B \in \mathcal{B}} B\right) \\
&= \operatorname{cl}\left(\operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A}} A + \operatorname{cl}\operatorname{conv} \bigcup_{B \in \mathcal{B}} B\right) \overset{\text{Prop. 2.3}}{=} \inf \mathcal{A} \oplus \inf \mathcal{B}.
\end{aligned}$$

By (1) (or directly) we deduce equality. The second part is completely the same. $\square$

The following definitions of convex and concave functions are clear having in mind that both the standard relation $\supseteq$ in $\mathcal{F}^\star$ and $\subseteq$ in $\mathcal{F}^\diamond$ have the meaning of "less or equal". Let X be a linear space and $C \subseteq X$ convex. A function $f : C \to \hat{\mathcal{F}}^\star$ is said to be *convex* if

$$\forall \lambda \in [0,1], \forall x, u \in C: \quad f(\lambda \cdot x + (1-\lambda) \cdot u) \supseteq \lambda\, f(x) \oplus (1-\lambda)\, f(u). \tag{2}$$

However, if a function $f : C \to \hat{\mathcal{F}}^\diamond$ satisfies (2) it is said to be *concave*. Analogously, a function $f : C \to \hat{\mathcal{F}}^\star$ is *concave* and a function $f : C \to \hat{\mathcal{F}}^\diamond$ is *convex* if the following dual condition is satisfied:

$$\forall \lambda \in [0,1], \forall x, u \in C: \quad f(\lambda \cdot x + (1-\lambda) \cdot u) \subseteq \lambda\, f(x) \oplus (1-\lambda)\, f(u). \tag{3}$$

At the first glance, a convex function $f : C \to \hat{\mathcal{F}}^\star$ does not subsume the important case of an extended real–valued convex function. In Example 4.3 below, however, we will see that extended real–valued problems are equivalent to special set–valued problems.

Up to now, all the operations used did not influence the orientation of a set. We want to express the change of the orientation of a set as follows: Given an oriented set A we denote by $\boxplus A$ the same set, but with opposite orientation. As usual, the negative of a convex set A is defined as

$$-A := \{y \in Y | \ -y \in A\}.$$

By convention, if A is an oriented set, this operation does not manipulate the orientation of A. In contrast to this, we introduce a second concept of a negative of a convex set which does so. Given an oriented set A we define $\boxminus A$ being the set $-A$, but with the opposite orientation. Instead of two signs, we now have four signs, namely $+, -, \boxplus, \boxminus$. Obviously, the following assertions hold true:

$$\begin{aligned} A &= \boxplus \boxplus A = \boxminus \boxminus A, & -A &= \boxplus \boxminus A = \boxminus \boxplus A, \\ \boxplus A &= + \boxplus A = - \boxminus A, & \boxminus A &= + \boxminus A = - \boxplus A. \end{aligned}$$

Clearly, an expression is independent of the order of the signs. Note that $\boxplus$ and $\boxminus$ are signs but not operations. This means, we must be careful with adding elements $A \in \mathcal{F}^\star$ with elements $B \in \mathcal{F}^\diamond$, up to now such an operation is not defined (later it will be defined for special cases). Nevertheless, we write $A \boxplus B := A + (\boxplus B)$ and $A \boxminus B := A + (\boxminus B)$, if these expressions are defined, i.e. A and B have opposite orientation. If $\emptyset^\star$ is the largest element in $\hat{\mathcal{F}}^\star$ and $\emptyset^\diamond$ is the smallest element in $\mathcal{F}^\diamond$, then let us use the convention $\boxminus \emptyset^\diamond = \emptyset^\star$. Thus we obtain the well–known convexity–concavity dualism also for convex functions with values in $\hat{\mathcal{F}}$. A function $f : C \to \hat{\mathcal{F}}^\star$ is convex (concave) if and only if $\boxminus f : C \to \hat{\mathcal{F}}^\diamond$ is concave (convex). For a given set $\mathcal{A} \subseteq \hat{\mathcal{F}}^\star$ and using the notation $\boxminus \mathcal{A} := \{\boxminus A | \ A \in \mathcal{A}\}$, it can be easily shown (Proposition 2.3) that

$$\boxminus \inf \mathcal{A} = \sup \boxminus \mathcal{A} \quad \text{and} \quad \boxminus \sup \mathcal{A} = \inf \boxminus \mathcal{A}. \tag{4}$$

A further motivation for the usage of the sign $\boxminus$ will be given in the next section.

3 Embedding the space $\mathcal{F}$ into a set of linear spaces

Embedding of spaces of convex sets into linear spaces was investigated by Rådström [17]. For further details, also compare [2] and the references therein.

The aim of this section is to embed the convex cone $\mathcal{F}_K \subseteq \mathcal{F}$ into a partially ordered linear space. In dependence of the orientation of the members of $\mathcal{F}_K$ we use different embedding maps. This procedure allows us to re–interprete the inverse element of the embedding map's image of a member of $\mathcal{F}_K$ as an element of $\mathcal{F}_{-K}$ having opposite orientation.

The following lemma is a refinement of [1, Theorem 13.1]. It is shown that only the set $\operatorname{ri}(0^+A)^\circ = \operatorname{ri}((0^+A)^\circ)$ (instead of the whole space $\mathbb{R}^p$) is essential for the description of a nonempty closed convex set via its support function.

Lemma 3.1 *Let A be a nonempty closed convex subset of $\mathbb{R}^p$. Then*

$$A = \bigcap_{y^* \in \mathrm{ri}\,(0^+A)^\circ} \{y \in Y |\ \langle y^*, y\rangle \le \delta^*\,(y^* |\, A\,)\}\,.$$

Proof. As a consequence of [1, Theorem 14.2] we have $\mathrm{cl}\,\mathrm{dom}\,\delta^*(\,\cdot\,|\,A\,) = (0^+A)^\circ$, compare [7, Theorem 2.2.4], too. Together with [1, Theorem 6.3] this yields

$$\mathrm{ri}\,(0^+A)^\circ = \mathrm{ri}\,\mathrm{cl}\,\mathrm{dom}\,\delta^*(\,\cdot\,|\,A\,) \subseteq \mathrm{dom}\,\delta^*(\,\cdot\,|\,A\,) \subseteq (0^+A)^\circ. \tag{5}$$

By [1, Theorem 13.1] and (5) we obtain

$$A = \bigcap_{y^* \in \mathbb{R}^p} \big\{y \in Y |\ \langle y^*, y\rangle \le \delta^*\,(y^* |\, A\,)\big\} = \bigcap_{y^* \in (0^+A)^\circ} \big\{y \in Y |\ \langle y^*, y\rangle \le \delta^*\,(y^* |\, A\,)\big\}.$$

It remains to show

$$Y_1 := \bigcap_{y^* \in (0^+A)^\circ} \big\{y \in Y |\ \langle y^*, y\rangle \le \delta^*\,(y^* |\, A\,)\big\} = \bigcap_{y^* \in \mathrm{ri}\,(0^+A)^\circ} \big\{y \in Y |\ \langle y^*, y\rangle \le \delta^*\,(y^* |\, A\,)\big\} =: Y_2.$$

The inclusion $Y_1 \subseteq Y_2$ is obvious. In order to show $Y_2 \subseteq Y_1$ let $y \in Y_2$ be arbitrarily chosen. It holds $\langle y^*, y\rangle \le \delta^*\,(y^* |\, A\,)$ for all $y^* \in \mathrm{ri}\,(0^+A)^\circ$. Let $\bar{y}^* \in (0^+A)^\circ$ and $y^* \in \mathrm{ri}\,(0^+A)^\circ$, then $\lambda\bar{y}^* + (1-\lambda)y^* \in \mathrm{ri}\,(0^+A)^\circ$ for all $\lambda \in [0,1)$ (compare [1, Theorem 6.1]). It follows

$$\langle \lambda\bar{y}^* + (1-\lambda)y^*, y\rangle \le \delta^*\,(\lambda\bar{y}^* + (1-\lambda)y^* |\, A\,) \le \lambda\,\delta^*\,(\bar{y}^* |\, A\,) + (1-\lambda)\delta^*\,(y^* |\, A\,)\,.$$

