STRONG WELL-POSEDNESS AND CONVERGENCE TO STEADY STATES FOR A GENERALIZED CAHN-HILLIARD EQUATION BASED ON A MICROFORCE BALANCE

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Abstract. In this paper the nonlinear Cahn-Hilliard equation based on a microforce balance in an isotropic medium is investigated. The corresponding model was proposed by M. E. Gurtin [9]. We show strong global well-posedness in the L_p - sense. Furthermore we use the Lojasiewicz-Simon inequality to show that each solution converges to a steady state as time tends to infinity, as soon as the potential Φ satisfies certain growth conditions.

1. Introduction and the Model

The *classical* Cahn-Hilliard equation

$$\partial_t\psi = \Delta\mu, \quad \mu = -\Delta\psi + \Phi'(\psi) \tag{1.1}$$

has been studied in a number of papers over the last decades. Here ψ is the so called order parameter, e.g. the mass density or the magnetic flux etc., μ is the chemical potential, which accounts for mass transport in the system, and Φ is the physical potential, which characterizes the two stable phases (e.g. water and ice) and is often assumed to be of a *double-well* type, i.e. $\Phi(s) = \Phi_0(s^2-1)^2$, where $\Phi_0 > 0$ is a constant.

The classical Cahn-Hilliard equation is derived as follows. We start with a free energy functional of the form

$$\mathcal{F}(\psi) = \int_\Omega \left(\frac{1}{2}|\nabla\psi|^2 + \Phi(\psi)\right)\, dx, \tag{1.2}$$

where Ω is a bounded, open and connected subset of $\mathbb{R}^n$ with boundary $\Gamma := \partial\Omega \in C^4$. We assume that the order parameter is a conserved quantity. The according conservation law reads

$$\partial_t\psi + \operatorname{div} h = 0, \tag{1.3}$$

where h is a vector field representing the phase flux of the order parameter. The next step is to combine the two quantities h and μ. Similar to Fourier's law in the derivation of the heat equation one typically assumes that h is given by

$$h = -\nabla\mu, \tag{1.4}$$

a *postulated* equation. Finally we have to derive an equation for μ. A view into the literature (see e.g. [20, Section V.6]) yields that μ is given by the *variational derivative* of $\mathcal{F}$, i.e.

$$\mu = \frac{\delta\mathcal{F}}{\delta\psi} = -\Delta\psi + \Phi'(\psi).$$

If $\mathcal{F}$ is of the form (1.2) this yields (1.1).

In the early nineties Fried & Gurtin [6, 7] and Gurtin [9] proposed a *generalized* Cahn-Hilliard equation, which is based on the following objections:

- Fundamental physical laws should account for the work associated with each operative kinematical process;

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- There is no clear separation of the balance law (1.3) and the constitutive equation (1.4);
- Forces that are associated with microscopic configurations of atoms are not considered in the derivation of the classical Cahn-Hilliard equation.

According to Fried & Gurtin [6, 7] there should exist so called 'microforces' whose work accompanies changes in the order parameter ψ. The microforce system is characterized by the microstress $\xi \in \mathbb{R}^n$ and a scalar quantity π which represents internal microforces. Here we neglect the presence of external microforces. The main assumption in [9] is that ξ and π satisfy the (local) *microforce balance*

$$\operatorname{div} \xi + \pi = 0, \tag{1.5}$$

which can be motivated from a static point of view, see [9] for more details. In a next step we want to derive constitutive equations, which relate the quantities h, ξ and π to the fields ψ and μ. The technique used in [9] for this derivation is based on the balance equation (1.5) and a (local) dissipation inequality, which is a direct consequence of the first and the second law of thermodynamics, see [9, Appendix A]. The result of this combination reads

$$\partial_t F(z) \leq -\nabla\mu \cdot h - \mu \operatorname{div} h + \mu m + \xi \cdot \nabla \partial_t \psi + \partial_t \psi \operatorname{div} \xi + \gamma \partial_t \psi, \tag{1.6}$$

where m is the external mass supply and F is a general free energy density, depending on $z = (\psi, \nabla\psi, \mu, \nabla\mu, \partial_t\psi)$, i.e.

$$\mathcal{F}(z) = \int_\Omega F(z)\ dx.$$

Here the term $\partial_t \psi$ accounts for the work that is associated with the kinematic process for the order parameter. Furthermore, in presence of external mass supply, (1.3) will be modified to

$$\partial_t \psi + \operatorname{div} h = m. \tag{1.7}$$

Applying (1.5), (1.7) and the chain rule to (1.6) this yields the equivalent inequality

$$(\partial_\psi F + \pi - \mu)\dot{\psi} + (\partial_{\nabla\psi} F - \xi) \cdot \nabla\dot{\psi} + \partial_\mu F \dot{\mu} + \partial_{\nabla\mu} F \nabla\dot{\mu} + \partial_{\dot{\psi}} F \ddot{\psi} + \nabla\mu \cdot h \leq 0, \tag{1.8}$$

where $\dot{u} = \partial_t u$ and $\ddot{u} = \partial_t^2 u$ for a smooth function u. This local inequality needs to be satisfied for all smooth fields ψ and μ. Hence we have necessarily

$$F(z) = F(\psi, \nabla\psi) \quad \text{and} \quad \xi(\psi, \nabla\psi) = \partial_{\nabla\psi} F(\psi, \nabla\psi)$$

and there remains the inequality

$$(\partial_\psi F + \pi - \mu)\dot{\psi} + \nabla\mu \cdot h \leq 0$$

whose general solution is given by (cf. [9, Appendix B])

$$\partial_\psi F + \pi - \mu = -\beta\dot{\psi} - c \cdot \nabla\mu \quad \text{and} \quad h = -a\dot{\psi} - B\nabla\mu,$$

with free parameters β (scalar), a, c (vectors) and B (matrix) and the constraint that the tensor

$$\begin{bmatrix} \beta & c^T \\ a & B \end{bmatrix} \tag{1.9}$$

is positive semidefinite. According to [9, Section 3.4] it holds that $c = 0$ if the underlying material is *isotropic*. In particular, if the free energy density F is given by $F(\psi, \nabla\psi) = \frac{1}{2}|\nabla\psi|^2 + \Phi(\psi)$ and $c = 0$ we obtain the following equations.

$$\begin{aligned} \partial_t\psi - \operatorname{div}(B(x)\nabla\mu) - \operatorname{div}(a(x)\partial_t\psi) &= f, \quad t > 0,\ x \in \Omega, \\ \mu + \Delta\psi - \partial_t\psi - \Phi'(\psi) &= 0, \quad t > 0,\ x \in \Omega. \end{aligned} \tag{1.10}$$

