Lyapunov functions and convergence to steady state for differential equations of fractional order

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Abstract

We study the asymptotic behaviour, as $t \to \infty$, of bounded solutions to certain integro-differential equations in finite dimensions which include differential equations of fractional order between 0 and 2. We derive appropriate Lyapunov functions for these equations and prove that any global bounded solution converges to a steady state of a related equation, if the nonlinear potential $\mathcal{E}$ occurring in the equation satisfies the Łojasiewicz inequality.

AMS subject classification: 45G05, 45M05

Keywords: integro-differential equations, fractional derivative, gradient system, Lyapunov function, convergence to steady state, Łojasiewicz inequality

1 Introduction

In this paper we study the asymptotic behaviour, as $t \to \infty$, of bounded solutions to integro-differential equations of two types:

- *Problems of order between 1 and 2:*

$$\frac{d}{dt}\,[k * (\dot{u} - u_1)](t) + \nabla\mathcal{E}(u(t)) = f(t),\ t > 0, \quad u(0) = u_0, \quad \dot{u}(0) = u_1; \tag{1}$$

- *Problems of order less than 1:*

$$\frac{d}{dt}\,[a * (u - u_0)](t) + \nabla\mathcal{E}(u(t)) = f(t),\ t > 0, \quad u(0) = u_0. \tag{2}$$

Here $u : \mathbb{R}_+ \to \mathbb{R}^n$ is the unknown and $\dot{u} = \frac{d}{dt}u$. Further, $k,\, a \in L_{1,\,loc}(\mathbb{R}_+)$ are scalar kernels that belong to certain kernel classes and $k * u$ stands for the convolution on the positive halfline, i.e. $(k * u)(t) = \int_0^t k(t-\tau)u(\tau)\,d\tau$, $t \geq 0$. The scalar nonlinearity $\mathcal{E}$ lies in $C^2(\mathbb{R}^n)$; by $\nabla\mathcal{E}$ we mean the gradient of $\mathcal{E}$. The vectors $u_0,\, u_1 \in \mathbb{R}^n$ as well as the function $f \in L_{1,\,loc}(\mathbb{R}_+;\mathbb{R}^n)$ are given data.

Typical examples for the kernels k and a we have in mind are given by

$$k(t) = g_{1-\alpha}(t)e^{-\mu t} + \mu(1 * [g_{1-\alpha}e^{-\mu\cdot}])(t), \quad t > 0,\ \alpha \in (0,1),\ \mu > 0, \tag{3}$$

and

$$a(t) = g_{1-\alpha}(t)e^{-\mu t}, \quad t > 0,\ \alpha \in (0,1),\ \mu > 0, \tag{4}$$

respectively, where g_β denotes the standard kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0.$$

In this case (1) and (2) amount to differential equations of fractional order $1+\alpha \in (1,2)$ and $\alpha \in (0,1)$, respectively. Recall that for a (sufficiently smooth) function v on $\mathbb{R}_+$, the Riemann-Liouville fractional derivative $D_t^\beta v$ of order $\beta \in (0,1)$ is defined by $D_t^\beta v = \frac{d}{dt}(g_{1-\beta} * v)$. Although we allow k and a to belong to more general classes of kernels (see (K1)-(K4) in Section 3.1 for k, and (K1)-(K3) in Section 4.1 for a), in the present paper we will refer to the problems (1) and (2) as *problems of order between 1 and 2*, respectively, *problems of order less than 1*.

As to applications, we primarily regard (1) and (2) as finite-dimensional model problems for more complex time fractional equations in infinite dimensions, which occur in mathematical physics, e.g., in the theory of viscoelasticity and in heat conduction with memory, see, e.g. [12].

The more simple first and second order systems

$$\dot{u}(t) + \nabla\mathcal{E}(u(t)) = 0, \ t > 0, \tag{5}$$

and

$$\ddot{u}(t) + \mu\dot{u}(t) + \nabla\mathcal{E}(u(t)) = 0, \ t > 0, \ \mu > 0, \tag{6}$$

as well as variants of it (e.g. with $f(t)$ on the right) have been studied extensively in the literature. A seminal contribution was made by Łojasiewicz ([17], [18]), who was able to prove that any bounded solution of (5) converges to an equilibrium provided that $\mathcal{E}$ is real analytic. His proof heavily relies on the following result.

Theorem 1.1 (Łojasiewicz [17, Thm. 4]) *Let $U \subset \mathbb{R}^n$ be open, $\mathcal{E} : U \to \mathbb{R}$ be real analytic, and let $u_* \in U$. Then there exist constants $\theta \in (0, 1/2]$, σ, $M > 0$ such that*

$$|\mathcal{E}(x) - \mathcal{E}(u_*)|^{1-\theta} \le M|\nabla\mathcal{E}(x)| \quad \textit{for all } x \in \mathbb{R}^n \textit{ with } |x - u_*| \le \sigma. \tag{7}$$

Note that compared with LaSalle's invariance principle, a significant advantage of the approach based on the so-called *Łojasiewicz inequality*, inequality (7), consists in the fact that this method also works if the set of equilibria is not discrete.

The corresponding result in the second order case was obtained by Haraux and Jendoubi [14]. By means of Theorem 1.1, they established the convergence to steady state of bounded solutions of (a slightly more general form of) (6) in case $\mathcal{E}$ is real analytic.

The aforementioned results have also been extended to the infinite dimensional case. A striking result was obtained by Simon [22], who was able to generalize the Łojasiewicz inequality to some analytic functionals defined on Hilbert spaces. Denoting by $\mathcal{E}'$ the Fréchet derivative of the functional $\mathcal{E}$ defined on a Hilbert space $\mathcal{V}$ which embeds densely into another Hilbert space $\mathcal{H}$, Simon used his generalized result, the so-called *Łojasiewicz-Simon inequality*, to prove the convergence to steady state of bounded solutions of the abstract first order equation $\dot{u} + \mathcal{E}'(u) = 0$ under natural regularity and compactness assumptions on $\mathcal{E}'$ and u. The corresponding result for bounded solutions of the abstract second order equation $\ddot{u} + \dot{u} + \mathcal{E}'(u) = 0$ was proved by Jendoubi [16]. Simon and Jendoubi also showed that it is sufficient to know that $\mathcal{E}$ satisfies the Łojasiewicz-Simon inequality only near some steady state u_* in the ω-limit set $\omega(u)$ of u to conclude that $\lim_{t\to\infty} u(t) = u_*$.

During the last decade the Łojasiewicz-Simon inequality has been used in the study of the asymptotic behaviour of bounded solutions of many different evolution equations, see e.g. [3], [5], [10], [13], [15], [20], and the references given therein. For a detailed study of the Łojasiewicz-Simon inequality we refer to [3].

As to integro-differential equations, there are only a few papers that contain results on convergence to steady state of the type described above; we mention here [1], [4], and [2]. These papers deal with first and second order problems with additional memory terms. For equations of fractional order (in time) such as e.g. (1) and (2) with k and a as in (3) and (4), respectively, corresponding results cannot be found in the literature, except for the first author's thesis [23], which uses the ideas of the present paper to obtain convergence to steady state for a phase-field model with memory that contains partial integro-differential equations of order between 1 and 2 (in time).

Concerning the asymptotic behaviour of solutions of integral and integro-differential equations we further refer to [12], the standard reference in the finite dimensional case, and to the paper [7], which investigates nonlinear Volterra equations in Banach spaces with accretive nonlinearity and a completely positive kernel. The latter concept also plays a crucial role in this paper, cf. Remark 3.1(iii) below.

The objective of this paper is to extend the convergence results described above for first and second order equations in finite dimensions to problems of order between 1 and 2 as well as of order less than 1. We will prove that any global bounded solution u of (1) and (2), respectively, with $u_* \in \omega(u)$ converges to u_*, if $\mathcal{E}$ satisfies the Łojasiewicz inequality near u_* (e.g. if $\mathcal{E}$ is real analytic). Here the vector $u_* \in \mathbb{R}^n$ solves a certain limiting equation, which in both cases, with $u_1 = 0$ in (1), takes the form $\nabla\mathcal{E}(u_*) = 0$, i.e. u_* is a steady state of the corresponding equations with $f = 0$.

In the formulation of our main results we state the validity of the Łojasiewicz inequality as a hypothesis. This is motivated by the fact that there exist examples of non-analytic functions that satisfy this inequality, see e.g. [3] and [15]. Note, however, that in general it is very difficult to verify validity of the Łojasiewicz inequality for a non-analytic function.

We point out that one of the main difficulties concerning (1) and (2) is to find appropriate Lyapunov functions for these equations. In both the first and the second order case a Lyapunov function (LF) is easily obtained: $V(t) = \mathcal{E}(u(t))$ is an LF for (5), while $V(t) = \frac{1}{2}|\dot{u}(t)|^2 + \mathcal{E}(u(t))$ is an LF for (6). The latter is still unsuitable for the approach via Łojasiewicz inequality but can be modified appropriately by adding the term $\delta\langle \dot{u}(t), \nabla\mathcal{E}(u(t))\rangle$ with $\delta > 0$ sufficiently small, see [14]. As for equations with memory terms, in [1], [2], and [4] Lyapunov functions are obtained by means of a technique that basically goes back to Dafermos [9]. The corresponding estimates are rather tedious and do not seem to work in the case of fractional order neither between 1 and 2 nor less than 1.

In this paper we construct Lyapunov functions for the equations (1) and (2), which are also appropriate for the approach via Łojasiewicz inequality. In both cases, the underlying energy estimates are derived from the basic inequality (9) (see below) for nonnegative nonincreasing kernels. It turns out that for (1) with k as in (3) our energy estimates, in some sense, 'interpolate' those known in the first (($\alpha \searrow 0$) in (3)) and the second order case (($\alpha \nearrow 1$) in (3)). We further remark that the estimates obtained in this paper are also extremely useful for proving global existence results for (1) and (2).

Our results can be generalized in various ways. In his thesis [23] the first author studies a variant of (1) in the infinite dimensional case and obtains a corresponding result on convergence to steady state. Furthermore, it is also possible to add some extra terms in (1), e.g. $b(\dot{u})$ on the left-hand side with $\langle b(y), y\rangle \geq 0$, $y \in \mathbb{R}^n$. Since all estimates are derived by multiplying the equation under study by $\dot{u}$ and using (9), our method also applies to first and second order equations with memory terms like e.g. $\ddot{u} + \dot{u} + a * \dot{u} + \nabla\mathcal{E}(u) = 0$, which have been studied before in the literature, see [4] for the latter equation.

The paper is organized as follows. Section 2 provides the basic inequalities which will be needed for the energy estimates. All results here are formulated in the abstract setting, i.e. for functions taking values in some Hilbert space. Section 3 deals with problems of order between 1 and 2 while Section 4 is devoted to problems of order less than 1.

