FRACTIONAL WHITE NOISE PERTURBATIONS OF PARABOLIC VOLTERRA EQUATIONS

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Abstract. Aim of this work is to extend the results of Clément, Da Prato & Prüss [3] on the fractional white noise perturbation with Hurst parameter $H \in (0,1)$. We will obtain similar results and it will turn out that the regularity of the solution $u(t)$ increases with Hurst parameter H.

1. Introduction and Notations

We are given a separable Hilbert space $\mathcal{H}$ with norm $|\cdot|_{\mathcal{H}}$ and inner product $(\cdot|\cdot)_{\mathcal{H}}$.

Let A be a closed linear densely defined operator in $\mathcal{H}$, and $b \in L_1(\mathbb{R}_+)$ a scalar kernel. As in [3] we consider the integro-differential equation

$$\begin{cases} \dot{u}(t) + \displaystyle\int_0^t b(t-\tau)Au(\tau)\mathrm{d}\tau = f(t), \quad t \geq 0, \\ \qquad\qquad\qquad\qquad u(0) = u_0. \end{cases} \tag{1.1}$$

Here the initial value u_0 is assumed to be an element of $\mathcal{H}$ and the forcing function f shall be of the form

$$f(t) = h(t) + Q^{1/2}\dot{B}^H(t),$$

with deterministic part $h \in L_{1,\mathrm{loc}}(\mathbb{R}_+;\mathcal{H})$ and B^H is a standard cylindrical fractional Brownian motion in $\mathcal{H}$ with Hurst parameter $H \in (0,1)$ (e.g. Grecksch and Anh [7]) with corresponding fractional white noise $\dot{B}^H$.

Because problem (1.1) is motivated from applications of linear viscoelastic material behavior, we consider $G \subset \mathbb{R}^N$ to be an open and bounded domain and the operator $-A$ to be an elliptic differential operator like the Laplacian, the elasticity operator, or the Stokes operator, together with appropriate boundary conditions

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(e.g. Prüss [10, Section I.5]). In the following we are particulary interested in the case $\mathcal{H} = L_2(G)$.

Hypothesis (A). *A is an unbounded, selfadjoint, positive definite operator in $\mathcal{H}$ with compact resolvent. Consequently, the eigenvalues μ_n of A form a nondecreasing sequence with $\lim_{n\to\infty} \mu_n = \infty$, the corresponding eigenvectors $(e_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ form an orthonormal basis of $\mathcal{H}$.*

Hypothesis (e). *There is a constant $C > 0$ such that*

$$|e_n(\xi)| \le C \quad \text{and} \quad |\nabla e_n(\xi)| \le C\mu_n^{1/2},$$

where $n \in \mathbb{N}$ and $\xi \in G$.

Hypothesis (b). *$b \in L_1(\mathbb{R}_+)$ is 3-monotone, i.e. b and $-\dot{b}$ are nonnegative, nonincreasing, convex; in addition,*

$$\lim_{t\to 0} \frac{\frac{1}{t}\int_0^t \tau b(\tau)\mathrm{d}\tau}{\int_0^t -\tau \dot{b}(\tau)\mathrm{d}\tau} < \infty. \tag{1.2}$$

Prüss proved in [10, Section I.1] that if **(A)** and **(b)** are valid, the integrated version of problem (1.1) admits a resolvent $S(t)$ (which is strongly continuous, uniformly bounded by 1, with $\lim_{t\to\infty} |S(t)|_{\mathcal{B}(\mathcal{H})} = 0$ and $S \in L_1(\mathbb{R}_+; \mathcal{B}(\mathcal{H}))$) such that the unique mild solution of (1.1) is given by the variation of parameters formula

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)\mathrm{d}\tau, \quad t \ge 0, \tag{1.3}$$

whenever $u_0 \in \mathcal{H}$ and $f \in L_{1,\mathrm{loc}}(\mathbb{R}_+; \mathcal{H})$.

By means of the spectral decomposition of A, the resolvent family $S(t)$ can be written explicitly as

$$S(t)x = \sum_{n=1}^{\infty} s_n(t)(x|e_n)e_n, \quad t \ge 0, \quad x \in \mathcal{H}, \tag{1.4}$$

where the scalar functions $s_n(t)$ are the solutions of the scalar problems

$$\dot{s}_n(t) + \mu_n \int_0^t b(t-\tau)s_n(\tau)\mathrm{d}\tau = 0, \quad t \ge 0, \quad s_n(0) = 1. \tag{1.5}$$

Next we want to give an abstract formulation of the assumptions on the covariance Q and the fractional white noise $\dot{B}^H$.

Hypothesis (B). *$Q \in \mathscr{L}_1(\mathcal{H})$ is selfadjoint, positive semi-definite and commutes with the operator A, i.e. there is a sequence $(\gamma_n) \in \ell_1(\mathbb{R}_+)$, such that $Qe_n = \gamma_n e_n$*

for all $n \in \mathbb{N}$. $B^H(t)$ *is of the form*

$$(B^H(t)|x) = \sum_{n=0}^{\infty} \beta_n^H(t)(x|e_n), \quad t \in \mathbb{R}, \quad x \in \mathcal{H}, \tag{1.6}$$

where $\beta_n^H(t)$ *are mutually independent real valued fractional Brownian motions with Hurst parameter* $H \in (0,1)$ *on the probability space* $(\Omega, \mathscr{F}, \mathbb{P})$.