For the sake of simplicity we have set $\beta = 1$; this does not change the analysis, as long as $\beta > 0$. To ensure that ψ is a conserved quantity, we take a Neumann boundary condition for μ, i.e. $B(x)\nabla\mu \cdot \nu = 0$ on Γ and we assume that $a(x) \cdot \nu(x) = 0,\ x \in \Gamma$. Since (1.10) is a fourth order system we need an additional boundary condition. Due to the presence of the Laplacian in $(1.10)_2$ and with a view on energy estimates (see Section 4), another natural boundary condition would be $\partial_\nu \psi = 0$. If we reduce (1.10) to a single equation for ψ, we obtain

$$\partial_t\psi - \operatorname{div}(a(x)\partial_t\psi) - \operatorname{div}(B(x)\nabla\partial_t\psi) + \operatorname{div}(B(x)\nabla\Delta\psi) - \operatorname{div}(B(x)\nabla\Phi'(\psi)) = f. \tag{1.11}$$

This raises the question which additional boundary condition on ψ we should take. In case of the classical Cahn-Hilliard equation we obtain from $(1.1)_2$ and $\partial_\nu \mu = \partial_\nu \psi = 0$ the condition $\partial_\nu \Delta\psi = 0$. Unfortunately, in the generalized equations and for an arbitrary matrix $B(x)$, this is not longer true. Therefore we require $B(x) = b(x)I$ for all $x \in \Gamma$, where $b : \Gamma \to \mathbb{R}$ is a given function, with $b(x) > 0$ for all $x \in \Gamma$. Thus, by $(1.10)_2$, the conditions $\partial_\nu \mu = \partial_\nu \psi = 0$ are equivalent to $\partial_\nu \Delta\psi = \partial_\nu \psi = 0$, hence we consider the system

$$
\begin{aligned}
\partial_t \psi - \operatorname{div}(B(x)\nabla\mu) - \operatorname{div}(a(x)\partial_t\psi) &= f, \quad t > 0,\ x \in \Omega, \\
\mu + \Delta\psi - \partial_t\psi - \Phi'(\psi) &= 0, \quad t > 0,\ x \in \Omega, \\
B(x)\nabla\psi \cdot \nu = B(x)\nabla\Delta\psi \cdot \nu &= 0, \quad t > 0,\ x \in \Gamma, \\
\psi(0) &= \psi_0 \quad t = 0,\ x \in \Omega,
\end{aligned} \tag{1.12}
$$

where we assume in the sequel

(H1) $B \in C_b^1(\Omega; \mathbb{R}^{n\times n})$, with $B(x) = B(x)^T,\ x \in \Omega$ and $B(x) = b(x)I,\ b(x) > 0,\ x \in \Gamma$;

(H2) $a \in C_b^1(\Omega; \mathbb{R}^n)$, with $\operatorname{div} a(x) \le 1,\ x \in \Omega$ and $a(x) \cdot \nu(x) = 0,\ x \in \Gamma$.

Since the tensor (1.9) (with $c = 0$) is assumed to be positive semidefinite, one should have

$$y^2 + a(x) \cdot zy + B(x)z \cdot z \ge 0,$$

for all $y \in \mathbb{R},\ z \in \mathbb{R}^n$ and $x \in \Omega$. Here we make the slightly stronger assumption

(H3) $y^2 + a(x) \cdot zy + B(x)z \cdot z \ge \varepsilon(y^2 + |z|^2)$, for all $y \in \mathbb{R},\ z \in \mathbb{R}^n$, $x \in \Omega$ and a fixed $\varepsilon \in (0, 1)$.

As it has already been mentioned at the beginning, there are quite a lot of papers which deal with the analysis of the classical Cahn-Hilliard equation. Therefore we only want to mention here the most important ones, from our point of view. Elliott & Zheng [5] proved global existence of solutions to (1.1), with homogeneous Dirichlet- or Neumann boundary conditions in an L_2 setting and Zheng [22] studied the asymptotic behaviour of (1.1) as time t tends to infinity. In [19], Racke & Zheng used spectral theory to show global well-posedness of (1.1) in L_2, if one considers a dynamic boundary condition of the form

$$\partial_t \psi - \Delta_\Gamma \psi + \partial_\nu \psi + \psi = 0, \tag{1.13}$$

where Δ_Γ stands for the Laplace-Beltrami operator on Γ. Such a boundary condition was suggested by the authors in [11]. In [17] the result of Racke & Zheng was extended to the L_p spaces ($p \ge 2$), by using the technique of maximal regularity. Furthermore the authors proved the existence of an global attractor in L_2. Hoffmann & Rybka [10] showed that every solution of (1.1) with Neumann boundary conditions converges to a steady state, as $t \to \infty$. To this end they applied the so called *Lojasiewicz-Simon inequality*. The same result for (1.1) with the dynamic boundary condition (1.13) and a general pysical potential Φ is due to Chill, Fašangová and Prüss [3] (see also [21] for a similar convergence result in case Φ is a double-well potential). Moreover in [18] we applied maximal regularity tools to prove the strong global well-posedness in L_p of the *non-isothermal* Cahn-Hilliard equation with a dynamic boundary condition of the form (1.13) (see [18] for more details). In addition we made also use of the Lojasiewicz-Simon inequality to prove convergence towards a steady state.

For the generalized Cahn-Hilliard equation (1.12) there is not such a large literature. As it has already been pointed out, the model (1.12) was proposed by Gurtin in [9]. One can say that most of the work concerning the generalized Cahn-Hilliard equation has been done by Miranville and several co-authors, see e.g. [1], [12], [13], [14], [15], [16] and the references cited therein. The difference to the present paper is that the authors consider also anisotropic mediums, i.e. $c \ne 0$ in (1.9) but they assume that $a = const$ and $B = const$. The fact that $c \ne 0$ makes the analysis harder, since the equivalence

$$\left.\begin{array}{l} B(x)\nabla\mu \cdot \nu = 0 \\ B(x)\nabla\psi \cdot \nu = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} 0 = B(x)\nabla\Delta\psi \cdot \nu \\ 0 = B(x)\nabla\psi \cdot \nu \end{array}\right. ,$$

which we have for $c = 0$ does not hold any longer and moreover the chemical potential μ is *not* explicitely given by $(1.10)_2$, in case $c \ne 0$.

In [1], [12], [13], [14] the authors use a variational approach to obtain global well-posedness of the generalized Cahn-Hilliard equation in an anisotropic medium in the sense of an L_2 setting. Due to this technique the solutions are weak with respect to the spatial variable. The existence of finite dimensional attractors in L_2 is e.g. shown in [1], [13] and [15].

The aim of the present paper is the following. Since all of the above results only deal with weak solutions in L_2, strong global well posedness in L_p is the subject of this paper. Furthermore, apart from the existence of attractors, there seems to be no result on the convergence of $\psi(t)$ as $t \to \infty$ to a steady state. This is also done in the present paper.

