T-SYMMETRICAL TENSOR DIFFERENTIAL FORMS WITH LOGARITHMIC POLES ALONG A HYPERSURFACE SECTION

P. BRÜCKMANN AND P. WINKERT

ABSTRACT. The aim of this paper is to investigate T-symmetrical tensor differential forms with logarithmic poles on the projective space $\mathbb{P}^N$ and on complete intersections $Y \subset \mathbb{P}^N$. Let $H \subset \mathbb{P}^N, N \geq 2$, be a nonsingular irreducible algebraic hypersurface which implies that $D = H$ is a prime divisor in $\mathbb{P}^N$. The main goal of this paper is the study of the locally free sheaves $\Omega^T_{\mathbb{P}^N}(\log D)$ and the calculation of their cohomology groups. In addition, for complete intersections $Y \subset \mathbb{P}^N$ we give some vanishing theorems and recursion formulas.

1. Introduction

The symmetry properties of tensors are important in physics and certain areas of mathematics. In the following, let k be the ground field which is assumed to be algebraically closed satisfying $\operatorname{char}(k) = 0$. We denote by $H \subset \mathbb{P}^N_k, N \geq 2$, a nonsingular, irreducible, algebraic hypersurface defined by the equation $F = 0$ where $\deg F = m$. Then $D = H$ gives a prime divisor of degree m in $\mathbb{P}^N_k$. The aim of this paper is the calculation of the dimension of the cohomology groups $H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t))$ with general twist $t \in \mathbb{Z}$, where T is a Young tableau specified later. $\Omega^T_{\mathbb{P}^N}(\log D)$ denotes the so-called sheaf of germs of T-symmetrical tensor differential forms with logarithmic poles along the prime divisor D (cf. [5], [8], [3]). In addition, we consider the associated cohomology groups of nonsingular, irreducible, n-dimensional complete intersections $Y \subset \mathbb{P}^N$, $n \geq 2$. In this case, let the prime divisor $D = Y \cap H$ be the intersection of Y and hypersurface H. As special cases, we investigate the alternating and the symmetric differential forms on $\mathbb{P}^N$ and on Y, respectively.

2. Notations and Preliminaries

Let Ω^1_X be the sheaf of germs of regular algebraic differential forms on a n-dimensional nonsingular, projective variety $X \subseteq \mathbb{P}^N$ and let $\Omega^r_X = \overset{r}{\wedge}\Omega^1_X$ and $S^r\Omega^1_X$ be the sheaves of alternating and symmetric differential forms on X, alternatively. We denote by $(\Omega^1_X)^{\otimes r}$ the $r-$th tensor power of Ω^1_X. The coherent sheaves Ω^1_X, Ω^r_X, $S^r\Omega^1_X$ and $(\Omega^1_X)^{\otimes r}$ are locally free on X with the rank n, $\binom{n}{r}$, $\binom{n+r-1}{r}$ and n^r, respectively.

2000 *Mathematics Subject Classification.* 14F10, 14M10, 14F17, 55N30.

Key words and phrases. Young tableaux, complete intersections, algebraic differential forms.

The irreducible representations of the symmetric group S_r correspond to the conjugacy classes of S_r. These are given by partitions $(l)\ :\ r = l_1 + \ldots + l_d$ with $l_i \in \mathbb{Z},\ l_1 \geq l_2 \geq \ldots \geq l_d \geq 1$. Partition (l) can be described by a so-called Young diagram T with r boxes and the row lengths $l_1, \ldots, l_d$. The column lengths of T will be denoted by $d_1, \ldots, d_l$ and we set $d = d_1 = \operatorname{depth} T$ and $l = l_1 = \operatorname{length} T$, respectively. Clearly, $d_1 \geq d_2 \geq \ldots \geq d_l \geq 1$ and the equations $\sum_{j=1}^{l} d_j = \sum_{i=1}^{d} l_i = r$ are fulfilled. Moreover, we put $l_i = 0$ for $i > d$ and $d_j = 0$ for $j > l$. The "hook-length" of the box inside the $i-$th row and the $j-$th column of the Young diagram is defined by $h_{i,j} = l_i - i + d_j - j + 1$ and the degree of the associated irreducible representation is equal to
$\nu_{(l)} = \frac{r!}{\prod h_{i,j}} = \frac{r!}{d!} \cdot \prod_{i=1}^{d} \frac{i!}{(l_i+d-i)!} \cdot \prod_{1 \leq i < j \leq d} \left(\frac{l_i - l_j}{j-i} + 1\right) =$
$= r! \cdot \det\left(\left(\frac{1}{\Gamma(l_i+1-i+j)}\right)\right)_{i,j=1,\ldots,d}$ (cf. [7]).
A numbering of the r boxes of a given Young diagram by the integers $1, 2, \ldots, r$ in any order is said to be a Young tableau which for simplicity again will be denoted by T. Now, one has an idempotent e_T in the group algebra $k \cdot S_r$ defined by

$$e_T = \frac{\nu_{(l)}}{r!} \cdot \left(\sum_{q \in Q_T} \operatorname{sgn}(q) \cdot q \right) \circ \left(\sum_{p \in P_T} p \right),$$

where the subgroups P_T and Q_T of S_r are given as follows:
$P_T = \{p \in S_r : p$ preserves each row of $T\}$,
$Q_T = \{q \in S_r : q$ preserves each column of $T\}$.
The idempotent e_T is called Young symmetrizer (cf. [7]). If the numbering of the boxes of the Young tableau generates inside every row and every column monotone increasing sequences, we speak of a standard tableau. The number of all standard tableaux to a given Young diagram is equal to the degree $\nu_{(l)}$. We denote by $D(r)$ the set of all standard tableaux to all Young diagrams with r boxes.
For a variety X the notation $\Omega_X^{\otimes r} = (\Omega_X^1)^{\otimes r}$ stands for the sheaf of germs of regular algebraic tensor differential forms. This implies that the symmetric group S_r and the related group algebra $k \cdot S_r$ act on $\Omega_X^{\otimes r}$ defined by
$p(a_1 \otimes \ldots \otimes a_r) = a_{p^{-1}(1)} \otimes \ldots \otimes a_{p^{-1}(r)}$ for all $p \in S_r$. That means, mapping p permutates the spots inside the tensor product. Furthermore, it holds

$$\Omega_X^{\otimes r} = \bigoplus_{T \in D(r)} \Omega_X^T$$

with $\Omega_X^T = e_T(\Omega_X^{\otimes r})$, where Ω_X^T is called the sheaf of germs of $T-$symmetrical tensor differential forms or simply the T-power of Ω_X^1 . If two Young tableaux T and $\widetilde{T}$ possess the same Young diagram, we have $\Omega_X^T \cong \Omega_X^{\widetilde{T}}$.
Under the assumption $\operatorname{depth} T \leq \dim X$ with a smooth n-dimensional variety X the belonging sheaf Ω_X^T is locally free of rank

$$\prod_{1 \leq i < j \leq n} \left(\frac{l_i - l_j}{j-i} + 1 \right) = \left(\prod_{i=1}^{n-1} i! \right)^{-1} \cdot \Delta(l_1 - 1, l_2 - 2, \ldots, l_n - n),$$

where $\Delta(t_1, t_2, \ldots, t_n) = \prod_{1 \le i < j \le n}(t_i - t_j)$ denotes the Vandermonde determinant. If $\operatorname{depth} T > \dim X$ then we have $\Omega_X^T = 0$. In the special cases $\Omega_X^r = \wedge^r \Omega_X^1$ and $S^r \Omega_X^1$ the Young tableau has only one column and one row, respectively. In the same way the T-power $\mathcal{F}^T$ of an arbitrary coherent algebraic sheaf $\mathcal{F}$ is defined. One has for instance $\Omega_X^T(\log D) = (\Omega_X^1(\log D))^T$.

Furthermore, we describe the T-power of an algebraic complex (cf. [3]):
Let R be a commutative ring which contains the algebraically closed ground field k fulfilling $\operatorname{char}(k) = 0$. We consider an algebraic complex K of $R-$modules given by $K : K_0 \xrightarrow{d} K_1 \xrightarrow{d} K_2 \xrightarrow{d} \ldots$ with $d^2 = 0$.
Then the $r-$th tensor power $P = K^{\otimes r}$ of K is defined by
$P = K^{\otimes r} : P_0 \xrightarrow{\delta} P_1 \xrightarrow{\delta} P_2 \xrightarrow{\delta} \ldots$ with $P_s = \bigoplus_{s_1+\ldots+s_r=s} K_{s_1} \otimes \ldots \otimes K_{s_r}$ and $\delta(b_1 \otimes \ldots \otimes b_r) = \sum_{i=1}^r (-1)^{s_1+\ldots+s_{i-1}} \cdot b_1 \otimes \ldots \otimes b_{i-1} \otimes d(b_i) \otimes b_{i+1} \otimes \ldots \otimes b_r$ where $b_j \in K_{s_j}$ for all j. Again the symmetric group S_r acts on this tensor power by permutation of the spots inside the tensor product. In order to obtain such an action of S_r on $P = K^{\otimes r}$, which commutates with δ, we introduce additionally a sign as follows:

(1) $\sigma(p\ ; s_1, \ldots, s_r) := \sum_{\substack{i<j \\ p(i)>p(j)}} s_i \cdot s_j$ for all $p \in S_r$

(2) $p(b_1 \otimes \ldots \otimes b_r) := (-1)^{\sigma(p\ ;s_1,\ldots,s_r)} \cdot b_{p^{-1}(1)} \otimes \ldots \otimes b_{p^{-1}(r)}$
where $b_j \in K_{s_j}$ for all $j \in \{1, \ldots, r\}$.

Then one has $P_s = \bigoplus_{T \in D(r)} K_s^{(T)}$, $K^{\otimes r} = \bigoplus_{T \in D(r)} K^{(T)}$,
$H^*(K^{\otimes r}) = \bigoplus_{T \in D(r)} H^*(K^{(T)})$ with $K_s^{(T)} = e_T(P_s)$ and
$K^{(T)} = e_T(K^{\otimes r}) :\ K_0^{(T)} \xrightarrow{\delta} K_1^{(T)} \xrightarrow{\delta} K_2^{(T)} \xrightarrow{\delta} \ldots$.
This complex $K^{(T)}$ is said to be the $T-$power of K. If two Young tableaux T and $\widetilde{T}$ possess the same Young diagram, one has $K^{(T)} \cong K^{(\widetilde{T})}$. For an exact sequence K the $T-$power $K^{(T)}$ of K is also an exact sequence.