2 Preliminaries

We begin by fixing some notation. In what follows $\mathcal{H}$ denotes a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. In case $\mathcal{H}=\mathbb{R}^n$ we omit the index. For an interval $J\subset\mathbb{R}$, $0<s\le 1$, and $1\le p<\infty$, by $H^s_p(J;\mathcal{H})$ we mean the Bessel potential space of $\mathcal{H}$-valued functions on J. We write $H^s_p(J)=H^s_p(J;\mathbb{R})$ for short. If $J=[0,T]$ with $T>0$, then ${}_0H^s_p(J;\mathcal{H})$ consists of all functions in $H^s_p(J;\mathcal{H})$ with vanishing trace at $t=0$ provided this trace exists. If F denotes some function space on $\mathbb{R}_+=[0,\infty)$, then by $f\in F_{loc}$ we mean that for any $T>0$ the function f belongs to the corresponding space on $[0,T]$.

Lemma 2.1 *Let $\mathcal{H}$ be a real Hilbert space and $T>0$. Suppose that $k\in L_{1,\,loc}(\mathbb{R}_+)$ is nonnegative. Then for any $v\in L_2([0,T];\mathcal{H})$ there holds*

$$|(k*v)(t)|^2_{\mathcal{H}}\le (k*|v|^2_{\mathcal{H}})(t)\,(1*k)(t),\quad a.a.\ t\in(0,T).$$

Proof: The asserted inequality is a consequence of the identity

$$(k*|v|^2_{\mathcal{H}})(t)=\frac{|(k*v)(t)|^2_{\mathcal{H}}}{(k*1)(t)}+\int_0^t k(t-\tau)\Big|\,v(\tau)-\frac{(k*v)(t)}{(k*1)(t)}\Big|^2_{\mathcal{H}}\,d\tau,$$

which holds for a.a. $t\in(0,T)$ provided that $(k*1)(t)>0$. □

Lemma 2.2 *Let $\mathcal{H}$ be a real Hilbert space and $T>0$. Then for any $k\in H^1_1([0,T])$ and any $v\in L_2([0,T];\mathcal{H})$ there holds*

$$\Big\langle v(t),\frac{d}{dt}\,(k*v)(t)\Big\rangle_{\mathcal{H}}=\frac{1}{2}\,\frac{d}{dt}\,(k*|v|^2_{\mathcal{H}})(t)+\frac{1}{2}\,k(t)|v(t)|^2_{\mathcal{H}}$$
$$+\;\frac{1}{2}\int_0^t[-\dot{k}(s)]\,|v(t)-v(t-s)|^2_{\mathcal{H}}\,ds,\quad a.a.\ t\in(0,T).\qquad(8)$$

Proof: The assertion follows from a straightforward computation. □

We remark that a more general version of (8) (with $\mathcal{H}=\mathbb{R}^n$) in integrated form can be found in [12, Lemma 18.4.1].

Theorem 2.1 *Let $\mathcal{H}$ be a real Hilbert space, $T>0$, and $k\in L_{1,\,loc}(\mathbb{R}_+)$ be nonnegative and nonincreasing such that $k*a=1$ in $(0,\infty)$ for some nonnegative $a\in L_{1,\,loc}(\mathbb{R}_+)$. Suppose that $v\in L_2([0,T];\mathcal{H})$ and that there exists $x\in\mathcal{H}$ such that $k*(v-x)\in{}_0H^1_2([0,T];\mathcal{H})$ as well as $k*|v-x|^2_{\mathcal{H}}\in{}_0H^1_1([0,T])$. Then*

$$\Big\langle v(t),\frac{d}{dt}\,(k*v)(t)\Big\rangle_{\mathcal{H}}\ge\frac{1}{2}\,\frac{d}{dt}(k*|v|^2_{\mathcal{H}})(t)+\frac{1}{2}\,k(t)|v(t)|^2_{\mathcal{H}},\quad a.a.\ t\in(0,T).\qquad(9)$$

Proof: **1. Reduction to the case $x=0$:** Suppose the theorem holds in the case $x=0$, that is, for any function $w\in L_2([0,T];\mathcal{H})$ satisfying $k*w\in{}_0H^1_2([0,T];\mathcal{H})$ and $k*|w|^2_{\mathcal{H}}\in{}_0H^1_1([0,T])$, we have

$$\Big\langle w(t),\frac{d}{dt}\,(k*w)(t)\Big\rangle_{\mathcal{H}}\ge\frac{1}{2}\,\frac{d}{dt}\,(k*|w|^2_{\mathcal{H}})(t)+\frac{1}{2}\,k(t)|w(t)|^2_{\mathcal{H}},\quad \text{a.a. } t\in(0,T).\qquad(10)$$

Let now $x \in \mathcal{H}$ and put $v = x + w$. Employing (10), it is then evident that

$$
\begin{aligned}
&\Big\langle v(t), \frac{d}{dt}(k * v)(t) \Big\rangle_{\mathcal{H}} = \Big\langle x + w(t), k(t)x + \frac{d}{dt}(k * w)(t) \Big\rangle_{\mathcal{H}} \\
&= k(t)|x|_{\mathcal{H}}^2 + k(t)\langle w(t), x\rangle_{\mathcal{H}} + \Big\langle x, \frac{d}{dt}(k * w) \Big\rangle_{\mathcal{H}} + \Big\langle w(t), \frac{d}{dt}(k * w)(t) \Big\rangle_{\mathcal{H}} \\
&\geq k(t)|x|_{\mathcal{H}}^2 + k(t)\langle w(t), x\rangle_{\mathcal{H}} + \Big\langle x, \frac{d}{dt}(k * w) \Big\rangle_{\mathcal{H}} + \frac{1}{2}\frac{d}{dt}(k * |w|_{\mathcal{H}}^2)(t) + \frac{1}{2}k(t)|w(t)|_{\mathcal{H}}^2 \\
&= \frac{1}{2}\frac{d}{dt}(k * |x + w|_{\mathcal{H}}^2)(t) + \frac{1}{2}k(t)|x + w(t)|_{\mathcal{H}}^2 = \frac{1}{2}\frac{d}{dt}(k * |v|_{\mathcal{H}}^2)(t) + \frac{1}{2}k(t)|v(t)|_{\mathcal{H}}^2, \qquad (11)
\end{aligned}
$$

for a.a. $t \in (0, T)$. Hence (9) is satisfied.

2. Approximating the kernel k**:** Let $w \in L_2([0,T];\mathcal{H})$ such that $k * w \in {}_0H_2^1([0,T];\mathcal{H})$ as well as $k * |w|_{\mathcal{H}}^2 \in {}_0H_1^1([0,T])$. Introducing the operators

$$
\begin{aligned}
B_1 u &= \frac{d}{dt} k * u, \;\; D(B_1) = \{u \in L_1(0,T) : \, k * u \in {}_0H_1^1([0,T])\}, \\
B_2 u &= \frac{d}{dt} k * u, \;\; D(B_2) = \{u \in L_2([0,T];\mathcal{H}) : \, k * u \in {}_0H_2^1([0,T];\mathcal{H})\},
\end{aligned}
$$

we have $w \in D(B_2)$ and $|w(\cdot)|_{\mathcal{H}}^2 \in D(B_1)$. The operators B_1 and B_2 are known to be m-accretive in $X_1 := L_1([0,T])$ and $X_2 := L_2([0,T];\mathcal{H})$, respectively, cf. [6], [8], and [11]. Their Yosida approximations $B_{i,n}$, $n \in \mathbb{N}$, $i = 1, 2$, defined by

$$
B_{i,n} = nB_i(n + B_i)^{-1}, \quad n \in \mathbb{N}, \; i = 1,\,2,
$$

enjoy the property that for any $u \in D(B_i)$, one has $B_{i,n}u \to B_i u$ in X_i as $n \to \infty$.

We will now derive representations for the operators $B_{i,n}$. To this purpose we define the kernels s_n as solutions of the scalar-valued Volterra equations

$$
s_n(t) + n(a * s_n)(t) = 1, \quad t > 0, \; n \in \mathbb{N}. \qquad (12)
$$

By assumption, the kernel a is completely positive. Therefore it follows from [19, Prop. 4.5] that s_n is nonnegative and nonincreasing in $(0, \infty)$ for every $n \in \mathbb{N}$. Furthermore, by differentiating (12), it is not difficult to see that $s_n \in H_1^1([0,T])$.

Let now $i \in \{1, 2\}$, $n \in \mathbb{N}$, and suppose that $u \in D(B_i)$. Then

$$
nu + \frac{d}{dt} k * u = f \quad \text{on } (0,T), \qquad (13)
$$

is equivalent to

$$
na * u + u = a * f \quad \text{on } (0,T), \qquad (14)
$$

since $k * a = 1$ and $(k * u)(0) = 0$. Convolving (14) with s_n and using (12) then yields $1 * u = s_n * a * f$, that is

$$
u = (n + B_i)^{-1} f = \frac{d}{dt}(s_n * a * f) \quad \text{on } (0,T). \qquad (15)
$$

In fact, the relation (15) with $f \in X_i$ also implies (13). To see this, convolve (15) separately with the functions k and n, add the resulting equations, and use $k * a = 1$ as well as (12).

From (15) we infer that

$$
B_{i,n} f = n \frac{d}{dt} k * \Big(\frac{d}{dt}(s_n * a * f) \Big) = \frac{d}{dt}(ns_n * f), \quad f \in X_i, \; i = 1,\,2,
$$

which shows that $B_{i,\,n}$ is obtained by replacing k in the definition of B_i with the more regular kernel $k_n := ns_n$.

3. Passing to the limit: Since k_n is nonincreasing and lies in $H^1_1([0,T])$, it follows from Lemma 2.2 that for every $n \in \mathbb{N}$,

$$\Big\langle w(t), \frac{d}{dt}(k_n * w)(t)\Big\rangle_{\mathcal{H}} \geq \frac{1}{2}\,\frac{d}{dt}(k_n * |w|^2_{\mathcal{H}})(t) + \frac{1}{2}\,k_n(t)|w(t)|^2_{\mathcal{H}}, \quad \text{a.a. } t \in (0,T). \tag{16}$$

Observe that $1 \in D(B_1)$ entails that

$$k_n \to k \;\text{ in }\; L_1([0,T]) \;\text{ as }\; n \to \infty. \tag{17}$$

Furthermore, since $w \in D(B_2)$ and $|w(\cdot)|^2_{\mathcal{H}} \in D(B_1)$, we have

$$\frac{d}{dt}(k_n * w) \;\to\; \frac{d}{dt}(k * w) \;\text{ in }\; L_2([0,T];\mathcal{H}) \;\text{ as }\; n \to \infty, \tag{18}$$

$$\frac{d}{dt}(k_n * |w|^2_{\mathcal{H}}) \;\to\; \frac{d}{dt}(k * |w|^2_{\mathcal{H}}) \;\text{ in }\; L_1([0,T]) \;\text{ as }\; n \to \infty. \tag{19}$$

By choosing an appropriate subsequence of (k_n), again denoted by (k_n), we may assume that the sequences in (17), (18), and (19) converge also pointwise a.e. in $(0,T)$. Using these properties, we may let $n \to \infty$ in (16), thereby obtaining the desired inequality (10) for w. The proof of Theorem 2.1 is complete. □

Remark 2.1 Under the same assumptions on the kernel k, the inequality (9) in Theorem 2.1 is also satisfied for any function $v \in H^1_1([0,T];\mathcal{H})$. This follows by an argument similar to the one in the proof of Theorem 2.1. Obviously, $v \in L_2([0,T];\mathcal{H})$ and thus (16) holds. Defining then the operator B_2 in this case by

$$B_2 u = \frac{d}{dt}\,k * u, \quad D(B_2) = \{u \in L_1([0,T];\mathcal{H}) :\, k * u \in {}_0H^1_1([0,T];\mathcal{H})\},$$

we have $v \in D(B_2)$ and $|v(\cdot)|^2_{\mathcal{H}} \in D(B_1)$. Consequently, (19) with w replaced by v and

$$\frac{d}{dt}(k_n * v) \;\to\; \frac{d}{dt}(k * v) \;\text{ in }\; L_1([0,T];\mathcal{H}) \;\text{ as }\; n \to \infty,$$

hold, which together with (17) allows to take the limit in (16), also in the a.e. sense, and to arrive at (9).