In the following section we will first study the corresponding linear system to (1.12) to obtain a unique solution in the optimal regularity class

$$\psi \in H_p^1(J_0; H_p^2(\Omega)) \cap L_p(J_0; H_p^4(\Omega)),\ p \in (1,\infty), \quad J_0 = [0, a_0],$$

where H_p^k denote the classical Sobolev spaces and $a_0 > 0$. This is done by using maximal regularity of type L_p and a Fredholm argument. In Section 3 we apply the contraction mapping principle and maximal L_p-regularity to handle the nonlinearity in (1.12). In the first instance this yields a possibly small intervall $J_a = [0, a] \subset J_0,\ a > 0$, such that

$$\psi \in H_p^1(J_a; H_p^2(\Omega)) \cap L_p(J_a; H_p^4(\Omega))$$

and by a standard continuation argument we obtain a maximal interval of existence $J_{max} = [0, a_{max}) \subset J_0$ for the solution ψ of (1.12). Section 4 is devoted to the global existence of ψ. The proof is based on some a priori estimates and the Gagliardo-Nirenberg inequality, where we use again maximal regularity tools. Finally we study the asymptotic behaviour of the solution to (1.12) in section 5. We show relative compactness of the orbit $\psi(\mathbb{R}_+)$ in an appropriate energy space and we make use of some differentiability properties of the underlying energy functional to (1.12). Lastly the Lojasiewicz-Simon inequality is the key, which leads to the convergence of $\psi(t)$ in $H_2^1(\Omega)$ towards a steady state, as $t \to \infty$.

2. The Linear Problem

As we have already seen, $(1.12)_{1,2}$ is equivalent to (1.11). Our aim is to write (1.11) as an abstract semilinear evolution equation. Therefore we set $X_0 = L_p(\Omega),\ 1 < p < \infty$, and define operators A_j in X_0 by

$$\begin{aligned} A_1\psi &= -\operatorname{div}(B(x)\nabla\psi), \quad D(A_1) = \{w \in H_p^2(\Omega) : B(x)\nabla w \cdot \nu = 0,\ x \in \Gamma\}, \\ A_2^0\psi &= -\operatorname{div}(a(x)\psi), \quad D(A_2^0) = H_p^1(\Omega) \quad \text{and} \\ A_3\psi &= -\Delta\psi, \quad D(A_3) = D(A_1). \end{aligned}$$

By D_{A_j} we denote the domains $D(A_j)$ endowed with the corresponding graph norms. Observe that by (H1) it holds that $|\cdot|_{D_{A_1}}$ and $|\cdot|_{D_{A_1 A_3}}$ are equivalent norms in $H_p^2(\Omega)$ and $H_p^4(\Omega)$, respectively.

We proceed with some preliminaries. Define a family $\mathcal{T} = \{T(t)\}_{t\in\mathbb{R}}$ of operators in X_0 by

$$(T(t)u)(x) = u(\phi(t;x))e^{\int_0^t \operatorname{div} a(\phi(s;x))\ ds}, \quad u \in X_0,\ t \in \mathbb{R},\ x \in \Omega$$

where $\phi = \phi(t;x),\ t \in \mathbb{R}$ is the unique global solution of the initial value problem

$$\dot{\phi}(t) = a(\phi(t)), \quad \phi(0) = x.$$

Due to (H2) it is easy to see that ϕ defines a global flow in Ω, hence $\mathcal{T}$ is a group of bounded linear operators. Denote by $-A_2$ the generator of $\mathcal{T}$ with domain

$$D(A_2) = \{u \in X_0 : \lim_{t\to 0} \frac{T(t)u - u}{t} \text{ exists in } X_0\}.$$

Clearly A_2 is an extension of A_2^0 in the sense that $D(A_2^0) \subset D(A_2)$ and $A_2^0 u = A_2 u$ for all $u \in D(A_2^0)$. We want to show that $-1 \in \rho(A_2)$, the resolvent set of A_2. Therefore we estimate with the help of the transformation formula and (H2) to the result

$$\begin{aligned}|T(t)u|_{X_0}^p &\le \int_\Omega |u(\phi(t;x))|^p e^{p\int_0^t \operatorname{div} a(\phi(s;x))\ ds}\ dx\\ &\le \int_\Omega |u(x)|^p e^{(p-1)\int_0^t \operatorname{div} a(\phi(s-t;x))\ ds}\ dx\\ &\le e^{(p-1)t}\int_\Omega |u(x)|^p\ dx = e^{(p-1)t}|u|_{X_0}^p,\end{aligned}$$

hence $|T(t)u|_{X_0} \le e^{\frac{1}{p'}t}|u|_{X_0}$. It follows that

$$\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > \frac{1}{p'}\} \subset \rho(-A_2),$$

since $p' > 1$. In particular it holds that $1 \in \rho(-A_2)$ or equivalently $-1 \in \rho(A_2)$. Next we make use of (H1) and (H2) to prove that the operator $(I + A_1 + A_2)$ is invertible, i.e. $-1 \in \rho(A_1 + A_2)$. First observe that the domain $D(A_1 + A_2) = D(A_1) \cap D(A_2) = D(A_1)$ is compactly embedded into X_0. Therefore the spectrum of $A_1 + A_2$ consists only of isolated eigenvalues. Assume that $-1 \in \sigma(A_1 + A_2)$, i.e.

$$-\psi = (A_1 + A_2)\psi, \quad 0 \ne \psi \in D(A_1).$$

Multiplying the latter equation by ψ and integrating by parts yields

$$\frac{1}{2}\int_\Omega (2 - \operatorname{div} a(x))\psi^2\ dx + \int_\Omega B(x)\nabla\psi\cdot\nabla\psi\ dx = 0,$$

where we also made use of (H2) and $(1.12)_3$. Since by (H3) the matrix $B(x)$ is positive definite, the second integral is nonnegative and if we apply (H2) one more time we see that the first integral is always positive, a contradiction. Thus $-1 \in \rho(A_1 + A_2)$. This in turn implies $(I + A_1 + A_2)^{-1} \in \mathcal{B}(X_0)$ and moreover $(I + A_1 + A_2)^{-1} \in \mathcal{B}(X_0; D(A_1))$.

After these preliminaries, consider the linear equation

$$(I_\omega + A_{1,\omega} + A_2)\partial_t u + A_{1,\omega}A_3 u = f,\ t > 0, \quad u(0) = u_0. \tag{2.1}$$

Here $A_{1,\omega} := A_1 + \omega I$, $I_\omega := (1-\omega)I$ and $\omega > 0$ is fixed but may be *arbitrarily small*. Note that (2.1) is equivalent to

$$\partial_t u + A_3 u = A_{1,\omega}^{-1}[f - (I_\omega + A_2)\partial_t u]. \tag{2.2}$$

If we could control the term $A_2\partial_t u$ in (2.2), we would obtain a smooth solution of (2.1). Therefore we will derive an equation for $A_2\partial_t u$. Since A_3 has the property of maximal L_p-regularity it is the generator of an analytic semigroup $e^{-A_3 t}$ in X_0. Hence the solution u of (2.2) may be represented by

$$u(t) = e^{-A_3 t}u_0 + \int_0^t e^{-A_3(t-s)}A_{1,\omega}^{-1}[f - ((I_\omega + A_2)\partial_t u]\ ds.$$

Differentiating the latter equation with respect to t and applying $(I_\omega + A_2)$ to the result yields

$$(I + A_1 + A_2)A_{1,\omega}^{-1}v(t) = (I_\omega + A_2)\int_0^t e^{-A_3(t-s)}A_3A_{1,\omega}^{-1}v(s)\ ds + g(t), \tag{2.3}$$

where $v := (I_\omega + A_2)\partial_t u$ and the function g is given by

$$g(t) = (I_\omega + A_2)[A_{1,\omega}^{-1}f(t) - A_3e^{-A_3t}u_0 - \int_0^t e^{-A_3(t-s)}A_3A_{1,\omega}^{-1}f(s)\ ds]. \tag{2.4}$$

We set

$$u_0 \in X_p := \{w \in W_p^{4-2/p}(\Omega) : \partial_\nu w = 0,\ (p > 1) \text{ and } \partial_\nu\Delta w = 0,\ (p > 3)\}$$

and make use of maximal L_p-regularity to obtain $g \in L_p(J_0; X_0)$ for each $f \in L_p(J_0; X_0)$. Note that in (2.3) and (2.4) we could have written A_2^0 instead of A_2, since $D(A_{1,\omega}) \subset D(A_2^0)$.

