The Attainment of the Solution of the Dual Program in Vertices for Vectorial Linear Programs

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Abstract

This article is a continuation of [14]. We developed in [14] a duality theory for convex vector optimization problems, which is different from other approaches in the literature. The main idea is to embed the image space $\mathbb{R}^q$ of the objective function into an appropriate complete lattice, which is a subset of the power set of $\mathbb{R}^q$. This leads to a duality theory which is very analogous to that of scalar convex problems. We applied these results to linear problems and showed duality assertions. However, in [14] we could not answer the question, whether the supremum of the dual linear program is attained in vertices of the dual feasible set. We show in this paper that this is, in general, not true but, it is true under additional assumptions.

1 Introduction

Vectorial linear programs play an important role in economics and finance and there have been many efforts to solve those problems with the aid of appropriate algorithms. There are several papers on variants of the simplex algorithm for the multiobjective case, see e.g. Armand, Malivert [1], Ecker, Hegner, Kouada [2], Ecker, Kouada [3], Evans, Steuer [5], Gal [6], Hartley [8], Isermann [9], Philip [16], [17], Yu, Zeleny [21] and Zeleny [22]. However, neither of these papers consider a dual simplex algorithm, which is in scalar linear programming a very important tool from the theoretical as well as the practical point of view. In the paper by Ehrgott, Puerto and Rodríguez-Chía [4] it was mentioned that "multi-objective duality theory cannot easily be used to develop a [dual or primal-dual simplex] algorithm". It is therefore our aim to consider an alternative approach to duality theory, which is appropriate for a dual simplex algoritm. This approach differs essentially from those in the literature (cf. Yu, Zeleny [21], Isermann [9] and Armand, Malivert [1]). In [14] we developed the basics of our theory and showed weak and strong duality assertions. The main idea is to embed the image space $\mathbb{R}^q$ of the objective function into a complete lattice, in fact into the space of

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self-infimal subsets of $\mathbb{R}^q \cup \{-\infty, +\infty\}$. As a result, many statements well-known from the case of scalar linear programming can be expressed analogously. However, in order to develop a dual simplex algorithm, it is important to have the property that the supremum of the dual problem is attained in vertices of the dual feasible set. This ensures that we only have to search a finite subset of feasible points. However, in [14] we could not show this attainment property, which is therefore the main subject of the present paper.

After a short introduction into the notation and the results of [14] we show that the attainment property is not true, in general. But, supposing some relatively mild assumptions, we can proof that for one of the three types of problems considered in [14], the supremum is indeed attained in vertices. This result is obtained by showing a kind of quasi-convexity of the (set-valued) dual objective function, which together with its concavity is a replacement for linearity. We further see that it is typical that the supremum is not attained in a single vertex (like in the scalar case) but in a set of possibly more than one vertex.

The application of these result in order to develop a dual simplex algirithm is presented in a forthcoming paper.

2 Preliminaries

We start to introduce the space of self-infimal sets, which plays an important role in the following. For a more detailed discussion of this space see [14].

Let $C \subsetneq \mathbb{R}^q$ be a closed convex cone with nonempty interior. The set of *minimal* or *weakly efficient* points of a subset $B \subseteq \mathbb{R}^q$ (with respect to C) is defined by

$$\operatorname{Min} B := \{y \in B |\ (\{y\} - \operatorname{int} C) \cap B = \emptyset\}.$$

The *upper closure* (with respect to C) of $B \subseteq \mathbb{R}^q$ is defined to be the set

$$\operatorname{Cl}_+ B := \{y \in \mathbb{R}^q |\ \{y\} + \operatorname{int} C \subseteq B + \operatorname{int} C\}.$$

Before we recall the definition of infimal sets, we want to extend the upper closure for subsets of the space $\overline{\mathbb{R}}^q := \mathbb{R}^q \cup \{-\infty, +\infty\}$. For a subset $B \subseteq \overline{\mathbb{R}}^q$ we set

$$\operatorname{Cl}_+ B := \begin{cases} \mathbb{R}^q & \text{if} \quad -\infty \in B \\ \emptyset & \text{if} \quad B = \{+\infty\} \\ \{y \in \mathbb{R}^q |\ \{y\} + \operatorname{int} C \subseteq B + \operatorname{int} C\} & \text{else.} \end{cases}$$