Now, let $X \subseteq \mathbb{P}^N$ be a projective variety satisfying $\omega_X \cong \mathcal{O}_X(n_X)$ for some $n_X \in \mathbb{Z}$, where ω_X stands for the canonical line bundle. This implies under the assumptions $d = \operatorname{depth} T = \dim X = n$ and $l = \operatorname{length} T > 1$ the isomorphism

$$\Omega_X^T \cong \Omega_X^{T'} \otimes \omega_X \cong \Omega_X^{T'}(n_X),$$

where T' arises from T by deleting the first column of T.
In the case $d = \operatorname{depth} T = \dim X = n$ and $l = \operatorname{length} T = 1$ (i.e. T has only one column) we have the isomorphism $\Omega_X^T \cong \Omega_X^n \cong \omega_X \cong \mathcal{O}_X(n_X)$.
An important tool in our considerations will be the Serre duality:
Suppose the Young tableau T has the column lengths $d_1, \ldots, d_l$ satisfying $d_1 = d = \operatorname{depth} T \le \dim X = n$. We get an associated Young tableau T^* by the column lengths $d_j^* = n - d_{l+1-j}$ for all $j = 1, \ldots, l$. One verifies readily that in case $\operatorname{depth} T < n$ holds $(T^*)^* = T$.
The next lemma delivers some duality relations about the dimensions of cohomology groups.

Lemma 2.1. *Let $Y = H_1 \cap \ldots \cap H_{N-n} \subseteq \mathbb{P}^N$ be a n-dimensional, non-singular, irreducible, complete intersection defined by algebraic hypersurfaces*

$H_i \subset \mathbb{P}^N$ satisfying $F_i = 0$ with $\deg F_i = m_i$. The dimension of Y is n. In this case, let the prime divisor $D = Y \cap H$ be the intersection of Y and hypersurface $H : F = 0$ of degree m. Assume that D also becomes a nonsingular irreducible complete intersection of dimension $n-1$. Then one has:

(i) $\dim H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) = \dim H^{N-q}(\mathbb{P}^N, \Omega^{N-r}_{\mathbb{P}^N}(\log D)(-t-m))$

(ii) $\dim H^q(Y, \Omega^r_Y(\log D)(t)) = \dim H^{n-q}(Y, \Omega^{n-r}_Y(\log D)(-t-m))$

(iii) $\dim H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t))$
$= \dim H^{N-q}(\mathbb{P}^N, \Omega^{T^*}_{\mathbb{P}^N}(\log D)(-t - l\cdot m + (l-1)(N+1)))$

(iv) $\dim H^q(Y, \Omega^T_Y(\log D)(t))$
$= \dim H^{n-q}(Y, \Omega^{T^*}_Y(\log D)(-t - l\cdot m - (l-1)(\sum_{i=1}^{N-n} m_i - N - 1)))$

(v) $\dim H^q(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t))$
$= \dim H^{N-q}(\mathbb{P}^N, \Omega^{T^*}_{\mathbb{P}^N}(\log D)(-t - r\cdot m + (r-1)(N+1)))$
where T^* denotes a rectangle with $N-1$ rows and r columns.

(vi) $\dim H^q(Y, S^r\Omega^1_Y(\log D)(t))$
$= \dim H^{n-q}(Y, \Omega^{T^*}_Y(\log D)(-t - r\cdot m - (r-1)(\sum_{i=1}^{N-n} m_i - N - 1)))$
where T^* denotes a rectangle with $n-1$ rows and r columns.

Proof. We consider the following exact sequence (cf. [5])

$$0 \longrightarrow \Omega^r_{\mathbb{P}^N}(\log D)(-m) \longrightarrow \Omega^r_{\mathbb{P}^N} \longrightarrow \Omega^r_D \longrightarrow 0.$$

For $r = N$ we have $\Omega^N_D = 0$, i.e. $\Omega^N_{\mathbb{P}^N}(\log D) \cong \Omega^N_{\mathbb{P}^N}(m) \cong \mathcal{O}_{\mathbb{P}^N}(m - N - 1)$. This implies a pairing $\Omega^r_{\mathbb{P}^N}(\log D)(t) \times \Omega^{N-r}_{\mathbb{P}^N}(\log D)(-t - m + N + 1) \longrightarrow \mathcal{O}_{\mathbb{P}^N}$, which means that the vector space $H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t))$ is dual to $H^{N-q}(\mathbb{P}^N, (\Omega^{N-r}_{\mathbb{P}^N}(\log D)(-t - m + N + 1)) \otimes \Omega^N_{\mathbb{P}^N})$.
Setting $\Omega^N_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^N}(-N-1)$ yields (i). The statement (ii) can be shown in a similar way. Note that $\Omega^n_Y(\log D) \cong \Omega^n_Y(m) \cong \mathcal{O}_Y(m + \sum_{i=1}^{N-n} m_i - N - 1)$. Now, let T be a Young tableau with r boxes, given by the row lengths $l_1, \ldots, l_d$ and the column lengths $d_1, \ldots, d_l$ where $d = d_1 = \operatorname{depth} T$ and $l = l_1 = \operatorname{length} T$. The Young tableau T^* has the column lengths $d^*_j = n - d_{l+1-j}$ for all $j \in \{1, \ldots, l\}$ and we have again $\Omega^N_{\mathbb{P}^N}(\log D) \cong \mathcal{O}_{\mathbb{P}^N}(m - N - 1)$. From the pairing $\Omega^T_{\mathbb{P}^N}(\log D)(t) \times \Omega^{T^*}_{\mathbb{P}^N}(\log D)(-t - l\cdot(m - N - 1)) \longrightarrow \mathcal{O}_{\mathbb{P}^N}$ follows $\operatorname{Hom}(\Omega^T_{\mathbb{P}^N}(\log D)(t), \mathcal{O}_{\mathbb{P}^N}) \cong \Omega^{T^*}_{\mathbb{P}^N}(\log D)(-t - l\cdot(m - N - 1))$, which shows assertion (iii). In order to show the formula for complete intersections Y instead of $\mathbb{P}^N$, we replace $-N-1$ by $\sum_{i=1}^{N-n} m_i - N - 1$. Choosing $l = r$ ($\operatorname{depth} T = 1$) in (iii) and (iv) proves (v) and (vi), respectively. $\square$

For a projective variety $X \subseteq \mathbb{P}^N$ and a coherent sheaf $\mathcal{F}$ on X the dimensions $\dim_k H^q(X, \mathcal{F})$ are finite and we have the so-called Euler-Poincare characteristic given by $\chi(X, \mathcal{F}) = \sum_{q=0}^{\dim X} (-1)^q \cdot \dim H^q(X, \mathcal{F})$. From a short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ with coherent sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ on X we obtain the equation $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$. Under the above assumptions we also know, that for a short exact sequence of coherent sheaves on X there exists a long exact sequence for the associated cohomology groups. For every coherent sheaf $\mathcal{F}$ on the projective variety $X \subset \mathbb{P}^N$ there exists a polynomial $P(X, \mathcal{F})(t) \in \mathbb{Q}[t]$ of degree $\dim X$ which fulfills $\chi(X, \mathcal{F}(t)) = P(X, \mathcal{F})(t)$ for all $t \in \mathbb{Z}$. $P(X, \mathcal{F})(t)$ is said to be the Hilbert polynomial of $\mathcal{F}$ (cf. [6], [4], [8]).

For example, the structure sheaf on $\mathbb{P}^N$ has the following Hilbert polynomial

$$P(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N})(t) = \chi(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = \frac{(t+N)\cdot \ldots \cdot (t+1)}{N!} \ . \tag{1}$$

3. The Projective Space $\mathbb{P}^N$

In the following, we change the meaning of the binomial coefficient setting $\binom{\alpha}{\beta} = 0$ for all $\alpha \in \mathbb{Z}, \beta \in \mathbb{N}$ satisfying $\alpha < \beta$, in particular: $\binom{\alpha}{\beta} = 0$ if $\alpha < 0$. For instance: $\dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = \binom{t+N}{N}$, $\dim H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = \binom{-t-1}{N}$, $H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = 0$ for $0 < q < N$.
Let $H \subset \mathbb{P}^N$ $(N \geq 2)$ be a nonsingular, irreducible, algebraic hypersurface defined by the equation $F = 0$, that means, $D = H$ is a prime divisor in $\mathbb{P}^N$. Both F and D are of degree m and $D = H$ has dimension $N-1$.

3.1. **Alternating Differential Forms.** We denote by $\Omega^r_{\mathbb{P}^N}$ the local free sheaf of germs of alternating differential forms on the projective space $\mathbb{P}^N$ and consider the following sequence $(t \in \mathbb{Z})$

$$0 \longrightarrow \Omega^r_{\mathbb{P}^N}(t) \longrightarrow \Omega^r_{\mathbb{P}^N}(\log D)(t) \longrightarrow \Omega^{r-1}_D(t) \longrightarrow 0, \tag{2}$$

which is known to be exact (cf. [5]). The dimensions of the cohomology groups $H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t))$ and $H^q(D, \Omega^{r-1}_D(t))$ are calculated in [1], where we also find the following exact sequences

$$0 \longrightarrow \Omega^r_{\mathbb{P}^N}(t-m) \longrightarrow \Omega^r_{\mathbb{P}^N}(t) \xrightarrow{\alpha} \mathcal{O}_D(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^r_{\mathbb{P}^N} \longrightarrow 0 \tag{3}$$

$$0 \longrightarrow \Omega^{r-1}_D(t-m) \longrightarrow \mathcal{O}_D(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^r_{\mathbb{P}^N} \xrightarrow{\beta} \Omega^r_D(t) \longrightarrow 0. \tag{4}$$

The mapping $\varphi^* := \beta \circ \alpha$ means the restriction of the differential forms on $\mathbb{P}^N$ to the hypersurface $D = H$. In the case $r = 1$, one has to replace the sheaf Ω^{r-1}_D by the structure sheaf $\mathcal{O}_D$. For $0 < q < N$ we have $\dim H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t)) = \delta_{q,r} \cdot \delta_{t,0}$ (Kronecker-δ) and we know by [1, Lemma 4] a base element of $H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$ which is given by the cohomology class of the cocycle $\omega^{(r)} \in \mathrm{C}^r(\mathfrak{U}, \Omega^r_{\mathbb{P}^N})$ defined by

$$\omega^{(r)}_{i_0,\ldots,i_r} = \frac{x_{i_0}}{x_{i_r}} \cdot \mathrm{d}\,\frac{x_{i_1}}{x_{i_0}} \wedge \mathrm{d}\,\frac{x_{i_2}}{x_{i_1}} \wedge \ldots \wedge \mathrm{d}\,\frac{x_{i_r}}{x_{i_{r-1}}}. \tag{5}$$

$\mathfrak{U}$ stands for the affine open covering of $\mathbb{P}^N$ by the affine spaces $U_i = \{x_i \neq 0\}$. For $r = 1$, in particular, $\omega^{(1)}_{i_0,i_1} = \dfrac{x_{i_0}}{x_{i_1}} \cdot \mathrm{d}\,\dfrac{x_{i_1}}{x_{i_0}}$ is a logarithmic differential. We may represent (5) by

$$\omega^{(r)}_{i_0,\ldots,i_r} = \omega^{(1)}_{i_0,i_1} \wedge \omega^{(1)}_{i_1,i_2} \wedge \ldots \wedge \omega^{(1)}_{i_{r-1},i_r} \ ,$$

which is an outer product of logarithmic differential forms. In the case $q = r = N$, $t = 0$ the cochain $\omega^{(N)}$ creates a base of $H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N})$ (cf. [1, Lemma 2]). Finally, we set $\omega^{(0)} = 1$.