We already know that the operator $(I + A_1 + A_2)$ is invertible, therefore we obtain from (2.3) an equation for v, namely $(I - S)v = g$, where $S := A_{1,\omega}(I + A_1 + A_2)^{-1}T$ and T is defined by

$$(Tv)(t) = (I_\omega + A_2)\int_0^t e^{-A_3(t-s)}A_3A_{1,\omega}^{-1}v(s)\ ds.$$

It is again a consequence of regularity theory that the operator $T : L_p(J_0; X_0) \to L_p(J_0; X_0)$ is compact. This in turn implies that $S : L_p(J_0; X_0) \to L_p(J_0; X_0)$ is compact, since $A_{1,\omega}(I + A_1 + A_2)^{-1}$ is L_p-bounded. Therefore $(I - S)$ is a Fredholm operator with index 0. Next we show that $(I - S)$ is injective. Assume that $0 \neq v \in L_p(J_0; X_0)$ is a solution of the equation $v = Sv$ and define

$$u(t) = \int_0^t (I_\omega + A_2)^{-1} v(s)\, ds.$$

The invertibility of $(I_\omega + A_2)$ for a sufficiently small $\omega > 0$ follows from the fact that $\rho(A_2)$ is open. Obviously it holds that $(I_\omega + A_2)\partial_t u = v$ and $u(0) = 0$. By construction the function u is a solution of (2.2), with $f = 0$ and $u_0 = 0$. One more time we make use of maximal regularity to obtain

$$u \in H_p^1(J_0; D_{A_1}) \cap L_p(J_0; D_{A_1 A_3}).$$

Furthermore u solves the system

$$\begin{aligned} (1-\omega)\partial_t u - \operatorname{div} B(x)\nabla\mu + \omega\mu - \operatorname{div}(a(x)\partial_t u) &= 0, \quad t > 0,\ x \in \Omega, \\ \mu + \Delta u - \partial_t u &= 0, \quad t > 0,\ x \in \Omega, \\ B(x)\nabla u \cdot \nu = B(x)\nabla\Delta u \cdot \nu &= 0, \quad t > 0,\ x \in \Gamma, \\ u(0) &= 0, \quad t = 0,\ x \in \Omega. \end{aligned}$$

To show that $u = 0$, we multiply the first equation by μ and integrate by parts to the result

$$\frac{d}{dt}\frac{(1-\omega)}{2}|\nabla u|_2^2 + (1-\omega)|u_t|_2^2 + (a\partial_t u|\nabla\mu)_2 + |B^{1/2}\nabla\mu|_2^2 + \omega|\mu|_2^2 = 0.$$

Here $(\cdot|\cdot)_2$ denotes the inner product in $L_2(\Omega)$ and $|w|_2^2 = (w|w)_2,\ w \in L_2(\Omega)$. Now we use (H3) with $y = \partial_t u$ and $z = \nabla\mu$ to see that $|\nabla u|_2^2 = 0$ if $\omega \le \varepsilon$, hence $u = 0$ and this in turn implies $v = 0$, a contradiction. Therefore the operator $(I - S)$ is injective and by the Fredholm alternative we obtain a unique solution $v \in L_p(J_0; X_0)$ of the equation $(I - S)v = g$ for each $g \in L_p(J_0; X_0)$. In particular, if g is given by (2.4) and

$$u(t) := e^{-A_3 t}u_0 + \int_0^t A_3 e^{-A_3 s}u_0\, ds + \int_0^t (I_\omega + A_2)^{-1} v(s)\, ds,$$

then as before

$$u \in H_p^1(J_0; D_{A_1}) \cap L_p(J_0; D_{A_1 A_3})$$

is the unique solution of (2.1) if and only if $f \in L_p(J_0; X_0)$ and $u_0 \in X_p$. We summarize these considerations in the following

Proposition 2.1. *Let $n \in \mathbb{N}$, $1 < p < \infty$ and $p \neq 3$. Suppose $\Omega \subset \mathbb{R}^n$ is bounded, open and connected with compact boundary $\Gamma = \partial\Omega \in C^4$ and let $J_0 = [0, a_0]$. Assume (H1)-(H3). Then* (2.1) *admits a unique solution*

$$u \in H_p^1(J_0; D_{A_1}) \cap L_p(J_0; D_{A_1 A_3}) =: Z$$

if and only if $f \in L_p(J_0; X_0)$ and $u_0 \in X_p$. The solution depends continously on the given data and there exists a constant $C = C(J_0) > 0$ such that the estimate

$$|u|_Z \le C(|f|_{L_p(J_0;X_0)} + |u_0|_{X_p}) \tag{2.5}$$

is valid.

Proof. What remains is to prove the estimate (2.5). But this is an immediate consequence of the open mapping theorem. □

3. Local Well-Posedness

In this section we will apply the contraction mapping principle to find a solution of the equation

$$(I_\omega + A_{1,\omega} + A_2)\partial_t u + A_{1,\omega}A_3 u = -A_1\Phi'(u) + \omega A_3 u + f,\ t > 0, \quad u(0) = u_0, \tag{3.1}$$

where $f \in L_p(J_0; X_0)$, $u_0 \in X_p$ and $\omega > 0$ may be arbitrarily small. Note that by Proposition 2.1 we have $A_2\partial_t u = A_2^0\partial_t u$.