By virtue of (5) we deduce that $\delta^*\,(y^* |\, A\,) < +\infty$. Letting $\lambda \to 1$ we obtain $\langle \bar{y}^*, y\rangle \le \delta^*\,(\bar{y}^* |\, A\,)$, i.e. $y \in Y_1$. □

With aid of the preceeding lemma we are able to give an equivalent characterization of the partially ordered conlinear spaces $\mathcal{F}_K^\star$ and $\mathcal{F}_K^\diamond$ (the axioms (S1) – (S8) are satisfied, if we replace $\{0\}$ by K, i.e. K is the neutral element). Let $\Gamma_K^\star$ be the space of all positively homogeneous concave functions from $\mathrm{ri}\,K^\circ$ into $\mathbb{R}$ and let $\Gamma_K^\diamond$ be the space of all positively homogeneous convex functions from $\mathrm{ri}\,K^\circ$ into $\mathbb{R}$. The spaces $\Gamma_K^\star$ and $\Gamma_K^\diamond$ are conlinear spaces, with respect to the addition and multiplication with nonnegative real numbers, defined pointwise using the corresponding operation in $\mathbb{R}$. Moreover, $\Gamma_K^\star$ and $\Gamma_K^\diamond$, equipped with the ordering relation $\le$ defined pointwise using the usual $\le$ relation in $\mathbb{R}$, are partially ordered conlinear spaces.

Theorem 3.2 *Let $Y = Y^* = \mathbb{R}^p$ and let $K \subseteq Y$ be a nonempty closed convex cone. Then the following assertions hold true:*

(i) *There exists a bijective map $j^\star : \mathcal{F}_K^\star \to \Gamma_K^\star$ such that for all $A, B \in \mathcal{F}_K^\star$ and all positive real numbers $\alpha > 0$:*

(a) $j^\star(A+B) = j^\star(A) + j^\star(B)$, (b) $j^\star(\alpha A) = \alpha j^\star(A)$,

(c) $j^\star(K) = 0_{\Gamma_K^\star}$, (d) $A \supseteq B \;\Leftrightarrow\; j^\star(A) \le j^\star(B)$.

(ii) *There exists a bijective map* $j^\diamond : \mathcal{F}^\diamond_{-K} \to \Gamma^\diamond_K$ *such that for all* $A, B \in \mathcal{F}^\diamond_{-K}$ *and all positive real numbers* $\alpha > 0$*:*

(a) $j^\diamond(A + B) = j^\diamond(A) + j^\diamond(B)$, (b) $j^\diamond(\alpha A) = \alpha j^\diamond(A)$,

(c) $j^\diamond(-K) = 0_{\Gamma^\diamond_K}$, (d) $A \subseteq B \;\Leftrightarrow\; j^\diamond(A) \leq j^\diamond(B)$.

Proof. (i) Consider the map $j^\star$ which assigns every $A \in \mathcal{F}^\star_K$ the negative support function of the set $A \subseteq Y$, restricted to the set $\operatorname{ri} K^\circ \subseteq Y^*$. More precisely $\gamma_A = j^\star(A)$ is defined as $\gamma_A : \operatorname{ri} K^\circ \to \mathbb{R} \cup \{\pm\infty\}$, $\gamma_A(y^*) := -\delta^*(y^* |\, A\,)$. Of course, $j^\star$ is a function. Moreover, $j^\star$ is a function from $\mathcal{F}^\star_K$ into $\Gamma^\star_K$. Indeed, let $A \in \mathcal{F}^\star_K$. Since A is nonempty, we have $\delta^*(y^* |\, A\,) > -\infty$ for all $y^* \in Y^*$. With aid of (5) we obtain $\delta^*(y^* |\, A\,) < +\infty$ for all $y^* \in \operatorname{ri} K^\circ$. Hence $\gamma_A = j^\star(A)$ only attains values in $\mathbb{R}$. Since support functions are sublinear and $\operatorname{ri} K^\circ$ is a convex cone, $\gamma_A = j^\star(A)$ is positively homogeneous and concave on $\operatorname{ri} K^\circ$. Hence $j^\star(A) \in \Gamma^\star_K$.

$j^\star : \mathcal{F}^\star_K \to \Gamma^\star_K$ is injective. Indeed, let $A, B \in \mathcal{F}^\star_K$ such that $j^\star(A) = j^\star(B)$. Note that A and B are nonempty, closed, convex and $0^+A = 0^+B = K$. Lemma 3.1 yields $A = B$.

$j^\star : \mathcal{F}^\star_K \to \Gamma^\star_K$ is surjective. Indeed, given an element $\gamma \in \Gamma^\star_K$. Define $d : Y^* \to \mathbb{R} \cup \{+\infty\}$ as

$$d(y^*) := \begin{cases} -\gamma(y^*) & \text{if} \quad y^* \in \operatorname{ri} K^\circ \\ +\infty & \text{else.} \end{cases}$$

Then d is convex, positively homogeneous, not identically $+\infty$ and $d > -\infty$. With aid of [1, Corollary 13.2.1.] we conclude $\operatorname{cl} d$ is the support function of the nonempty closed convex set

$$A_\gamma := \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in Y |\; \langle y^*, y\rangle \leq d(y^*)\} = \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in Y |\; \langle y^*, y\rangle \leq -\gamma(y^*)\}\,.$$

Applying [1, Corollary 8.3.3.], taking into account the considerations in [1, page 62] and applying Lemma 3.1 it follows

$$0^+A_\gamma = \bigcap_{y^* \in \operatorname{ri} K^\circ} 0^+ \{y \in Y |\; \langle y^*, y\rangle \leq d(y^*)\} = \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in Y |\; \langle y^*, y\rangle \leq 0\} = K.$$

By definition, we have $j^\star(A_\gamma)(y^*) = -\operatorname{cl} d(y^*)$ for all $y^* \in \operatorname{ri} K^\circ$. With aid of [1, Theorem 7.4.] we have $\operatorname{cl} d(y^*) = d(y^*)$ for all $y^* \in \operatorname{ri} K^\circ$. Hence $j^\star(A_\gamma) = \gamma$.

(i)(a) and (i)(b) follow from elementary properties of the supremum in $\mathbb{R}$, compare [1, page 113], too. (i)(c) follows from the definition of the polar cone and of the support function. (i)(d) is a consequence of [1, Corollary 13.1.1].

(ii) Define $j^\diamond(A) := -j^\star(\boxminus A)$ and use (i). $\square$

We now extend the space $\mathcal{F}^\star_K$ by a largest (supremum oriented) element $\emptyset$ in the same way as described in Section 2. The resulting space is denoted by $\hat{\mathcal{F}}^\star_K$. We also extend the space $\Gamma^\star_K$ by a largest element $+\infty$, which is defined to be a function from $\operatorname{ri} K^\circ$ into $\mathbb{R} \cup \{+\infty\}$ which is identically $+\infty$. Analogously, we define the spaces $\hat{\mathcal{F}}^\diamond_{-K}$ and $\hat{\Gamma}^\diamond_K$ using smallest elements $\emptyset$ (infimum oriented) and $-\infty$, respectively.

Corollary 3.3 *Theorem 3.2 remains valid if $\mathcal{F}^\star_K$, $\mathcal{F}^\diamond_{-K}$, $\Gamma^\star_K$ and $\Gamma^\diamond_K$ are replaced by $\hat{\mathcal{F}}^\star_K$, $\hat{\mathcal{F}}^\diamond_{-K}$, $\hat{\Gamma}^\star_K$ and $\hat{\Gamma}^\diamond_K$, respectively.*

Proof. Extend the isomorphisms $j^\star$ and $j^\diamond$ of Theorem 3.2 using the conventions $j^\star(\emptyset) = +\infty$ and $j^\diamond(\emptyset) = -\infty$, respectively. $\square$

The following proposition collects some simple auxiliary assertions.