Note that the upper closure of a subset of $\overline{\mathbb{R}}^q$ is always a subset in $\mathbb{R}^q$. The *infimal set* of $B \subseteq \overline{\mathbb{R}}^q$ (with respect to C) is defined by

$$\operatorname{Inf}\ B := \begin{cases} \operatorname{Min}\operatorname{Cl}_+ B & \text{if} \quad \emptyset \neq \operatorname{Cl}_+ B \neq \mathbb{R}^q \\ \{-\infty\} & \text{if} \quad \operatorname{Cl}_+ B = \mathbb{R}^q \\ \{+\infty\} & \text{if} \quad \operatorname{Cl}_+ B = \emptyset. \end{cases}$$

This means that the *infimal set of B with respect to C* coincides essentially with the set of weakly efficient elements of the set $\operatorname{cl}(B+C)$ with respect to C. The supremal set of a set $B \subseteq \overline{\mathbb{R}}^q$ is defined analogously and is denoted by $\operatorname{Sup} B$. To this end, we have $\operatorname{Sup} B = -\operatorname{Inf}(-B)$.

In the sequel we need the following assertions due to Nieuwenhuis [15]. For $B \subseteq \mathbb{R}^q$ with $\emptyset \neq B + \operatorname{int} C \neq \mathbb{R}^q$ it holds

$$\operatorname{Inf} B = \{y \in \mathbb{R}^q |\ y \notin B + \operatorname{int} C,\ \{y\} + \operatorname{int} C \subseteq B + \operatorname{int} C\}\,, \tag{1}$$

$$\operatorname{Inf} B \cap B = \operatorname{Min} B. \tag{2}$$

Let $\mathcal{I}$ be the family of all self-infimal subsets of $\overline{\mathbb{R}}^q$, i.e., all sets $B \subseteq \overline{\mathbb{R}}^q$ satisfying $\operatorname{Inf} B = B$. In $\mathcal{I}$ we introduce an order relation $\preccurlyeq$ as follows:

$$B_1 \preccurlyeq B_2 :\Longleftrightarrow\ \operatorname{Cl}_+ B_1 \supseteq \operatorname{Cl}_+ B_2.$$

As shown in [14], there is an isotone bijection j between the space $(\mathcal{I}, \preccurlyeq)$ and the space $(\mathcal{F}, \supseteq)$ of upper closed subsets of $\mathbb{R}^q$ ordered by set inclusion. Indeed, one can choose

$$j : \mathcal{I} \to \mathcal{F}, \quad j(\,\cdot\,) = \operatorname{Cl}_+(\,\cdot\,), \quad j^{-1}(\,\cdot\,) = \operatorname{Inf}(\,\cdot\,).$$

Note that j is also isomorphic for an appropriate definition of an addition and a multiplication by nonnegative real numbers. Moreover, $(\mathcal{I}, \preccurlyeq)$ is a complete lattice and for nonempty sets $\mathcal{B} \subseteq \mathcal{I}$ it holds [14, Theorem 3.5]

$$\inf \mathcal{B} = \operatorname{Inf} \bigcup_{B \in \mathcal{B}} B, \qquad \sup \mathcal{B} = \operatorname{Sup} \bigcup_{B \in \mathcal{B}} B.$$

This shows that the infimum and the supremum in $\mathcal{I}$ are closely related to the usual solution concepts in vector optimization.

In [14] we considered the following three linear vector optimization problems. As usual in vector optimization we use the abbreviation $f[S] := \bigcup_{x \in S} f(x)$.

$$(\mathrm{LP}^1) \qquad \bar{P} = \operatorname{Inf} M[S], \qquad S := \{x \in \mathbb{R}^n |\ Ax \geq b\}\,,$$

$$(\mathrm{LP}^2) \qquad \bar{P} = \operatorname{Inf} M[S], \qquad S := \{x \in \mathbb{R}^n |\ x \geq 0,\ Ax \geq b\}\,,$$

$$(\mathrm{LP}^3) \qquad \bar{P} = \operatorname{Inf} M[S], \qquad S := \{x \in \mathbb{R}^n |\ x \geq 0,\ Ax = b\}\,,$$

where $M \in \mathbb{R}^{q \times n}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