We assume in the sequel that $\Phi \in C^{4-}(\mathbb{R})$. Using Proposition 2.1 we denote by $u^* \in Z(a_0)$ the unique solution to

$$(I_\omega + A_{1,\omega} + A_2)\partial_t u^* + A_{1,\omega}A_3 u^* = f,\ t > 0, \quad u^*(0) = u_0$$

Let $J_a = [0, a]$, set $Z(a) := H_p^1(J_a; D_{A_1}) \cap L_p(J_a; D_{A_1A_3})$ and define a ball $\mathbb{B}_R \subset Z(a)$ by means of

$$\mathbb{B}_R := \{w \in Z(a) : |w - u^*|_{Z(a)} \le R,\ w|_{t=0} = u_0\},\ R \in (0, 1].$$

Observe that by trace theory the embeddings

$$Z(a) \hookrightarrow C(J_a; X_p) \hookrightarrow C(J_a; C^1(\overline{\Omega})), \tag{3.2}$$

are valid, provided $p > n/3 + 2/3$. Furthermore it holds that

$$v \in H_p^1(J_a; H_p^2(\Omega)) \hookrightarrow C(J_a; H_p^2(\Omega)), \tag{3.3}$$

for all $v \in Z(a)$ and all $p > 1$. In this sense all functions $v \in \mathbb{B}_R$ are uniformly bounded in $C(J_a; C^1(\overline{\Omega}))$ and $C(J_a; H_p^2(\Omega))$. Indeed, by (3.2) and (3.3) we have

$$|v - u^*|_{C(J_a;C^1(\overline{\Omega}))} \le M_1|v - u^*|_{Z(a)} \le R$$

and

$$|v - u^*|_{C(J_a;H_p^2(\Omega)))} \le M_2|v - u^*|_{Z(a)} \le R,$$

where the constants $M_1, M_2 > 0$ do not depend on a, since $(v - u^*)|_{t=0} = 0$.

In the sequel we denote by $||\cdot||_p$ and $||\cdot||_{\infty,p}$ the norms in $L_p(J_a \times \Omega)$ and $L_\infty(J_a; L_p(\Omega))$, respectively. Consider the equation

$$(I_\omega + A_{1,\omega} + A_2)\partial_t u + A_{1,\omega}A_3 u = -A_1\Phi'(v) + \omega A_3 v + f,\ t > 0, \quad u(0) = \psi_0, \tag{3.4}$$

with $v \in \mathbb{B}_R$ and define an operator $T : \mathbb{B}_R \to Z(a)$ by $Tv = u$, where $u \in Z(a)$ is the unique solution of (3.4). By Proposition 2.1 T is well-defined, since

$$||A_1\Phi'(v)||_p \le C(||\Phi'(v)||_p + ||A_3\Phi'(v)||_p) \le C(1 + ||A_3 v||_p + ||\nabla v||_{2p}^2) < \infty$$

and due to (3.2), (3.3).

We will now show that T defines a *self mapping*, i.e. $T\mathbb{B}_R \subset \mathbb{B}_R$, and a *strict contraction* on $\mathbb{B}_R$, i.e. for all $v, \bar{v} \in \mathbb{B}_R$ there exists a number $\gamma < 1$, such that

$$|Tv - T\bar{v}|_{Z(a)} \le \gamma|v - \bar{v}|_{Z(a)}$$

holds. To prove the contraction mapping property we make use of (2.5), (3.2), (3.3) and Hölder's inequality.

$$\begin{aligned}
|Tv - T\bar{v}|_{Z(a)} = |u - \bar{u}|_{Z(a)} &\le C(||A_1(\Phi'(v) - \Phi'(\bar{v}))||_p + ||A_3(v - \bar{v})||_p) \\
&\le C(||\Phi'(v) - \Phi'(\bar{v})||_p + ||\Delta(\Phi'(v) - \Phi'(\bar{v}))||_p + a^{1/p}||\Delta(v - \bar{v})||_{\infty,p}) \\
&\le C(a^{1/p}|v - \bar{v}|_{L_\infty(J_a;H_p^2(\Omega))} + a^{1/p}||(\Delta v - \Delta\bar{v})\Phi''(v)||_{\infty,p} + \\
&\quad + a^{1/p}||(\Phi''(v) - \Phi''(\bar{v}))\Delta\bar{v}||_{\infty,p} + a^{1/p}||(|\nabla v|^2 - |\nabla\bar{v}|^2)\Phi'''(v)||_{\infty,p} + \\
&\quad + a^{1/p}||(\Phi'''(v) - \Phi'''(\bar{v}))|\nabla\bar{v}|^2||_{\infty,p}) \\
&\le Ca^{1/p}|v - \bar{v}|_{Z(a)},
\end{aligned}$$

where $C > 0$ depends on R and u^*. Since R and u^* are fixed the claim follows if a is sufficiently small. The self mapping property is proven in a similar way.

$$\begin{aligned}|Tv - u^*|_{Z(a)} = |u - u^*|_{Z(a)} &\leq C(||A_1\Phi'(v)||_p + ||A_3 v||_p)\\ &\leq C(||\Phi'(v)||_p + ||\Delta\Phi'(v)||_p + a^{1/p}||\Delta v||_{\infty,p})\\ &\leq C(a^{1/p} + ||\Delta v\Phi''(v)||_p + |||\nabla v|^2\Phi'''(v)||_p + a^{1/p}||\Delta v||_{\infty,p})\\ &\leq C(a^{1/p} + a^{1/p}||\Delta v||_{\infty,p})\\ &\leq Ca^{1/p} \leq R,\end{aligned}$$

if a is again sufficiently small. Now we may apply the contraction mapping principle and obtain a unique fixed point $\tilde{u} \in \mathbb{B}_R$ of T, i.e. $T\tilde{u} = \tilde{u}$. Thus $\tilde{u}$ is the unique local solution of (3.1) with initial value $u_0 \in X_p$. We summarize the preceeding calculations in the following

Proposition 3.1. *Let $n \in \mathbb{N}$, $p > n/3 + 2/3$ and $p \neq 3$. Suppose $\Omega \subset \mathbb{R}^n$ is bounded, open and connected with compact boundary $\Gamma = \partial\Omega \in C^4$ and let $J_0 = [0, a_0]$. Assume furthermore that (H1)-(H3) hold and let $\Phi \in C^{4-}(\mathbb{R})$. Then there exists an $a \in (0, a_0]$ such that the nonlinear problem* (1.12) *admits a unique solution ψ on $J_a = [0, a]$ in the regularity class*

$$\psi \in H^1_p(J_a; D_{A_1}) \cap L_p(J_a; D_{A_1A_3})$$

if $f \in L_p(J_0; X_0)$ and $\psi_0 \in X_p$. The solution depends continously on the given data and the mapping $\psi_0 \mapsto \psi$ defines a local semiflow on the linear manifold X_p, if f does not depend on t.

4. Global Well-Posedness

A successive application of Proposition 3.1 yields a maximal interval of existence $J_{max} := [0, a_{max})$. To prove the global existence of the local solution $\psi \in Z(a)$ of (1.10) on every compact interval $J_0 = [0, a_0]$, we have to show that $|\psi|_{Z(a)}$ is uniformly bounded with respect to $a \in [0, a_0]$. To this end we have to derive an a priori estimate for ψ. As in section 2 we multiply $(1.12)_1$ by μ and integrate by parts to the result

$$\frac{d}{dt}\left(\frac{1}{2}|\nabla\psi|_2^2 + \int_\Omega \Phi(\psi)\ dx\right) + |\partial_t\psi|_2^2 + (a\partial_t\psi|\nabla\mu)_2 + |B^{1/2}\nabla\mu|_2^2 = (f|\mu)_2. \tag{4.1}$$

For the term on the right hand side we use Hölder's inequality, the Poincaré-Wirtinger inequality and last but not least Young's inequality to obtain

$$(f|\mu)_2 \leq |f|_2|\mu|_2 \leq C|f|_2\left(|\nabla\mu|_2 + |\int_\Omega \mu\ dx|\right) \leq C(\varepsilon)|f|_2^2 + \varepsilon|\nabla\mu|_2^2 + C|f|_2\ |\int_\Omega \mu\ dx|.$$

To proceed, we need two additional assumptions:

- **(H4)** $\Phi(s) \geq -\frac{\eta}{2}s^2 - c_0$, for all $s \in \mathbb{R}$, where $c_0 > 0$, $\eta < \lambda_1$ and λ_1 is the first nontrivial eigenvalue of the Neumann-Laplacian;
- **(H5)** $|\Phi'(s)| \leq \left(c_1\Phi(s) + c_2(s^2+1)\right)^\theta$, $\theta \in (0,1)$, for all $s \in \mathbb{R}$ and some $c_1, c_2 > 0$.