The assumption $k * |v - x|^2_{\mathcal{H}} \in {}_0H^1_1([0,T])$ in Theorem 2.1 seems to be a little awkward from the application's point of view. We will now describe a rather wide class of kernels k for which we have

$$w \in L_2([0,T];\mathcal{H}) \;\text{ and }\; k * w \in {}_0H^1_2([0,T];\mathcal{H}) \Rightarrow k * |w|^2_{\mathcal{H}} \in {}_0H^1_1([0,T]). \tag{20}$$

In what follows $\hat{f}$ stands for the Laplace transform of f. A function $a \in L_{1,\,loc}(\mathbb{R}_+)$ is called to be of *subexponential growth* if for all $\varepsilon > 0$, $\int_0^\infty e^{-\varepsilon t}|a(t)|\,dt < \infty$. Following [19, Def. 3.3] we say that a kernel $a \in L_{1,\,loc}(\mathbb{R}_+)$ of subexponential growth is 1*-regular* if there is a constant $c > 0$ such that $|\lambda \hat{a}'(\lambda)| \leq c\,|\hat{a}(\lambda)|$ for all $\mathrm{Re}\,\lambda > 0$. A kernel $a \in L_{1,\,loc}(\mathbb{R}_+)$ of subexponential growth satisfying $\hat{a}(\lambda) \neq 0$, $\mathrm{Re}\,\lambda > 0$, is called θ*-sectorial* ($\theta > 0$) if $|\arg \hat{a}(\lambda)| \leq \theta$ for all $\mathrm{Re}\,\lambda > 0$ (cp. [19, Def. 3.2]). The subsequent class of kernels has been introduced in [25, Def. 2.6.3], see also [24, Def. 2.1].

Definition 2.1 *Let $a \in L_{1,\,loc}(\mathbb{R}_+)$ be of subexponential growth, and let $\theta_a > 0$, and $\alpha \geq 0$. Then a is said to belong to the class $\mathcal{K}^1(\alpha, \theta_a)$ if a is 1-regular and θ_a-sectorial, and satisfies*

$$\limsup_{\mu\to\infty} |\,\hat{a}(\mu)|\,\mu^\alpha < \infty,\; \liminf_{\mu\to\infty} |\,\hat{a}(\mu)|\,\mu^\alpha > 0,\; \liminf_{\mu\to 0} |\,\hat{a}(\mu)| > 0.$$

We have now the following result.

Proposition 2.1 *Let $\mathcal{H}$ be a real Hilbert space, $T > 0$, and $k \in L_{1,\,loc}(\mathbb{R}_+)$ be such that $k * a = 1$ in $(0, \infty)$ for some $a \in \mathcal{K}^1(\alpha, \theta)$ with $\alpha \in (0, 1)$ and $\theta < \pi$. Then the implication (20) holds true.*

Proof: Suppose that $w \in L_2([0, T]; \mathcal{H})$ and $k * w \in {}_0H_2^1([0, T]; \mathcal{H})$. Setting $f = \frac{d}{dt}(k * w)$, we then have $f \in L_2([0, T]; \mathcal{H})$. Convolving the last equation with a and using $k * a = 1$ as well as $(k * w)(0) = 0$, we see that $w = a * f$. This implies that $w \in {}_0H_2^\alpha([0, T]; \mathcal{H})$, by [24, Cor. 2.1]). Therefore $|w(\cdot)|_{\mathcal{H}} \in {}_0H_2^\alpha([0, T])$. It follows then from [21, Section 5.4.3] that

$$|w(\cdot)|_{\mathcal{H}}^2 \in \begin{cases} {}_0H^\alpha_{\frac{1}{1-\alpha}}([0, T]) & : \quad 0 < \alpha < \frac{1}{2} \\ {}_0H_q^\alpha([0, T]),\ 1 < q < 2 & : \quad \alpha = \frac{1}{2} \\ {}_0H_2^\alpha([0, T]) & : \quad \frac{1}{2} < \alpha < 1, \end{cases} \tag{21}$$

that is, in any case we have $|w(\cdot)|_{\mathcal{H}}^2 \in {}_0H_p^\alpha([0, T])$ with some $p \in (1, 2]$. Using [24, Cor. 2.1]) once more, we conclude that there exists a function $h \in L_p([0, T])$ such that $|w(\cdot)|_{\mathcal{H}}^2 = a * h$. Hence

$$k * |w(\cdot)|_{\mathcal{H}}^2 = k * a * h = 1 * h \in {}_0H_p^1([0, T]) \hookrightarrow {}_0H_1^1([0, T]),$$

which proves the proposition. □

Example 2.1 Let $\alpha \in (0, 1)$ and $\mu \geq 0$. Set

$$a(t) = g_\alpha(t)e^{-\mu t} \quad \text{and} \quad k(t) = g_{1-\alpha}(t)e^{-\mu t} + \mu(1 * [g_{1-\alpha}e^{-\mu \cdot}])(t), \quad t > 0. \tag{22}$$

Then a and k are strictly positive and decreasing; observe that $\dot{k}(t) = \dot{g}_{1-\alpha}(t)e^{-\mu t} < 0$, $t > 0$. The Laplace transforms are given by

$$\hat{a}(\lambda) = \frac{1}{(\lambda + \mu)^\alpha}, \quad \hat{k}(\lambda) = \frac{1}{(\lambda + \mu)^{1-\alpha}}\Big(1 + \frac{\mu}{\lambda}\Big), \quad \mathrm{Re}\,\lambda > 0,$$

which shows that $a \in \mathcal{K}^1(\alpha, \alpha\frac{\pi}{2})$ as well as $k * a = 1$ on $(0, \infty)$. Hence Proposition 2.1 is applicable to k, and so the implication (20) is valid for this kernel. Notice as well that $k \in \mathcal{K}^1(\alpha, (1 + \alpha)\frac{\pi}{2})$, and hence (20) holds also for the kernel a, again by Proposition 2.1.

3 Problems of order between 1 and 2

3.1 Setting

In this section we investigate problems of the form

$$\frac{d}{dt}\,[k * (\dot{u} - u_1)](t) + \nabla\mathcal{E}(u(t)) = f(t),\ t > 0, \quad u(0) = u_0, \quad \dot{u}(0) = u_1, \tag{23}$$

where $u_0,\ u_1 \in \mathbb{R}^n$. We will suppose that the subsequent assumptions are satisfied.

(K1) $k \in L_{1,loc}(\mathbb{R}_+)$ is nonnegative and nonincreasing;

(K2) there exists a nonnegative kernel $a \in L_{1,loc}(\mathbb{R}_+)$ such that $k * a = 1$ on $(0, \infty)$;

(K3) there is a constant $\mu > 0$ and $b \in L_1(\mathbb{R}_+)$ strictly positive and nonincreasing such that

$$k(t) = b(t) + \mu(1 * b)(t), \quad t > 0; \tag{24}$$

(K4) there is $T_0 \geq 0$ such that $\tilde{k} := k - k_\infty$ with $k_\infty := \lim_{t\to\infty} k(t)$ belongs to $L_2(T_0, \infty)$ and

$$\int_{T_0}^{\infty} (\int_s^\infty |\tilde{k}(\tau)|^2 \, d\tau)^{1/2} \, ds < \infty;$$

(H$\mathcal{E}$) the function $\mathcal{E}$ belongs to $C^2(\mathbb{R}^n)$;

(Hf) $f \in L_2(\mathbb{R}_+; \mathbb{R}^n)$ and there is $T_1 \geq 0$ such that

$$\int_{T_1}^{\infty} (\int_s^\infty |f(\tau)|^2 \, d\tau)^{1/2} \, ds < \infty.$$

Remarks 3.1 (i) If we weaken the assumption (K3) by replacing 'strictly positive' with 'nonnegative', then by decreasing μ, we obtain again a decomposition of the form (24) with b strictly positive and nonincreasing. In fact, if $k = b_0 + \mu_0(1 * b_0)$ with $\mu_0 > 0$ and $b_0 \in L_1(\mathbb{R}_+)$ nonnegative and nonincreasing, then $k = b_\mu + \mu(1 * b_\mu)$, $\mu \in (0, \mu_0)$, where $b_\mu = b_0 + (\mu_0 - \mu)(e^{-\mu \cdot} * b_0)$. By Young's inequality, $b_\mu \in L_1(\mathbb{R}_+)$. Also $b_\mu = k - \mu(1 * b_\mu)$ is nonincreasing, since k is nonincreasing, due to (K1).

(ii) Note that (K1) and (K3) imply that $k(t) \geq k_\infty = \lim_{s\to\infty} k(s) > 0$, $t > 0$.

(iii) By (K1) and (K2), the kernel a in (K2) is *completely positive*, see [12] and [19].

(iv) Observe that the kernel k in Example 2.1 with $\mu > 0$ satisfies (K1)-(K4).

(v) The assumption (Hf) entails that $f \in L_1(\mathbb{R}_+; \mathbb{R}^n)$, see [15, Chap. 3, Sec. 4].

Definition 3.1 *For $T > 0$ we say that a function $u \in C^1([0,T]; \mathbb{R}^n)$ is a* **solution** *of (23) on $[0,T]$ if $k * (\dot{u} - \dot{u}(0)) \in {}_0H_2^1([0,T]; \mathbb{R}^n)$ and (23) holds a.e. on $[0,T]$. A function $u \in C^1([0,\infty); \mathbb{R}^n)$ is called a* **global solution** *of (23) if for any $J = [0,T]$, $T > 0$, the function $u|_J$ is a solution of (23) on J. A global solution u of (23) is called* **global bounded solution** *if $|u|_{L_\infty(\mathbb{R}_+;\mathbb{R}^n)} < \infty$.*

3.2 Lyapunov functions: the case $u_1 = 0$, $f = 0$

Our first objective consists in deriving appropriate energy estimates for global solutions of (23), which allow us to construct a proper Lyapunov function for (23).

It is instructive to begin with the special case where $u_1 = 0$ and $f = 0$. Letting u be a global solution of (23), we have in this case

$$\frac{d}{dt} (k * \dot{u})(t) + \nabla \mathcal{E}(u(t)) = 0, \quad t > 0. \tag{25}$$

We take the inner product of (25) and $\dot{u}$ to find that

$$\langle \dot{u}, \frac{d}{dt} (k * \dot{u}) \rangle + \frac{d}{dt} \mathcal{E}(u) = 0, \quad t > 0. \tag{26}$$

We would next like to apply Theorem 2.1 to the first term on the left-hand side of (26). By (K1) and (K2), the kernel k is admissible. So it remains to verify the required regularity assumptions on u.