It is well known that $B^H(t)$ (as in (1.6)) is not a well defined $\mathcal{H}$-valued random variable. However, due to $B^H(t) : \Omega \to \mathcal{H}_{Q^{-1/2}}$, where $\mathcal{H}_{Q^{-1/2}}$ is the completion of $\mathcal{H}$ with respect to the norm $|x|^2_{Q^{-1/2}} := |Q^{-1/2}x|_{\mathcal{H}}$, $x \in \mathcal{H}$, the forcing function f is well defined since $Q^{1/2}B^H(t)$ is a mapping with values in $\mathcal{H}$.

In the sequel an upper index $\langle t \rangle$, $t > 0$ at a function $f : \mathbb{R}_+ \to \mathbb{R}$ means

$$f^{\langle t \rangle}(\tau) := \begin{cases} f(t-\tau) & : \tau \le t; \\ 0 & : \tau > t. \end{cases}$$

Moreover, we will make use of the theory of integration with respect to fractional Brownian motions, which is provided by Pipiras and Taqqu [9]. Hence we denote the fractional integral of order $\alpha > 0$ of a function ϕ by $\mathcal{I}^\alpha \phi$, precisely this means

$$(\mathcal{I}^\alpha \phi)(r) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(\tau)(\tau - r)_+^{\alpha-1} \mathrm{d}\tau, \quad r \in \mathbb{R},$$

where $(x)_+ = 0$ if $x \le 0$, and $(x)_+ = x$ if $x > 0$. By means of the Marchaud fractional derivative we introduce the left inverse of $\mathcal{I}^\alpha$ as $\mathcal{D}^\alpha$ for $\alpha > 0$, i.e. for appropriate functions ϕ it holds that

$$\mathcal{D}^\alpha(\mathcal{I}^\alpha \phi) \equiv \phi. \tag{1.7}$$

Next we want to characterize the class of integrands f with respect to a fractional Brownian motion, such that the integral $\int_{\mathbb{R}} f(\tau)\mathrm{d}B^H(\tau)$ is well defined. In order to study the most general case, we consider the space Λ_H for integrands in the time domain which arises as

$$\Lambda_H := \left\{ f : \int_{\mathbb{R}} \left[(\mathcal{D}^{\frac{1}{2}-H} f)(r) \right]^2 \mathrm{d}r < \infty \right\} \quad \text{for} \quad 0 < H < \tfrac{1}{2}, \tag{1.8}$$

or alternatively as

$$\Lambda_H := \left\{ f : \int_{\mathbb{R}} \left[(\mathcal{I}^{H-\frac{1}{2}} f)(r) \right]^2 \mathrm{d}r < \infty \right\} \quad \text{for} \quad \tfrac{1}{2} < H < 1. \tag{1.9}$$

In both cases Λ_H is a linear space with inner product

$$(f|g)_{\Lambda_H} = \frac{\Gamma^2(H + \frac{1}{2})}{\zeta^2(H - \frac{1}{2})} \int_{\mathbb{R}} (\mathcal{D}^{\frac{1}{2}-H} f)(r)(\mathcal{D}^{\frac{1}{2}-H} g)(r)\mathrm{d}r,$$

or accordingly

$$(f|g)_{\Lambda_H} = \frac{\Gamma^2(H+\frac{1}{2})}{\zeta^2(H-\frac{1}{2})}\int_{\mathbb{R}} (\mathcal{I}^{H-\frac{1}{2}}f)(r)(\mathcal{I}^{H-\frac{1}{2}}g)(r)\mathrm{d}r,$$

where

$$\zeta(H) = \left[\int_0^\infty \left[(1+\tau)^H - \tau^H\right]^2 \mathrm{d}\tau + \frac{1}{2H+1}\right]^{1/2}.$$

Pipiras and Taqqu proved in [9, Proposition 3.2], that the embeddings

$$L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \hookrightarrow L_{1/H}(\mathbb{R}) \hookrightarrow \Lambda_H, \tag{1.10}$$

hold true for $H \in (\frac{1}{2}, 1)$.

In the spectral domain we are interested in integrands being a member of the homogeneous Bessel potential space of order $\frac{1}{2} - H$,

$$\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}) = \left\{ f \in \mathcal{S}^*(\mathbb{R}) : \int_{\mathbb{R}} |\mathcal{F}f(\tau)|^2 |\tau|^{-2H+1}\mathrm{d}\tau < \infty \right\},$$

where $\mathcal{S}^*$ is the space of tempered distributions. It is well known, that for $f \in \dot{H}_2^{\frac{1}{2}-H}(\mathbb{R})$ the Fourier transform of $\mathcal{I}^{H-\frac{1}{2}}f$ or $\mathcal{D}^{\frac{1}{2}-H}f$ is

$$\psi_{H-\frac{1}{2}}(x)(\mathcal{F}f)(x)|x|^{\frac{1}{2}-H} = (\mathcal{F}f)(x)(ix)^{\frac{1}{2}-H}, \tag{1.11}$$

where

$$\psi_\alpha(x) = e^{-i\pi\alpha/2}\chi_{\{x>0\}} + e^{i\pi\alpha/2}\chi_{\{x<0\}}, \quad x \in \mathbb{R}.$$