Proposition 3.4 *Let $A, B, K \subseteq \mathbb{R}^p$ be nonempty closed convex sets and let K additionally be a cone. Then the following statements hold true:*

(i) *$A \subseteq B$ implies $0^+A \subseteq 0^+B$.*

(ii) *$0^+(A \oplus B \oplus K) = K$ implies $0^+(A \oplus K) = 0^+(B \oplus K) = K$.*

(iii) *If $\mathcal{A} \subseteq \mathcal{F}^\star$ is nonempty and $K := 0^+(\inf \mathcal{A})$, then $0^+(A \oplus K) = K$ for all $A \in \mathcal{A}$.*

Proof. (i) Let $A \subseteq B$. [1, Corollary 8.3.3.] yields $0^+A = 0^+(A \cap B) = 0^+A \cap 0^+B \subseteq 0^+B$. (ii) Let $a \in A$ and $b \in B$. Then we have $\{a\} + \{b\} + K \subseteq \{a\} + B \oplus K \subseteq A \oplus B \oplus K$. By (i) it follows $K \subseteq 0^+(B \oplus K) \subseteq K$. Hence $0^+(B \oplus K) = K$ and, analogously, $0^+(A \oplus K) = K$. (iii) Let $a \in A \in \mathcal{A}$. By [1, Theorem 8.1] we have $\{a\} + K \subseteq A \oplus K \subseteq (\inf \mathcal{A}) \oplus K = \inf \mathcal{A}$. By (i) it follows $K \subseteq 0^+(A \oplus K) \subseteq K$. $\square$

Corollary 3.5 *Let $\mathcal{A} \subseteq \mathcal{F}^\star$ be nonempty with $K := 0^+(\inf \mathcal{A})$ and let $j^\star : \mathcal{F}^\star_K \to \Gamma^\star_K$ the isomorphism of Theorem 3.2. Then it holds*

$$j^\star\big(\inf_{A \in \mathcal{A}} (A \oplus K)\big) = \inf_{A \in \mathcal{A}} j^\star\,(A \oplus K)\,.$$

Proof. Let $\bar{A} \in \mathcal{A}$ be arbitrarily given. With Proposition 3.4 (iii) we obtain $0^+(\bar{A} \oplus K) = K$, i.e. $\bar{A} \oplus K \in \mathcal{F}^\star_K$. With aid of Theorem 3.2 (i)(d) we get

$$\begin{aligned}
& \forall \bar{A} \in \mathcal{A}: \ \inf_{A \in \mathcal{A}} (A \oplus K) \supseteq \bar{A} \oplus K \\
\Rightarrow\ & \forall \bar{A} \in \mathcal{A}: \ j^\star\big(\inf_{A \in \mathcal{A}} (A \oplus K)\big) \leq j^\star\big(\bar{A} \oplus K\big) \\
\Rightarrow\ & j^\star\big(\inf_{A \in \mathcal{A}} (A \oplus K)\big) \leq \inf_{A \in \mathcal{A}} j^\star\big(A \oplus K\big).
\end{aligned}$$

On the other hand, Theorem 3.2 yields

$$\begin{aligned}
& \forall \bar{A} \in \mathcal{A}: \ \inf_{A \in \mathcal{A}} j^\star(A \oplus K) \leq j^\star(\bar{A} \oplus K) \\
\Rightarrow\ & \forall \bar{A} \in \mathcal{A}: \ (j^\star)^{-1} \inf_{A \in \mathcal{A}} j^\star(A \oplus K) \supseteq \bar{A} \oplus K \\
\Rightarrow\ & (j^\star)^{-1} \inf_{A \in \mathcal{A}} j^\star(A \oplus K) \supseteq \inf_{A \in \mathcal{A}} (A \oplus K) \\
\Rightarrow\ & \inf_{A \in \mathcal{A}} j^\star(A \oplus K) \leq j^\star\big(\inf_{A \in \mathcal{A}} (A \oplus K)\big).
\end{aligned}$$

$\square$

Let Γ_K be the space of all positively homogeneous (and not necessarily convex or concave) functions $\gamma : \operatorname{ri} K^\circ \to \mathbb{R}$. Let Γ_K be equipped with an addition and a scalar multiplication, defined pointwise using the corresponding operation in $\mathbb{R}$, and with an ordering relation $\leq$, defined pointwise using the usual $\leq$ relation in $\mathbb{R}$. Then the space Γ_K is a partially ordered linear space. Theorem 3.2 yields that $(\mathcal{F}^\star_K, \supseteq)$ and $(\mathcal{F}^\diamond_{-K}, \subseteq)$ are isomorphic to convex cones in the partially ordered linear space $(\Gamma_K, \leq)$. Let $j^\star : \mathcal{F}^\star_K \to \Gamma_K$ be the injective homomorphism which embeds $\mathcal{F}^\star_K$ into Γ_K and let $j^\diamond : \mathcal{F}^\diamond_{-K} \to \Gamma_K$ be the injective homomorphism which embeds $\mathcal{F}^\diamond_{-K}$ into Γ_K. Then it easily follows that

$$\forall A \in \mathcal{F}^\star_K : \;\; j^\star(A) + j^\diamond(\boxminus A) = 0, \qquad \forall A \in \mathcal{F}^\diamond_{-K} : \;\; j^\diamond(A) + j^\star(\boxminus A) = 0. \tag{6}$$

In this sense $\boxminus A$ can be regarded as the "inverse element" of a nonempty closed convex set A. However, this does not imply that $\mathcal{F}^\star_K \cup \mathcal{F}^\diamond_{-K}$ is a linear space, because it is not a conlinear space.

The next Proposition tells us that the supremum and infimum in Γ_K coincide, respectively, with the supremum and infimum taken pointwise.

Proposition 3.6 *The space $(\Gamma_K, \leq)$ is Dedekind complete. For $\emptyset \neq \mathcal{A} \subseteq (\Gamma_K, \leq)$ it holds*

$$\begin{aligned} &\mathcal{A} \textit{ bounded above} &\Rightarrow\;\; &\forall y^* \in \operatorname{ri} K^\circ : \;\; \big(\sup_{\gamma \in \mathcal{A}} \gamma\big)(y^*) = \sup_{\gamma \in \mathcal{A}} \big(\gamma(y^*)\big), \\ &\mathcal{A} \textit{ bounded below} &\Rightarrow\;\; &\forall y^* \in \operatorname{ri} K^\circ : \;\; \big(\inf_{\gamma \in \mathcal{A}} \gamma\big)(y^*) = \inf_{\gamma \in \mathcal{A}} \big(\gamma(y^*)\big). \end{aligned}$$

Proof. The proof is immediate. □

We introduce the following notation: Let $\mathcal{A} \subseteq (\Gamma^\star_K, \leq)$. Clearly, it follows $\mathcal{A} \subseteq (\Gamma_K, \leq)$. We write $\inf^\star \mathcal{A}$ and $\sup^\star \mathcal{A}$ if $\mathcal{A}$ is considered to be subset of $(\Gamma^\star_K, \leq)$ and we write $\inf \mathcal{A}$ and $\sup \mathcal{A}$ if $\mathcal{A}$ is considered to be subset of $(\Gamma_K, \leq)$. Analogously, we define $\inf^\diamond \mathcal{A}$ and $\sup^\diamond \mathcal{A}$ for a subset $\mathcal{A} \subseteq (\Gamma^\diamond_K, \leq) \subseteq (\Gamma_K, \leq)$. Theorem 2.2 yields the following implications:

$$\emptyset \neq \mathcal{A} \subseteq (\Gamma^\star_K, \leq) \text{ bounded below} \quad \Rightarrow \quad \overset{\star}{\inf}\, \mathcal{A} \leq \inf \mathcal{A}, \tag{7}$$

$$\emptyset \neq \mathcal{A} \subseteq (\Gamma^\diamond_K, \leq) \text{ bounded below} \quad \Rightarrow \quad \overset{\diamond}{\inf}\, \mathcal{A} \leq \inf \mathcal{A}, \tag{8}$$

$$\emptyset \neq \mathcal{A} \subseteq (\Gamma^\star_K, \leq) \text{ bounded above} \quad \Rightarrow \quad \sup \mathcal{A} \leq \overset{\star}{\sup}\, \mathcal{A}, \tag{9}$$

$$\emptyset \neq \mathcal{A} \subseteq (\Gamma^\diamond_K, \leq) \text{ bounded above} \quad \Rightarrow \quad \sup \mathcal{A} \leq \overset{\diamond}{\sup}\, \mathcal{A}. \tag{10}$$

The following proposition shows (7) and (10) are even satisfied with equality.