Due to the boundary conditions $(1.12)_3$ we have $\int_\Omega \Delta\psi\ dx = \int_\Omega \partial_t\psi\ dx = 0$, hence

$$\begin{aligned}|\int_\Omega \mu\ dx| &\leq |\int_\Omega \Delta\psi\ dx| + |\int_\Omega \partial_t\psi\ dx| + |\int_\Omega \Phi'(\psi)\ dx|\\ &\leq \left(|f|_2 + \int_\Omega |\Phi'(\psi)|\ dx\right) \leq \left(|f|_2 + \int_\Omega \left(c_1\Phi(\psi) + c_2(\psi^2+1)\right)^\theta\ dx\right).\end{aligned}$$

Therefore it follows from Hölder's and Young's inequality that

$$|f|_2\ |\int_\Omega \mu\ dx| \leq C\left(|f|_2^2 + |f|_2^{\frac{1}{1-\theta}} + \int_\Omega \left(\Phi(\psi) + \psi^2 + 1\right)\ dx\right).$$

Inserting the above estimates into (4.1) yields

$$\frac{d}{dt}E(\psi) + |\partial_t\psi|_2^2 + (a\partial_t\psi|\nabla\mu)_2 + |B^{1/2}\nabla\mu|_2^2 - \varepsilon|\nabla\mu|_2^2 \leq C_1E(\psi) + C_2(|f|_2^q + |f|_2^2 + 1), \tag{4.2}$$

where we used again the Poincaré-Wirtinger inequality and the fact that

$$|\int_\Omega \psi\ dx| \le |f|_{L_1(J_0\times\Omega)}.$$

Furthermore $q = \max\{2, \frac{1}{1-\theta}\}$ and $E(\psi)$ is defined by

$$E(\psi) = \frac{1}{2}|\nabla\psi|_2^2 + \int_\Omega \Phi(\psi)\ dx.$$

In order to apply Gronwall's lemma to (4.2) we need to know that $E(\psi)$ is bounded from below. But this is another consequence of the Poincaré-Wirtinger inequality and (H4).

Finally, by Gronwall's lemma we obtain the result

$$E(\psi(t)) \le C(a_0)(E(\psi_0)+1), \quad \text{for all } t\in[0,a_0],$$

which by the Poincaré-Wirtinger inequality implies that

$$u \in L_\infty([0,a_{max}]; H_2^1(\Omega)). \tag{4.3}$$

Now we go back to equation (3.1). The following inequality is a consequence of (2.5).

$$\begin{aligned} |\psi|_{Z(a)} &\le C(a_0)(|A_1\Phi'(\psi)|_p + |A_3\psi|_p + |f|_p + |\psi_0|_{X_p}) \le C(a_0)(1+|A_1\Phi'(\psi)|_p + |A_3\psi|_p) \\ &\le C(A_0)(1+|\Phi'(\psi)|_{L_p(J_a;H_p^2(\Omega))} + |\psi|_{L_p(J_a;H_p^2(\Omega))}). \end{aligned} \tag{4.4}$$

To estimate $\Phi'(\psi)$ and ψ in $L_p(J_a;H_p^2(\Omega))$ we will apply the Gagliardo-Nirenberg inequality and the a priori estimate from above. This yields

$$|\psi|_{H_p^2(\Omega)} \le C|\psi|^\kappa_{H_p^4(\Omega)}|\psi|^{1-\kappa}_{L_q(\Omega)}, \tag{4.5}$$

where $\psi = \psi(t)$ and κ, p and q are linked by

$$\frac{2}{n} + \frac{1}{q} - \frac{1}{p} = \kappa\left(\frac{4}{n} + \frac{1}{q} - \frac{1}{p}\right)$$

and $q \le 6$. It is easy to see that $\kappa\in(0,1)$ if we choose e.g. $q = 6$. Integrating (4.5) with respect to t and using Hölder's inequality yields

$$|\psi|_{L_p(J_a;H_p^2(\Omega))} \le C_\kappa|\psi|^\kappa_{L_p(J_a;H_p^4(\Omega))} \le C_\kappa|\psi|^\kappa_{Z(a)}, \tag{4.6}$$

with a constant $C_\kappa > 0$ which is independent of $a \in J_{max}$. For $\Phi'(u)$ we need an additional assumption.

(H6) $|\Phi'''(s)| \le c(1+|s|^\gamma)$, for all $s\in\mathbb{R}$ and some $c_1, c_2 > 0$, where $\gamma < 3$ in case $n = 3$.

Then, by using the fact that $\Delta\Phi'(\psi) = \Delta\psi\Phi''(\psi) + |\nabla\psi|^2\Phi'''(\psi)$ we have

$$|\Delta\psi\Phi''(\psi)|_p \le |\psi|^{\gamma+1}_{2(\gamma+1)p}|\Delta\psi|_{2p} \le C|\psi|^{a+b(\gamma+1)}_{H_p^4}|\psi|_q^{1-a+(1-b)(\gamma+1)}, \tag{4.7}$$

where $\psi = \psi(t)$ and q will be chosen in such a way that $H_2^1(\Omega)\hookrightarrow L_q$, i.e. $\frac{n}{q} \ge \frac{n}{2} - 1$ and

$$(a+(\gamma+1)b)\left(4 - \frac{n}{p} + \frac{n}{q}\right) = 2 - \frac{n}{p} + \frac{n}{q}(\gamma+2).$$

The second term $|\nabla u|^2\Phi'''(\psi)$ will be treated in a similar way.

$$|\ |\nabla\psi|^2\Phi'''(\psi)|_p \le |\psi|^\gamma_{2\gamma p}|\nabla\psi|^2_{4p} \le C|\psi|^{2a+b\gamma}_{H_p^4}|\psi|_q^{2-2a+(1-b)\gamma}, \tag{4.8}$$

with $\frac{n}{q} \ge \frac{n}{2} - 1$ and

$$(2a+\gamma b)\left(4 - \frac{n}{p} + \frac{n}{q}\right) = 2 - \frac{n}{p} + (\gamma+2)\frac{n}{q}.$$