Here χ_M denotes the indicator function of the set M. Hence by Plancherel's Theorem it holds that

$$\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}) \cong \Lambda_H. \tag{1.12}$$

An easy calculus shows that the identity

$$(f|g)_{\Lambda_H} = \mathbb{E}\left[\left(\int_{\mathbb{R}} f(\tau)\mathrm{d}\beta^H(\tau)\right)\left(\int_{\mathbb{R}} g(\tau)\mathrm{d}\beta^H(\tau)\right)\right] \tag{1.13}$$

holds for all $f, g \in \mathcal{E}$, where $\mathcal{E}$ denotes the set of all elementary functions. Since $\mathcal{E}$ is dense in Λ_H (see [9, Theorems 3.2 resp. 3.3]) equation (1.13) holds for all $f, g \in \Lambda_H$.

Remark.

(i) Since by (1.11) $\mathcal{F}(\mathcal{I}^{-\kappa} f) \equiv \mathcal{F}(\mathcal{D}^{\kappa} f)$ and by Plancherel's Theorem the norm in $\dot{H}_2^{\kappa}(\mathbb{R})$ can be rewritten as

$$|f|_{\dot{H}_2^{\kappa}(\mathbb{R})} = \begin{cases} |\mathcal{D}^{\kappa} f|_{L_2(\mathbb{R})} & : \kappa \geq 0 \\ |\mathcal{I}^{-\kappa} f|_{L_2(\mathbb{R})} & : \kappa < 0 \end{cases} \tag{1.14}$$

(ii) Observe that equation (1.13) also holds on an arbitrary set $M \subset \mathbb{R}$. This can be seen by replacing f and g by $f\chi_M$ and $g\chi_M$, respectively.

The plan of our paper is as follows. In Section 2 we state the main results about fractional white noise perturbations of equations in linear viscoelasticity, i.e. equation (1.1), assuming the Hypotheses **(A)**, **(b)**, and **(B)** explained above. These results are proved in Section 3 by means of the methods introduced in the monograph by Da Prato and Zabczyk [5], adapted to evolutionary integral equations in Clément and Da Prato [1], [2]. The required estimates were already available and taken from Monniaux and Prüss [8] and Clément, Da Prato & Prüss [3].
Section 4 is devoted to a study of the equation

$$u + g_\alpha * Au = g_\beta * Q^{1/2} \dot{B}^{\mathsf{H}}$$

on the halfline, where $g_\kappa(t) = t^{\kappa-1}/\Gamma(\kappa)$, $t > 0$ for $\kappa > 0$ denotes the Riemann-Liouville kernel of fractional integration.

2. Main results

Concentrating on the stochastic case we let $h(t) = 0$, i.e. $f(t) = Q^{1/2}\dot{B}^{\mathsf{H}}(t)$; w.l.o.g. we set $u_0 = 0$. This means that we have to investigate the stochastic convolution

$$u(t) = \int_0^t S(t-\tau)\mathrm{d}(Q^{1/2}B^{\mathsf{H}})(\tau), \quad t \geq 0. \tag{2.1}$$

In virtue of the spectral decompositions of A and Q we may rewrite

$$u(t) = \sum_{n=1}^{\infty} \sqrt{\gamma_n} \int_0^t s_n(t-\tau) e_n \mathrm{d}\beta_n^{\mathsf{H}}(\tau), \quad t \geq 0. \tag{2.2}$$

Our main result on problem (1.1) reads as follows.

Theorem 1. *Let $\mathsf{H} \in (0,1)$. Assume that Hypotheses **(A)**, **(b)**, **(B)** are valid and suppose*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{-\frac{2\mathsf{H}}{\rho}} < \infty, \tag{2.3}$$

where

$$\rho := 1 + \frac{2}{\pi} \sup\{|\arg \widehat{b}(\lambda)| : \operatorname{Re} \lambda > 0\}. \tag{2.4}$$

Then the series (2.2) converges in $L_2(\Omega; \mathcal{H})$*, uniformly in* t *on bounded subsets of* $\mathbb{R}_+$ *and* $u \in C_b(\mathbb{R}_+; L_2(\Omega; \mathcal{H}))$*.* $u(t)$ *is a Gaussian random variable with mean zero and covariance operator* Q_t*, defined by*

$$Q_t x = \sum_{n=1}^{\infty} \left\| s_n^{\langle t \rangle} \sqrt{\gamma_n} \right\|_{\Lambda_H}^2 (x|e_n) e_n, \quad x \in \mathcal{H}, \tag{2.5}$$

and we have $\operatorname{Tr}[Q_t] \leq c_H \operatorname{Tr}[Q A^{-2H/\rho}]$*.*
If in addition, there is $\theta \in (0,1)$ *such that*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2H(\theta-1)}{\rho}} < \infty, \tag{2.6}$$

then for each $\alpha \in (0, \theta H)$*, the trajectories of* $u(t)$ *are almost everywhere* α*-Hölder-continuous, i.e.* $u \in C_b^{\alpha}(\mathbb{R}_+; L_2(\Omega; \mathcal{H}))$*.*
In case $\mathcal{H} = L_2(G)$ *and Hypothesis* ***(e)*** *as well as*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{\theta-2H}{\rho}} < \infty, \tag{2.7}$$

are met, the trajectories of $u(t, \xi)$ *are almost surely* α*-Hölder-continuous in* ξ*, for each exponent* $\alpha \in (0, \theta)$*, i.e.* $u \in C_b(\mathbb{R}_+; C^{\alpha}(G; L_2(\Omega)))$*.*

Here and in the sequel we denote by $c_H > 0$ a generic constant depending on H.