Proposition 3.7 *If $\mathcal{A} \subseteq (\Gamma^\star_K, \leq)$ is nonempty and bounded below and $\mathcal{B} \subseteq (\Gamma^\diamond_K, \leq)$ is nonempty and bounded above, then $\inf^\star \mathcal{A} = \inf \mathcal{A}$ and $\sup^\diamond \mathcal{B} = \sup \mathcal{B}$, respectively.*

Proof. Taking into account Theorem 2.2 it remains to show that $\inf \mathcal{A} \in \Gamma^\star_K$ and $\sup \mathcal{B} \in \Gamma^\diamond_K$, respectively. This is true because the pointwise infimum (supremum) over a set of concave (convex) functions is concave (convex). □

The following example shows that, in general, (8) is not satisfied with equality.

Example 3.8 Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $\mathcal{A} \subseteq \Gamma^\star_K$, $\mathcal{A} = \{\gamma_1, \gamma_2\}$, where

$$\forall y^* \in \operatorname{int} \mathbb{R}^2_- :\ \gamma_1(y^*) = -\left\langle y^*, \{(0,1)^T\}\right\rangle = -y_2^*, \qquad \gamma_2(y^*) = -\left\langle y^*, \{(1,0)^T\}\right\rangle = -y_1^*.$$

Then, we have $\sup \mathcal{A} = \bar{\gamma}$, where $\bar{\gamma} : \operatorname{int} \mathbb{R}^2_- \to \mathbb{R}$ is given by $\bar{\gamma}(y^*) = \max_{i=1,2} \{\gamma_i(y^*)\} = \max\{-y_2^*, -y_1^*\}$. However, we have $\sup^\star \mathcal{A} = \hat{\gamma}$, where $\hat{\gamma} : \operatorname{int} \mathbb{R}^2_- \to \mathbb{R}$ is given by

$$\hat{\gamma}(y^*) = -\delta^*\left(y^* \middle| \left(\{(0,1)^T\} + \mathbb{R}^2_+\right) \cap \left(\{(1,0)^T\} + \mathbb{R}^2_+\right)\right) = -\left\langle y^*, \{(1,1)^T\}\right\rangle = -(y_1^* + y_2^*).$$

Thus, $\sup \mathcal{A} \neq \sup^\star \mathcal{A}$.

An analogous example could be given to show that, in general, (9) does not hold with equality. We close this section with some auxiliary assertions.

Lemma 3.9 *Let $Y = \mathbb{R}^p$ and let $K \subseteq Y$ be a nonempty closed convex cone which is not a linear subspace of $\mathbb{R}^p$. Then it holds:*

$$(y \in \operatorname{ri} K \wedge y^* \in \operatorname{ri} K^\circ) \ \Rightarrow\ \langle y, y^* \rangle < 0.$$

Proof. Assume the contrary. Taking into account the definition of the polar cone this means there exists $\bar{y} \in \operatorname{ri} K$ and $\bar{y}^* \in \operatorname{ri} K^\circ$ such that $\langle \bar{y}, \bar{y}^* \rangle = 0$. It follows

$$\forall y \in K :\ \langle y, \bar{y}^* \rangle = 0. \tag{11}$$

Assuming that (11) is not true, there is some $\tilde{y} \in K$ such that $\langle \tilde{y}, \bar{y}^* \rangle < 0$. Since $\bar{y} \in \operatorname{ri} K$ there exists some $\mu > 1$ such that $\hat{y} := \mu\bar{y} + (1-\mu)\tilde{y} \in K$. Hence $\langle \hat{y}, \bar{y}^* \rangle > 0$ which contradicts $\bar{y}^* \in K^\circ$.

With aid of (11) we obtain $-\bar{y}^* \in K^\circ$. Since K° is a convex cone and $\bar{y}^* \in K^\circ \cap (-K^\circ)$ we have $K^\circ = K^\circ + \{-\bar{y}^*\}$. With $\bar{y}^* \in \operatorname{ri} K^\circ$ we conclude $0 \in \operatorname{ri} K^\circ + \{-\bar{y}^*\} = \operatorname{ri}(K^\circ + \{-\bar{y}^*\}) = \operatorname{ri} K^\circ$. This implies $\operatorname{lin} K^\circ = \operatorname{aff} K^\circ = K^\circ$, i.e. K° is a linear subspace of $\mathbb{R}^p$. From this we deduce that the bipolar cone $K^{\circ\circ}$ is also a linear subspace of $\mathbb{R}^p$. The bipolar theorem [1, Theorem 14.1] yields $K = K^{\circ\circ}$, i.e. K a is linear subspace of $\mathbb{R}^p$, a contradiction. $\square$

Proposition 3.10 *Let $X = \mathbb{R}^n$ and let $f : X \to \hat{\mathcal{F}}^\star$ be convex. If $x_0 \in \operatorname{ri} \operatorname{dom} f$, then $0^+ f(x) \subseteq 0^+ f(x_0)$ for all $x \in \operatorname{dom} f$.*

Proof. Note that $\operatorname{dom} f$ is convex. Let $x \in \operatorname{dom} f$ be arbitrarily given and, by hypothesis, $x_0 \in \operatorname{ri} \operatorname{dom} f$. By [1, Theorem 6.4], there exists $\mu > 1$ such that $\bar{x} := \mu x_0 + (1-\mu)x \in \operatorname{dom} f$. Set $\lambda := 1/\mu \in (0,1)$. The convexity of f yields $f(x_0) \supseteq \lambda f(x) \oplus (1-\lambda) f(\bar{x})$. Since $\bar{x} \in \operatorname{dom} f$ we can choose some $\bar{y} \in f(\bar{x})$, hence $f(x_0) \supseteq \lambda f(x) + (1-\lambda)\{\bar{y}\} := C$. By hypothesis, $f(x) \subseteq \mathbb{R}^p$ is a nonempty closed convex set for each $x \in \operatorname{dom} f$. Therefore Proposition 3.4 (i) yields $0^+ C \subseteq 0^+ f(x_0)$. With aid of [1, Theorem 8.1] we conclude that $0^+ C = 0^+ f(x)$, hence $0^+ f(x) \subseteq 0^+ f(x_0)$. $\square$

In this form, the preceeding proposition will be used in the proof of the duality theorem. Moreover, as an easy consequence (see [12] for analogous assertions) we obtain that $x \longmapsto 0^+ f(x)$ must be constant on $\mathrm{ri}\,\mathrm{dom}\, f$. In view of Theorem 3.2, this means that, with respect to the set $\mathrm{ri}\,\mathrm{dom}\, f$, the values of a convex function $f : X \to \hat{\mathcal{F}}^\star$ can be embedded into a linear space.

4 Conjugate Duality

Let $X = X^* = \mathbb{R}^n$, $U = U^* = \mathbb{R}^m$, $Y = \mathbb{R}^p$ and let $f : X \to \hat{\mathcal{F}}^\star$ be a function. Recall that both the standard relation $\supseteq$ in $\hat{\mathcal{F}}^\star$ the standard relation $\subseteq$ in $\hat{\mathcal{F}}^\diamond$ have the meaning of "less or equal". Therefore we will write $\leq$ instead of $\supseteq$ and $\subseteq$ in order to emphasize the analogy of our statements to the well–known scalar case. So, the interpretation of $\leq$ depends on the orientation of the sets being compared.