For any $T > 0$ we know from Definition 3.1 that $k * \dot{u} = k * (\dot{u} - \dot{u}(0)) \in {}_0H_2^1([0,T]; \mathbb{R}^n)$. Moreover, by (H$\mathcal{E}$) and since (25) together with (K2) and $(k * \dot{u})(0) = 0$ yields

$$\dot{u}(t) = -[a * \nabla \mathcal{E}(u)](t), \quad t \geq 0, \tag{27}$$

we see that $\dot{u} \in H_1^1([0,T]; \mathbb{R}^n)$, which in turn implies $k * (|\dot{u}|^2) \in {}_0H_1^1([0,T])$.

Thus we may apply Theorem 2.1 to obtain

$$\frac{d}{dt}\left[\frac{1}{2}(k*|\dot{u}|^2)(t)+\mathcal{E}(u(t))\right]\le -\frac{1}{2}k(t)\,|\dot{u}(t)|^2,\quad t>0. \tag{28}$$

Employing assumption (K3), from (28) we deduce the estimate

$$\frac{d}{dt}\left[\frac{1}{2}(b*|\dot{u}|^2)(t)+\mathcal{E}(u(t))\right]\le -\frac{1}{2}k(t)\,|\dot{u}(t)|^2-\frac{\mu}{2}(b*|\dot{u}|^2)(t),\quad t>0. \tag{29}$$

Summarizing, we have proved

Proposition 3.1 *Suppose that the assumptions (K1), (K2), (K3), and (H$\mathcal{E}$) are fulfilled. Let $u_0\in\mathbb{R}^n$ and assume that $u\in C^1([0,\infty);\mathbb{R}^n)$ is a global solution of (23) with $u_1=0$ and $f=0$. Then the function V defined by*

$$V(t)=\frac{1}{2}(b*|\dot{u}|^2)(t)+\mathcal{E}(u(t)),\quad t\ge 0,$$

is locally absolutely continuous and nonincreasing on $\mathbb{R}_+$, and there holds the estimate

$$\dot{V}(t)\le -\frac{1}{2}k(t)\,|\dot{u}(t)|^2-\frac{\mu}{2}(b*|\dot{u}|^2)(t),\quad a.a.\ t>0. \tag{30}$$

Remarks 3.2 (i) The assertion of Proposition 3.1 remains true if the assumption (H$\mathcal{E}$) is replaced by the weaker condition $\mathcal{E}\in C^1(\mathbb{R}^n)$ and u is additionally assumed to satisfy $k*(|\dot{u}|^2)\in {}_0H^1_{1,\,loc}([0,\infty))$.

(ii) The function V in Proposition 3.1 is a *strict* Lyapunov function in the sense that if V is constant on some interval $[t_0,t_1]\subset[0,\infty)$ $(t_0<t_1)$, then this implies that u is constant on $[0,t_1]$. This follows immediately from (30) and (K3).

We will now consider a global bounded solution of (23). It turns out that the function V of Proposition 3.1 can be modified appropriately to get a stronger version of (30).

By (H$\mathcal{E}$) and global boundedness of u, we evidently have $|\nabla^2\mathcal{E}(u)|_{L_\infty(\mathbb{R}_+;\mathbb{R}^{n\times n})}\le C_1$ for some constant $C_1>0$. From this estimate and (25) we obtain by using (K3), Lemma 2.1, and Young's inequality,

$$\begin{aligned}\frac{d}{dt}\langle\nabla\mathcal{E}(u),b*\dot{u}\rangle&=\langle\nabla^2\mathcal{E}(u)\dot{u},b*\dot{u}\rangle+\langle\nabla\mathcal{E}(u),\frac{d}{dt}(b*\dot{u})\rangle\\&=\langle\nabla^2\mathcal{E}(u)\dot{u},b*\dot{u}\rangle-\langle\nabla\mathcal{E}(u),\mu b*\dot{u}\rangle-|\nabla\mathcal{E}(u)|^2\\&\le\frac{C_1}{2}|\dot{u}|^2+\frac{C_1}{2}|b*\dot{u}|^2+\frac{1}{2}|\nabla\mathcal{E}(u)|^2+\frac{\mu^2}{2}|b*\dot{u}|^2-|\nabla\mathcal{E}(u)|^2\\&\le\frac{C_1}{2}|\dot{u}|^2+\frac{1}{2}|b|_{L_1(\mathbb{R}_+)}(C_1+\mu^2)\,b*|\dot{u}|^2-\frac{1}{2}|\nabla\mathcal{E}(u)|^2,\quad t>0.\end{aligned} \tag{31}$$

We then set

$$\tilde{V}(t)=V(t)+\delta\langle\nabla\mathcal{E}(u(t)),(b*\dot{u})(t)\rangle,\quad t\ge 0,$$

where δ is a small positive number. Thus, by Remark 3.1(ii), and combining (30) and (31) we get

$$\begin{aligned}\dot{\tilde{V}}(t)\le&-\frac{1}{2}\Big(\mu-\delta|b|_{L_1(\mathbb{R}_+)}[C_1+\mu^2]\Big)(b*|\dot{u}|^2)(t)\\&-\frac{1}{2}(k_\infty-\delta C_1)|\dot{u}(t)|^2-\frac{\delta}{2}|\nabla\mathcal{E}(u(t))|^2,\end{aligned}$$

which immediately yields the next result.

Proposition 3.2 *Suppose that (K1), (K2), (K3), and (H$\mathcal{E}$) are fulfilled. Let $u_0 \in \mathbb{R}^n$ and assume that $u \in C^1([0,\infty);\mathbb{R}^n)$ is a global bounded solution of (23) with $u_1 = 0$ and $f = 0$. Then there exist constants $\delta > 0$ and $C > 0$ such that the function $\tilde{V}$ defined by*

$$\tilde{V}(t) = V(t) + \delta\langle \nabla\mathcal{E}(u(t)), (b * \dot{u})(t)\rangle, \quad t \geq 0,$$

is locally absolutely continuous and nonincreasing on $\mathbb{R}_+$, and there holds

$$\dot{\tilde{V}}(t) \leq -C\Big(|\dot{u}(t)|^2 + (b * |\dot{u}|^2)(t) + |\nabla\mathcal{E}(u(t))|^2\Big), \quad a.a.\ t > 0. \tag{32}$$

$\tilde{V}$ is a strict Lyapunov function in the sense of Remark 3.2(ii).

3.3 Lyapunov functions: the general case

We now study the full problem (23). The strategy will be the same as in the preceding subsection. However, some modifications are necessary to treat the new terms coming from the data f and u_1, which are now present.

We begin by reformulating (23) in an appropriate way. Recall that $\tilde{k}(t) = k(t) - k_\infty$, $t > 0$, and set

$$\tilde{\mathcal{E}}(y) = \mathcal{E}(y) - k_\infty\langle u_1, y\rangle, \quad y \in \mathbb{R}^n. \tag{33}$$

Then the integro-differential equation in (23) can be written as

$$\frac{d}{dt}\,[k * \dot{u}](t) + \nabla\tilde{\mathcal{E}}(u(t)) = \tilde{k}(t)u_1 + f(t), \quad t > 0. \tag{34}$$

Let now $u \in C^1([0,\infty);\mathbb{R}^n)$ be a global solution of (23) with $k * (|\dot{u} - u_1|^2) \in {}_0H^1_{1,\,loc}([0,\infty))$. Then we may multiply (34) by $\dot{u}$ and apply Theorem 2.1 with $\mathcal{H} = \mathbb{R}^n$ and $x = u_1$ to the function $v = \dot{u}$ to the result

$$\frac{d}{dt}\left[\frac{1}{2}\,(k * |\dot{u}|^2)(t) + \tilde{\mathcal{E}}(u(t))\right] \leq -\frac{1}{2}\,k(t)|\dot{u}(t)|^2 + \langle \tilde{k}(t)u_1, \dot{u}(t)\rangle + \langle f(t), \dot{u}(t)\rangle, \quad t > 0. \tag{35}$$

By (K3) and since $k(t) \geq k_\infty > 0$, $t > 0$, as well as

$$\langle \tilde{k}u_1, \dot{u}\rangle + \langle f, \dot{u}\rangle \leq \frac{2\tilde{k}^2|u_1|^2 + 2|f|^2}{k_\infty} + \frac{k_\infty}{4}\,|\dot{u}|^2,$$

it follows from (35) that

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\,(b * |\dot{u}|^2)(t) + \tilde{\mathcal{E}}(u(t))\right] \leq &-\frac{k_\infty}{4}\,|\dot{u}(t)|^2 - \frac{\mu}{2}\,(b * |\dot{u}|^2)(t)\\ &+ \frac{2\tilde{k}(t)^2|u_1|^2 + 2|f(t)|^2}{k_\infty}, \quad t > 0.\end{aligned} \tag{36}$$

In view of (K4) and (Hf), the function

$$V(t) = \frac{1}{2}\,(b * |\dot{u}|^2)(t) + \tilde{\mathcal{E}}(u(t)) + 2k_\infty^{-1}\int_t^\infty ([\tilde{k}(\tau)]^2|u_1|^2 + |f(\tau)|^2)\,d\tau, \quad t \geq T_0, \tag{37}$$

is well-defined on $[T_0,\infty)$, and (36) shows that

$$\dot{V}(t) \leq -\frac{k_\infty}{4}\,|\dot{u}(t)|^2 - \frac{\mu}{2}\,(b * |\dot{u}|^2)(t), \quad t > T_0. \tag{38}$$

If u is a global bounded solution of (23), we may proceed similarly as in Subsection 3.2. Using the bound $|\nabla^2\mathcal{E}(u)|_{L_\infty(\mathbb{R}_+;\mathbb{R}^{n\times n})} \le C_1$ we have

$$\begin{aligned}\frac{d}{dt}\langle\nabla\tilde{\mathcal{E}}(u),b*\dot{u}\rangle &= \langle\nabla^2\tilde{\mathcal{E}}(u)\dot{u}, b*\dot{u}\rangle - \langle\nabla\tilde{\mathcal{E}}(u), \mu b*\dot{u}\rangle - |\nabla\tilde{\mathcal{E}}(u)|^2 \\ &\quad + \langle\nabla\tilde{\mathcal{E}}(u), \tilde{k}u_1 + f\rangle \\ &\le \frac{C_1}{2}\,|\dot{u}|^2 + \frac{1}{2}\,|b|_{L_1(\mathbb{R}_+)}(C_1+2\mu^2)\,b*|\dot{u}|^2 - \frac{1}{2}\,|\nabla\tilde{\mathcal{E}}(u)|^2 \\ &\quad + 2\tilde{k}^2|u_1|^2 + 2|f|^2, \quad t>0. \end{aligned} \tag{39}$$