Lemma 3.1. *Let $0 < r \leq N$. Then the homomorphism $d : H^{r-1}(D, \Omega^{r-1}_D) \longrightarrow H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$ in the long homology sequence with respect to the exact sequence*

$$0 \longrightarrow \Omega^r_{\mathbb{P}^N} \longrightarrow \Omega^r_{\mathbb{P}^N}(\log D) \longrightarrow \Omega^{r-1}_D \longrightarrow 0$$

is epimorphic. If in addition $2(r-1) \neq N-1$ is valid, then d is an isomorphism.

Proof. We calculate the image of the cohomology class of $\omega^{(r-1)}$ at the composition

$$H^{r-1}(\mathbb{P}^N, \Omega^{r-1}_{\mathbb{P}^N}) \xrightarrow{\varphi^*} H^{r-1}(D, \Omega^{r-1}_D) \xrightarrow{d} H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$$

and denote $\varphi^*(\omega^{(r-1)})$ again by $\omega^{(r-1)}$. Let $\mathfrak{U}$ be the affine, open covering of $\mathbb{P}^N$ given by the affine spaces $U_i = \{x_i \neq 0\}$. We consider the following commutative diagram

$$\begin{array}{ccccccc}
0 \to & \mathrm{C}^{r-1}(\mathfrak{U}, \Omega^r_{\mathbb{P}^N}) & \to & \mathrm{C}^{r-1}(\mathfrak{U}, \Omega^r_{\mathbb{P}^N}(\log D)) & \to & \mathrm{C}^{r-1}(\mathfrak{U}, \Omega^{r-1}_D) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{C}^{r}(\mathfrak{U}, \Omega^r_{\mathbb{P}^N}) & \to & \mathrm{C}^{r}(\mathfrak{U}, \Omega^r_{\mathbb{P}^N}(\log D)) & \to & \mathrm{C}^{r}(\mathfrak{U}, \Omega^{r-1}_D) & \to 0
\end{array}$$

where the cocycle $\omega^{(r-1)} \in \mathrm{C}^{r-1}(\mathfrak{U}, \Omega^{r-1}_D)$ possesses in $\mathrm{C}^{r-1}(\mathfrak{U}, \Omega^r_{\mathbb{P}^N}(\log D))$ the preimage ϱ defined by $\varrho_{i_0,\ldots,i_{r-1}} = \omega^{(r-1)}_{i_0,\ldots,i_{r-1}} \wedge \frac{x^m_{i_0}}{F} \cdot \mathrm{d}\, \frac{F}{x^m_{i_0}}$ (cf. [5]). Elementary calculations show that $d\omega^{(r-1)} = (-1)^r \cdot m \cdot \omega^{(r)} \in \mathrm{C}^r(\mathfrak{U}, \Omega^r_{\mathbb{P}^N})$. Therefore, the cocycle $d\omega^{(r-1)} \in \mathrm{C}^r(\mathfrak{U}, \Omega^r_{\mathbb{P}^N})$ is nonzero and the associated cohomology class is a base of $H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$. Thus, the homomorphism $d : H^{r-1}(D, \Omega^{r-1}_D) \longrightarrow H^r(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N})$ is epimorphic. In the case $2(r-1) \neq N-1$, we obtain $\dim H^{r-1}(D, \Omega^{r-1}_D) = 1$ by [1, Satz 2 and Lemma 5], which implies that d is an isomorphism. □

Theorem 3.2.
Let $D \subset \mathbb{P}^N$ be a smooth algebraic hypersurface of degree m $(N \geq 2)$.

(a) *For each $r \in \{1, \ldots, N-1\}$ one has:*
$$\dim H^0(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) = \sum_{i=0}^{r} (-1)^i \cdot \binom{N+1}{r-i} \cdot \binom{t+N-i\cdot(m-1)-r}{N}$$

(b) *For all $r \in \{1, \ldots, N-1\}$ holds: $H^0(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) \neq 0 \;\Leftrightarrow\; t \geq r$*

(c) *In the case $r = N$ one has:* $\dim H^0(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(t)) = \binom{t+m-1}{N}$

(d) *If $D \subset \mathbb{P}^N$ is a hyperplane $(m = 1)$, then holds:*
$$\dim H^0(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) = \binom{N}{r} \cdot \binom{t+N-r}{N}$$

Proof. The formula (a) follows directly from the long exact cohomology sequence related to the exact sequence in (2) by applying Lemma 3.1. For $r = N$ we obtain $\Omega^N_{\mathbb{P}^N}(\log D) \cong \Omega^N_{\mathbb{P}^N}(m) \cong \mathcal{O}_{\mathbb{P}^N}(m-N-1)$ which yields (c). (a) obviously implies (b) and (d). □

Theorem 3.3.

(a) *Let $0 < q < N$, $q + r \neq N$ and $r \geq 1$.*
Then we obtain $H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) = 0$ for all $t \in \mathbb{Z}$.

(b) *For* $1 \le r \le N-1$ *it follows:*

$$\begin{aligned}
&\dim H^{N-r}(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) \\
&\quad = \sum_{i=0}^{N+1} (-1)^i \cdot \binom{N+1}{i} \cdot \binom{t+(N-r)\cdot m-(i-1)\cdot(m-1)}{N} \\
&\quad = \sum_{i=0}^{N+1} (-1)^i \cdot \binom{N+1}{i} \cdot \binom{-t+(r-1)\cdot m-(i-1)\cdot(m-1)}{N}
\end{aligned}$$

That means: If D *is a hyperplane* ($m=1$), *then we have* $H^{N-r}(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) = 0$ *for all* $t \in \mathbb{Z}$.

(c) *For* $1 \le r \le N-1$ *one has:*

$$\begin{aligned}
&\dim H^N(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) \\
&\quad = \sum_{i=0}^{N-r} (-1)^i \cdot \binom{N+1}{N-r-i} \cdot \binom{-t-m-i\cdot(m-1)+r}{N}
\end{aligned}$$

If D *is a hyperplane* ($m=1$), *then we get:*

$$\dim H^N(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) = \binom{N}{r} \cdot \binom{-t-1+r}{N}$$

(d) $\dim H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}(\log D)(t)) = \binom{-t-m+N}{N}$

Proof. We consider the following exact sequence

$$\begin{aligned}
\ldots \longrightarrow H^{q-1}(D, \Omega^{r-1}_D(t)) &\xrightarrow{d_1} H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t)) \longrightarrow \\
\longrightarrow H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) &\longrightarrow H^q(D, \Omega^{r-1}_D(t)) \xrightarrow{d_2} \\
\xrightarrow{d_2} H^{q+1}(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t)) &\longrightarrow \ldots,
\end{aligned} \tag{6}$$

and assume $0 < q$, $0 < r$ and $q + r < N$. By Lemma 3.1 the mappings d_1 and d_2 are epimorphic for all $t \in \mathbb{Z}$ and from (6) we get the exact sequence

$$0 \to H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) \to H^q(D, \Omega^{r-1}_D(t)) \xrightarrow{d_2} H^{q+1}(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t)) \to 0.$$

Under these assumptions holds $H^q(D, \Omega^{r-1}_D(t)) = 0$ if $q \neq r-1$ or $t \neq 0$ (cf. [1]). In case $q = r-1, t = 0$ we know that d_2 is an isomorphism by Lemma 3.1 since $2(r-1) < N-1$. Therefore, one has

$$H^q(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) = 0 \text{ for } 0 < q \ , \ 0 < r \text{ and } q + r < N.$$

For $q < N, r < N, q + r > N$ we use the Serre duality to show statement (a). The case $r = N$ is trivial since $\Omega^N_{\mathbb{P}^N}(\log D) \cong \mathcal{O}_{\mathbb{P}^N}(m - N - 1)$.
If $r \ge 2$ and $q + r = N$ then the mappings d_1 and d_2 are epimorphic, i.e.

$$\begin{aligned}
&\dim H^{N-r}(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(\log D)(t)) \\
&\qquad = \dim H^{N-r}(D, \Omega^{r-1}_D(t)) - \dim H^{N-r+1}(\mathbb{P}^N, \Omega^r_{\mathbb{P}^N}(t)).
\end{aligned}$$

In the case $r = 1$ and $q = N - 1$ one has

$\dim H^{N-1}(\mathbb{P}^N, \Omega^1_{\mathbb{P}^N}(\log D)(t))$

$$= \dim H^{N-1}(D, \mathcal{O}_D(t)) - \dim H^N(\mathbb{P}^N, \Omega^1_{\mathbb{P}^N}(t)) + H^N(\mathbb{P}^N, \Omega^1_{\mathbb{P}^N}(\log D)(t)).$$

Applying Theorem 3.2, Lemma 2.1 and the results in [1] delivers (b) and (c). $\square$

3.2. **T-symmetric Tensor Differential Forms.** Let T be a Young tableau with r boxes. We study the sheaf $\Omega^T(\log D) = (\Omega^1(\log D))^T$ on $\mathbb{P}^N$ and begin with a free resolution of the sheaf $\Omega^1(\log D)$.

Lemma 3.4. *Let $D \subset \mathbb{P}^N$ be a nonsingular, irreducible, algebraic hypersurface of degree $m \geq 2$ defined by the equation $F = 0$.*
Then there exists a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-m) \longrightarrow \overset{N+1}{\oplus} \mathcal{O}_{\mathbb{P}^N}(-1) \longrightarrow \Omega^1_{\mathbb{P}^N}(\log D) \longrightarrow 0. \tag{7}$$

If D is a hyperplane, i.e. $m = 1$, we have $\Omega^1_{\mathbb{P}^N}(\log D) \cong \overset{N}{\oplus} \mathcal{O}_{\mathbb{P}^N}(-1)$.