It turns out that the condition $\gamma < 3$ in case $n = 3$ ensures that either $a + (\gamma+1)b < 1$ and $2a+\gamma b < 1$ in (4.7) and (4.8), respectively. Integrating (4.7) and (4.8) with respect to t and using Hölders inequality we obtain as above

$$|\Phi'(\psi)|_{L_p(J_a;H_p^2(\Omega))} \le C_\kappa|\psi|^\kappa_{Z(a)}, \tag{4.9}$$

where $C_\kappa > 0$ does not depend on $a \in J_{max}$. Using (4.6) and (4.9) in connection with (4.4) we see that $|\psi|_{Z(a)}$ is uniformly bounded with respect to $a \in J_{max}$. We summarize our previous considerations in

Theorem 4.1. *Let $n \in \mathbb{N}$, $p \geq 2$ and $p \neq 3$. Suppose $\Omega \subset \mathbb{R}^n$ is bounded, open and connected with compact boundary $\Gamma = \partial\Omega \in C^4$ and let $J_0 = [0, a_0]$. Assume furthermore that (H1)-(H6) hold. Then there exists a unique solution ψ of* (3.1) *in the maximal regularity class*

$$\psi \in H_p^1(J_0; D_{A_1}) \cap L_p(J_0; D_{A_1 A_3}),$$

if $f \in L_p(J_0; X_0)$ and $\psi_0 \in X_p$. The solution depends continously on the given data and the mapping $\psi_0 \mapsto \psi$ defines a global semiflow on the linear manifold X_p, if f does not depend on t.

Remark. It is easily seen that (H4)-(H6) are fulfilled for polynomials of the form

$$\Phi(s) = \sum_{k=0}^{2m} a_k s^k, \quad \text{with } a_{2m} > 0.$$

The number $m \in \mathbb{N}$ is restricted by (H6) in case $n = 3$ and (H5) holds with $\theta = 1 - 1/2m$. In particular (H4)-(H6) are satisfied for the double-well potential $\Phi(s) = \Phi_0(s^2 - 1)^2,\ \Phi_0 > 0$.

5. Asymptotic Behaviour

In this section we investigate the asymptotic behaviour of the unique solution of the system

$$\begin{aligned} \partial_t\psi - \operatorname{div} B(x)\nabla\mu - \operatorname{div}(a(x)\partial_t\psi) &= 0, \quad t > 0,\ x \in \Omega, \\ \mu + \Delta\psi - \partial_t\psi - \Phi'(\psi) &= 0, \quad t > 0,\ x \in \Omega, \\ B(x)\nabla\psi \cdot \nu = B(x)\nabla\Delta\psi \cdot \nu &= 0, \quad t > 0,\ x \in \Gamma, \\ \psi(0) &= \psi_0 \quad t = 0,\ x \in \Omega, \end{aligned} \tag{5.1}$$

as time tends to infinity. The main tool to show convergence of the solution ψ of (5.1) will be the *Lojasiewicz-Simon inequality* (see Proposition 5.4). We recall the energy-functional $E(\psi)$ introduced before

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega \Phi(u)\, dx.$$

A suitable energy space for E will be

$$W := \{u \in H_2^1(\Omega) : \int_\Omega u\, dx = 0\}.$$

Note that the condition $\int_\Omega u\, dx = 0$ is compatible with our system. This might be seen by integrating $(5.1)_1$ and invoking the boundary conditions $(1.10)_3$ and (H2). We obtain $\int_\Omega \psi\, dx = \int_\Omega \psi_0\, dx$. If we replace the solution ψ by $\tilde{\psi} = \psi - c$, with $c = \frac{1}{|\Omega|}\int_\Omega \psi_0$ we see that $\int_\Omega \tilde{\psi}\, dx = 0$, if $\Phi(s)$ is replaced by $\tilde{\Phi}(s) := \Phi(s + c)$.

With the same methods as in [3, Lemma 6.2] (using (H6)), we can compute the Fréchet-derivative of E with respect to u, which is given by the formula

$$\langle E'(u), h\rangle_{W^*,W} = \int_\Omega \nabla u \nabla h\, dx + \int_\Omega \Phi'(u) h\, dx \tag{5.2}$$

for all $u, h \in W$. Furthermore, using (4.1) we see that the time derivative of $E(\psi(t))$ reads

$$\frac{d}{dt}\, E(\psi(t)) = -|\partial_t\psi|_2^2 - (a\partial_t\psi|\nabla\mu)_2 - |B^{1/2}\nabla\mu|_2^2 \leq -\varepsilon(|\nabla\mu|_2^2 + |\partial_t\psi|_2^2), \tag{5.3}$$

where we also used (H3). Applying (H4) and the Poincaré-Wirtinger inequality this yields $\psi \in C_b(\mathbb{R}_+; W)$.

Proposition 5.1. *Let ψ be the global solution of* (5.1) *with initial value $\psi_0 \in X_2$. Then $\psi(\mathbb{R}_+)$ is relatively compact in W.*

Proof. Since $\int_\Omega \partial_t\psi\ dx = \int_\Omega \Delta\psi\ dx = 0$ the operator A_1 is invertible, if we take

$$\mathbb{X} := \left\{ w \in L_p(\Omega) : \int_\Omega w\ dx = 0 \right\}$$

instead of $L_p(\Omega)$ as the ground space. Hence we obtain the equation

$$\partial_t\psi + A_3\psi = -\Phi'(\psi) - A_1^{-1}(I + A_2)\partial_t\psi.$$

The solution $\psi(t)$ to this equation is given by the variation of parameters formula

$$\psi(t) = e^{-A_3 t}\psi_0 - \int_0^t e^{-A_3(t-s)} \left(\Phi'(\psi(s)) + A_1^{-1}(I + A_2)\partial_t\psi(s)\right)\ ds.$$

We integrate the second integral in the latter equation by parts. This yields

$$\begin{aligned}\int_0^t e^{-A_3(t-s)} A_1^{-1}(I + A_2)\partial_t\psi(s)\ ds = A_1^{-1}(I + A_2)\psi(t) - e^{-A_3 t}A_1^{-1}(I + A_2)\psi_0 \\ - \int_0^t e^{-A_3(t-s)} A_3 A_1^{-1}(I + A_2)\psi(s)\ ds.\end{aligned}$$

The analytic semigroup $e^{-A_3 t}$, generated by A_3, is exponentially stable on $\mathbb{X}$. Choosing $q = \min\{2, 6/(\gamma + 2)\}$ we may apply semigroup theory in connection with (H6) and the estimate $\psi \in C_b(\mathbb{R}_+; W)$ to obtain $\psi \in C_b(\mathbb{R}_+; H_q^{2r}(\Omega))$ for each $r \in (0, 1)$. The space $H_q^{2r}(\Omega)$ is compactly embedded into $H_2^1(\Omega)$ if r is sufficiently close to 1, therefore the claim is proved. □

Now we collect some properties of the ω-limit set of the solution ψ.

Proposition 5.2. *Let ψ be the global solution of* (5.1) *with initial value $\psi_0 \in X_2$. Then the ω-limit set*

$$\omega(\psi) := \{\varphi \in W :\ \exists\ (t_n) \nearrow \infty \text{ s.t. } \psi(t_n) \to \varphi \text{ in } W\}$$

is non-empty, connected and compact. Moreover the energy-functional E is constant on $\omega(\psi)$.