In the following let a vector $c \in \operatorname{int} C$ be fixed. In [14] we calculated the dual problems to $(\mathrm{LP}^{\mathrm{i}})$ $(i = 1, 2, 3)$ (depending on c) as

$$(\mathrm{LD}^1_{\mathrm{c}}) \qquad \begin{cases} \bar{D}_c = \operatorname{Sup} \bigcup_{u \in T_c} \big(c\, u^T b\ +\ \operatorname{Inf}(M - c\, u^T A)[\mathbb{R}^n]\big) \\ T_c := \big\{u \in \mathbb{R}^m |\ u \geq 0,\ \exists c^* \in B_c : A^T u = M^T c^*\big\}\,, \end{cases}$$

$$(\mathrm{LD}^2_{\mathrm{c}}) \qquad \begin{cases} \bar{D}_c = \operatorname{Sup} \bigcup_{u \in T_c} \big(c\, u^T b\ +\ \operatorname{Inf}(M - c\, u^T A)[\mathbb{R}^n_+]\big) \\ T_c := \big\{u \in \mathbb{R}^m |\ u \geq 0,\ \exists c^* \in B_c : A^T u \leq M^T c^*\big\}\,. \end{cases}$$

$$(\mathrm{LD}^3_{\mathrm{c}}) \qquad \begin{cases} \bar{D}_c = \operatorname{Sup} \bigcup_{u \in T_c} \big(c\, u^T b\ +\ \operatorname{Inf}(M - c\, u^T A)[\mathbb{R}^n_+]\big) \\ T_c := \big\{u \in \mathbb{R}^m |\ \exists c^* \in B_c : A^T u \leq M^T c^*\big\}\,. \end{cases}$$

where the compact and convex set $B_c := \{c^* \in -C^\circ |\ \langle c, c^* \rangle = 1\}$ is used to express the dual side conditions. We obseverd in [14] that the set T_c in $(\mathrm{LD}^1_{\mathrm{c}})$-$(\mathrm{LD}^3_{\mathrm{c}})$ is always a closed convex subset of $\mathbb{R}^m$ and, if C is polyhedral, then T_c is polyhedral, too. Moreover, we have shown the following duality result.

Theorem 2.1 ([14]) *It holds weak and strong duality between* $(\mathrm{LP^i})$ *and* $(\mathrm{LD^i_c})$ $(i = 1, 2, 3)$. *More precisely we have*

(i) $\bar{D}_c = \bar{P} \subseteq \mathbb{R}^q$ *if* $S \neq \emptyset$ *and* $T_c \neq \emptyset$, *where "*Sup*" can be replaced by "*Max*" in this case,*

(ii) $\bar{D}_c = \bar{P} = \{-\infty\}$ *if* $S \neq \emptyset$ *and* $T_c = \emptyset$,

(iii) $\bar{D}_c = \bar{P} = \{+\infty\}$ *if* $S = \emptyset$ *and* $T_c \neq \emptyset$.

The following example [14] illustrates the dual problem and the strong duality. Moreover, in this example we have the attainment of the supremum of the dual problem in the (three) vertices of the dual feasible set T.

Example 2.2 ([14]) (see Figure 1) Let $q = m = n = 2$, $C = \mathbb{R}^2_+$ and consider the problem (LP^2) with the data

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

The dual feasible set for the choice $c = (1,1)^T \in \operatorname{int} \mathbb{R}^2_+$ is $T_c = \{u_1, u_2 \geq 0 |\ u_1 + u_2 \leq 1/3\}$. The vertices of T_c are the points $v_1 = (0,0)^T$, $v_2 = (1/3, 0)^T$ and $v_3 = (0, 1/3)^T$. We obtain the values of the dual objective function at v_1, v_2, v_3 as $D_c(v_1) = \operatorname{bd} \mathbb{R}^2_+$, $D_c(v_2) = \{y \in \mathbb{R}^2 |\ y_1 + 2y_2 = 2\}$ and $D_c(v_3) = \{y \in \mathbb{R}^2 |\ 2y_1 + y_2 = 2\}$. We see that the three dual feasible points $v_1, v_2, v_3 \in T_c$ are already sufficient for strong duality.

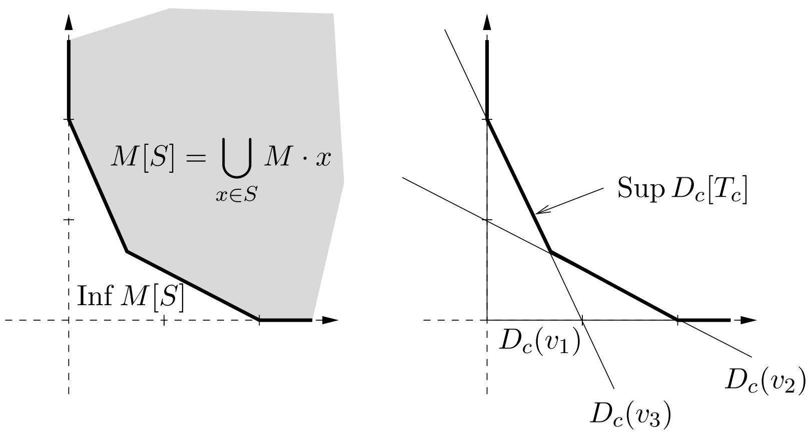

Figure 1: The primal and dual values in Example 2.2.