Let $c \in Y$ and let $\{c\}$ be infimum oriented. A function $f_c^* : X^* \to \hat{\mathcal{F}}^\diamond$, defined by

$$f_c^*(x^*) := \sup_{x \in X} \left\{ \langle x^*, x \rangle \cdot \{c\} \boxminus f(x) \right\}$$

is said to be the *conjugate of* f with respect to c. From (1) we conclude that f_c^* is convex (even if f is not). As an easy consequence of the definition of f_c^* we obtain the Fenchel–Young inequality

$$\forall x \in X,\; x^* \in X^*,\; c \in Y : \;\; f_c^*(x^*) \geq \langle x^*, x \rangle \cdot \{c\} \boxminus f(x). \tag{12}$$

For functions $f : X \to \hat{\mathcal{F}}^\star$ and $h : X \to \hat{\mathcal{F}}^\star$ it is evident that

$$\big(\forall x \in X : \; f(x) \leq h(x)\big) \quad \Rightarrow \quad \big(\forall x^* \in X^*, \forall c \in Y : \; f_c^*(x^*) \geq h_c^*(x^*)\big). \tag{13}$$

It follows the main result of this paper, a duality theorem for functions with values in the space of closed convex subsets of $Y = \mathbb{R}^p$.

Theorem 4.1 (Duality theorem) *For given functions $f : \; X \to \hat{\mathcal{F}}^\star$ and $g : \; U \to \hat{\mathcal{F}}^\star$, a linear map $A : \; X \to U$ and a vector $c \in Y$, let*

$$p : X \to \hat{\mathcal{F}}^\star \qquad \text{and} \qquad d_c : U^* \to \hat{\mathcal{F}}^\star$$

be defined, respectively, by

$$p(x) = f(x) \oplus g(Ax) \qquad \text{and} \qquad d_c(u^*) = \boxminus\big(f_c^*(A^T u^*) \oplus g_c^*(-u^*)\big).$$

These functions satisfy the weak duality inequality

$$D_c := \sup_{u^* \in U^*} d_c(u^*) \leq \inf_{x \in X} p(x) =: P. \tag{14}$$

Furthermore, let f and g be convex, let

$$0 \in \mathrm{ri}\,(\mathrm{dom}\, g - A \;\mathrm{dom}\, f) \tag{15}$$

and, in dependence of $K := 0^+ P$, let the element $c \in Y$ be chosen as follows:

(i) $c \in \operatorname{ri} K$, *if* $K \subsetneq Y$ *is not a linear subspace of* Y *or* $K = Y$,

(ii) $c \in Y \setminus K$, *if* $K \subsetneq Y$ *is a linear subspace of* Y,

Then, we have strong duality, i.e. $D_c = P$.

Proof. Let $x \in X$, $u^* \in U^*$ and $c \in Y$ be arbitrarily given and set $u := Ax$. With aid of the Fenchel–Young inequality (12) we obtain the weak duality inequality (14) as follows:

$$\begin{aligned} d_c(u^*) &= \boxminus\big(f_c^*(A^T u^*) \oplus g_c^*(-u^*)\big) \\ &\leq \big(f(x) \boxminus \langle A^T u^*, x\rangle \cdot \{c\}\big) \oplus \big(g(Ax) \boxminus \langle -u^*, Ax\rangle \cdot \{c\}\big) \\ &= \big(f(x) \oplus g(Ax)\big) \boxminus \langle u^*, Ax\rangle \cdot \{c\} \boxplus \langle u^*, Ax\rangle \cdot \{c\} \;=\; p(x). \end{aligned}$$

The proof of the strong duality assertion is organized as follows. We start with case (i). Then we show that case (ii) is a consequence of case (i).

(i) In case of $K = Y$ there is nothing to prove because the strong duality immediately follows from the weak duality assertion. Therefore, let $K \subsetneq Y$ be not a linear subspace of Y and let $c \in \operatorname{ri} K$. Proposition 2.5 yields

$$P = P + 0^+ P = P + K = \inf_{x \in X} p(x) + K = \inf_{x \in X} (p(x) \oplus K).$$

By Proposition 3.4 we know that $f(x) \oplus K \in \mathcal{F}_K^\star$ for all $x \in \operatorname{dom} f$. The analogous assertion for g is more complicated. With the same arguments we can only conclude that $g(u) \oplus K \in \mathcal{F}_K^\star$ for all $u \in \operatorname{dom} g \cap \operatorname{rg} A$. With aid of (15) and [1, Theorem 6.6] we obtain the existence of some $u_0 \in \operatorname{ri} \operatorname{dom} g \cap \operatorname{ri}(A \operatorname{dom} f) \subseteq \operatorname{ri} \operatorname{dom} g \cap \operatorname{rg} A$, this means $u_0 \in \operatorname{ri} \operatorname{dom} g$ and $0^+(g(u_0) \oplus K) = K$. Proposition 3.10 yields that $0^+(g(u) \oplus K) \subseteq K$ for all $u \in \operatorname{dom} g$. Hence $g(u) \oplus K \in \mathcal{F}_K^\star$ for all $u \in \operatorname{dom} g$. Define the functions $\bar f : X \to \hat{\mathcal{F}}_K^\star$ and $\bar g : U \to \hat{\mathcal{F}}_K^\star$ by $\bar f := f \oplus K$ and $\bar g := g \oplus K$, respectively. By Corollary 3.5 and Corollary 3.3 we obtain

$$j^\star(P) = j^\star\big(\inf_{x \in X} \big(p(x) \oplus K\big)\big) = \inf_{x \in X} j^\star\big(p(x) \oplus K\big) = \inf_{x \in X} \left\{ j^\star \circ \bar f(x) + j^\star \circ \bar g(Ax) \right\}.$$

Corollary 3.3 yields that $(j^\star \circ \bar f) : X \to \hat\Gamma_K^\star$ and $(j^\star \circ \bar g) : U \to \hat\Gamma_K^\star$ are convex. In Proposition 3.7 it is shown that the infimum of a bounded subset of $\mathcal{F}_K^\star$ does not change while embedding $\mathcal{F}_K^\star$ into the linear space Γ_K. As shown in Proposition 3.6 the infimum of a subset of Γ_K can be calculated pointwise for all $y^* \in \operatorname{ri} K^\circ$. Hence, we can write

$$\forall y^* \in \operatorname{ri} K^\circ : \quad j^\star\big(P\big)(y^*) = \inf_{x \in X} \left\{ \big(\bar f_{y^*}(x)\big) + \big(\bar g_{y^*}(Ax)\big) \right\}, \tag{16}$$

where the extended real–valued functions $\bar f_{y^*} : X \to \mathbb{R} \cup \{+\infty\}$, $\bar f_{y^*}(x) := j^\star\big(\bar f(x)\big)(y^*)$ and $\bar g_{y^*} : U \to \mathbb{R} \cup \{+\infty\}$, $\bar g_{y^*}(u) := j^\star\big(\bar g(u)\big)(y^*)$ are convex for all $y^* \in \operatorname{ri} K^\circ$. Clearly, for all $y^* \in \operatorname{ri} K^\circ$ it holds $\operatorname{dom} f = \operatorname{dom} \bar f_{y^*}$ and $\operatorname{dom} g = \operatorname{dom} \bar g_{y^*}$. Hence, (15) implies that $0 \in \operatorname{ri}(\operatorname{dom} \bar g_{y^*} - A \operatorname{dom} \bar f_{y^*})$ for all $y^* \in \operatorname{ri} K^\circ$. A scalar duality theorem, for instance [3, Theorem 3.3.5], now yields that

$$\forall y^* \in \operatorname{ri} K^\circ : \quad j^\star\big(P\big)(y^*) = \sup_{u^* \in U^*} \left\{ -\bar f_{y^*}^*(A^T u^*) - \bar g_{y^*}^*(-u^*) \right\}. \tag{17}$$