Proof. Let $U_i = \{x_i \neq 0\} \subset \mathbb{P}^N$ and let $U \subseteq \mathbb{P}^N$ be an arbitrary open affine subset. We are going to show that there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_{\mathbb{P}^N}(-m)) \overset{\alpha}{\to} \overset{N+1}{\oplus} \Gamma(U, \mathcal{O}_{\mathbb{P}^N}(-1)) \overset{\beta}{\to} \Gamma(U, \Omega^1_{\mathbb{P}^N}(\log D)) \to 0.$$

For sections $f_0, \ldots, f_N \in \Gamma(U, \mathcal{O}(-1))$ we put $g := -\frac{1}{m} \cdot \sum_{\mu=0}^N x_\mu f_\mu \in \Gamma(U, \mathcal{O})$. Let $F_j = \frac{\partial F}{\partial x_j}$ denotes the partial derivatives of F. The mapping β is defined by $(f_0, \ldots, f_N) \longmapsto \omega$, where the differential form ω on $U \cap U_i$ is given by

$$\omega = \omega_i := \sum_{\substack{\nu=0 \\ \nu \neq i}}^{N} \left(f_\nu + g \cdot \frac{F_\nu}{F} \right) \cdot x_i \cdot \mathrm{d}\, \frac{x_\nu}{x_i} \ .$$

One easily verifies that ω is a section of $\Omega^1_{\mathbb{P}^N}(\log D)$ on U and it holds, in particular, $\omega_i = \omega_j$ for any $i, j \in \{0, 1, \ldots, N\}$. For a section $\delta \in \Gamma(U, \mathcal{O}(-m))$ let $f_\nu = \delta \cdot F_\nu$ for all $\nu = 0, 1, \ldots, N$ which implies that $f_\nu \in \Gamma(U, \mathcal{O}(-1))$ and $g = -\delta \cdot F$. Finally, we have $\ker \beta = \{(\delta \cdot F_0, \ldots, \delta \cdot F_N)\} \cong \Gamma(U, \mathcal{O}_{\mathbb{P}^N}(-m))$, which yields the claim for $m \geq 2$.
In the last part we have to show the statement of Lemma 3.4 in case $m = 1$. Let $D \subset \mathbb{P}^N$ be the hyperplane satisfying the equation $x_N = 0$, and let $U \subseteq \mathbb{P}^N$ be an open subset. For given sections $f_0, \ldots, f_{N-1} \in \Gamma(U, \mathcal{O}(-1))$ let ω be the differential form, which has on $U \cap U_i$, $i = 0, , \ldots, N-1$, the representation

$$\omega = \omega_i = \sum_{\substack{\nu=0 \\ \nu \neq i}}^{N-1} f_\nu \cdot x_i \cdot \mathrm{d}\, \frac{x_\nu}{x_i} - \left(\sum_{\mu=0}^{N-1} f_\mu \cdot x_\mu \right) \cdot \frac{x_i}{x_N}\, \mathrm{d}\, \frac{x_N}{x_i},$$

respectively on $U \cap U_N$,

$$\omega = \omega_N = \sum_{\nu=0}^{N-1} f_\nu \cdot x_N \cdot \mathrm{d}\, \frac{x_\nu}{x_N}.$$

Then ω is a section of $\Omega^1_{\mathbb{P}^N}(\log D)$ on U, and the mapping $(f_0, \ldots, f_{N-1}) \mapsto \omega$ becomes an isomorphism of $\Gamma(U, \overset{N}{\oplus} \mathcal{O}_{\mathbb{P}^N}(-1))$ onto $\Gamma(U, \Omega^1_{\mathbb{P}^N}(\log D))$. $\square$

Lemma 3.5. *Let T be a Young tableau with r boxes and the row lengths $l_1, l_2, \ldots, l_d$, set $t_i := r + l_i - i$ for all $i \geq 1$ ($l_i = 0$ if $i > d$) and assume $d = \operatorname{depth} T \leq N$. Then the following sequence is exact for $m \geq 2$:*

$$0 \longrightarrow \bigoplus_{b_d} \mathcal{O}_{\mathbb{P}^N}(d \cdot (1-m) - r) \xrightarrow{\alpha_d} \bigoplus_{b_{d-1}} \mathcal{O}_{\mathbb{P}^N}((d-1) \cdot (1-m) - r) \xrightarrow{\alpha_{d-1}}$$
$$\xrightarrow{\alpha_{d-1}} \bigoplus_{b_{d-2}} \mathcal{O}_{\mathbb{P}^N}((d-2) \cdot (1-m) - r) \xrightarrow{\alpha_{d-2}} \ldots \xrightarrow{\alpha_2} \bigoplus_{b_1} \mathcal{O}_{\mathbb{P}^N}(1-m-r) \xrightarrow{\alpha_1} \tag{8}$$
$$\xrightarrow{\alpha_1} \bigoplus_{b_0} \mathcal{O}_{\mathbb{P}^N}(-r) \xrightarrow{\alpha_0} \Omega^T_{\mathbb{P}^N}(\log D) \longrightarrow 0$$

with the integers

$$b_s = \left(\prod_{i=1}^{N} i! \right)^{-1} \cdot \sum_{1 \leq i_1 < \ldots < i_s \leq d} \Delta(t_1, t_2, \ldots, t_{i_1} - 1, \ldots, t_{i_s} - 1, \ldots, t_N, t_{N+1}) \tag{9}$$

where Δ denotes the Vandermonde determinant.
In the case $s = 0$ we have

$$b_0 = \left(\prod_{i=1}^{N} i! \right)^{-1} \cdot \prod_{1 \leq i < j \leq N+1} (l_i - l_j + j - i) = \left(\prod_{i=1}^{N} i! \right)^{-1} \cdot \Delta(t_1, t_2, \ldots, t_N, t_{N+1}).$$

For $m = 1$ (D is a hyperplane) holds

$$\Omega^T_{\mathbb{P}^N}(\log D) \cong \bigoplus^{\operatorname{rk}(\Omega^T_{\mathbb{P}^N})} \mathcal{O}_{\mathbb{P}^N}(-r)$$

with

$$\begin{aligned} \operatorname{rk}(\Omega^T_{\mathbb{P}^N}) &= \prod_{1 \leq i < j \leq N} \left(\frac{l_i - l_j}{j - i} + 1 \right) = \sum_{i=0}^{d} (-1)^i \cdot b_i \\ &= \left(\prod_{i=1}^{N-1} i! \right)^{-1} \cdot \Delta(t_1, t_2, \ldots, t_N) \ . \end{aligned}$$

Proof. The T-Power of (7) yields the claim for $m \geq 2$ (cf. [3]) and Lemma 3.4 shows the case $m = 1$. □

Theorem 3.6. *Let T be a Young tableau with r boxes and with $d = \operatorname{depth} T$ rows.*

(a) $\chi(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) = \dfrac{1}{N!} \cdot \sum_{i=0}^{d} (-1)^i \cdot b_i \cdot \prod_{j=1}^{N} (t - i \cdot (m-1) + j - r)$

(b) *For* $\operatorname{depth} T < N$ *one has:*

$$\dim H^0(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) = \sum_{i=0}^{d} (-1)^i \cdot b_i \cdot \binom{t - i \cdot (m-1) + N - r}{N}$$

and therefore: $H^0(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) \neq 0 \iff t \geq r$

(c) *Let* $d = \operatorname{depth} T = N$ *and let* l_N *be the number of columns of* T *with the length* N*. We denote by* T' *the Young tableau which is given by* T *without these columns of length* N*. Then* $\operatorname{depth} T' < N$ *and it holds* $\Omega^T_{\mathbb{P}^N}(\log D)(t) \cong \Omega^{T'}_{\mathbb{P}^N}(\log D)(t + l_N \cdot (m - N - 1))$.
If T *is a rectangle with* N *rows and* l *columns, then we have* $\Omega^T_{\mathbb{P}^N}(\log D)(t) \cong \mathcal{O}_{\mathbb{P}^N}(t + l \cdot (m - N - 1))$.
(d) *For* $1 \leq q < N - d$ *we get* $H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) = 0$ *for all* $t \in \mathbb{Z}$.
(e) *Let* d_l *be the length of the last column of* T*. Then holds:* $H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) = 0$ *for* $N - d_l < q < N$ *and* $\forall t \in \mathbb{Z}$.

Proof. The short exact sequences of (8) yields

$$\begin{aligned}
0 \quad &\longrightarrow \bigoplus_{b_d} \mathcal{O}_{\mathbb{P}^N}(d \cdot (1-m) - r) \longrightarrow \bigoplus_{b_{d-1}} \mathcal{O}_{\mathbb{P}^N}((d-1) \cdot (1-m) - r) \longrightarrow \\
&\longrightarrow \operatorname{Im} \alpha_{d-1} \longrightarrow 0 \\
0 \quad &\longrightarrow \operatorname{Im} \alpha_{d-1} \longrightarrow \bigoplus_{b_{d-2}} \mathcal{O}_{\mathbb{P}^N}((d-2) \cdot (1-m) - r) \longrightarrow \operatorname{Im} \alpha_{d-2} \longrightarrow 0 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
0 \quad &\longrightarrow \operatorname{Im} \alpha_2 \longrightarrow \bigoplus_{b_1} \mathcal{O}_{\mathbb{P}^N}(1 - m - r) \longrightarrow \operatorname{Im} \alpha_1 \longrightarrow 0 \\
0 \quad &\longrightarrow \operatorname{Im} \alpha_1 \longrightarrow \bigoplus_{b_0} \mathcal{O}_{\mathbb{P}^N}(-r) \longrightarrow \Omega^T_{\mathbb{P}^N}(\log D) \longrightarrow 0
\end{aligned}$$

where $H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = 0$ for $1 \leq q \leq N-1$ and for all $t \in \mathbb{Z}$. This implies $H^q(\mathbb{P}^N, \operatorname{Im} \alpha_i(t)) = 0$ for $1 \leq q \leq N - 1 + i - d$ and hence, we have in case $d < N$

$$\begin{aligned}
&\dim H^0(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) \\
&= b_0 \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t - r)) - \dim H^0(\mathbb{P}^N, \operatorname{Im} \alpha_1(t)) \\
&\dim H^0(\mathbb{P}^N, \operatorname{Im} \alpha_1(t)) \\
&= b_1 \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t + 1 - m - r)) - \dim H^0(\mathbb{P}^N, \operatorname{Im} \alpha_2(t)) \\
&\vdots \\
&\dim H^0(\mathbb{P}^N, \operatorname{Im} \alpha_{d-1}(t)) = b_{d-1} \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t + (d-1) \cdot (1-m) - r)) \\
&- b_d \cdot \dim H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t + d \cdot (1-m) - r)).
\end{aligned}$$