Remark.

(i) $\Lambda_{\frac{1}{2}}$ is isometrically isomorphic to $L_2(\mathbb{R})$. In this sense Theorem 1 is a generalization of [3, Theorem 2.1].
(ii) Note that by Hypothesis **(A)** and **(b)** the problem under consideration is *parabolic*, i.e. $\rho \in [1, 2)$.

3. Proof of the main results

The idea of the proof is, of course, similar to Clément et al. [3] and follows the arguments for the Cauchy problem presented in Da Prato and Zabczyk [5].

Let us cite a useful lemma which was proven in [3, Lemma 3.1]:

Lemma 3.1. *Suppose the kernel* $b(t)$ *is subject to Hypothesis* ***(b)****, and let* $\rho \in (1, 2)$ *be defined by (2.4). Then for every* $n \in \mathbb{N}$ *it is*

(i) $|s_n(t)| \leq 1$ *for all* $t, \mu_n > 0$*;*

(ii) $|\dot{s}_n|_{L_1(\mathbb{R}_+)} \le C$ *for all* $\mu_n > 0$*;*

(iii) $|t\dot{s}_n|_{L_1(\mathbb{R}_+)} \le C\mu_n^{-1/\rho}$ *for all* $\mu_n > 0$*;*

(iv) $|s_n|_{L_1(\mathbb{R}_+)} \le C\mu_n^{-1/\rho}$ *for all* $\mu_n > 0$*,*

where $C > 0$ *denotes a constant which is independent of* $\mu_n > 0$*.*

Now, let the hypotheses of Theorem 1 be fulfilled. Observe that for $H \in (\frac{1}{2}, 1)$ by (iv) and (i) of Lemma 3.1 the functions $s_n^{\langle t\rangle}$ belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and hence by embedding (1.10) to Λ_H. So by identity (1.13) we obtain

$$\mathbb{E}|u(t)|_{\mathcal{H}}^2 = c_H \sum_{n=1}^{\infty} \gamma_n \int_{\mathbb{R}} \left[(\mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle})(r) \right]^2 \mathrm{d}r = c_H \sum_{n=1}^{\infty} \gamma_n \left| \mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle} \right|_{L_2(\mathbb{R})}^2 . \tag{3.1}$$

As a result of [11, Theorem 5.3] the operator $\mathcal{I}^{H-\frac{1}{2}}$ is bounded from $L_{1/H}(\mathbb{R})$ into $L_2(\mathbb{R})$. Thus we have by (i) and (iv) of Lemma 3.1

$$\mathbb{E}|u(t)|_{\mathcal{H}}^2 \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|_{L_{1/H}(\mathbb{R})}^2 \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|_{L_1(\mathbb{R})}^{2H} \le c_H \sum_{n=1}^{\infty} \gamma_n \mu_n^{-2H/\rho} \tag{3.2}$$

which is finite by assumption. In the case $H \in (0, \frac{1}{2})$ one may argue as in the latter situation to obtain

$$\mathbb{E}|u(t)|_{\mathcal{H}}^2 = c_H \sum_{n=1}^{\infty} \gamma_n \left| \mathcal{D}^{\frac{1}{2}-H} s_n^{\langle t\rangle} \right|_{L_2(\mathbb{R})}^2 \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|^2_{H_2^{\frac{1}{2}-H}(\mathbb{R})} \tag{3.3}$$

and hence by interpolation and Lemma 3.1

$$\mathbb{E}|u(t)|_{\mathcal{H}}^2 \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|_{L_1(\mathbb{R})}^{2H} \cdot |s_n^{\langle t\rangle}|_{H_1^1(\mathbb{R})}^{2(1-H)} \le c_H \sum_{n=1}^{\infty} \gamma_n \mu_n^{-2H/\rho} \tag{3.4}$$

holds.

Thus $u(t)$ is a zero mean $\mathcal{H}$-valued Gaussian random variable. Let Q_t be its covariance operator, then for $H \in (\frac{1}{2}, 1)$

$$\begin{aligned}
(Q_t x|y)_{\mathcal{H}} &= \mathbb{E}\left[(u(t)|x)(u(t)|y)\right] \\
&= \sum_{n=1}^{\infty} \gamma_n (e_n|x)(e_n|y) \mathbb{E} \left| \int_{\mathbb{R}} s_n^t(\tau) \mathrm{d}\beta_n^H(\tau) \right|^2 \\
&= \frac{\Gamma^2(H+\frac{1}{2})}{\zeta^2(H-\frac{1}{2})} \sum_{n=1}^{\infty} (e_n|x)(e_n|y) \int_{\mathbb{R}} \left[(\mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle} \sqrt{\gamma_n})(\tau) \right]^2 \mathrm{d}\tau \\
&= \frac{\Gamma^2(H+\frac{1}{2})}{\zeta^2(H-\frac{1}{2})} \sum_{n=1}^{\infty} \left(\int_{\mathbb{R}} \left[(\mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle} \sqrt{\gamma_n}) \right]^2 (\tau) \mathrm{d}\tau (e_n|x) e_n \middle| y \right),
\end{aligned}$$

and with help of (3.1) and (3.2) $\mathrm{Tr}[Q_t] \le c_H \,\mathrm{Tr}[QA^{-2H/\rho}]$ follows. Replacing $\mathcal{I}^{H-\frac{1}{2}}$ by $\mathcal{D}^{\frac{1}{2}-H}$ yields the claim for $H \in (0, \frac{1}{2})$.