By hypothesis, we have given $c \in \operatorname{ri} K$, where the set $\{c\}$ is understood to be infimum oriented. Hence $\boxplus\{c\} + K \in \mathcal{F}_K^\star$. Let $y^* \in \operatorname{ri} K^\circ$. Since K is not a linear space, Lemma 3.9 yields that $\langle y^*, c\rangle < 0$. Hence there exists $\alpha_{y^*} > 0$ such that $\langle \alpha_{y^*} y^*, c\rangle = -1$. By the definition of $j^\star$ in the proof of Theorem 3.2 this can be rewritten as

$$\forall t \in \mathbb{R}: \quad j^\star\big(\boxplus\{t\cdot c\} + K\big)(\alpha_{y^*}y^*) = -\langle \alpha_{y^*}y^*, t\cdot c\rangle - \sup_{y\in K}\langle \alpha_{y^*}y^*, y\rangle = t \tag{18}$$

For all $y^* \in \operatorname{ri} K^\circ$ and $\alpha := \alpha_{y^*} > 0$ we have

$$\begin{aligned}
\alpha\, j^\star\big(P\big)(y^*) &\overset{\text{Def. }\Gamma_K^\star}{=} j^\star\big(P\big)(\alpha y^*) \overset{(17)}{=} \sup_{u^*\in U^*}\left\{-\bar f^*_{\alpha y^*}(A^T u^*) - \bar g^*_{\alpha y^*}(-u^*)\right\} \\
&\overset{\text{Def. Conjugate}}{=} \sup_{u^*\in U^*}\left\{\inf_{x\in X}\left\{-\langle A^T u^*, x\rangle + \bar f_{\alpha y^*}(x)\right\} + \inf_{u\in U}\left\{\langle u^*, u\rangle + \bar g_{\alpha y^*}(u)\right\}\right\} \\
&\overset{(18)}{=} \sup_{u^*\in U^*}\left\{\inf_{x\in X}\left\{j^\star\big(\boxplus\{-\langle A^T u^*, x\rangle\cdot c\} + K\big)(\alpha y^*) + j^\star \bar f(x)(\alpha y^*)\right\}\right. \\
&\qquad\qquad \left. + \inf_{u\in U}\left\{j^\star\big(\boxplus\{\langle u^*, u\rangle\cdot c\} + K\big)(\alpha y^*) + j^\star \bar g(u)(\alpha y^*)\right\}\right\} \\
&\overset{\text{Cor. 3.3, Pr. 3.6,3.7}}{=} \sup_{u^*\in U^*}\left\{j^\star\left(\inf_{x\in X}\left\{\boxminus\langle A^T u^*, x\rangle\{c\} + \bar f(x)\right\}\right)(\alpha y^*)\right. \\
&\qquad\qquad \left. + j^\star\left(\inf_{u\in U}\left\{\boxminus\langle -u^*, u\rangle\{c\} + \bar g(u)\right\}\right)(\alpha y^*)\right\} \\
&\overset{(4)}{=} \sup_{u^*\in U^*}\left\{j^\star\left(\boxminus\sup_{x\in X}\left\{\langle A^T u^*, x\rangle\{c\} \boxminus \bar f(x)\right\}\right)(\alpha y^*)\right. \\
&\qquad\qquad \left. + j^\star\left(\boxminus\sup_{u\in U}\left\{\langle -u^*, u\rangle\{c\} \boxminus \bar g(u)\right\}\right)(\alpha y^*)\right\} \\
&\overset{\text{Cor. 3.3, Pr. 3.6, (9)}}{\leq} j^\star\left(\sup_{u^*\in U^*}\left\{\boxminus\sup_{x\in X}\left\{\langle A^T u^*, x\rangle\{c\} \boxminus \bar f(x)\right\}\right.\right. \\
&\qquad\qquad \left.\left. \oplus \boxminus\sup_{u\in U}\left\{\langle -u^*, u\rangle\{c\} \boxminus \bar g(u)\right\}\right\}\right)(\alpha y^*) \\
&\overset{\text{Def. Conjugate}}{=} j^\star\left(\sup_{u^*\in U^*}\left\{\boxminus \bar f^*_c(A^T u^*) \oplus \boxminus \bar g^*_c(-u^*)\right\}\right)(\alpha y^*) \\
&\overset{\text{Def. }\Gamma_K^\star}{=} \alpha\, j^\star\left(\sup_{u^*\in U^*}\left\{\boxminus \bar f^*_c(A^T u^*) \oplus \boxminus \bar g^*_c(-u^*)\right\}\right)(y^*).
\end{aligned}$$

Since $y^* \in \operatorname{ri} K^\circ$ was arbitrarily chosen and $\alpha > 0$ in the last quantity, Theorem 3.2 implies

$$P \leq \sup_{u^*\in U^*}\left\{\boxminus \bar f^*_c(A^T u^*) \oplus \boxminus \bar g^*_c(-u^*)\right\}.$$

Note that K is a nonempty closed convex cone, hence $0 \in K$. In case of supremum orientation, this means $K \leq \{0\}$. It follows that $\bar f(x) \leq f(x)$ for all $x \in X$ and $\bar g(u) \leq g(u)$ for all $u \in U$. By (13), we deduce that $\bar f^*_c(x^*) \geq f^*_c(x^*)$ for all $x^* \in X^*$ and $c \in Y$ and $\bar g^*_c(u^*) \geq g^*_c(u^*)$ for all $u^* \in U^*$ and $c \in Y$. Hence, for all $u^* \in U^*$ we have

$$\boxminus \bar f^*_c(A^T u^*) \oplus \boxminus \bar g^*_c(-u^*) \leq \boxminus f^*_c(A^T u^*) \oplus \boxminus g^*_c(-u^*).$$

Taking the supremum over all $u^* \in U^*$ we obtain

$$P \le \sup_{u^* \in U^*} \left\{ \boxminus \bar{f}_c^*(A^T u^*) \oplus \boxminus \bar{g}_c^*(-u^*) \right\} \le \sup_{u^* \in U^*} \left\{ \boxminus f_c^*(A^T u^*) \oplus \boxminus g_c^*(-u^*) \right\} = D_c.$$

Together with the weak duality inequality this yields $P = D_c$.

(ii) Let $K \subsetneq Y$ be a linear subspace of Y and let $c \in Y \setminus K$. Consider the supremum oriented set $B := \boxplus \{c\} \mathbb{R}_+$. We define a new objective function as $\tilde{p} : X \to \hat{\mathcal{F}}^\star$, $\tilde{p}(x) := p(x) + B = f(x) \oplus (g(Ax) + B)$. By Proposition 2.5, we have $\tilde{P} := \inf_{x \in X} \tilde{p}(x) = \big(\inf_{x \in X} p(x)\big) + B = P + B$. With aid of [1, Corollary 9.1.2] we conclude that $\tilde{K} := 0^+ \tilde{P} = 0^+ P + B = K + B$. Clearly, $\tilde{K}$ is not a linear space and $c \in \operatorname{ri} \tilde{K}$. It is an easy task to show that $\tilde{g} : U \to \hat{\mathcal{F}}^\star$, $\tilde{g} := g + B$ is convex and (15) remains true for this new problem. Hence, by part (i) of this theorem, we have strong duality for this new problem. Let us calculate the conjugate function $\tilde{g}_c^* : U^* \to \hat{\mathcal{F}}^\diamond$ for the function $\tilde{g}$. By the definition of the conjugate and by Proposition 2.5 it follows

$$\begin{aligned} \tilde{g}_c^*(u^*) &= \sup_{u \in U} \{ \langle u, u^* \rangle \{c\} \boxminus (g(u) + B) \} \\ &= \sup_{u \in U} \left\{ \big(\langle u, u^* \rangle \{c\} \boxminus g(u) \big) \boxminus B \right\} = g_c^*(u^*) \oplus \boxminus B. \end{aligned}$$

Hence, the dual objective function $\tilde{d} : U^* \to \hat{\mathcal{F}}^\star$ for the problem $\inf_{x \in X} \tilde{p}(x)$ is given by

$$\tilde{d}_c(u^*) = \boxminus f_c^*(A^T u^*) \oplus \boxminus g_c^*(-u^*) \oplus B.$$

With aid of (1) and since $0 \in B$ we deduce

$$\tilde{d}_c := \sup_{u^* \in U^*} \tilde{d}_c(u^*) \le \sup_{u^* \in U^*} d_c(u^*) \oplus B = D_c \oplus B \le D_c.$$

The strong duality assertion for the problem $\inf_{x \in X} \tilde{p}(x)$ yields $P + B = \tilde{P} \le D_c$. Analogously (replace B by $-B$) it follows $P - B \le D_c$. Hence, we obtain $(P + B) \cap (P - B) \le D_c$.