Setting

$$\begin{aligned}\tilde{V}(t) &= \frac{1}{2}\,(b*|\dot{u}|^2)(t) + \tilde{\mathcal{E}}(u(t)) + \delta\langle\nabla\tilde{\mathcal{E}}(u(t)),(b*\dot{u})(t)\rangle \\ &\quad + 2(\delta + k_\infty^{-1})\int_t^\infty([\tilde{k}(\tau)]^2|u_1|^2 + |f(\tau)|^2)\,d\tau, \quad t\ge T_0, \end{aligned} \tag{40}$$

and choosing $\delta>0$ sufficiently small, we then obtain in view of (37), (38), and (39),

$$\dot{\tilde{V}}(t) \le -C\Big(|\dot{u}(t)|^2 + (b*|\dot{u}|^2)(t) + |\nabla\tilde{\mathcal{E}}(u(t))|^2\Big), \quad t>T_0, \tag{41}$$

where $C>0$ is a constant. We have thus proved

Proposition 3.3 *Suppose the assumptions (K1), (K2), (K3), (K4), (H$\mathcal{E}$), and (Hf) are fulfilled. Let u_0, $u_1\in\mathbb{R}^n$ and assume that $u\in C^1([0,\infty);\mathbb{R}^n)$ is a global solution of (23) with $k*(|\dot{u}-u_1|^2)\in {}_0H^1_{1,\,loc}([0,\infty))$. Then the function V given by (37) is locally absolutely continuous and nonincreasing on $[T_0,\infty)$, and the estimate (38) holds in the a.e. sense.*

If $u\in C^1([0,\infty);\mathbb{R}^n)$ is a global bounded solution of (23) with $k*(|\dot{u}-u_1|^2)\in {}_0H^1_{1,\,loc}([0,\infty))$, then there exist constants $\delta>0$ and $C>0$ such that the function $\tilde{V}$ defined by (40) is locally absolutely continuous and nonincreasing on $[T_0,\infty)$, and the estimate (41) is satisfied in the a.e. sense.

Remarks 3.3 (i) The second part of both (K4) and (Hf) is not used in the above proof of Proposition 3.3.

(ii) Observe that the condition $k*(|\dot{u}-u_1|^2)\in {}_0H^1_{1,\,loc}([0,\infty))$ in Proposition 3.3 is automatically satisfied if $a*f\in H^1_{1,\,loc}([0,\infty))$, cf. the remarks prior to (28).

(iii) The functions V and $\tilde{V}$ in Proposition 3.3 are strict Lyapunov functions: If V ($\tilde{V}$) is constant on some interval $[t_0,t_1]\subset[T_0,\infty)$ ($t_0<t_1$), then u is constant on $[0,t_1]$.

3.4 Properties of the ω-limit set

We recall that the ω-limit set of a global solution u of (23) is defined by

$$\omega(u) = \{u_*\in\mathbb{R}^n:\ \text{there exist } t_n\nearrow\infty \text{ s.t. } \lim_{n\to\infty} u(t_n) = u_*\}.$$

For every global bounded solution u of (23), $\omega(u)$ is nonempty, compact and connected.

Proposition 3.4 *Suppose (K1), (K2), (K3), (K4), (H$\mathcal{E}$), and (Hf) are fulfilled. Let u_0, $u_1\in\mathbb{R}^n$ and assume that $u\in C^1([0,\infty);\mathbb{R}^n)$ is a global bounded solution of (23) with $k*(|\dot{u}-u_1|^2)\in {}_0H^1_{1,\,loc}([0,\infty))$. Then* *(i)* $\dot u \in L_2(\mathbb{R}_+;\mathbb{R}^n)$ *and* $b * |\dot u|^2 \in L_1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$.

(ii) The potential $\tilde{\mathcal{E}}$ *is constant on* $\omega(u)$ *and* $\lim_{t\to\infty} \tilde{\mathcal{E}}(u(t))$ *exists.*

(iii) $\nabla\mathcal{E}(u_*) - k_\infty u_1 = 0$ *for every* $u_* \in \omega(u)$.

Proof: Since u is a global bounded solution of (23), the function $\tilde{\mathcal{E}}(u(\cdot))$ is bounded from below and thus $V : [T_0,\infty) \to \mathbb{R}$ defined in (37) is bounded from below. Furthermore, the function V is nonincreasing, by Proposition 3.3, and therefore $\lim_{t\to\infty} V(t) = \inf_{t\geq T_0} V(t) =: V_\infty$ exists. The first part of assertion (i) is then a direct consequence of estimate (38).

Let now $u_* \in \omega(u)$ and $t_n \nearrow \infty$ such that $\lim_{n\to\infty} u(t_n) = u_*$. Since $\dot u \in L_2(\mathbb{R}_+;\mathbb{R}^n)$, we have for every $n \in \mathbb{N}$ and any $s \in [0,1]$,

$$\begin{aligned} |u(t_n+s) - u_*| &\leq |u(t_n) - u_*| + \int_{t_n}^{t_n+s} |\dot u(\tau)|\,d\tau \\ &\leq |u(t_n) - u_*| + (\int_{t_n}^{t_n+s} |\dot u(\tau)|^2\,d\tau)^{1/2}, \end{aligned} \tag{42}$$

with both terms on the right-hand side of (42) tending to zero as $n \to \infty$. Therefore $u(t_n+s) \to u_*$ as $n \to \infty$ for all $s \in [0,1]$. By continuity of $\tilde{\mathcal{E}}$, this in turn entails $\tilde{\mathcal{E}}(u(t_n+s)) \to \tilde{\mathcal{E}}(u_*)$ as $n \to \infty$ for all $s \in [0,1]$, and thus

$$\tilde{\mathcal{E}}(u_*) = \lim_{n\to\infty} \int_0^1 \tilde{\mathcal{E}}(u(t_n+s))\,ds, \tag{43}$$

by the dominated convergence theorem. Integrating then $V(t_n + \cdot)$ (with $t_n \geq T_0$) defined in (37) over $[0,1]$, we obtain

$$\begin{aligned} \int_0^1 V(t_n+s)\,ds &= \int_0^1 \tilde{\mathcal{E}}(u(t_n+s))\,ds \\ &+ \int_{t_n}^{t_n+1} \Big[\frac{1}{2}\,(b*|\dot u|^2)(s) + 2k_\infty^{-1} \int_s^\infty ([\tilde k(\tau)]^2|u_1|^2 + |f(\tau)|^2)\,d\tau\Big]\,ds, \end{aligned}$$

which shows that

$$V_\infty = \lim_{n\to\infty} \int_0^1 V(t_n+s)\,ds = \tilde{\mathcal{E}}(u_*), \tag{44}$$

in virtue by (43), (i), (K4), and (Hf).

Since u_* was chosen arbitrarily in $\omega(u)$, (44) implies that $\tilde{\mathcal{E}}$ is constant on $\omega(u)$.

We next show that $\lim_{t\to\infty}(b*|\dot u|^2)(t) = 0$. If the contrary was true, there would be $\varepsilon > 0$ and a sequence (t_n) converging to ∞ such that $(b*|\dot u|^2)(t_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. By compactness, there exists a subsequence (t_{n_k}) and $u_* \in \omega(u)$ such that $u(t_{n_k}) \to u_*$ as $k \to \infty$, and therefore $\tilde{\mathcal{E}}(u(t_{n_k})) \to V_\infty$ as $k \to \infty$. But this together with the structure of V implies that $(b*|\dot u|^2)(t_{n_k}) \to 0$ as $k \to \infty$, a contradiction. Hence $\lim_{t\to\infty}(b*|\dot u|^2)(t) = 0$.

By (K4), (Hf), and the last property, we infer from the structure of V that $\lim_{t\to\infty} \tilde{\mathcal{E}}(u(t)) = V_\infty$. Hence (i) and (ii) are established.

In order to prove (iii), let $u_* \in \omega(u)$ and select $t_n \nearrow \infty$ such that $\lim_{n\to\infty} u(t_n) = u_*$. We have already seen that this entails $u(t_n+s) \to u_*$ as $n \to \infty$ for all $s \in [0,1]$. Hence $\nabla\tilde{\mathcal{E}}(u(t_n+s)) \to \nabla\tilde{\mathcal{E}}(u_*)$ as $n \to \infty$ for all $s \in [0,1]$. Using the dominated convergence theorem and (34) as well as (K3), we have

$$
\begin{aligned}
\nabla\tilde{\mathcal{E}}(u_*) &= \lim_{n\to\infty}\int_0^1 \nabla\tilde{\mathcal{E}}(u(t_n+s))\,ds \\
&= \lim_{n\to\infty}\Big[-\int_0^1 \frac{d}{dt}\,[k*\dot{u}](t_n+s)\,ds + \int_{t_n}^{t_n+1}(\tilde{k}(s)u_1+f(s))\,ds\Big] \qquad (45)\\
&= -\lim_{n\to\infty}[(k*\dot{u})(t_n+1)-(k*\dot{u})(t_n)] \\
&= -\lim_{n\to\infty}\Big[(b*\dot{u})(t_n+1)-(b*\dot{u})(t_n)+\mu\int_{t_n}^{t_n+1}(b*\dot{u})(s)\,ds\Big] = 0. \qquad (46)
\end{aligned}
$$

In fact, the second integral in (45) vanishes due to (K4), (Hf), and Hölder's inequality . As to the last step, we know that $b \in L_1(\mathbb{R}_+)$ and $b*|\dot{u}|^2 \in L_1(\mathbb{R}_+)\cap C_0(\mathbb{R}_+)$, and hence $b*\dot{u} \in L_2(\mathbb{R}_+)\cap C_0(\mathbb{R}_+)$, by Lemma 2.1. Assertion (iii) follows now in view of (33). This completes the proof. □

3.5 Convergence to steady state

We will now show that global bounded solutions of (23) converge to solutions $u_* \in \mathbb{R}^n$ of $\nabla\mathcal{E}(u_*) - k_\infty u_1 = 0$ as $t \to \infty$. The proof relies on Propositions 3.3, 3.4, and the Łojasiewicz inequality.

Theorem 3.1 *Suppose (K1), (K2), (K3), (K4), (H$\mathcal{E}$), and (Hf) are fulfilled. Let $u_0, u_1 \in \mathbb{R}^n$ and $u \in C^1([0,\infty);\mathbb{R}^n)$ be a global bounded solution of (23) with $k*(|\dot{u}-u_1|^2) \in {}_0H^1_{1,\,loc}([0,\infty))$. Assume further that there exists some $u_* \in \omega(u)$ such that $\tilde{\mathcal{E}}$ defined in (33) satisfies the Łojasiewicz inequality near u_*, i.e. there are constants $\theta \in (0,1/2]$ and σ, $M > 0$ such that*

$$
|\tilde{\mathcal{E}}(x)-\tilde{\mathcal{E}}(u_*)|^{1-\theta} \le M|\nabla\tilde{\mathcal{E}}(x)| \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x-u_*| \le \sigma.
$$

Then $\lim_{t\to\infty} u(t) = u_*$, and $\nabla\mathcal{E}(u_*) - k_\infty u_1 = 0$.