This shows (b). For $d = N$ we already know that

$$\begin{aligned}
\Omega^T_{\mathbb{P}^N}(\log)(t) &\cong \Omega^N_{\mathbb{P}^N}(\log D) \otimes \ldots \otimes \Omega^N_{\mathbb{P}^N}(\log D) \otimes \Omega^{T'}_{\mathbb{P}^N}(\log D)(t) \\
&\cong \Omega^{T'}_{\mathbb{P}^N}(\log D)(t + l_N \cdot (m - N - 1))
\end{aligned}$$

which proves assertion (c). In order to prove (d), we consider again the short exact sequences of (8) and obtain

$$\begin{aligned}
&H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) = 0 \;\text{ if} \\
&H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t - r)) = 0 \;\text{ and }\; H^{q+1}(\mathbb{P}^N, \operatorname{Im} \alpha_1(t)) = 0
\end{aligned}$$

$$
\begin{aligned}
& H^{q+1}(\mathbb{P}^N, \operatorname{Im}\alpha_1(t)) = 0 \text{ if} \\
& \quad H^{q+1}(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t+1-m-r)) = 0 \text{ and } H^{q+2}(\mathbb{P}^N, \operatorname{Im}\alpha_2(t)) = 0 \\
& \qquad\qquad \vdots \\
& H^{q+d-1}(\mathbb{P}^N, \operatorname{Im}\alpha_{d-1}(t)) = 0 \text{ if} \\
& \quad H^{q+d-1}(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t+(d-1)\cdot(1-m)-r)) = 0 \\
& \text{and } H^{q+d}(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t+d\cdot(1-m)-r)) = 0.
\end{aligned}
$$

This implies $H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) = 0$ for $1 \le q \le N-d-1$ and $\forall t \in \mathbb{Z}$. The last statement can be proven by Serre duality which means

$$
\begin{aligned}
& \dim H^q(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t)) \\
& = \dim H^{N-q}(\mathbb{P}^N, \Omega^{T^*}_{\mathbb{P}^N}(\log D)(-t-m-(l-1)\cdot(m-N-1)))
\end{aligned}
$$

where $\operatorname{depth} T^* = N - d_l < N$. Note if we use (b) with T^* instead of T, we obtain a formula for $\dim H^N(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D)(t))$. □

3.3. **Symmetric Differential Forms.** Let T be a Young tableau with r boxes and only one row, i.e. $\operatorname{depth} T = 1$. We will specify the dimensions of $H^q(\mathbb{P}^N, S^r\Omega^1(\log D)(t))$ and consider the following exact sequence (cf. Lemma 3.5)

$$
0 \longrightarrow \bigoplus_{b_1} \mathcal{O}_{\mathbb{P}^N}(-m+1-r) \longrightarrow \bigoplus_{b_0} \mathcal{O}_{\mathbb{P}^N}(-r) \longrightarrow S^r\Omega^1_{\mathbb{P}^N}(\log D) \longrightarrow 0 \tag{10}
$$

with the integers $b_0 = \binom{N+r}{N}$ and $b_1 = \binom{N+r-1}{N}$.

Theorem 3.7. *Let $N \ge 2$. Then one has:*

(a) $\chi(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t))$

$$
= \frac{1}{N!}\cdot\binom{N+r}{N}\cdot\prod_{j=1}^{N}(t-r+j) - \frac{1}{N!}\cdot\binom{N+r-1}{N}\cdot\prod_{i=1}^{N}(t-m+1-r+i)
$$

(b) $\dim H^0(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t))$

$$
= \binom{N+r}{N}\cdot\binom{t-r+N}{N} - \binom{N+r-1}{N}\cdot\binom{t-m+1-r+N}{N}
$$

(c) *For $1 \le q \le N-2$ holds: $H^q(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) = 0$ for all $t \in \mathbb{Z}$*

(d) $\dim H^{N-1}(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t))$

$$
\begin{aligned}
& = \sum_{i=0}^{N-1}(-1)^i\cdot\widetilde{b}_i\cdot\binom{-t-r(m-2)-i\cdot(m-1)-1}{N} \\
& \quad -\binom{N+r}{N}\cdot\binom{-t+r-1}{N} + \binom{N+r-1}{N}\cdot\binom{-t+m+r-2}{N}
\end{aligned}
$$

$\dim H^N(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t))$

$$= \sum_{i=0}^{N-1} (-1)^i \cdot \widetilde{b}_i \cdot \binom{-t-r(m-2)-i\cdot(m-1)-1}{N}$$

with the integers

$$\widetilde{b}_i = \frac{1}{N+r} \cdot \binom{N+r}{N-1-i} \cdot \binom{N+r}{N} \cdot \binom{r+i-1}{i} \tag{11}$$

Proof. (a) follows directly from (10) and the additivity of the Euler characteristic. We consider (10) together with the corresponding cohomology sequence and know that $H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) = 0$ for any $q \in \{1, \dots, N-1\}$ and for all $t \in \mathbb{Z}$, which implies (b) and (c). Using the Serre Duality yields

$$\dim H^N(\mathbb{P}^N, S^r\Omega^1(\log D)(t))$$
$$= \dim H^0(\mathbb{P}^N, \Omega^{T^*}(\log D)(-t + (r-1)\cdot(N+1) - r\cdot m)) \ ,$$

where T^* is a rectangle with depth $T^* = N-1$ rows and length $T^* = r$ columns and with the associated integers $\widetilde{b}_i$ in (11) (cf. Lemma 3.5). Theorem 3.6(b) delivers the formula for $\dim H^0(\mathbb{P}^N, \Omega^{T^*}(\log D)(-t+(r-1)\cdot(N+1)-r\cdot m))$. Finally, one gets easily the dimension $\dim H^{N-1}(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t))$ from the long cohomology sequence. ☐

Corollary 3.8. *For $N \geq 2$ we obtain:*

(a) $H^0(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) \neq 0 \ \Leftrightarrow \ t \geq r$
(b) $\dim H^0(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) = \chi(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t))$ *if* $t \geq m + r - N - 1$
(c) $H^N(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) = 0 \ \Leftrightarrow \ t \geq -r(m-2) - N$
(d) $H^{N-1}(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) = 0$ *if* $t \geq m + r - N - 1$

Proof. Obviously, the proof follows from Theorem 3.7. ☐

Theorem 3.9. *Let $N \geq 2$ and let D be a hyperplane, that is, $m = 1$. Then one has*

(a) $\dim H^0(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) = \binom{N+r-1}{N-1} \cdot \binom{t-r+N}{N}$

(b) *For* $1 \leq q \leq N-1$ *holds:* $H^q(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) = 0 \ \forall t \in \mathbb{Z}$

(c) $\dim H^N(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D)(t)) = \binom{N+r-1}{N-1} \cdot \binom{-t+r-1}{N}$

Proof. $S^r\Omega^1_{\mathbb{P}^N}(\log D) \cong \bigoplus_{\binom{N+r-1}{N-1}} \mathcal{O}_{\mathbb{P}^N}(-r)$. ☐

4. Complete Intersections $Y \subset \mathbb{P}^N$

Let $Y = H_1 \cap \ldots \cap H_{N-n} \subseteq \mathbb{P}^N$ be a nonsingular, irreducible, complete intersection of algebraic hypersurfaces $H_i \subset \mathbb{P}^N$, where H_i is given by the equation $F_i = 0$ with $\deg F_i = m_i$. We denote by n the dimension of Y. Let D be a prime divisor on Y, which is defined by the equation $D = Y \cap H$

with a hypersurface $H : F = 0$. The degree of H is m. In the following, we abbreviating denote $c = N - n = \operatorname{codim} Y$ and assume $n \geq 2$. Let X be a further complete intersection which is described by $X = H_1 \cap \ldots \cap H_{c-1}$. Here $\dim X = n+1$ and $Y = X \cap H_c$. There exists also a divisor $D^* = X \cap H$ on X. Assume that the hypersurfaces $H_1, \ldots, H_{N-n}$ and H lie in general position, i.e. for instance $X = H_1 \cap \ldots \cap H_{c-1} \subseteq \mathbb{P}^N$ and the prime divisors D on Y and D^* on X are nonsingular, irreducible, complete intersections, too.

4.1. **Alternating Differential Forms.** In case $r = n$ we obtain $\Omega_Y^n = \omega_Y \cong \mathcal{O}_Y\left(\sum_{i=1}^c m_i - N - 1\right)$ which implies

$$\Omega_Y^n(\log D) \cong \Omega_Y^n(m) \cong \mathcal{O}_Y(\sum_{i=1}^c m_i - N - 1 + m) \ ,$$

where $D = Y \cap H$ with $\deg H = m$. The dimensions of $H^q(Y, \Omega_Y^n(\log D)(t)) = H^q(Y, \mathcal{O}_Y(\sum_{i=1}^c m_i - N - 1 + m + t))$ are well known:

If $1 \leq q \leq n-1$ then $H^q(Y, \mathcal{O}_Y(t)) = 0 \ \forall t \in \mathbb{Z}$.

$$\dim H^0(Y, \mathcal{O}_Y(t)) = \binom{t+N}{N} +$$

$$+ \sum_{j=1}^c (-1)^j \cdot \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq c} \binom{t + N - m_{i_1} - m_{i_2} - \ldots - m_{i_j}}{N}$$

$$\dim H^n(Y, \mathcal{O}_Y(t)) = \dim H^0(Y, \mathcal{O}_Y(-t + m_1 + m_2 + \ldots + m_c - N - 1))$$

(cf. e.g. [1] or the proof of Lemma (4.4) in the present paper).
We study the cohomology groups $H^q(Y, \Omega_Y^r(\log D)(t))$ with $r < \dim Y = n$:

Lemma 4.1. *The following sequences are exact.*

$$(a)\ 0 \longrightarrow \mathcal{O}_X(-m_c) \longrightarrow \mathcal{O}_X \xrightarrow{\varphi^*} \mathcal{O}_Y \longrightarrow 0 \tag{12}$$

$$(b)\ 0 \to \Omega_X^r(\log D^*)(-m_c) \xrightarrow{\alpha} \Omega_X^r(\log D^*) \xrightarrow{\beta} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X^r(\log D^*) \to 0 \tag{13}$$

$$(c)\ 0 \to \Omega_Y^{r-1}(\log D)(-m_c) \xrightarrow{\gamma} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X^r(\log D^*) \xrightarrow{\delta} \Omega_Y^r(\log D) \to 0 \tag{14}$$

Proof. Notice, for $r = 1$ we have to substitute $\Omega_Y^{r-1}(\log D)$ by the structure sheaf $\mathcal{O}_Y$. The composition $\delta \circ \beta$ is the restriction of the differential forms on X to the subvariety $Y \subset X$. Obviously, the sequence (12) is exact and (13) results by multiplication of (12) with the locally free sheaf $\Omega_X^r(\log D^*)$. We will show that (14) is also an exact sequence. Let $U \subseteq X$ be an open subset of X and let $V = Y \cap U$ be an open, nonempty subset of Y. Without loss of generality we assume $U \subseteq U_i = \{x_i \neq 0\}$. Moreover, we suppose the existence of local parameters $u_1, \ldots, u_{n-1}, u_n = \frac{F}{x_i^m}, u_{n+1} = \frac{F_c}{x_i^{m_c}}$ of X on U such that their restriction to Y are also local parameters
$v_1 = \varphi^*(u_1), \ldots, v_{n-1} = \varphi^*(u_{n-1}), v_n = \varphi^*(u_n) = \frac{F}{x_i^m}$ of Y on V. Then