3 Dual Attainment in Vertices

We start with an example that shows that the supremum of the dual problem is, in general, not attained in vertices of the dual feasible set. Then we show that the dual attainmant in vertices can be ensured under certain additional assumptions.

Example 3.1 Let $q = m = n = 2$, $C = \mathbb{R}^2_+$ and consider the problem (LP^2) with the data

$$M = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -2 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

As above, we set $c = (1,1)^T \in \operatorname{int} \mathbb{R}^2_+$, hence $B_c = \{c_1^*, c_2^* \geq 0 \mid c_1^* + c_2^* = 1\}$. One easily verifies that the dual feasible set is the set $T_c = \operatorname{conv}\{v_1, v_2, v_3, v_4\}$, where $v_1 = (0,0)^T$, $v_2 = (1,0)^T$, $v_3 = (5,2)^T$ and $v_4 = (0, 1/3)^T$. However, the four vertices of T_c don't generate the supremum, in fact we have (see Figure 2)

$$\operatorname{Sup} \bigcup_{i=1}^{4} D_c(v_i) = D_c(v_1) = \operatorname{Inf} M[\mathbb{R}^2_+] = \mathbb{R}_+(0,1)^T \cup \mathbb{R}_+(1,-1)^T.$$

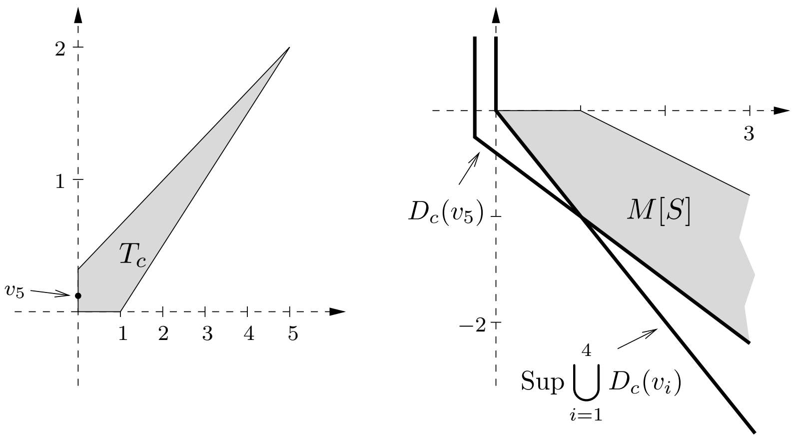

Figure 2: The dual feasible set and certain values of the dual objective in Example 3.1.

One can show that $v_1 = (0,0)^T$ together with $v_5 = (0, 1/8)^T$ generate the supremum, i.e.,

$$\bar{D}_c = \operatorname{Sup}\Big(D_c\big((0,0)^T\big) \cup D_c\big((0,1/8)^T\big)\Big),$$

but v_5 is not a vertice of T_c.

It is natural to ask for additional assumptions to ensure that the supremum of the dual problem is always generated by the vertices (or extreme points) of the dual feasible set T_c. We can give a positive answer for the problem (LD^1_c) by the following considerations. Moreover, we show that problem (LD^1_c) can be simplified under relatively mild assumptions.

Proposition 3.2 *Let $M \in \mathbb{R}^{q \times n}$, $\operatorname{rank} M = q$, $u \in \mathbb{R}^q$, $v \in \mathbb{R}^n$. Then, for the matrix $H := M - uv^T$ it holds $\operatorname{rank} H \geq q - 1$.*