Proof: Let $u \in C^1([0,\infty);\mathbb{R}^n)$ be a global bounded solution of (23) with $k*(|\dot{u}-u_1|^2) \in {}_0H^1_{1,\,loc}([0,\infty))$ and suppose that $u_* \in \omega(u)$ is as in the statement of Theorem 3.1. Define the function W by

$$
\begin{aligned}
W(t) &= \tilde{V}(t) - \tilde{\mathcal{E}}(u_*) \\
&= V(t) - \tilde{\mathcal{E}}(u_*) + \delta\langle\nabla\tilde{\mathcal{E}}(u(t)),(b*\dot{u})(t)\rangle \\
&\quad + 2\delta\int_t^\infty ([\tilde{k}(\tau)]^2|u_1|^2 + |f(\tau)|^2)\,d\tau, \quad t \ge T_0,
\end{aligned}
$$

where $\delta > 0$ is as in Proposition 3.3. By the latter result, both V and $\tilde{V}$ are locally absolutely continuous and nonincreasing on $[T_0,\infty)$. From the proof of Proposition 3.4 (see (44)) we further know that $\lim_{t\to\infty} V(t) = \tilde{\mathcal{E}}(u_*)$ and that $\lim_{t\to\infty}(b*\dot{u})(t) = 0$. These properties together with (K4) and (Hf) show that W is nonincreasing and that $\lim_{t\to\infty} W(t) = 0$. Moreover, by Proposition 3.3, there exists a constant $C > 0$ such that

$$
\dot{W}(t) \le -C\Big(|\dot{u}(t)|^2 + (b*|\dot{u}|^2)(t) + |\nabla\tilde{\mathcal{E}}(u(t))|^2\Big), \quad \text{a.a. } t > T_0. \qquad (47)
$$

If $W(t_0) = 0$ for some $t_0 \ge T_0$, then $W(t) = 0$ for all $t \ge t_0$, and hence $u(t) = u_*$, by Remark 3.3(iii). So we may assume that $W(t) > 0$ for all $t \ge T_0$.

We next consider the function $W(t)^{1-\theta}$. By the definitions of V and W, Young's inequality, and Lemma 2.1 together with $b \in L_1(\mathbb{R}_+)$, we have

$$\begin{aligned} W(t)^{1-\theta} \le C_1 \Big\{ & |\tilde{\mathcal{E}}(u(t)) - \tilde{\mathcal{E}}(u_*)|^{1-\theta} + [(b * |\dot{u}|^2)(t)]^{\frac{2(1-\theta)}{2}} \\ & + |\nabla \tilde{\mathcal{E}}(u(t))| + [(b * |\dot{u}|^2)(t)]^{\frac{1-\theta}{2\theta}} \\ & + \Big(\int_t^\infty ([\tilde{k}(\tau)]^2 |u_1|^2 + |f(\tau)|^2)\, d\tau \Big)^{\frac{2(1-\theta)}{2}} \Big\}, \quad t \ge T_0, \end{aligned} \tag{48}$$

for some constant $C_1 > 0$. Observe that $\theta \in (0, 1/2]$ implies $2(1-\theta) \ge 1$ as well as $(1-\theta)/\theta \ge 1$. Using this and the fact that $(b * |\dot{u}|^2)(t)$ and the integral in (48) tend to zero as $t \to \infty$, we obtain

$$\begin{aligned} W(t)^{1-\theta} \le C_2 \Big\{ & |\tilde{\mathcal{E}}(u(t)) - \tilde{\mathcal{E}}(u_*)|^{1-\theta} + |\nabla \tilde{\mathcal{E}}(u(t))| + [(b * |\dot{u}|^2)(t)]^{\frac{1}{2}} \\ & + \Big(\int_t^\infty ([\tilde{k}(\tau)]^2 |u_1|^2 + |f(\tau)|^2)\, d\tau \Big)^{\frac{1}{2}} \Big\}, \quad t \ge t_*, \end{aligned} \tag{49}$$

where $C_2 > 0$ is some constant and $t_* \ge \max\{T_0, T_1\}$ is chosen sufficiently large.

Define then the set $\Omega_\sigma \subset (t_*, \infty)$ by

$$\Omega_\sigma = \{ t \in (t_*, \infty) : \ |u(t) - u_*| < \sigma \}.$$

By continuity of u, Ω_σ is an open set in $\mathbb{R}$. Restricting t in (49) to Ω_σ, we may use the Łojasiewicz inequality for $\tilde{\mathcal{E}}$ near u_* to get

$$\begin{aligned} W(t)^{1-\theta} \le C_3 \Big\{ & |\nabla \tilde{\mathcal{E}}(u(t))| + [(b * |\dot{u}|^2)(t)]^{\frac{1}{2}} \\ & + \Big(\int_t^\infty ([\tilde{k}(\tau)]^2 |u_1|^2 + |f(\tau)|^2)\, d\tau \Big)^{\frac{1}{2}} \Big\}, \quad t \in \Omega_\sigma, \end{aligned} \tag{50}$$

for some constant $C_3 > 0$. From (47) and (50) we infer that

$$\begin{aligned} -\frac{d}{dt}[W(t)^\theta] &= -\theta W(t)^{\theta-1} \dot{W}(t) \\ &\ge \frac{\theta C \{ |\dot{u}(t)|^2 + (b * |\dot{u}|^2)(t) + |\nabla \tilde{\mathcal{E}}(u(t))|^2 \}}{C_3 \{ |\nabla \tilde{\mathcal{E}}(u(t))| + [(b * |\dot{u}|^2)(t)]^{\frac{1}{2}} + (\int_t^\infty ([\tilde{k}(\tau)]^2 |u_1|^2 + |f(\tau)|^2)\, d\tau)^{\frac{1}{2}} \}} \\ &\ge C_4 \Big(|\dot{u}(t)|^2 + (b * |\dot{u}|^2)(t) + |\nabla \tilde{\mathcal{E}}(u(t))|^2 \Big)^{\frac{1}{2}} \\ &\quad - C_5 \Big(\int_t^\infty ([\tilde{k}(\tau)]^2 |u_1|^2 + |f(\tau)|^2)\, d\tau \Big)^{\frac{1}{2}} \\ &\ge C_6 \Big(|\dot{u}(t)| + [(b * |\dot{u}|^2)(t)]^{\frac{1}{2}} + |\nabla \tilde{\mathcal{E}}(u(t))| \Big) \\ &\quad - C_5 \Big(\int_t^\infty ([\tilde{k}(\tau)]^2 |u_1|^2 + |f(\tau)|^2)\, d\tau \Big)^{\frac{1}{2}}, \quad \text{a.a. } t \in \Omega_\sigma, \end{aligned} \tag{51}$$

where $C_i > 0$, $i = 4, 5, 6$, are constants. Integrating (51) over Ω_σ and employing (K4) as well as (Hf) then yields $\dot{u} \in L_1(\Omega_\sigma; \mathbb{R}^n)$. In fact, since W is nonincreasing and $\lim_{t\to\infty} W(t) = 0$, we have

$$\int_{\Omega_\sigma} -\frac{d}{dt}[W(t)^\theta]\, dt \le \int_{t_*}^\infty -\frac{d}{dt}[W(t)^\theta]\, dt = W(t_*)^\theta.$$

We use now a standard argument (cf. e.g. [15]) to see that $\dot{u} \in L_1(\mathbb{R}_+;\mathbb{R}^n)$. Choose $t_n \nearrow \infty$ such that $\lim_{n\to\infty} u(t_n) = u_*$. We may assume that $t_n \in \Omega_\sigma$ for all $n \in \mathbb{N}$. Define next exit times s_n by means of

$$s_n = \sup\{t > t_n :\ [t_n, t] \subset \Omega_\sigma\}, \quad n \in \mathbb{N}. \tag{52}$$

Then there exists $N \in \mathbb{N}$ such that $s_N = \infty$. To see this, suppose the contrary is true. By continuity of u and the definition of Ω_σ it follows then that $|u(s_n) - u_*| = \sigma > 0$ for all $n \in \mathbb{N}$. On the other hand, using $\dot{u} \in L_1(\Omega_\sigma;\mathbb{R}^n)$, we have

$$\begin{aligned} |u(s_n) - u_*| &\le |u(t_n) - u_*| + \int_{t_n}^{s_n} |\dot{u}(\tau)|\, d\tau \\ &\le |u(t_n) - u_*| + \int_{(t_n,\infty)\cap\Omega_\sigma} |\dot{u}(\tau)|\, d\tau \quad \to 0 \ \text{ as } \ n \to \infty, \end{aligned}$$

a contradiction. Therefore $s_N = \infty$ for some $N \in \mathbb{N}$, and thus $[t_N, \infty) \subset \Omega_\sigma$. Hence $\dot{u} \in L_1(\mathbb{R}_+;\mathbb{R}^n)$, which immediately implies that $\lim_{t\to\infty} u(t) = u_*$.

Finally, from Proposition 3.4 we see that $\nabla\mathcal{E}(u_*) - k_\infty u_1 = 0$. The proof is complete. □

4 Problems of order less than 1

4.1 Setting

In this section we study problems of the form

$$\frac{d}{dt}\,[a * (u - u_0)](t) + \nabla\mathcal{E}(u(t)) = f(t),\ t > 0, \quad u(0) = u_0, \tag{53}$$

where $u_0 \in \mathbb{R}^n$. We will suppose that the subsequent assumptions are satisfied.

(K1) $a \in L_{1,loc}(\mathbb{R}_+)$ is nonnegative;

(K2) there exists a nonnegative and nonincreasing kernel $k \in L_{1,loc}(\mathbb{R}_+)$ such that $k * a = 1$ on $(0, \infty)$;

(K3) there is a constant $\mu > 0$ and $b \in L_1(\mathbb{R}_+)$ strictly positive and nonincreasing such that

$$k(t) = b(t) + \mu(1 * b)(t), \quad t > 0;$$

(H$\mathcal{E}$) the function $\mathcal{E}$ belongs to $C^2(\mathbb{R}^n)$;

(Hf) $f \in L_1(\mathbb{R}_+;\mathbb{R}^n) \cap H^1_2(\mathbb{R}_+;\mathbb{R}^n)$ and there is $T_1 \ge 0$ such that

$$\int_{T_1}^{\infty} (\int_s^{\infty} (|f(\tau)|^2 + |\dot{f}(\tau)|^2\, d\tau)^{1/2}\, ds < \infty.$$

Remark 4.1 Observe that the kernel a in Example 2.1 with $\mu > 0$ satisfies (K1)-(K3).