$\Gamma(V, \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X^r(\log D^*))$ is a free $\Gamma(V, \mathcal{O}_Y)$-module whose rank is equal to $\binom{n+1}{r}$. Let $\omega \in \Gamma(V, \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X^r(\log D^*))$ be a section of the form

$$\begin{aligned}\omega = \sum_{i_\nu=1}^{n-1} & f_{i_1,\ldots,i_r} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_r} \\ & + \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-1},n} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_{r-1}} \wedge \frac{\mathrm{d}\, u_n}{u_n} \\ & + \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-1},n+1} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_{r-1}} \wedge \mathrm{d}\, u_{n+1} \\ & + \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-2},n,n+1} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_{r-2}} \wedge \frac{\mathrm{d}\, u_n}{u_n} \wedge \mathrm{d}\, u_{n+1} \ ,\end{aligned}$$

where $f_{i_1,\ldots,i_r} \in \Gamma(V, \mathcal{O}_Y)$. The homomorphism δ is defined as follows:

$$\delta(\omega) = \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_r} \,\mathrm{d}\, v_{i_1} \wedge \ldots \wedge \mathrm{d}\, v_{i_r} + \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-1},n} \,\mathrm{d}\, v_{i_1} \wedge \ldots \wedge \mathrm{d}\, v_{i_{r-1}} \wedge \frac{\mathrm{d}\, v_n}{v_n},$$

which means that $\delta(\omega) \in \Gamma(V, \Omega_Y^r(\log D))$. The kernel of δ is given by

$$\begin{aligned}\ker \delta = \Bigg\{ & \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-1},n+1} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_{r-1}} \wedge \mathrm{d}\, u_{n+1} \\ & + \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-2},n,n+1} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_{r-2}} \wedge \frac{\mathrm{d}\, u_n}{u_n} \wedge \mathrm{d}\, u_{n+1} \Bigg\},\end{aligned}$$

where $\ker \delta \subseteq \Gamma(V, \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega_X^r(\log D^*))$. In order to show that the kernel of δ is isomorphic to $\Gamma(V, \Omega_Y^{r-1}(\log D)(-m_c))$, we consider the following homomorphisms

$$\ker \delta \xrightarrow{\widetilde{\alpha}} \Gamma(V, \mathcal{O}_Y(-m_c) \otimes_{\mathcal{O}_X} \Omega_X^{r-1}(\log D^*)) \xrightarrow{\widetilde{\beta}} \Gamma(V, \Omega_Y^{r-1}(\log D)(-m_c)).$$

Let $\xi \in \ker \delta$ be any element. The mappings $\widetilde{\alpha}$ and $\widetilde{\beta}$ are illustrated by

$$\begin{aligned}\widetilde{\alpha}(\xi) = \frac{1}{x_i^{m_c}} & \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-1},n+1} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_{r-1}} \\ & + \frac{1}{x_i^{m_c}} \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-2},n,n+1} \,\mathrm{d}\, u_{i_1} \wedge \ldots \wedge \mathrm{d}\, u_{i_{r-2}} \wedge \frac{\mathrm{d}\, u_n}{u_n},\end{aligned}$$

respectively,

$$\begin{aligned}\widetilde{\beta}(\widetilde{\alpha}(\xi)) = \frac{1}{x_i^{m_c}} & \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-1},n+1} \,\mathrm{d}\, v_{i_1} \wedge \ldots \wedge \mathrm{d}\, v_{i_{r-1}} \\ & + \frac{1}{x_i^{m_c}} \sum_{i_\nu=1}^{n-1} f_{i_1,\ldots,i_{r-2},n,n+1} \,\mathrm{d}\, v_{i_1} \wedge \ldots \wedge \mathrm{d}\, v_{i_{r-2}} \wedge \frac{\mathrm{d}\, v_n}{v_n} \ .\end{aligned}$$

Since $x_i^{m_c} \cdot \mathrm{d}\, u_{n+1} = x_i^{m_c} \cdot \mathrm{d}\, \frac{F_c}{x_i^{m_c}}$ is a global section of the sheaf $\mathcal{O}_Y(m_c) \otimes_{\mathcal{O}_X} \Omega^1_X$ the functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ are independent of the index i with $U \subseteq U_i$ and independent of the choice of the local parameters $u_1, \dots, u_{n-1}$. One can easily see that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are monomorphic. The mapping $\widetilde{\beta}$ is the restriction from X to Y which obviously is epimorphic. While $\widetilde{\alpha}$ is generally not epimorphic, any element of $\Gamma(V, \Omega^{r-1}_Y(\log D)(-m_c))$ has a preimage in $\ker \delta$. We can represent an element of $\Gamma(V, \Omega^{r-1}_Y(\log D)(-m_c))$ by the form $\widetilde{\beta}(\widetilde{\alpha}(\xi))$ with functions $f_{i_1,\dots,i_r} \in \Gamma(V, \mathcal{O}_Y)$. In order to find a preimage in $\ker \delta$, we use the same functions $f_{i_1,\dots,i_r}$, and in place of v_i we take the local parameters u_i on X and multiply with $x_i^{m_c} \cdot \mathrm{d}\, u_{n+1}$. This proves that the composition $\widetilde{\beta} \circ \widetilde{\alpha}$ is isomorphic, the sequence (14) is exact. □

By means of these exact sequences we are going to prove recursion formulas about the dimensions of the cohomology groups $H^q(Y, \Omega^r_Y(\log D)(t))$. As above mentioned, for $r = n$ these dimensions are known.

Theorem 4.2.

(a) $\chi(Y, \Omega^r_Y(\log D)(t)) = \chi(X, \Omega^r_X(\log D^*)(t))$
$-\chi(X, \Omega^r_X(\log D^*)(t - m_c)) - \chi(Y, \Omega^{r-1}_Y(\log D)(t - m_c))$ *for* $r \geq 1$
In the case $r = 1$ *one has to substitute* $\Omega^{r-1}_Y(\log D)$ *by the structure sheaf* $\mathcal{O}_Y$.

(b) *Let* $0 < q < n$, $q + r \neq n$ *and* $r \geq 0$.
Then one has $H^q(Y, \Omega^r_Y(\log D)(t)) = 0$ *for any* $t \in \mathbb{Z}$.

(c) $\dim H^0(Y, \Omega^r_Y(\log D)(t))$
$= \dim H^0(X, \Omega^r_X(\log D^*)(t)) - \dim H^0(X, \Omega^r_X(\log D^*)(t - m_c))$
$- \dim H^0(Y, \Omega^{r-1}_Y(\log D)(t - m_c))$ *for* $0 < r < n$

(d) $\dim H^n(Y, \Omega^r_Y(\log D)(t)) = \dim H^0(X, \Omega^{n-r}_X(\log D^*)(-t - m))$
$- \dim H^0(X, \Omega^{n-r}_X(\log D^*)(-t - m_c - m))$
$- \dim H^0(Y, \Omega^{n-r-1}_Y(\log D)(-t - m_c - m))$

(e) $\dim H^1(Y, \Omega^{n-1}_Y(\log D)(t))$
$= \dim H^0(Y, \Omega^{n-1}_Y(\log D)(t)) + \dim H^0(Y, \Omega^n_Y(\log D)(t + m_c))$
$+ \dim H^0(X, \Omega^n_X(\log D^*)(t)) - \dim H^0(X, \Omega^n_X(\log D^*)(t + m_c))$
$- \dim H^1(X, \Omega^n_X(\log D^*)(t)) + \dim H^1(X, \Omega^n_X(\log D^*)(t + m_c))$

(f) $\dim H^{n-r}(Y, \Omega^r_Y(\log D)(t))$
$= \dim H^{n-r-1}(Y, \Omega^{r+1}_Y(\log D)(t+m_c)) - \dim H^{n-r}(X, \Omega^{r+1}_X(\log D^*)(t))$
$+ \dim H^{n-r}(X, \Omega^{r+1}_X(\log D^*)(t + m_c))$ *for* $2 \leq r < n$

Proof. Under the additional condition $q + r < n$ the proof of (b) will be shown by complete induction with respect to $c = \operatorname{codim} Y$ and r. Then the case $q + r > n$ follows directly from the Serre duality. If $c = 0$, i.e. $Y = \mathbb{P}^N$, Theorem 3.3 implies $H^q(Y, \Omega^r_Y(\log D)(t)) = 0$ for $0 < q < N$ and $q + r \neq N$. If $r = 0$ then we get $H^q(Y, \mathcal{O}_Y(t)) = 0$ for $0 < q < n$ (cf. e.g. [2, Lemma 1]). In particular, we have the following induction assumption $(c - 1 = \operatorname{codim} X)$:

(i) From $q, r \in \mathbb{N}$, $0 < q,\ 0 \leq r$ and $q + r < n + 1$ it follows $H^q(X, \Omega^r_X(\log D^*)(t)) = 0$ for all $t \in \mathbb{Z}$.

Now assume $0 < q,\ 0 \leq r$ and $q + r < n$. From (13) we get the exact sequence

$$\ldots \longrightarrow H^q(X, \Omega^r_X(\log D^*)(t)) \longrightarrow H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X(\log D^*)) \longrightarrow \\ \longrightarrow H^{q+1}(X, \Omega^r_X(\log D^*)(t - m_c)) \longrightarrow \ldots\ .$$

Since $0 < q$, $q + 1 + r < n + 1$ we have by induction assumption (i) : $H^q(X, \Omega^r_X(\log D^*)(t)) = 0$ and $H^{q+1}(X, \Omega^r_X(\log D^*)(t - m_c)) = 0$. Hence, $H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X(\log D^*)) = 0$ for $0 < q$, $q + r < n$ and any $t \in \mathbb{Z}$. Now, let $r > 0$ be a fixed integer. We use the following induction assumption:

(ii) If $0 < q$ and $q + r - 1 < n$ then $H^q(Y, \Omega^{r-1}_Y(\log D)(t)) = 0$ for all $t \in \mathbb{Z}$.