Proof. We suppose $q \geq 2$, otherwise the assertion is obvious. The matrix consisting of the first k columns $\{a_1, a_2, ..., a_k\}$ of a matrix A is denoted by $A^{(k)}$. Without loss of generality we can suppose $\operatorname{rank} M^{(q)} = q$. Assume that $\operatorname{rank} H^{(q)} =: k \leq q - 2$. Without loss of generality we have $\operatorname{rank} H^{(k)} = k$. Since $\operatorname{rank} H^{(k+1)} = k$, there exist $w \in \mathbb{R}^{k+1} \setminus \{0\}$ such that $H^{(k+1)}w = 0$, hence $M^{(k+1)}w = u(v^T)^{(k+1)}w$. We have $\operatorname{rank} M^{(k+1)} = k+1$, hence $(v^T)^{(k+1)}w \neq 0$. It follows that $u \in M_{k+1}[\mathbb{R}^{k+1}] = \operatorname{lin}\{m_1, m_2, ..., m_{k+1}\} =: L$. Thus, for all $x \in \mathbb{R}^{k+1}$, we have $H^{(k+1)}x = M^{(k+1)}x - u(v^T)^{(k+1)}x \in L + L = L$. From $\operatorname{rank} M^{(q)} = q$ we conclude that $m_{k+2} \notin L$, hence $h_{k+2} = m_{k+2} + uv_{k+2} \notin L$. It follows that $\operatorname{rank} H^{(q)} \geq k+1$, a contradiction. Thus, we have $\operatorname{rank} H \geq \operatorname{rank} H^{(q)} \geq q-1$. □

Proposition 3.3 *Let $M \in \mathbb{R}^{q\times n}$, $\operatorname{rank} M = q$, $c \in \operatorname{int} C$, $c^* \in B_c$ and $H_{c^*} := M - cc^{*T}M$. It holds*

(i) $\operatorname{rank} H_{c^*} = q-1$,

(ii) *$H_{c^*}[\mathbb{R}^n]$ is a hyperplane in $\mathbb{R}^q$ orthogonal to c^*,*

(iii) $\operatorname{Inf} H_{c^*}[\mathbb{R}^n] = H_{c^*}[\mathbb{R}^n]$.

Proof. (i) We easily verify that $c^{*T}H_{c^*} = 0$, and so $\operatorname{rank} H_{c^*} < q$. Thus, the statement follows from Proposition 3.2. (ii) is immediate. (iii) We first show that $H_{c^*}[\mathbb{R}^n] + \operatorname{int} C = \{y \in \mathbb{R}^q |\ c^{*T}y > 0\}$. Of course, for $y \in H_{c^*}[\mathbb{R}^n] + \operatorname{int} C$ we have $c^{*T}y > 0$. Conversely, let $c^{*T}y > 0$ for some $y \in \mathbb{R}^q$. Then, there exists some $\lambda > 0$ such that $c^{*T}\lambda y = 1 = c^{*T}c$. It follows $\lambda y - c \in H_{c^*}[\mathbb{R}^n]$ and hence $y \in H_{c^*}[\mathbb{R}^n] + \operatorname{int} C$.

a) $H_{c^*}[\mathbb{R}^n] \subseteq \operatorname{Inf} H_{c^*}[\mathbb{R}^n]$. Assume that $y \in H_{c^*}[\mathbb{R}^n]$, but $y \notin \operatorname{Inf} H_{c^*}[\mathbb{R}^n]$. Then by (1) we have $y \in H_{c^*}[\mathbb{R}^n] + \operatorname{int} C$, hence $c^{*T}y > 0$, a contradiction.

b) $\operatorname{Inf} H_{c^*}[\mathbb{R}^n] \subseteq H_{c^*}[\mathbb{R}^n]$. Let $y \in \operatorname{Inf} H_{c^*}[\mathbb{R}^n]$ and take into account (1). On the one hand this means $y \notin H_{c^*}[\mathbb{R}^n] + \operatorname{int} C$ and hence $c^{*T}y \leq 0$. On the other hand we have $\{y\} + \operatorname{int} C \subseteq H_{c^*}[\mathbb{R}^n] + \operatorname{int} C$, i.e., for all $\lambda > 0$ it holds $c^{*T}(y + \lambda c) > 0$, whence $c^{*T}y \geq 0$. Thus, $c^{*T}y = 0$, i.e., $y \in H_{c^*}[\mathbb{R}^n]$. □

Theorem 3.4 *Consider problem* $(\mathrm{LD}^1_\mathrm{c})$*, where $M \in \mathbb{R}^{q\times n}$, $\operatorname{rank} M = q$, $c \in \operatorname{int} C$. Let the matrix $L \in \mathbb{R}^{q\times m}$ be defined by $L := (MM^T)^{-1}MA^T$. Then it holds*

(i) *For $u \in \mathbb{R}^m$, $c^* \in \mathbb{R}^q$: ($A^Tu = M^Tc^* \quad \Longrightarrow \quad c^* = Lu$).*

(ii) *$L[T_c] \subseteq B_c$.*

(iii) *The dual objective $D_c : T_c \to \mathcal{I}$, $D_c(u) := c\,u^Tb \,+\, \operatorname{Inf}(M - c\,u^TA)[\mathbb{R}^n]$ can be expressed as*

$$D_c(u) = \{y \in \mathbb{R}^q |\ \langle Lu, y\rangle = \langle u, b\rangle\}.$$