Concerning Hölder-continuity we will use the following two estimates with the convention $s_n(\tau) = 0$ for $\tau < 0$.

Lemma 3.2. *Suppose that the kernel $b(t)$ is subject to Hypothesis **(b)** and let $\kappa \in (1,2)$. Then for each $\theta \in (0,1)$ there is a constant $C_\theta > 0$ such that*

$$\int_x^t |s_n(t-\tau)|^\kappa \mathrm{d}\tau \le C_\theta \mu_n^{(\theta-1)/\rho} |t-x|^\theta, \quad 0 < x < t, \tag{3.5}$$

and

$$\int_{-\infty}^x |s_n(t-\tau) - s_n(x-\tau)|^\kappa \mathrm{d}\tau \le C_\theta \mu_n^{(\theta-1)/\rho} |t-x|^\theta, \quad x < t. \tag{3.6}$$

The proof of Lemma 3.2 follows exactly the lines of [3, Proof of Lemma 3.1]. Therefore we omit it.

For $H \in (\frac{1}{2}, 1)$ we use the identity (1.13) to obtain

$$\begin{aligned}\mathbb{E}|u(t)-u(x)|^2_{\mathcal{H}} &= \mathbb{E}(u(t)-u(x)|u(t)-u(x))_{\mathcal{H}} = \sum_{n=1}^\infty \gamma_n (s_n^{\langle t\rangle} - s_n^{\langle x\rangle} | s_n^{\langle t\rangle} - s_n^{\langle x\rangle})_{\Lambda_H} \\ &= c_H \sum_{n=1}^\infty \gamma_n \left| \mathcal{I}^{H-\frac{1}{2}} (s_n^{\langle t\rangle} - s_n^{\langle x\rangle}) \right|^2_{L_2(\mathbb{R})} \le c_H \sum_{n=1}^\infty \gamma_n |s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|^2_{L_{1/H}(\mathbb{R})}\end{aligned}$$

and we have

$$|s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|_{L_{1/H}} = \left[\int_{-\infty}^x |s_n(t-\tau) - s_n(x-\tau)|^{1/H} \mathrm{d}\tau + \int_x^t |s_n(t-\tau)|^{1/H} \mathrm{d}\tau \right]^H.$$

For $H \in (0, \frac{1}{2})$ it is

$$\mathbb{E}|u(t)-u(x)|^2_{\mathcal{H}} = c_H \sum_{n=1}^\infty \gamma_n \left| \mathcal{D}^{\frac{1}{2}-H} (s_n^{\langle t\rangle} - s_n^{\langle x\rangle}) \right|^2_{L_2(\mathbb{R})}$$

and the estimate

$$\left| \mathcal{D}^{\frac{1}{2}-H} (s_n^{\langle t\rangle} - s_n^{\langle x\rangle}) \right|^2_{L_2(\mathbb{R})} \le c |s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|^2_{H_2^{\frac{1}{2}-H}(\mathbb{R})} \le \tilde{c} |s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|^{2H}_{L_1(\mathbb{R})} \tag{3.7}$$

holds for sufficient large $n \in \mathbb{N}$, by interpolation and Lemma 3.1. Thus by employing Lemmata 3.1 and 3.2 this yields

$$\mathbb{E}|u(t)-u(x)|^2_{\mathcal{H}} \le c_H |t-x|^{2\theta H} \sum_{n=1}^\infty \gamma_n \mu_n^{2H(\theta-1)/\rho},$$

for $H \in (0,1)$ and we may conclude Hölder-continuity of $u(t)$ as in the proofs given in Clément and Da Prato [1] or [2]. Similarly, in case **(e)** holds, we obtain Hölder-continuity in space from the identities

$$\mathbb{E}|u(t,\xi) - u(t,\eta)|^2 = c_H \sum_{n=1}^\infty \gamma_n \left| \mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle} \right|^2_{L_2(\mathbb{R})} |e_n(\xi) - e_n(\eta)|^2$$

for $H \in (\frac{1}{2}, 1)$ and

$$\mathbb{E}|u(t,\xi) - u(t,\eta)|^2 = c_H \sum_{n=1}^{\infty} \gamma_n \left| \mathcal{D}^{\frac{1}{2}-H} s_n^{\langle t \rangle} \right|^2_{L_2(\mathbb{R})} |e_n(\xi) - e_n(\eta)|^2$$

for $H \in (0, \frac{1}{2})$ respectively.