To prove: If $0 < q$ and $q + r < n$ then $H^q(Y, \Omega^r_Y(\log D)(t)) = 0$ for all $t \in \mathbb{Z}$. Let $0 < q$, $q + r < n$. We consider the exact sequence which is given by (14)

$$\ldots \longrightarrow H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X(\log D^*)) \longrightarrow H^q(Y, \Omega^r_Y(\log D)(t) \longrightarrow \\ \longrightarrow H^{q+1}(Y, \Omega^{r-1}_Y(\log D)(t - m_c)) \longrightarrow \ldots\ .$$

By (ii) one has $H^{q+1}(Y, \Omega^{r-1}_Y(\log D)(t - m_c)) = 0$ for all $t \in \mathbb{Z}$ since $q + 1 + r - 1 = q + r < n$ (and $q + 1 < n$). Furthermore, we know that $H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X(\log D^*)) = 0$ for any $t \in \mathbb{Z}$ because of $0 < q$, $q + r < n$. This implies $H^q(Y, \Omega^r_Y(\log D)(t) = 0$ for $0 < q < n$ and $q + r < n$ for any $t \in \mathbb{Z}$. For the proof of (c) we first consider the exact sequence from (13)

$$0 \longrightarrow H^0(X, \Omega^r_X(\log D^*)(t - m_c)) \longrightarrow H^0(X, \Omega^r_X(\log D^*)(t)) \longrightarrow \\ \longrightarrow H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X(\log D^*) \longrightarrow H^1(X, \Omega^r_X(\log D^*)(t - m_c)) \longrightarrow \ldots, \tag{15}$$

and apply (i) which yields $H^1(X, \Omega^r_X(\log D^*)(t - m_c)) = 0$ as $1 + r < n + 1 = \dim X$. Because of (14) one gets the exact sequence

$$0 \longrightarrow H^0(Y, \Omega^{r-1}_Y(\log D)(t - m_c)) \longrightarrow H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_X} \Omega^r_X(\log D^*)) \longrightarrow \\ H^0(Y, \Omega^r_Y(\log D)(t)) \longrightarrow H^1(Y, \Omega^{r-1}_Y(\log D)(t - m_c)) \longrightarrow \ldots, \tag{16}$$

and due to $1 + r - 1 = r < n$ one has $H^1(Y, \Omega^{r-1}_Y(\log D)(t - m_c)) = 0$. Statement (c) can be read from (15) and (16). Assertion (d) can easily be shown by Serre duality. The Euler-Poincare characteristic can be calculated by the exact sequences (12)–(14). This allows us to specify finally the dimension of $H^{n-r}(Y, \Omega^r_Y(\log D)(t))$. (e) and (f) also can be shown using the exact cohomology sequences. □

4.2. **T-symmetric Differential Forms.** Let $Y \subseteq \mathbb{P}^N$ be the n-dimensional complete intersection of multidegree $(m) = (m_1, m_2, \ldots, m_c)$. $c = N - n$ denotes the codimension of Y.

We consider a Young tableau T with r boxes, the row lengths $l_1 \geq l_2 \geq \ldots \geq l_d > 0$ and the column lengths $d_1 \geq d_2 \geq \ldots \geq d_l > 0$. We denote $l = l_1 = \operatorname{length} T$ and $d = d_1 = \operatorname{depth} T$. Let $M(T)$ be the set of all integer matrices $A = ((d_{i,j})) \in \mathbb{N}^{(c+1,l)}$ with $c+1$ rows, l columns and with the following properties:

(1) $d_{1,j} = d_j \ \forall j \in \{1, \ldots, l\}$,
(2) $d_{i,l} \geq d_{i+1,l} \geq 0 \ \forall i \in \{1, \ldots, c\}$,
(3) $d_{i,j} \geq d_{i+1,j} \geq d_{i,j+1} \ \forall i \in \{1, \ldots, c\} \ \forall j \in \{1, \ldots, l-1\}$.

Let $\varrho_i(A) = \sum_{j=1}^{l} d_{ij}$ be the i-th row sum of A and we put $\varrho(A) = \varrho_{c+1}(A)$. We denote by

$$\mu = \sum_{j=1}^{c} d_j \tag{17}$$

the number of boxes in the first c columns of T, where $d_j = 0$ for $j > l$. One can easily see that $r - \mu \leq \varrho(A) \leq r$ for all $A \in M(T)$. Finally, we define the subset $M_s(T)$ of $M(T)$ by $M_s(T) := \{A \in M(T) \ : \ \varrho(A) = r - s\}$ for all $s \in \{0, 1, \ldots, \mu\}$. For simplification we set furthermore:

$$\Omega^{T'}_{\mathbb{P}^N|Y}(\log D^*) = \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*) \ , \ E^s_T = \bigoplus_{A \in M_s(T)} \Omega^{T'(A)}_{\mathbb{P}^N|Y}(\log D^*)(t(A))$$

$$\text{with } t(A) = \sum_{i=1}^{c} (\varrho_{i+1}(A) - \varrho_i(A)) \cdot m_i.$$

Here $T'(A)$ denotes a Young tableau with $\varrho(A)$ boxes and the column lengths $d_{c+1,1}, \ldots, d_{c+1,l}$, that is, $T'(A)$ depends only on the last row of A. If $\varrho(A) = 0$ we need to replace the sheaf $\Omega^{T'(A)}_{\mathbb{P}^N|Y}(\log D^*)$ by the structure sheaf $\mathcal{O}_Y$.

Lemma 4.3. *There exists following exact sequence:*

$$0 \longrightarrow E^{\mu}_T \xrightarrow{\beta_\mu} E^{\mu-1}_T \xrightarrow{\beta_{\mu-1}} \ldots \xrightarrow{\beta_2} E^1_T \xrightarrow{\beta_1} \Omega^T_{\mathbb{P}^N|Y}(\log D^*) \xrightarrow{\beta_0} \Omega^T_Y(\log D) \longrightarrow 0. \tag{18}$$

Proof. (18) is the T-Power of the following short exact sequence (cf. [3]):

$$0 \longrightarrow \bigoplus_{i=1}^{c} \mathcal{O}_Y(-m_i) \xrightarrow{\alpha} O_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^1_{\mathbb{P}^N}(\log D^*) \xrightarrow{\beta} \Omega^1_Y(\log D) \longrightarrow 0 \tag{19}$$

We need to show that (19) is an exact sequence. Let $U \subseteq \mathbb{P}^N$ be an open subset. Without loss of generality we put $U \subseteq U_i = \{x_i \neq 0\}$. Assume that there exist local parameters $\frac{F_1}{x_i^{m_1}}, \ldots, \frac{F_c}{x_i^{m_c}}, u_1, \ldots, u_{n-1}, \frac{F}{x_i^m}$ of $\mathbb{P}^N$ on U such that the restrictions $v_1 = \varphi^*(u_1), \ldots, v_{n-1} = \varphi^*(u_{n-1}), v_n = \varphi^*\left(\frac{F}{x_i^m}\right)$ are local parameters of Y on $U \cap Y$. We know that $\Gamma(U \cap Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^1_{\mathbb{P}^N}(\log D^*))$ is a free $\Gamma(U \cap Y, \mathcal{O}_Y)-$module defined by the span

$$\mathrm{d}\,\frac{F_1}{x_i^{m_1}}, \ldots, \mathrm{d}\,\frac{F_c}{x_i^{m_c}}, \mathrm{d}\,u_1, \ldots, \mathrm{d}\,u_{n-1}, \frac{x_i^m}{F} \cdot \mathrm{d}\,\frac{F}{x_i^m} .$$

$\Gamma(U \cap Y, \Omega^1_Y(\log D))$ is a free $\Gamma(U \cap Y, \mathcal{O}_Y)-$module with the span $\mathrm{d}\,v_1, \ldots, \mathrm{d}\,v_n$. Let $\omega \in \Gamma(U \cap Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^1_{\mathbb{P}^N}(\log D^*))$ be any element given by

$$\omega = \sum_{j=1}^{c} f_j \cdot x_i^{m_j} \cdot \mathrm{d}\,\frac{F_j}{x_i^{m_j}} + \sum_{k=1}^{n-1} g_k \cdot \mathrm{d}\,u_k + h \cdot \frac{x_i^m}{F} \cdot \mathrm{d}\,\frac{F}{x_i^m}.$$

The homomorphism β maps ω to $\beta(\omega) = \sum_{k=1}^{n-1} g_k \cdot \mathrm{d}\,v_k + h \cdot \frac{\mathrm{d}\,v_n}{v_n}$ where the kernel of this mapping is given by $\ker \beta = \left\{ \sum_{j=1}^{c} f_j \cdot x_i^{m_j} \cdot \mathrm{d}\,\frac{F_j}{x_i^{m_j}} \right\}$. We obtain the following homomorphism

$$\gamma \ : \ \bigoplus_{j=1}^{c} \Gamma(U \cap Y, \mathcal{O}_Y(-m_j)) \longrightarrow \ker \beta \ \text{ with } \ (f_1, \ldots, f_c) \longmapsto \sum_{j=1}^{c} f_j \cdot x_i^{m_j} \cdot \mathrm{d}\,\frac{F_j}{x_i^{m_j}}$$

which is isomorphic and independent of the index i with $U \subseteq U_i$. $\square$

Lemma 4.4. *For an arbitrary Young tableau T' there exists the following exact sequence*

$$\begin{aligned}
&0 \longrightarrow \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-\sum_{i=1}^{c} m_i) \xrightarrow{\alpha_c} \bigoplus_{1 \le i \le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-\sum_{j=1}^{c} m_j + m_i) \xrightarrow{\alpha_{c-1}} \ldots \\
&\ldots \xrightarrow{\alpha_3} \bigoplus_{1 \le i_1 < i_2 \le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-m_{i_1} - m_{i_2}) \xrightarrow{\alpha_2} \bigoplus_{1 \le i \le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-m_i) \xrightarrow{\alpha_1} \\
&\xrightarrow{\alpha_1} \Omega^{T'}_{\mathbb{P}^N}(\log D^*) \xrightarrow{\alpha_0} \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*) \longrightarrow 0.
\end{aligned} \tag{20}$$

Proof. We consider the following exact sequence which is called the Koszul complex:

$$\begin{aligned}
&0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-\sum_{i=1}^{c} m_i) \xrightarrow{\alpha_c} \bigoplus_{1 \le i \le c} \mathcal{O}_{\mathbb{P}^N}(-\sum_{j=1}^{c} m_j + m_i) \xrightarrow{\alpha_{c-1}} \ldots \\
&\xrightarrow{\alpha_3} \bigoplus_{1 \le i_1 < i_2 \le c} \mathcal{O}_{\mathbb{P}^N}(-m_{i_1} - m_{i_2}) \xrightarrow{\alpha_2} \bigoplus_{1 \le i \le c} \mathcal{O}_{\mathbb{P}^N}(-m_i) \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^N} \xrightarrow{\alpha_0} \mathcal{O}_Y \longrightarrow 0.
\end{aligned}$$

Multiplying this exact sequence with the local free sheaf $\Omega^{T'}_{\mathbb{P}^N}(\log D^*)$ yields the assertion. $\square$