(iv) *For $u_1, u_2, ..., u_r \in T_c$, $\lambda_i \geq 0$ $(i = 1, ..., r)$ with $\sum_{i=1}^r \lambda_i = 1$ it holds*

$$D_c\Bigg(\sum_{i=1}^r \lambda_i u_i\Bigg) \preccurlyeq \operatorname{Sup} \bigcup_{i=1}^r D_c(u_i).$$

(v) *If $\bar{D}_c \subseteq \mathbb{R}^q$, the supremum of* $(\mathrm{LD}^1_\mathrm{c})$ *is generated by the set* $\operatorname{ext} T_c$ *of extreme points of T_c, i.e.,*

$$\operatorname{Sup} \bigcup_{u\in T_c} D_c(u) = \operatorname{Sup} \bigcup_{u\in \operatorname{ext} T_c} D_c(u).$$

Proof. (i) Since $\operatorname{rank} M = q$, $MM^T \in \mathbb{R}^{q\times q}$ is invertible, and so the statement is easy to verify.

(ii) Let $u \in T_c$. Hence there exists $c^* \in B_c$ such that $A^T u = M^T c^*$. By (i) it follows that $c^* = Lu$, whence $Lu \in B_c$.

(iii) Let $u \in T_c$. By (i) we have $A^T u = M^T Lu$. From Proposition 3.3 we obtain $D_c(u) = c\,u^T b + \{y \in \mathbb{R}^q |\ \langle Lu, y\rangle = 0\} =: B_1$. Of course, we have $B_1 \subseteq B_2 := \{y \in \mathbb{R}^q |\ \langle Lu, y\rangle = \langle u, b\rangle\}$. To see the opposite inclusion, let $y \in B_2$, i.e., $\langle Lu, y\rangle = \langle u, b\rangle$. It follows $c\,(Lu)^T y = cu^T b$. By (ii), we have $c^* := Lu \in B_c$. With the aid of Proposition 3.3, we obtain $y = cu^T b + y - cc^{*T} y \in \left\{cu^T b\right\} + (I - cc^{*T} I)[\mathbb{R}^n] = B_1$.

(iv) Consider the function $d_c : T_c \to \mathcal{F}$, defined by $d_c(u) := j(D_c(u)) = \operatorname{Cl}_+ D_c(u)$. Proceeding as in the proof of Proposition 3.3 (iii), we obtain $d_c(u) = \{y \in \mathbb{R}^q |\ \langle Lu, y\rangle \geq \langle u, b\rangle\}$ for all $u \in T_c$. One easily verifies the inclusion $d_c\big(\sum_{i=1}^r \lambda_i u_i\big) \supseteq \bigcap_{i=1}^r d_c(u_i)$. Since j is an isotone bijection between $(\mathcal{I}, \preccurlyeq)$ and $(\mathcal{F}, \supseteq)$, this is equivalent to the desired assertion.

(v) Let $u \in T_c$ be given. Since T_c is closed and convex and contains no lines, [18, Theorem 18.5] yields that there are extreme points $u_1, ..., u_k$ and extreme directions $u_{k+1}, ..., u_l$ of T_c such that

$$u = \sum_{i=1}^{l} \lambda_i u_i, \quad \text{with} \quad \lambda_i \geq 0 \quad (i = 1, ..., l), \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1,$$

(see [18, Sections 17 and 18]). Of course, we have $v := \sum_{i=k+1}^{l} \lambda_i u_i \in 0^+ T_c$ and $u - v \in T_c$. From (iv) we obtain

$$D_c(u - v) = D_c\left(\sum_{i=1}^{k} \lambda_i u_i\right) \preccurlyeq \sup_{i=1,...,k} D_c(u_i) \preccurlyeq \sup_{u \in \operatorname{ext} T_c} D_c(u).$$

It remains to show that $D_c(u) \preccurlyeq D_c(u - v)$.