Definition 4.1 *For $T > 0$ we say that a function $u \in C([0,T];\mathbb{R}^n)$ is a* **solution** *of (53) on $[0,T]$ if $u \in H^1_1([0,T];\mathbb{R}^n)$ and (53) holds on $[0,T]$. A function $u \in C([0,\infty);\mathbb{R}^n)$ is called a* **global solution** *of (53) if for any $J = [0,T]$, $T > 0$, the function $u|_J$ is a solution of (53) on J. A global solution u of (53) is called* **global bounded solution** *if $|u|_{L_\infty(\mathbb{R}_+;\mathbb{R}^n)} < \infty$.*

Remark 4.2 Under the above assumptions, problem (53) admits a unique *local* solution in the described regularity class for all $u_0 \in \mathbb{R}^n$. The maximal interval of existence $[0, \tau(u_0))$ w.r.t. this class is characterized by the condition $\lim_{t\nearrow\tau(u_0)} |u(t)| = \infty$. This can be proved by means of a standard fixed point argument; observe that the smoothness of $\mathcal{E}$ and f allows to write (53) in the form

$$\dot{u}(t) = -k * (\nabla^2\mathcal{E}(u)\dot{u})(t) - k(t)\nabla\mathcal{E}(u_0) + k * \dot{f} + k(t)f(0),\ t > 0, \quad u(0) = u_0.$$

4.2 Lyapunov function

We begin by constructing a proper Lyapunov function for (53). Let $u \in H^1_{1,\,loc}([0,\infty);\mathbb{R}^n)$ be a global solution of (53) and put

$$v = \frac{d}{dt}\,[a * (u - u_0)].$$

We take the inner product of the integro-differential equation in (53) with $\dot{u}$ to find that

$$\langle v, \dot{u}\rangle + \frac{d}{dt}\,\mathcal{E}(u) = \langle f, \dot{u}\rangle, \quad t > 0. \tag{54}$$

Thanks to (K2), there exists a nonnegative and nonincreasing kernel $k \in L_{1,loc}(\mathbb{R}_+)$ such that $k * a = 1$ on $(0, \infty)$. Thus we may write

$$\dot{u} = \frac{d}{dt}\,(u - u_0) = \frac{d^2}{dt^2}\,[k * a * (u - u_0)] = \frac{d}{dt}\,(k * v), \tag{55}$$

which together with (54) yields

$$\langle v, \frac{d}{dt}\,(k * v)\rangle + \frac{d}{dt}\,\mathcal{E}(u) = \langle f, \dot{u}\rangle, \quad t > 0. \tag{56}$$

Since $v = -\nabla\mathcal{E}(u) + f$ and by the assumptions (H$\mathcal{E}$) and (Hf), the function v belongs to the class $H^1_{1,\,loc}([0,\infty);\mathbb{R}^n)$. Hence (see Remark 2.1) we may apply inequality (9) to the first term in (56) to the result

$$\frac{d}{dt}\Big[\frac{1}{2}\,(k * |v|^2)(t) + \mathcal{E}(u(t))\Big] \le -\,\frac{1}{2}\,k(t)|v(t)|^2 + \langle f(t), \dot{u}(t)\rangle, \quad t > 0. \tag{57}$$

Employing assumption (K3) we infer from (57) that

$$\frac{d}{dt}\Big[\frac{1}{2}\,(b * |v|^2)(t) + \mathcal{E}(u(t))\Big] \le -\,\frac{1}{2}\,k(t)|v(t)|^2 - \frac{\mu}{2}\,(b * |v|^2)(t) + \langle f(t), \dot{u}(t)\rangle, \quad t > 0. \tag{58}$$

In order to treat the term $\langle f, \dot{u}\rangle$ note that by Lemma 2.1 and Young's inequality we have

$$\begin{aligned}
\langle f, \dot{u}\rangle &= \langle f, \frac{d}{dt}\,(k * v)\rangle = \langle f, \frac{d}{dt}\,(b * v)\rangle + \mu\langle f, b * v\rangle \\
&= \frac{d}{dt}\langle f, b * v\rangle - \langle \dot{f}, b * v\rangle + \mu\langle f, b * v\rangle \\
&\le \frac{d}{dt}\,\langle f, b * v\rangle + 2|b|_{L_1(\mathbb{R}_+)}\Big(\mu|f|^2 + \mu^{-1}|\dot{f}|^2\Big) + \frac{\mu}{4|b|_{L_1(\mathbb{R}_+)}}\,|b * v|^2 \\
&\le \frac{d}{dt}\langle f, b * v\rangle + M\Big(|f|^2 + |\dot{f}|^2\Big) + \frac{\mu}{4}\,b * (|v|^2), \quad t > 0,
\end{aligned} \tag{59}$$

where $M = 2|b|_{L_1(\mathbb{R}_+)} \max\{\mu, \mu^{-1}\}$. Setting

$$V(t) = \frac{1}{2}(b * |v|^2)(t) + \mathcal{E}(u(t)) - \langle f(t), (b * v)(t)\rangle + M \int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2)\, d\tau, \quad t \ge 0, \tag{60}$$

we obtain from (58) and (59) that

$$\dot{V}(t) \le -\frac{1}{2}\, k(t)|v(t)|^2 - \frac{\mu}{4}(b * |v|^2)(t), \quad t > 0. \tag{61}$$

Proposition 4.1 *Suppose the assumptions (K1), (K2), (K3), (H$\mathcal{E}$), and (Hf) are fulfilled. Let $u_0 \in \mathbb{R}^n$ and assume that $u \in H^1_{1,loc}([0,\infty); \mathbb{R}^n)$ is a global solution of (53). Then the function V given by (60) is locally absolutely continuous and nonincreasing on $[0,\infty)$, and the estimate (61) holds in the a.e. sense.*

Remark 4.3 (i) The second part of (Hf) is not used in the above proof of Proposition 4.1.

(ii) The function V in Proposition 4.1 is a strict Lyapunov function in the sense of Remark 3.2. In fact, if V is constant on $[t_0, t_1] \subset [0,\infty)$ ($t_0 < t_1$), then by (61), this means that $(b * |v|^2)(t) = 0$ for a.a. $t \in (t_0, t_1)$, which in turn implies $v = 0$ in $[0, t_1]$, by (K3). By definition of v, we have $k * v = u - u_0$, and hence $u = u_0$ in $[0, t_1]$.

4.3 Properties of the ω-limit set

We consider now global bounded solutions of (53). For any such solution u, the ω-limit set $\omega(u)$ defined as in Subsection 3.4 is nonempty, compact and connected.

Proposition 4.2 *Suppose (K1), (K2), (K3), (H$\mathcal{E}$), and (Hf) are fulfilled. Let $u_0 \in \mathbb{R}^n$ and assume that $u \in H^1_{1,loc}([0,\infty); \mathbb{R}^n)$ is a global bounded solution of (53). Then*

(i) $v = \frac{d}{dt}[a * (u - u_0)] \in L_2(\mathbb{R}_+; \mathbb{R}^n)$ *and* $b * |v|^2 \in L_1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$.

(ii) The potential $\mathcal{E}$ is constant on $\omega(u)$ and $\lim_{t\to\infty} \mathcal{E}(u(t))$ exists.

(iii) $\nabla\mathcal{E}(u_*) = 0$ *for every* $u_* \in \omega(u)$.

Proof: Let u be a global bounded solution of (53). Evidently, $\mathcal{E}(u)$ is bounded from below. Furthermore, by Young's inequality and Lemma 2.1,

$$\langle f(t), (b * v)(t)\rangle \le |b|_{L_1(\mathbb{R}_+)}|f(t)|^2 + \frac{1}{4}(b * |v|^2)(t), \quad t \ge 0,$$

and so it is clear that $V : \mathbb{R}_+ \to \mathbb{R}$ defined in (60) is bounded from below. By Proposition 4.1, the function V is nonincreasing and therefore $\lim_{t\to\infty} V(t) = \inf_{t\ge 0} V(t) =: V_\infty$ exists. The first part of assertion (i) follows then directly from estimate (61).

In order to see $b * |v|^2 \in C_0(\mathbb{R}_+)$, note first that $|v|_{L_\infty(\mathbb{R}_+;\mathbb{R}^n)} \le C$ for some constant $C > 0$. In fact, this follows from $v = -\nabla\mathcal{E}(u) + f$ and the boundedness of u and f. Continuing, given $\varepsilon > 0$ we can choose $\delta > 0$ such that $|b|_{L_1(0,\delta)} C^2 \le \varepsilon$. Define the function b_δ by $b_\delta(t) = b(t)$, $t \in (0,\delta)$, and $b_\delta(t) = 0$, $t \ge \delta$. Using Young's inequality we may then estimate

$$\begin{aligned}(b * |v|^2)(t) &\le |b_\delta * |v|^2|_{L_\infty(\mathbb{R}_+)} + ((b - b_\delta) * |v|^2)(t) \\ &\le \varepsilon + ((b - b_\delta) * |v|^2)(t), \quad t \ge 0.\end{aligned} \tag{62}$$

Since $0 \le (b-b_\delta)(t) \le b(\delta)$ for a.a. $t>0$, we have further

$$|((b-b_\delta)*|v|^2)(t)| = \int_0^{t/2} (b-b_\delta)(s)|v(t-s)|^2\,ds + \int_{t/2}^t (b-b_\delta)(s)|v(t-s)|^2\,ds$$
$$\le b(\delta)\int_{t/2}^t |v(s)|^2\,ds + C^2 \int_{t/2}^t (b-b_\delta)(s)\,ds,$$

which together with $|v(\cdot)|^2 \in L_1(\mathbb{R}_+)$ and $b \in L_1(\mathbb{R}_+)$ shows that $\lim_{t\to\infty}((b-b_\delta)*|v|^2)(t) = 0$. The latter property, (62), and the fact that ε can be chosen arbitrarily small, then imply $\lim_{t\to\infty}(b*|v|^2)(t) = 0$. Hence (i) is established.

Let now $u_* \in \omega(u)$ and $t_n \nearrow \infty$ such that $\lim_{n\to\infty} u(t_n) = u_*$. Since

$$u(t) - u_0 = (k*v)(t), \quad t \ge 0, \tag{63}$$

cp. (55), it follows that $\lim_{m\to\infty}(k*v)(t_m)$ exists and that

$$u_* = u_0 + \lim_{m\to\infty}(k*v)(t_m). \tag{64}$$

Let again $k_\infty := \lim_{t\to\infty} k(t) > 0$ and $\tilde{k} := k - k_\infty$. Using (63) and (64) we have for $t_n \le t$, $n \in \mathbb{N}$,

$$\begin{aligned} u(t) - u_* &= (k*v)(t) - \lim_{m\to\infty}(k*v)(t_m) \\ &= (k*v)(t) - (k*v)(t_n) + \Big((k*v)(t_n) - \lim_{m\to\infty}(k*v)(t_m)\Big) \\ &= (\tilde{k}*v)(t) - (\tilde{k}*v)(t_n) + k_\infty \int_{t_n}^t v(\tau)\,d\tau \\ &\quad + \Big((k*v)(t_n) - \lim_{m\to\infty}(k*v)(t_m)\Big). \end{aligned} \tag{65}$$

Observe that

$$0 \le \tilde{k}(t) = b(t) - \mu \int_t^\infty b(\tau)\,d\tau \le b(t), \quad t > 0,$$

and thus, by Lemma 2.1,

$$|(\tilde{k}*v)(t)| \le [(b*|v|^2)(t)]^{1/2} |b|_{L_1(\mathbb{R}_+)}^{1/2}. \tag{66}$$