4. Fractional derivatives and fractional white noise

In the remaining part of this paper we take up a different viewpoint to equations with fractional noise. We consider the problems

$$u + g_\alpha * Au = g_\beta * Q^{1/2} \dot{B}^H \tag{4.1}$$

in the Hilbert space $\mathcal{H}$, where the operator A is subject to Hypothesis **(A)** and also to **(e)** if appropriate, the covariance Q and the fractional Brownian motion B^H are subject to **(B)**, and g_κ denotes the fractional integration kernel

$$g_\kappa(t) = \frac{t^{\kappa-1}}{\Gamma(\kappa)}, \quad t > 0,$$

where $\kappa > 0$. Note that if we set $\beta = 1$ and $a = g_\alpha$ for $\alpha \in (0, 1]$ equation (4.1) is a special case of problem (1.1).

For $\alpha \in (0, 2)$, $\beta > 0$, define the scalar fundamental solution of (4.1) by

$$\widehat{r}_n(\lambda) = \frac{\widehat{g}_\beta(\lambda)}{1 + \mu_n \widehat{g}_\alpha(\lambda)} = \frac{\lambda^\alpha}{\lambda^\beta(\lambda^\alpha + \mu_n)}, \quad \operatorname{Re}\lambda \geq 0, \quad \lambda \neq 0, \quad \mu_n > 0, \tag{4.2}$$

where $\widehat{r}_n$ denotes the Laplace transform of r_n. Furthermore with the convention $r_n(\tau) = 0$ for $\tau < 0$ we have by the Paley-Wiener Theorem

$$\begin{aligned} |r_n|_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R})} &= \int_{\mathbb{R}} |(\mathcal{F}r_n)(\rho)|^2 |\rho|^{1-2H} \mathrm{d}\rho \\ &\leq c_\alpha \int_{\mathbb{R}} \Big[\frac{|\rho|^\alpha}{|\rho|^\beta (|\rho|^\alpha + \mu_n)} \Big]^2 |\rho|^{1-2H} \mathrm{d}\rho \\ &= 2c_\alpha \int_0^\infty \Big[\frac{\rho^\alpha}{\rho^\beta(\rho^\alpha + \mu_n)} \Big]^2 \rho^{1-2H} \mathrm{d}\rho \\ &= 2c_\alpha \mu_n^{\frac{2(1-\beta-H)}{\alpha}} \int_0^\infty \Big[\frac{\tau^{\alpha-\beta-H+\frac{1}{2}}}{1+\tau^\alpha} \Big]^2 \mathrm{d}\tau, \end{aligned} \tag{4.3}$$

and the right integral is finite if and only if $1 - H < \beta < 1 - H + \alpha$. Thus by isomorphism (1.12) r_n belongs to Λ_H whenever $\alpha \in (0, 2)$ and $\beta \in (1-H, 1-H+\alpha)$. The solution of (4.1) can be rewritten as

$$u(t) = \sum_{n=1}^{\infty} \sqrt{\gamma_n} \int_0^t r_n(t-\tau) \mathrm{d}\beta_n^H(\tau) e_n, \quad t > 0, \tag{4.4}$$

and therefore as in Section 3 it is by means of representation (1.14)

$$\mathbb{E}|u(t)|^2_{\mathcal{H}} = c_H \sum_{n=1}^{\infty} \gamma_n |r_n^{\langle t\rangle}|^2_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R})} \tag{4.5}$$

as well as

$$\mathbb{E}|u(t)-u(x)|^2_{\mathcal{H}} = c_H \sum_{n=1}^{\infty} \gamma_n |r_n^{\langle t\rangle} - r_n^{\langle x\rangle}|^2_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R})} \tag{4.6}$$

and in case $\mathcal{H} = L_2(G)$ and **(e)** is valid

$$\mathbb{E}|u(t,\xi)-u(t,\eta)|^2 = c_H \sum_{n=1}^{\infty} \gamma_n |r_n^{\langle t\rangle}|^2_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R})} |e_n(\xi)-e_n(\eta)|^2. \tag{4.7}$$

Moreover

$$|r_n^{\langle t\rangle}|_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \le |r_n^{\langle t\rangle}|_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R})} = |r_n^{\langle 0\rangle}|_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R})} = |r_n|_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}$$

holds $t \ge 0$, as soon as $r_n \in \dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)$. Identities (4.5) and (4.6) show that the solution $u(t)$ of (4.1) exists and is continuous in $L_2(\Omega;\mathcal{H})$ if and only if

$$\sigma_1 := \sum_{n=1}^{\infty} \gamma_n |r_n|^2_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} < \infty. \tag{4.8}$$

Next observe that we have for $H \in (\frac{1}{2}, 1)$

$$\begin{aligned}
\left|\mathcal{I}^{H-\frac{1}{2}}(r_n^{\langle t\rangle} - r_n^{\langle x\rangle})\right|^2_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} \left|(\mathcal{I}^{H-\frac{1}{2}} r_n^{\langle t\rangle})(\tau) - (\mathcal{I}^{H-\frac{1}{2}} r_n^{\langle x\rangle})(\tau)\right|^2 \mathrm{d}\tau \\
&= \int_{\mathbb{R}} \left|(\mathcal{I}^{H-\frac{1}{2}} r_n^{\langle 0\rangle})(\tau - t) - (\mathcal{I}^{H-\frac{1}{2}} r_n^{\langle 0\rangle})(\tau - x)\right|^2 \mathrm{d}\tau \\
&= \left|(\mathcal{I}^{H-\frac{1}{2}} r_n^{\langle 0\rangle})(x-t+\cdot) - (\mathcal{I}^{H-\frac{1}{2}} r_n^{\langle 0\rangle})(\cdot)\right|^2_{L_2(\mathbb{R})} \\
&\le \left|\mathcal{I}^{H-\frac{1}{2}} r_n^{\langle 0\rangle}\right|^2_{B^\theta_{2,\infty}(\mathbb{R})} |t-x|^{2\theta}
\end{aligned}$$