We next show that $P \le (P + B) \cap (P - B)$. By the definition of the space $\mathcal{F}$, P is a convex subset of Y. Let $y \in (P + B) \cap (P - B)$ be given. This means $y = p_1 + r_1 c = p_2 - r_2 c$ for some elements $p_1, p_2 \in P$ and real number $r_1, r_2 \ge 0$. If $r_1 + r_2 = 0$ there is nothing to prove. For $r_1 + r_2 > 0$ it follows

$$y = \frac{r_2}{r_1 + r_2} y + \frac{r_1}{r_1 + r_2} y = \frac{r_2}{r_1 + r_2}(p_1 + r_1 c) + \frac{r_1}{r_1 + r_2}(p_2 - r_2 c) = \frac{r_2}{r_1 + r_2} p_1 + \frac{r_1}{r_1 + r_2} p_2 \in P.$$

Hence, $P \supseteq (P + B) \cap (P - B)$, this means $P \le (P + B) \cap (P - B)$.

Together we have shown that $P \le D_c$. Taking into account the weak duality assertion it follows $P = D_c$ and the theorem is also proven for the case (ii). □

We next express the preceeding theorem by conventional notations. Although the analogy to the scalar theory is more difficult to see, this form is more convenient for applications. Let $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set–valued map. As usual, the set $\operatorname{gr} f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p |\ y \in f(x)\}$ is called the *graph* of f. We say f has closed (convex) values if $f(x) \subseteq \mathbb{R}^p$ is closed (convex) for all $x \in \mathbb{R}^n$. Clearly, if f has a closed (convex) graph, then f has closed (convex) values. The opposite implication is not true, in general. The map f has closed values and a convex graph if and only if f can be interpreted as convex function $f : \mathbb{R}^n \to \hat{\mathcal{F}}^\star$.

Corollary 4.2 *For given set–valued maps $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $g : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$, a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $c \in \mathbb{R}^p$, we have*

$$\bigcup_{x\in\mathbb{R}^n} \{f(x) + g(Ax)\} \subseteq \bigcap_{u^*\in\mathbb{R}^m} \left\{ \bigcup_{x\in\mathbb{R}^n} \{f(x) - \langle A^T u^*, x\rangle \{c\}\} + \bigcup_{u\in\mathbb{R}^m} \{g(u) + \langle u^*, u\rangle \{c\}\} \right\}.$$

If, furthermore, f and g have convex graphs and closed values and satisfy the condition $0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f)$ and, in dependence of $K := 0^+(\operatorname{cl} \bigcup_{x\in X} \big(f(x)+g(Ax)\big)$, the vector $c \in \mathbb{R}^p$ is chosen as in Theorem 4.1, we have strong duality, i.e.

$$\operatorname{cl} \bigcup_{x\in\mathbb{R}^n} \{f(x) + g(Ax)\} = \bigcap_{u^*\in\mathbb{R}^m} \operatorname{cl} \left\{ \bigcup_{x\in\mathbb{R}^n} \{f(x) - \langle A^T u^*, x\rangle \{c\}\} + \bigcup_{u\in\mathbb{R}^m} \{g(u) + \langle u^*, u\rangle \{c\}\} \right\}.$$

Proof. For all $u^* \in \mathbb{R}^m$, we have

$$\begin{aligned} \bigcup_{x\in\mathbb{R}^n} \{f(x) + g(Ax)\} &= \bigcup_{x\in\mathbb{R}^n} \{f(x) - \langle A^T u^*, x\rangle \{c\} + g(Ax) + \langle u^*, Ax\rangle \{c\}\} \\ &\subseteq \bigcup_{x\in\mathbb{R}^n} \{f(x) - \langle A^T u^*, x\rangle \{c\}\} + \bigcup_{u\in\mathbb{R}^m} \{g(u) + \langle u^*, u\rangle \{c\}\}. \end{aligned}$$

Taking the intersection over all $u^* \in \mathbb{R}^m$ we obtain the weak duality inclusion.

Let f and g have convex graphs and closed values. This means f and g can be interpreted as convex functions $f : \mathbb{R}^n \to \hat{\mathcal{F}}^\star$ and $g : \mathbb{R}^m \to \hat{\mathcal{F}}^\star$. The expression $\inf_{x\in X} \{f(x) \oplus g(Ax)\}$ in Theorem 4.1 has the meaning of $\operatorname{cl}\operatorname{conv} \bigcup_{x\in X} \operatorname{cl} \{f(x) + g(Ax)\}$. We next show that this expression can be simplified in the present case, namely

$$\inf_{x\in X} \{f(x) \oplus g(Ax)\} = \operatorname{cl} \bigcup_{x\in X} \{f(x) + g(Ax)\}. \tag{19}$$

The set $P := \bigcup_{x\in X} \operatorname{cl} \{f(x) + g(Ax)\}$ is convex. Indeed, let $y_1, y_2 \in P$. Hence there exist $x_i \in X$ $(i = 1, 2)$ such that $y_i \in \operatorname{cl} \big(f(x_i) + g(Ax_i)\big)$. For all $\lambda \in [0, 1]$ it follows

$$\begin{aligned} \lambda y_1 + (1-\lambda)y_2 &\in \lambda \operatorname{cl} \big(f(x_1) + g(Ax_1)\big) + (1-\lambda) \operatorname{cl} \big(f(x_2) + g(Ax_2)\big) \\ &= \operatorname{cl} \big(\lambda f(x_1) + (1-\lambda)f(x_2) + \lambda g(Ax_1) + (1-\lambda)g(Ax_2)\big) \\ &\subseteq \operatorname{cl} \big(f(\lambda x_1 + (1-\lambda)x_2) + g(A(\lambda x_1 + (1-\lambda)x_2))\big) \subseteq P. \end{aligned}$$

We show that $\operatorname{cl} \bigcup_{x\in X} \operatorname{cl} p(x) = \operatorname{cl} \bigcup_{x\in X} p(x)$, where $p(x) = f(x) + g(Ax)$. Clearly, we have $\operatorname{cl} \bigcup_{x\in X} \operatorname{cl} p(x) \supseteq \operatorname{cl} \bigcup_{x\in X} p(x)$. Let $\bar{y} \in \bigcup_{x\in X} \operatorname{cl} p(x)$. Then there exists some $\bar{x} \in X$ and a sequence $y_i \to \bar{y}$ such that $y_i \in p(\bar{x}) \subseteq \bigcup_{x\in X} p(x)$. Hence $\bar{y} \in \operatorname{cl} \bigcup_{x\in X} p(x)$. This means we have $\bigcup_{x\in X} \operatorname{cl} p(x) \subseteq \operatorname{cl} \bigcup_{x\in X} p(x)$ and taking the closure on both sides the desired assertion follows. Together we obtain (19).

By analogous arguments the right–hand side of the strong duality equality equals the expression $\sup_{u^* \in U^*} \boxminus f_c^*(A^T u^*) \oplus \boxminus g_c^*(-u^*)$ in Theorem 4.1. Hence, this equality holds true. $\square$

In the next example we show that the duality theorem for extended real–valued convex functions (which was used in the proof) follows from the set–valued duality theorem by a simple reformulation of the problem.