From (66) and (i) we deduce that

$$\lim_{t\to\infty}(\tilde{k}*v)(t) = 0. \tag{67}$$

Using (65) and Hölder's inequality, we have for any $n \in \mathbb{N}$ and any $s \in [0,1]$,

$$|u(t_n+s) - u_*| \le |(\tilde{k}*v)(t_n+s)| + |(\tilde{k}*v)(t_n)| + k_\infty \Big(\int_{t_n}^{t_n+s} |v(\tau)|^2\,d\tau\Big)^{1/2}$$
$$+ \Big((k*v)(t_n) - \lim_{m\to\infty}(k*v)(t_m)\Big).$$

Since $v \in L_2(\mathbb{R}_+;\mathbb{R}^n)$ and by (67), it follows that $u(t_n+s) \to u_*$ as $n\to\infty$ for all $s \in [0,1]$. By continuity of $\tilde{\mathcal{E}}$, this in turn implies $\mathcal{E}(u(t_n+s)) \to \mathcal{E}(u_*)$ as $n\to\infty$ for all $s\in[0,1]$, and thus

$$\mathcal{E}(u_*) = \lim_{n\to\infty}\int_0^1 \mathcal{E}(u(t_n+s))\,ds, \tag{68}$$

by the dominated convergence theorem. Integrating then $V(t_n+\cdot)$ defined in (60) over $[0,1]$, we get

$$\int_0^1 V(t_n+s)\,ds = \int_0^1 \mathcal{E}(u(t_n+s))\,ds + \frac{1}{2}\int_{t_n}^{t_n+1} (b*|v|^2)(s)\,ds$$
$$+ \int_{t_n}^{t_n+1} \Big[-\langle f(s),(b*v)(s)\rangle + M\int_s^\infty (|f(\tau)|^2+|\dot f(\tau)|^2)\,d\tau\Big]\,ds,$$

which shows that

$$V_\infty = \lim_{n\to\infty}\int_0^1 V(t_n+s)\,ds = \mathcal{E}(u_*), \tag{69}$$

in virtue by (68), (i), (Hf), and the simple estimate

$$|\int_{t_n}^{t_n+1}\langle f(s),(b*v)(s)\rangle\,ds|^2 \le |b|_{L_1(\mathbb{R}_+)}\int_{t_n}^{t_n+1}|f(s)|^2\,ds\int_{t_n}^{t_n+1}(b*|v|^2)(s)\,ds.$$

Since u_* was chosen arbitrarily in $\omega(u)$, (69) implies that $\mathcal{E}$ is constant on $\omega(u)$. In view of (i), (Hf), and the structure of V, we also see that $\lim_{t\to\infty}\mathcal{E}(u(t)) = V_\infty$. Hence (ii) is proven.

Finally, to prove (iii), let again $u_*\in\omega(u)$ and $t_n\nearrow\infty$ such that $\lim_{n\to\infty}u(t_n)=u_*$. Recall that we know already that then $u(t_n+s)\to u_*$ as $n\to\infty$ for all $s\in[0,1]$. Therefore $\nabla\mathcal{E}(u(t_n+s))\to\nabla\mathcal{E}(u_*)$ as $n\to\infty$ for all $s\in[0,1]$. Since $\nabla\mathcal{E}(u)=-v+f$ and by the dominated convergence theorem, we have

$$\nabla\mathcal{E}(u_*) = \lim_{n\to\infty}\int_0^1\nabla\mathcal{E}(u(t_n+s))\,ds$$
$$= \lim_{n\to\infty}\int_{t_n}^{t_n+1}\Big[-v(s)+f(s)\Big]\,ds = 0,$$

where the last step follows from $v\in L_2(\mathbb{R}_+;\mathbb{R}^n)$, (Hf), and Hölder's inequality. □

4.4 Convergence to steady state

We will now prove that any global bounded solution of (53) converges to a solution $u_*\in\mathbb{R}^n$ of $\nabla\mathcal{E}(u_*)=0$ as $t\to\infty$. To this end we will use Propositions 4.1, 4.2, and the Łojasiewicz inequality.

Theorem 4.1 *Suppose (K1), (K2), (K3), (H$\mathcal{E}$), and (Hf) are fulfilled. Let $u_0\in\mathbb{R}^n$ and $u\in H^1_{1,\,loc}([0,\infty);\mathbb{R}^n)$ be a global bounded solution of (53). Assume further that there exists some $u_*\in\omega(u)$ such that $\mathcal{E}$ satisfies the Łojasiewicz inequality near u_*, i.e. there are constants $\theta\in(0,1/2]$ and σ, $M>0$ such that*

$$|\mathcal{E}(x)-\mathcal{E}(u_*)|^{1-\theta}\le M|\nabla\mathcal{E}(x)|\quad \textit{for all } x\in\mathbb{R}^n \textit{ with } |x-u_*|\le\sigma.$$

Then $\lim_{t\to\infty}u(t)=u_*$, and $\nabla\mathcal{E}(u_*)=0$.

Proof: Let $u\in H^1_{1,\,loc}([0,\infty);\mathbb{R}^n)$ be a global bounded solution of (53) and $u_*\in\omega(u)$ as in the statement of Theorem 4.1. Setting $W(t)=V(t)-\mathcal{E}(u_*)$, $t\ge0$, where V is as in (60), we know that W is nonnegative, nonincreasing, and locally absolutely continuous on $\mathbb{R}_+$, and that $\lim_{t\to\infty}W(t)=0$. Furthermore,

$$\dot W(t)\le -\frac{1}{2}\,k_\infty|v(t)|^2-\frac{\mu}{4}\,(b*|v|^2)(t),\quad \text{a.a. } t>0, \tag{70}$$

where $v = \frac{d}{dt}[a * (u - u_0)]$. All these properties follow from Proposition 4.1 and the proof of Proposition 4.2.

If $W(t_0) = 0$ for some $t_0 \geq 0$, then $W(t) = 0$ for all $t \geq t_0$, and hence, by Remark 4.3(ii), $u(t) = u_0 = u_*$, $t \geq 0$. So we may assume that $W(t)$ is strictly positive on $\mathbb{R}_+$.

Using Young's inequality and Lemma 2.1, we deduce from the definitions of V and W that

$$\begin{aligned} W(t)^{1-\theta} \leq C_1\Big\{ &|\mathcal{E}(u(t)) - \mathcal{E}(u_*)|^{1-\theta} + [(b * |v|^2)(t)]^{\frac{2(1-\theta)}{2}} \\ &+ |f(t)| + [(b * |v|^2)(t)]^{\frac{1-\theta}{2\theta}} \\ &+ \Big(\int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2)\, d\tau\Big)^{\frac{2(1-\theta)}{2}} \Big\}, \quad t \geq 0, \end{aligned} \tag{71}$$

for some constant $C_1 > 0$. Since $\theta \in (0, 1/2]$, we have $2(1-\theta) \geq 1$ and $(1-\theta)/\theta \geq 1$. Using this and the fact that $(b * |v|^2)(t)$ and the integral in (71) tend to zero as $t \to \infty$, we obtain

$$\begin{aligned} W(t)^{1-\theta} \leq C_2\Big\{ &|\mathcal{E}(u(t)) - \mathcal{E}(u_*)|^{1-\theta} + |f(t)| + [(b * |v|^2)(t)]^{\frac{1}{2}} \\ &+ \Big(\int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2)\, d\tau\Big)^{\frac{1}{2}} \Big\}, \quad t \geq t_*, \end{aligned} \tag{72}$$

where $C_2 > 0$ is a constant and $t_* \geq T_1$ is selected sufficiently large.

As in the proof of Theorem 3.1 we introduce the open set

$$\Omega_\sigma = \{t \in (t_*, \infty) : \ |u(t) - u_*| < \sigma\}.$$

Restricting t in (72) to Ω_σ, we may employ the Łojasiewicz inequality for $\mathcal{E}$ near u_* and use $\nabla\mathcal{E}(u) = -v + f$ to get

$$\begin{aligned} W(t)^{1-\theta} \leq C_3\Big\{ &|v(t)| + |f(t)| + [(b * |v|^2)(t)]^{\frac{1}{2}} \\ &+ \Big(\int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2)\, d\tau\Big)^{\frac{1}{2}} \Big\}, \quad t \in \Omega_\sigma, \end{aligned} \tag{73}$$

with some constant $C_3 > 0$. From (70) and (73) we then obtain for a.a. $t \in \Omega_\sigma$

$$\begin{aligned} -\frac{d}{dt}[W(t)^\theta] &= -\theta W(t)^{\theta-1}\dot{W}(t) \\ &\geq \frac{\theta\{\frac{1}{2}k_\infty |v(t)|^2 + \frac{\mu}{4}(b * |v|^2)(t)\}}{C_3\{|v(t)| + |f(t)| + [(b * |v|^2)(t)]^{\frac{1}{2}} + (\int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2)\, d\tau)^{\frac{1}{2}}\}} \\ &\geq C_4\Big(|v(t)| + [(b * |v|^2)(t)]^{\frac{1}{2}}\Big) - C_5\Big(|f(t)| + \Big(\int_t^\infty ([\tilde{k}(\tau)]^2 |u_1|^2 + |f(\tau)|^2)\, d\tau\Big)^{\frac{1}{2}}\Big), \end{aligned} \tag{74}$$

where $C_4, C_5 > 0$ are constants. Integrating (74) over Ω_σ and using (Hf) yields $v \in L_1(\Omega_\sigma; \mathbb{R}^n)$.

We show now that $v \in L_1(\mathbb{R}_+; \mathbb{R}^n)$. To this end, choose $t_n \nearrow \infty$ such that $\lim_{n\to\infty} u(t_n) = u_*$. We may assume that $t_n \in \Omega_\sigma$ for all $n \in \mathbb{N}$. Let s_n, $n \in \mathbb{N}$, be the corresponding exit times defined in (52). Then there exists $N \in \mathbb{N}$ such that $s_N = \infty$. If the contrary was true, we would have $|u(s_n) - u_*| = \sigma > 0$ for all $n \in \mathbb{N}$. On the other hand, we get from (65) that

$$\begin{aligned} |u(s_n) - u_*| &\leq |(\tilde{k} * v)(s_n)| + |(\tilde{k} * v)(t_n)| + k_\infty \int_{t_n}^{s_n} |v(\tau)|\, d\tau \\ &\quad + |(k * v)(t_n) - \lim_{m\to\infty}(k * v)(t_m)| \\ &\leq |(\tilde{k} * v)(s_n)| + |(\tilde{k} * v)(t_n)| + k_\infty \int_{(t_n,\infty)\cap\Omega_\sigma} |v(\tau)|\, d\tau \\ &\quad + |(k * v)(t_n) - \lim_{m\to\infty}(k * v)(t_m)| \to 0 \text{ as } n \to \infty, \end{aligned}$$

due to $v \in L_1(\Omega_\sigma;\mathbb{R}^n)$ and (67). So we have a contradiction, and therefore $s_N = \infty$ for some $N \in \mathbb{N}$. Hence $v \in L_1(\mathbb{R}_+;\mathbb{R}^n)$, which together with (65) and (67) entails that $\lim_{t\to\infty} u(t) = u_*$.

Finally, from Proposition 4.2 we see that $\nabla\mathcal{E}(u_*) = 0$. The proof is complete. □

Acknowledgements. The authors are grateful to Jan Prüss for many fruitful discussions and valuable suggestions.

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