and analogue for $H \in (0, \frac{1}{2})$

$$\left|\mathcal{D}^{\frac{1}{2}-H}(r_n^{\langle t\rangle} - r_n^{\langle x\rangle})\right|^2_{L_2(\mathbb{R})} \le \left|\mathcal{D}^{\frac{1}{2}-H} r_n^{\langle 0\rangle}\right|^2_{B^\theta_{2,\infty}(\mathbb{R})} |t-x|^{2\theta},$$

where $B^\theta_{2,\infty}(\mathbb{R})$ denotes a Besov space. Now we have the embedding

$$H_2^\theta(\mathbb{R}) \hookrightarrow B^\theta_{2,\infty}(\mathbb{R}),$$

cf. [12, Theorem 2.3.2 (c)], and the apparent relation

$$|f|_{\dot{H}_2^\kappa(\mathbb{R})} + |f|_{\dot{H}_2^{\theta+\kappa}(\mathbb{R})} = \begin{cases} |\mathcal{D}^\kappa f|_{H_2^\theta} & : \kappa \ge 0, \\ |\mathcal{I}^{-\kappa} f|_{H_2^\theta} & : \kappa < 0. \end{cases} \tag{4.9}$$

So the condition

$$\sigma_2 := \sum_{n=1}^{\infty} \gamma_n \left[|r_n|_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} + |r_n|_{\dot{H}_2^{\theta+\frac{1}{2}-H}(\mathbb{R}_+)} \right]^2 < \infty \tag{4.10}$$

implies Hölder continuity of $u(t)$ in time of order θ. Finally from **(e)** we obtain by interpolation

$$|e_n(\xi) - e_n(\eta)| \leq C|\xi - \eta|^{\theta} \mu_n^{\theta/2},$$

hence

$$\sigma_3 := \sum_{n=1}^{\infty} \gamma_n \mu_n^{\theta} |r_n|^2_{\dot{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} < \infty \tag{4.11}$$

yields Hölder-continuity of $u(t,\xi)$ in space ξ of order θ. Therefore the goal is to estimate the $\dot{H}_2^{\theta+\frac{1}{2}-H}(\mathbb{R}_+)$-norms of r_n, where the functions $r_n(t)$ are the fundamental solutions of the scalar problems

$$r_n + \mu_n g_\alpha * r_n = g_\beta. \tag{4.12}$$

This will be done by the following Lemma.

Lemma 4.1. *Suppose $\alpha \in (0,2)$, $\beta > 0$, $\theta \in [0,1]$, and let $r_n(t)$ denote the solution of (4.12). Then*

$$|r_n|^2_{\dot{H}_2^{\theta+\frac{1}{2}-H}(\mathbb{R}_+)} \leq C_{\alpha,\beta,\theta} \mu_n^{\frac{2(1-\beta+\theta-H)}{\alpha}}, \quad \mu_n > 0,$$

whenever $\beta \in (1 - H + \theta, 1 - H + \alpha)$.

Proof. Again we extend the functions r_n trivially on negative halfline. Let $H \in (\frac{1}{2}, 1)$. We first consider the case $\theta = 0$. Then by the Paley-Wiener theorem, $\mathcal{I}^{H-\frac{1}{2}} r_n \in L_2(\mathbb{R})$ if and only if $\widehat{\mathcal{I}^{H-\frac{1}{2}} r_n} \in \mathscr{H}_2(\mathbb{C}_+)$, the Hardy space of exponent 2 and $|\mathcal{I}^{H-\frac{1}{2}} r_n|_{L_2(\mathbb{R})} = (1/\sqrt{2\pi}) |\widehat{\mathcal{I}^{H-\frac{1}{2}} r_n}|_{\mathscr{H}_2(\mathbb{C}_+)}$. Applying the Paley-Wiener theorem one more time, it suffices to show that $\mathcal{F}(\mathcal{I}^{H-\frac{1}{2}} r_n) \in L_2(\mathbb{R})$. Now we may use (1.11) to compute

$$\int_{\mathbb{R}} \left| \mathcal{F}(\mathcal{I}^{H-\frac{1}{2}} r_n)(\rho) \right|^2 \mathrm{d}\rho \leq \int_0^\infty \left[\frac{\rho^\alpha}{\rho^\beta(\rho^\alpha + \mu_n)} \right]^2 \rho^{-2H+1} \mathrm{d}\rho$$

and we have seen in (4.3) that the right integral converges if and only if $\beta \in (1 - H, 1 - H + \alpha)$. In case $\theta \neq 0$, observe that $|\cdot|_{L_2(\mathbb{R})} + |D^\theta \cdot|_{L_2(\mathbb{R})}$ defines an equivalent norm in $H_2^\theta(\mathbb{R})$, hence replacing β by $\beta - \theta$ the result follows by Plancherel's Theorem. For $H \in (0, \frac{1}{2})$ one may proceed as above with replacing $\mathcal{I}^{H-\frac{1}{2}}$ by $\mathcal{D}^{\frac{1}{2}-H}$. □

Now we are in the position to state our result on (4.1).