Consider the set $V := \{u - v\} + \mathbb{R}_+ v \subseteq T_c$. By (ii), it holds $L[V] = L(u-v) + L[\mathbb{R}_+ v] \subseteq B_c$. Since $L[\mathbb{R}_+ v]$ is a cone, but B_c is bounded it follows that $L[\mathbb{R}_+ v] = \{0\}$. This implies $L(u - v + \lambda v) = c^*$ for all $\lambda \geq 0$, in particular, $L(u - v) = Lu = c^*$. From (iii), we now conclude that exactly one of the following assertions is true:

$$D_c(u) \preccurlyeq D_c(u - v) \quad \text{or} \quad \big(D_c(u - v) \preccurlyeq D_c(u) \ \wedge \ D_c(u - v) \neq D_c(u)\big).$$

We show that the second assertion yields a contradiction. Since $c^* = Lu = L(u - v) \in B_c \subseteq -C^\circ \setminus \{0\}$, we have $\langle u - v, b\rangle < \langle u, b\rangle$ and so $\langle v, b\rangle > 0$ in this case, hence $\langle u - v + \lambda v, b\rangle \to +\infty$ for $\lambda \to +\infty$. It follows that

$$\bar{D}_c = \sup_{u \in T_c} D_c(u) \succcurlyeq \sup_{u \in V} D_c(u) = \sup_{\lambda \geq 0} \{y \in \mathbb{R}^q |\ \langle c^*, y\rangle = \langle u - v + \lambda v, b\rangle\} = \{+\infty\}\,.$$

This contradicts the assumption $\bar{D}_c \subseteq \mathbb{R}^m$. $\square$

Corollary 3.5 *Consider problem* $(\mathrm{LD}^1_{\mathrm{c}})$, *where* $M \in \mathbb{R}^{q\times n}$, $\operatorname{rank} M = q$, $c \in \operatorname{int} C$ *and* C *polyhedral.*

If the supremum $\bar{D}_c$ of (LD_c^1) *is a subset of* $\mathbb{R}^q$*, then it is generated by the finitely many vertices* $u_1, ..., u_k$ *of (the nonempty polyhedral set)* T_c*. Moreover, we have*

$$\bar{D}_c = \mathrm{Sup} \bigcup_{i=1,...,k} D_c(u_i) = \mathrm{Max} \bigcup_{i=1,...,k} D_c(u_i).$$

Proof. The set T_c is polyhedral [14, Proposition 7.4]. Hence, $\mathrm{ext}\, T_c$ consists of finitely many points, called the vertices of T_c. The first equality follows from Theorem 3.4 (v).

To show the second equality let $y \in \mathrm{Sup} \bigcup_{i=1,...,k} D_c(u_i) \subseteq \mathbb{R}^q$ be given. By an assertion analogous to (1) this means $y \notin \bigcup_{i=1,...,k} D_c(u_i) - \mathrm{int}\, C$ and $\{y\} - \mathrm{int}\, C \subseteq \bigcup_{i=1,...,k} D_c(u_i) - \mathrm{int}\, C$. From the last inclusion we conclude that $y \in \mathrm{cl} \bigcup_{i=1,...,k} (D_c(u_i) - \mathrm{int}\, C) = \bigcup_{i=1,...,k} \mathrm{cl}\, (D_c(u_i) - \mathrm{int}\, C) = \bigcup_{i=1,...,k} (D_c(u_i) - C)$. Hence there exists some $i \in \{1, ..., k\}$ such that $y \in (D_c(u_i) - C) \setminus (D_c(u_i) - \mathrm{int}\, C)$. By the same arguments as used in the proof of Proposition 3.3 (iii) we can show the last statement means $y \in D_c(u_i)$, i.e. we have $y \in \bigcup_{i=1,...,k} D_c(u_i)$. The statement now follows from an assertion analogous to (2). $\square$

It is typical that more than one vertice is necessary to generate the infimum or supremum in case of vectorial linear programming. It remains the question how to determine a minimal subset of vertices of S and T_c that generates the infimum and supremum, respectively.

4 Comparison with duality based on scalarization

Duality assertions for linear vector optimization problems are derived by many authors (compare Isermann [10] and Jahn [11]). In these approaches the dual problem is constructed in such a way that the dual variables are linear mappings from $\mathbb{R}^m \to \mathbb{R}^q$, whereas in our approach the dual variables are vectors belonging to $\mathbb{R}^m$. In order to show strong duality assertions these authors suppose that $b \neq 0$. As shown in Theorem 2.1 we do not need such an assumption in order to prove strong duality assertions. However, there are several relations between our dualtity statements and those given by Jahn [11]. First, we recall an assertion given by Jahn [11] in order to compare our results wi th corresponding duality statements given by Jahn and others. In the following we consider (LD_c^1) for some $c \in \mathrm{int}\, C$.