Proof. The fact that $\omega(\psi)$ is non-empty, connected and compact follows from Proposition 5.1 and well known results in the theory of dynamical systems (see e.g. [2]). □

We remark that the corresponding stationary system to (5.1) is given by

$$\begin{cases} -\Delta\varphi + \Phi'(\varphi) = c^*, & x \in \Omega, \\ \partial_\nu\varphi = 0, & x \in \Gamma, \end{cases} \tag{5.4}$$

where $c^* = const$.

Proposition 5.3. *Let ψ be the global solution of* (5.1) *with initial value $\psi_0 \in X_2$. Then each $\varphi \in \omega(\psi)$ is a solution of the stationary system* (5.4) *and every $\varphi \in \omega(\psi)$ is a critical point of E, i.e. $\langle E'(\varphi), h\rangle_{W^*,W} = 0$ for all $h \in W$.*

Proof. We follow the lines of the proof of [3, Proposition 3.2 (iv)]. Let $\varphi \in \omega(\psi)$. Then there exists a sequence $(t_n) \nearrow \infty$ such that $\psi(t_n) \to \varphi$ in W as $n \to \infty$. From (5.3) we obtain $\partial_t\psi \in L_2(\mathbb{R}_+; L_2(\Omega))$, hence for every $s \in [0, 1]$ we have

$$\lim_{n\to\infty} \psi(t_n + s) = \varphi \text{ in } L_2(\Omega).$$

By Proposition 5.1 the above limit even exists in W. Integrating the equations $(5.1)_{1,2}$ from t_n to $t_n + 1$ we obtain that $\mu = c^* = const$ and φ is a solution to (5.4). The last claim of the proposition follows from (5.2) and integration by parts. □

Remark. By the same arguments as in [3, Proposition 6.4] it holds that every critical point φ of E is a solution of (5.4), i.e. $\varphi \in H_p^2(\Omega)$. This can be done by using a bootstrap argument and integration by parts. In other words this means that every weak solution of (5.4) is already a strong one and vice versa.

Given that each point which belongs to $\omega(\psi)$ is already a solution of (5.4), this raises the question, whether the limit $\lim_{t\to\infty}\psi(t)$ exists or not. This is a formidable problem, since in general the set $\mathcal{E}$ may be highly non-discrete. The *Lojasiewicz-Simon inequality* has turned out to be a very useful tool to answer this question in the affirmative.

Proposition 5.4 (Lojasiewicz-Simon inequality)**.** *Let $\varphi \in W$ be a critical point of the functional E. Assume that Φ is real analytic and satisfies (H6). Then there exist constants $C, \delta > 0,\ s \in (0, \frac{1}{2}]$, such that*

$$|E(u) - E(\varphi)|^{1-s} \leq C|E'(u)|_{W^*},$$

whenever $|u - \varphi|_W \leq \delta$.

Proof. See [3, Proposition 6.6] for a proof. □

Now we are ready to prove the main result of this section.

Theorem 5.5. *Let ψ be the global solution of the Cahn-Hilliard-Gurtin equation* (5.1) *with initial value $\psi_0 \in X_p$ and suppose that Φ satisfies the conditions (H4), (H6). Assume furthermore that Φ is real analytic. Then the limit*

$$\lim_{t\to\infty} \psi(t) =: \varphi$$

exists in W and φ is a solution of the stationary problem (5.4).

Proof. Proposition 5.4 yields, that for every $\varphi \in \omega(\psi)$ there exist constants $s \in (0, \frac{1}{2}]$, $C > 0$ and $\delta > 0$ such that

$$|E(u) - E(\varphi)|^{1-s} \leq C|E'(u)|_{W^*},$$

whenever $|u - \varphi|_W \leq \delta$. By Proposition 5.2 the ω-limit set $\omega(\psi)$ is compact, hence we may cover it by a union of *finitely* many balls with center φ_i and radius δ_i, $i = 1, \ldots, N$. Since $E(\psi) \equiv E_\infty$ on $\omega(\psi)$, there are *uniform* constants $s \in (0, \frac{1}{2}]$, $C > 0$ and an open set $U \supset \omega(u)$, such that

$$|E(u) - E_\infty|^{1-s} \leq C|E'(u)|_{W^*}, \tag{5.5}$$

for all $u \in U$. To prove convergence of $\psi(t)$ as $t \to \infty$, define a function $H : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$H(t) := (E(\psi(t)) - E_\infty)^s.$$

Note that $\lim_{t\to\infty} H(t) = 0$, since $E(\psi(t)) \searrow E_\infty$. A well known result in the theory of dynamical systems implies that $\lim_{t\to\infty} \operatorname{dist}(\psi(t), \omega(\psi)) = 0$, i.e. there exists $t^* \geq 0$, such that $\psi(t) \in U$, whenever $t \geq t^*$. Next by (5.3) and (5.5) we obtain

$$-\frac{d}{dt}\ H(t) = s\left(-\frac{d}{dt}\ E(\psi(t))\right)|E(\psi(t)) - E_\infty|^{s-1} \geq C\frac{\varepsilon(|\nabla\mu|_2^2 + |\partial_t\psi|_2^2)}{|E'(\psi(t))|_{W^*}}\ . \tag{5.6}$$

Now we have to estimate $E'(\psi(t))$ in W^*. From (5.2) it follows that

$$\begin{aligned}
|\langle E'(\psi), h\rangle_{W^*,W}| &= |\int_\Omega \nabla\psi\nabla h\ dx + \int_\Omega \Phi'(\psi)h\ dx| = |\int_\Omega (-\Delta\psi + \Phi'(u))h\ dx| \\
&= |\int_\Omega (\mu - \partial_t\psi)h\ dx| \leq c\left(|\int_\Omega (\mu - \bar{\mu})h\ dx| + |\int_\Omega \partial_t\psi h\ dx|\right) \\
&\leq C(|\mu - \bar{\mu}|_2 + |\partial_t\psi|_2)|h|_2 \leq C(|\nabla\mu|_2 + |\partial_t\psi|_2)|h|_W,
\end{aligned}$$

where $\bar{\mu} := \frac{1}{|\Omega|}\int_\Omega \mu\ dx$. Here we used the Poincaré-Wirtinger inequality and the fact that $\int_\Omega h\ dx = 0$. Thus we have

$$|E'(\psi(t))|_{W^*} = \sup_{|h|_W \leq 1} |\langle E'(\psi(t)), h\rangle_{W^*,W}| \leq C(|\nabla\mu|_2 + |\partial_t\psi|_2).$$

We insert this estimate into (5.6) to obtain

$$-\frac{d}{dt}\ H(t) \geq C(|\nabla\mu(t)|_2 + |\partial_t\psi|_2).$$

This implies $\partial_t\psi \in L_1(\mathbb{R}_+; L_2(\Omega))$, hence the limit $\lim_{t\to\infty} \psi(t) =: \varphi \in \omega(\psi)$ exists in $L_2(\Omega)$. Therefore the claim follows from relative compactness of $\psi(\mathbb{R}_+)$ in W and Proposition 5.3. □

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