Theorem 2. *Let $\alpha \in (0,2)$, $\beta > 0$, $\theta \in [0,1]$ such that $\beta \in (1-H+\theta, 1-H+\alpha)$. Assume that **(A)** and **(B)** are satisfied.*

(i) *If*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2(1-\beta-H)}{\alpha}} < \infty$$

then the solution u of (4.1) exists and belongs to $C_b(\mathbb{R}_+; L_2(\Omega;\mathcal{H}))$.

(ii) *If*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2(1-\beta+\theta-H)}{\alpha}} < \infty$$

then $u \in C_b^{\theta}(\mathbb{R}_+; L_2(\Omega;\mathcal{H}))$.

(iii) *If $\mathcal{H} = L_2(G)$, **(e)** holds, and*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2(1-\beta-H)+\alpha\theta}{\alpha}} < \infty$$

then $u \in C_b(\mathbb{R}_+; C^{\theta}(G; L_2(\Omega)))$.

Proof. Use Lemma 4.1 to estimate the quantities σ_i, $i = 1,2,3$. □

Example. Let $\mathcal{H} = L_2(0,\pi)$, $A = A_0^m$, where $A_0 = -(\mathrm{d}/\mathrm{d}x)^2$ with domain $D(A_0) = H_2^2(0,\pi) \cap \mathring{H}_2^1(0,\pi)$. It is obvious that A is subject to Hypothesis **(A)** and it is well known that eigenvalues of A are $\mu_k = k^{2m}$ for $k \in \mathbb{N}$. The covariance Q is given by its spectral decomposition

$$Qx = \sum_{k=1}^{\infty} \gamma_k (x|e_k) e_k,$$

with $(\gamma_k)_{k\in\mathbb{N}} \subset (0,1]$ such that $\sum_{k=1}^{\infty} \gamma_k < \infty$. For our example we choose $\gamma_k = k^{-l}$, $l > 1$, and we obtain

$$\sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{2(1-\beta-H)}{\alpha}} < \infty \iff \beta > 1 - H - \frac{\alpha(l-1)}{4m};$$

$$\sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{2(1-\beta+\theta-H)}{\alpha}} < \infty \iff \beta > 1 - H + \theta - \frac{\alpha(l-1)}{4m};$$

$$\sum_{k=1}^{\infty} \gamma_k \mu_k^{\frac{2(1-\beta-H)+\alpha\theta}{\alpha}} < \infty \iff \beta > 1 - H + \frac{\alpha\theta}{2} - \frac{\alpha(l-1)}{4m}.$$

Obviously the latter series converge for all $\beta \in (1-H+\theta, 1-H+\alpha)$, hence Theorem 2 applies independently from the choice of l and m. Observe that the spatial regularity is better than in time and that for $H \in (\frac{1}{2}, 1)$ the regularity in space and in time is better than in case $H = \frac{1}{2}$. On the other hand regularity degrade for $H \in (0, \frac{1}{2})$.

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References

[1] Ph. Clément and G. DaPrato. Some results on stochastic convolutions arising in volterra equations perturbed by noise. *Atti Accad. Naz. Lincei (9) Mat. Appl.*, 7(3):147–153, 1996.

[2] Ph. Clément and G. DaPrato. White noise perturbations of the heat equation in materials with memory. *Dynam. Systems Appl.*, 6(4):441–460, 1997.

[3] Ph. Clément, G. DaPrato, and J. Prüss. White noise perturbation of the linear parabolic viscoelasticity. *Rend. Istit. Mat. Univ. Trieste*, 29(1-2):207–220, 1997.

[4] G. DaPrato. *An introduction to infinite-dimensional analysis.* Universitext. Springer, 2006.

[5] G. DaPrato and J. Zabczyk. *Stochastic equations in infinite dimensions.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1992.

[6] T.E. Duncan, B. Maslowski, and B. Pasik-Duncan. Fractional Brownian motion and stochastic equations in Hilbert spaces. *Stochastics and Dynamics*, 2(2):225–250, 2002.

[7] W. Grecksch and V.V. Anh. A parabolic stochastic differential equation with fractional Brownian motion input. *Stochastic & Probability Letters*, 41(4):337–346, 1999.

[8] S. Monniaux and J. Prüss. A theorem of the Dore-Venni type for noncommuting operators. *Trans. Amer. Math. Soc.*, 349(12):4787–4814, 1997.

[9] V. Pipiras and M.S. Taqqu. Integration queations related to fractional Brownian motion. *Probab. Theory Relat. Fields*, 118(2):251–291, 2000.

[10] J. Prüss. *Evolutionary integral equations and applications*, volume 87 of *Monographs in Mathematics.* Birkhäuser, 1993.

[11] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives.* Gordon and Breach Science Publishers, 1994.

[12] H. Triebel. *Interpolation theory, function spaces, differential operators.* Johann Ambrosius Barth, Heidelberg, 1995.

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