Theorem 4.1 *(Jahn [11], Theorem 2.3)*
Assume that $\mathbb{V}$ *and* $\mathbb{Y}$ *are real separated locally convex linear spaces and* $b \in \mathbb{V}$, $u^* \in \mathbb{V}^*$, $y \in \mathbb{Y}$, $\lambda^* \in \mathbb{Y}^*$.

(i) *If there exists a linear mapping* $Z : \mathbb{V} \to \mathbb{Y}$ *with* $y = Z(b)$ *and* $u^* = Z^*(\lambda^*)$*, then* $\lambda^*(y) = u^*(b)$.

(ii) *If* $b \neq 0$, $\lambda^* \neq 0$ *and* $\lambda^*(y) = u^*(b)$*, then there exists a continuous linear mapping* $Z : \mathbb{V} \to \mathbb{Y}$ *with* $y = Z(b)$ *and* $u^* = Z^*(\lambda^*)$.

Usually, one considers in linear vector optimization dual problems of the form (Isermann [10] and Jahn [11])

$$(\mathrm{LD^o}) \qquad \mathrm{Max} \bigcup_{Z \in T_c^o} Zb,$$

where
$$T_c^o := \{Z \in \mathbb{R}^{q\times m} \mid \exists c^* \in B_c :\ Z^T c^* \geq 0, (ZA)^T c^* = M^T c^*\}. \tag{3}$$
We have the following relationships between (LD$_c^1$) and (LD^o):

Theorem 4.2 *Let $M \in \mathbb{R}^{q\times n}$, $\operatorname{rank} M = q$, $c \in \operatorname{int} C$ and $L := (MM^T)^{-1}MA^T \in \mathbb{R}^{q\times m}$. Then it holds*
$$D_1 := \bigcup_{u\in T_c} \{y \in \mathbb{R}^q \mid \langle Lu, y\rangle = \langle u, b\rangle\} \supseteq \bigcup_{Z\in T_c^o} Zb =: D_2$$
In the case of $b \neq 0$ we have equality, i.e., $D_1 = D_2$.

Proof. a) We show $D_2 \subseteq D_1$. Assume $y \in D_2$. Then there exists $Z \in T_c^o$ with a corresponding $c^* \in B_c$, i.e.,
$$Z^T c^* \geq 0, \quad (ZA)^T c^* = M^T c^* \tag{4}$$
and $y = Zb$. Put $u = Z^T c^*$, then Theorem 4.1 (i) yields
$$\langle c^*, y\rangle = \langle u, b\rangle. \tag{5}$$
Furthermore, taking into account (4) we obtain
$$u = Z^T c^* \geq 0$$
and
$$A^T u = A^T Z^T c^* = (ZA)^T c^* = M^T c^*.$$
By Theorem 3.4 (i) and (5), we conclude that $y \in D_1$.

b) We show $D_1 \subseteq D_2$ under the assumption $b \neq 0$. Suppose $y \in D_1$. Then there exists $u \in T_c$ with the corresponding $c^* \in B_c$, i.e.
$$u \geq 0, \tag{6}$$
$$A^T u = M^T c^* \tag{7}$$
and
$$\langle Lu, y\rangle = \langle u, b\rangle.$$
From Theorem 3.4 (i), (ii) we get $\langle c^*, y\rangle = \langle u, b\rangle$ and $c^* \in B_c$. So the assumptions $c^* \neq 0$, $b \neq 0$ of Theorem 4.1 (ii) are fulfilled and we conclude that there exists $Z \in \mathbb{R}^{q\times m}$ with $y = Zb$ and $u = Z^T c^*$. Moreover, we obtain by (7)
$$(ZA)^T c^* = A^T Z^T c^* = A^T u = M^T c^*$$
and from (6) we get
$$Z^T c^* = u \geq 0.$$
This yields $y \in D_2$, which completes the proof.

$\square$

Remark 4.3 Theorem 4.2 shows, that for linear vector optimization problems under the assumption $b \neq 0$ our dual problems coincides with the dual problems given in the papers [10] and [11].

The following example shows that the assumption $b \neq 0$ cannot be omitted in order to have the equality $D_1 = D_2$.

Example 4.4 Let $q = m = n = 2$, $C = \mathbb{R}^2_+$, $c = (1,1)^T$,

$$A = M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we have $L = M = A$, $T_c = B_c$, $D_2 = \left\{(0,0)^T\right\}$ and

$$D_1 = \left\{ y \in \mathbb{R}^2 |\ (y_1 \geq 0 \wedge y_2 \leq 0) \vee (y_1 \leq 0 \wedge y_2 \geq 0) \right\},$$

i.e., $D_1 \neq D_2$.

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