Example 4.3 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be convex functions satisfying the condition $0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f)$. A given extended real–valued problem $P = \inf_{x \in \mathbb{R}^n}\big(f(x) + g(Ax)\big)$ can be rewritten as $P = \inf \bigcup_{x \in \mathbb{R}^n}\big(\{f(x)\} + \{g(Ax)\} + \mathbb{R}_+\big)$. Consider the inner (set–valued) problem. It is easy to see that (15) is satisfied for this problem. If P is finite, we have $K = \mathbb{R}^+$. Hence the choice $c = 1$ is possible in order to obtain strong duality by Corollary 4.2. With some simple calculations (using Corollary 4.2) we obtain $P = \inf \bigcup_{x \in \mathbb{R}^n}\big(\{f(x)\} + \{g(Ax)\} + \mathbb{R}_+\big) = \inf \bigcap_{u^* \in \mathbb{R}^m}\big(\{-f^*(A^T u^*)\} - \{g^*(-u^*)\} + \mathbb{R}_+\big)$, where f^* and g^* are the classical conjugate functions of f and g. It is an easy task to show that $\inf \bigcap_{u^* \in \mathbb{R}^m}\big(\{-f^*(A^T u^*)\} - \{g^*(-u^*)\} + \mathbb{R}_+\big) = \sup_{u^* \in \mathbb{R}^m}\big(-f^*(A^T u^*) - g^*(-u^*)\big)$. The latter expression is exactly the classical dual problem for the extended real-valued problem.

Optimization problems with set relations naturally occur in vector optimization (see for instance [15], [4]) and in set–valued optimization in the sense of [8]. Let $p : X \to Y \cup \{+\infty\}$ or $p : X \rightrightarrows Y$, and let $C \subseteq Y$ be a closed convex and pointed ($C \cap -C = \{0\}$) cone. Usually one asks for the set $\operatorname{Eff}[P; C]$ of efficient points of the set $P := \bigcup_{x \in X} p(x)$ with respect to C. Using the set–valued duality assertions of Corollary 4.2, this problem can be equivalently expressed as

$$\operatorname{Eff}\left[\bigcup_{x \in X} p(x); C\right] = \operatorname{Eff}\left[\bigcap_{u^* \in U^*} d_c(u^*); C\right].$$

In this manner it is also interesting to investigate the relation to duality assertions in vector optimization, for instance those of Tanino [18]. For this, we refer to forthcoming papers.

5 Some special cases

We consider the problem of minimizing a convex function $f : X \to \hat{\mathcal{F}}^\star$ with respect to a nonempty closed convex set $D \subseteq X$. This problem can be formulated as follows.

$$\inf_{x \in X}\big(f(x) + g(Ax)\big),$$

where $A : X \to X$ is the identity map, and $g : X \to \hat{\mathcal{F}}^\star$ is the "set–valued indicator function", defined as

$$g(x) := \begin{cases} \{0\} & \text{if} \quad x \in D \\ \emptyset & \text{else.} \end{cases}$$

The conjugate function $g_c^*: X^* \to \mathcal{F}^\diamond$ of g can be understood as "set–valued support function" of the set D. It is given by

$$g_c^*(x^*) = \mathrm{cl} \bigcup_{x\in D} \{\langle x^*, x\rangle \{c\}\} = \left[-\delta^*(x^*, -D), \delta^*(x^*, D)\right] \cdot \{c\}.$$

As a special case for the function f, let us consider $f(x) = \{C \cdot x\}$ where C is a real $p \times n$ matrix. An easy computation shows that

$$f_c^*(x^*) = \bigcup_{x\in X} \left\{(c \cdot (x^*)^T - C) \cdot x\right\} = \left(c \cdot (x^*)^T - C\right) \cdot \mathbb{R}^n.$$

In the special case of $X = Y = \mathbb{R}^n$ and $C := I$ being the $n \times n$ unit matrix, Corollary 4.2 yields the following dual description of a nonempty closed convex set $D \subseteq \mathbb{R}^n$. If 0^+D is not a linear subspace of $\mathbb{R}^n$, for all $c \in \mathrm{ri}\,(0^+D)$ it holds

$$D = \bigcap_{x^*\in\mathbb{R}^n} \left((I - (x^*)^T \cdot c) \cdot \mathbb{R}^n + \left[-\delta^*(x^*, -D), \delta^*(x^*, D)\right] \cdot \{c\}\right). \tag{20}$$

Moreover, if 0^+D is a linear subspace of $\mathbb{R}^n$, (20) is valid for all $c \in \mathbb{R}^n \setminus 0^+D$. Note that in (20) the constraint qualification (15) is superfluous, see Remark 5.1 below.

We next turn to the case of linear inequality constraints. Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$ a given vector. We write $u \leq v$ if $v - u \in \mathbb{R}^m_+$. Consider the problem

$$\inf_{x\in D} \{Cx\}, \quad D = \{x \in X \mid Ax \geq b\}. \tag{21}$$

Recall that $\inf_{x\in D} \{Cx\}$ has the meaning of $\mathrm{cl\,conv} \bigcup_{x\in D} \{Cx\} = \bigcup_{x\in D} \{Cx\} =: C \cdot D$. We introduce $g: U \to \hat{\mathcal{F}}^\star$ as

$$g(u) := \begin{cases} \{0\} & \text{if} \quad u \geq b \\ \emptyset & \text{else.} \end{cases}$$

Thus, (21) can be rewritten as $\inf_{x\in X}\big(f(x) + g(Ax)\big)$. Some simple calculations yield

$$\begin{aligned} g^*(u^*) &= \bigcup_{u\geq b} \{\langle u^*, u\rangle \{c\}\} = \bigcup_{u\geq 0} \{\langle u^*, u + b\rangle \{c\}\} \\ &= \left\{c \cdot (u^*)^T \cdot b\right\} + \bigcup_{u\geq 0} \{\langle u^*, u\rangle \{c\}\} = \left\{c \cdot (u^*)^T \cdot b\right\} + h(u^*), \end{aligned}$$

where

$$h(u^*) := \bigcup_{u\geq 0} \{\langle u^*, u\rangle \{c\}\} = \begin{cases} \{c\} \cdot \mathbb{R}_- & \text{if} \quad u^* \in \mathbb{R}^m_- \setminus \{0\} \\ \{c\} \cdot \mathbb{R}_+ & \text{if} \quad u^* \in \mathbb{R}^m_+ \setminus \{0\} \\ \{0\} & \text{if} \quad u^* = 0 \\ \{c\} \cdot \mathbb{R} & \text{else.} \end{cases}$$

Note that $h(u^*) = -h(-u^*)$, hence, the dual objective function is given by

$$d_c(u^*) = (C - c \cdot (A^* \cdot u^*)^T) \cdot \mathbb{R}^n + \left\{c \cdot (u^*)^T \cdot b\right\} + h(u^*).$$

By Theorem 4.1 we obtain the following strong duality assertion. Let $K := 0^+(C \cdot D)$. If there exists some $x \in X$ such that $Ax \geq c$, then, respectively, for all $c \in \operatorname{ri} K$ if K is not a linear space and for all $c \in Y \setminus K$ if K is a linear subspace of Y it is true that

$$C \cdot D = \bigcap_{x^* \in X^*} \Big(\big(C - c \cdot (A^T \cdot u^*)^T\big) \cdot \mathbb{R}^n + \big\{c \cdot (u^*)^T \cdot b\big\} + h(u^*) \Big). \tag{22}$$

Remark 5.1 In Theorem 4.1 (duality theorem) we suppose the constraint qualification (15). In the proof of this theorem we use this condition two times. First, it is used to show that $0^+(g(u) \oplus K) \subseteq K$ for all $u \in \operatorname{dom} g$. In all the examples of this section this is given directly. Secondly, we use (15) in order to obtain the corresponding condition for the scalar problems (16). If all these problems are polyhedral, (15) can be replaced by $\operatorname{dom} g \cap A\operatorname{dom} f \neq \emptyset$, compare e.g. [3, Corollary 5.1.9]. Hence, in (20) and (22) the constraint qualification reduces to $D \neq \emptyset$.

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