Theorem 4.5. *Under the assumption $1 \le q < n - \operatorname{depth} T - \mu$ one gets*

$$H^q(Y, \Omega^T_Y(\log D)(t)) = 0 \ \textit{for all } t \in \mathbb{Z}.$$

Proof. We write instead of (20) short exact sequences and obtain

$$
\begin{aligned}
&0 \longrightarrow \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-\sum_{i=1}^{c} m_i) \longrightarrow \bigoplus_{1\le i\le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-\sum_{j=1}^{c} m_j + m_i) \longrightarrow\\
&\quad \longrightarrow \operatorname{Im}\alpha_{c-1} \longrightarrow 0\\
&0 \longrightarrow \operatorname{Im}\alpha_{c-1} \longrightarrow \bigoplus_{1\le i_1<i_2\le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-\sum_{j=1}^{c} m_j + m_{i_1} + m_{i_2}) \longrightarrow\\
&\quad \longrightarrow \operatorname{Im}\alpha_{c-2} \longrightarrow 0\\
&\qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&0 \longrightarrow \operatorname{Im}\alpha_2 \longrightarrow \bigoplus_{1\le i\le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(-m_i) \longrightarrow \operatorname{Im}\alpha_1 \longrightarrow 0\\
&0 \longrightarrow \operatorname{Im}\alpha_1 \longrightarrow \Omega^{T'}_{\mathbb{P}^N}(\log D^*) \longrightarrow \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*) \longrightarrow 0.
\end{aligned}
$$

Using the long exact cohomology sequences yields a vanishing criterion for $H^q(Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(t))$. We have

$$
\begin{aligned}
H^q(Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(t)) = 0 \text{ if } &\ H^q(\mathbb{P}^N, \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(t)) = 0\\
&\text{and } H^{q+1}(\mathbb{P}^N, \operatorname{Im}\alpha_1(t)) = 0\\
H^{q+1}(\mathbb{P}^N, \operatorname{Im}\alpha_1(t)) = 0 \text{ if } &\ H^{q+1}(\mathbb{P}^N, \bigoplus_{1\le i\le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(t - m_i)) = 0\\
&\text{and } H^{q+2}(\mathbb{P}^N, \operatorname{Im}\alpha_2(t)) = 0\\
&\vdots\\
H^{q+c-1}(\mathbb{P}^N, \operatorname{Im}\alpha_{c-1}(t)) = 0 &\\
\text{if } &\ H^{q+c-1}(\mathbb{P}^N, \bigoplus_{1\le i\le c} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(t - \sum_{j=1}^{c} m_j + m_i)) = 0\\
&\text{and } H^{q+c}(\mathbb{P}^N, \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(t - \sum_{i=1}^{c} m_i) = 0.
\end{aligned}
$$

Applying Theorem 3.6 (d) yields $H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)) = 0$ for $1 \le q < n - \operatorname{depth} T'$. Now we study $H^q(Y, \Omega^T_Y(\log D)(t))$ with the aid of (18). Decomposing (18) in short exact sequences delivers

$$
\begin{aligned}
&0 \longrightarrow E^{\mu}_T \longrightarrow E^{\mu-1}_T \longrightarrow \operatorname{Im}\beta_{\mu-1} \longrightarrow 0\\
&0 \longrightarrow \operatorname{Im}\beta_{\mu-1} \longrightarrow E^{\mu-2}_T \longrightarrow \operatorname{Im}\beta_{\mu-2} \longrightarrow 0\\
&\vdots \qquad\qquad\qquad\qquad \vdots\\
&0 \longrightarrow \operatorname{Im}\beta_2 \longrightarrow E^1_T \longrightarrow \operatorname{Im}\beta_1 \longrightarrow 0\\
&0 \longrightarrow \operatorname{Im}\beta_1 \longrightarrow \Omega^T_{\mathbb{P}^N|Y}(\log D^*) \longrightarrow \Omega^T_Y(\log D) \longrightarrow 0.
\end{aligned}
$$

With $E^s_T(t) = \bigoplus\limits_{A\in M_s(T)} \Omega^{T'(A)}_{\mathbb{P}^N|Y}(\log D^*)(t+t(A))$ one has

$$\begin{aligned} H^q(Y,\Omega^T_Y(\log D)(t)) = 0 \text{ if } & H^q(Y,\Omega^T_{\mathbb{P}^N|Y}(\log D^*)(t)) = 0 \\ & \text{and } H^{q+1}(Y, \operatorname{Im}\beta_1(t)) = 0 \\ \vdots \qquad & \\ H^{q+\mu-1}(Y,\operatorname{Im}\beta_{\mu-1}(t)) = 0 \text{ if } & H^{q+\mu-1}(Y,E^{\mu-1}_T(t)) = 0 \\ & \text{and } H^{q+\mu}(Y,E^\mu_T(t)) = 0. \end{aligned}$$

This implies $H^q(Y,\Omega^T_Y(\log D)(t)) = 0$ for $1 \le q < n - \operatorname{depth} T - \mu$. □

Now assume for instance $\mu < n - \operatorname{depth} T$. Then for each $t \in \mathbb{Z}$ it follows from our exact sequences: $H^q(\mathbb{P}^N, \operatorname{Im}\alpha_i(t)) = 0$ if $1 \le q \le \mu + i$,
$H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)) = 0$ if $1 \le q \le \mu + c$,
$H^q(Y, E^j_T(t)) = 0$ if $1 \le q \le j$, $H^q(Y, \operatorname{Im}\beta_j(t)) = 0$ if $1 \le q \le j$.
In particular, the cohomology groups $H^1(\ldots)$ of all these sheaves vanish. Therefore, we have the opportunity to calculate the dimensions of their cohomology groups $H^0(\ldots)$:
Let $h^T(t)$ abbreviating denotes the dimension $\dim H^0(\mathbb{P}^N, \Omega^T_{\mathbb{P}^N}(\log D^*)(t))$ as an integer function of t . Remember that $(m) = (m_1, m_2, \ldots, m_c)$ is the multidegree of the complete intersection Y . We set

$$h^T_{(m)}(t) := h^T(t) + \sum_{s=1}^{c} (-1)^s \cdot \sum_{1\le i_1<i_2<\ldots<i_s\le c} h^T(t - m_{i_1} - m_{i_2} - \ldots - m_{i_s}) \ .$$

Because of (20) we have $\dim H^0(Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{T'}_{\mathbb{P}^N}(\log D^*)(t)) = h^{T'}_{(m)}(t)$ and using (18) we get the following formula:

Theorem 4.6. *If $\mu < \dim Y - \operatorname{depth} T$ then*

$$\dim H^0(Y,\Omega^T_Y(\log D)(t)) = \sum_{A\in M(T)} (-1)^{r-\varrho(A)} \cdot h^{T'(A)}_{(m)}(t + t(A))$$

with $t(A) = \sum_{i=1}^c (\varrho_{i+1}(A) - \varrho_i(A)) \cdot m_i$.
In particular for $t = 0$: $H^0(Y,\Omega^T_Y(\log D)) = 0$ if $\mu < \dim Y - \operatorname{depth} T$.

Remark 4.7. For regular T-symmetrical tensor differential forms one has $H^0(Y,\Omega^T_Y) = 0$ if $\mu < \dim Y$.

4.3. **Symmetric Differential Forms.** We consider symmetrical differential forms with logarithmic poles as a special case, that means, T is a Young tableau with r boxes and only one row ($\operatorname{depth} T = 1$, $l = \operatorname{length} T = r$).
Let $D^* = H$ be the prime divisor on projective space $\mathbb{P}^N$ and let D be the prime divisor on the n-dimensional complete intersection Y as above ($n \ge 2$). Distinguishing the cases $r \le c$ and $c < r$ we obtain two exact sequences as

symmetrical power of (19) : Assume at first $r \leq c$:

$$0 \longrightarrow \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_r \leq c} \mathcal{O}_Y(-m_{i_1} - m_{i_2} - \ldots - m_{i_r}) \longrightarrow$$

$$\longrightarrow \bigoplus_{1 \leq i_1 < \ldots < i_{r-1} \leq c} \mathcal{O}_Y(-m_{i_1} - m_{i_2} - \ldots - m_{i_{r-1}}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^1_{\mathbb{P}^N}(\log D^*) \longrightarrow \ldots$$

$$\ldots \longrightarrow \bigoplus_{1 \leq i \leq c} \mathcal{O}_Y(-m_i) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^{r-1}\Omega^1_{\mathbb{P}^N}(\log D^*) \longrightarrow$$

$$\longrightarrow \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*) \longrightarrow S^r\Omega^1_Y(\log D) \longrightarrow 0$$

In the case $c < r$ the following sequence is exact:

$$0 \longrightarrow \mathcal{O}_Y(-\sum_{j=1}^{c} m_j) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{r-c}_{\mathbb{P}^N}(\log D^*) \longrightarrow$$

$$\longrightarrow \bigoplus_{1 \leq i \leq c} \mathcal{O}_Y(-\sum_{j=1}^{c} m_j + m_i) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \Omega^{r-c+1}_{\mathbb{P}^N}(\log D^*) \longrightarrow \ldots$$

$$\ldots \longrightarrow \bigoplus_{1 \leq i \leq c} \mathcal{O}_Y(-m_i) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^{r-1}\Omega^1_{\mathbb{P}^N}(\log D^*) \longrightarrow$$

$$\longrightarrow \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*) \longrightarrow S^r\Omega^1_Y(\log D) \longrightarrow 0$$

Furthermore, we have Lemma 4.4 with the sheaf $S^r\Omega^1_{\mathbb{P}^N}(\log D^*)$ instead of $\Omega^{T'}_{\mathbb{P}^N}(\log D^*)$. With the corresponding cohomology sequences we get:

Theorem 4.8. *Assume* $n = \dim Y \geq 2$.

(a) *If* $1 \leq q \leq n-2$ *then* $H^q(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*)) = 0 \ \forall t \in \mathbb{Z}$

(b)

$$\dim H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*)) = \dim H^0(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D^*)(t)) +$$

$$+ \sum_{j=1}^{c} (-1)^j \cdot \sum_{1 \leq i_1 < \ldots < i_j \leq c} \dim H^0(\mathbb{P}^N, S^r\Omega^1_{\mathbb{P}^N}(\log D^*)(t - m_{i_1} - \ldots - m_{i_j}))$$

(c) $H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*)) \neq 0 \ \Leftrightarrow \ t \geq r$

(d) *In case* $t = 0$ *:* $H^0(Y, \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*)) = 0$ *for all* $r > 0$

Theorem 4.9.

(a) *If* $r \leq c$ *and* $1 \leq q < n - r$ *then* $H^q(Y, S^r\Omega^1_Y(\log D)(t)) = 0 \ \forall t \in \mathbb{Z}$

(b) *If* $c < r$ *and* $1 \leq q < n - c - 1$ *then* $H^q(Y, S^r\Omega^1_Y(\log D)(t)) = 0 \ \forall t \in \mathbb{Z}$

Proof. By Theorem 4.5 we know $H^q(Y, \Omega^T_Y(\log D)(t)) = 0$ for all $t \in \mathbb{Z}$ if $1 \leq q < n-\operatorname{depth} T - \mu$. For symmetric differential forms we have depth $T = 1$ and $\mu = \sum_{i=1}^{c} d_i = \min\{c, r\}$, where $d_i = 1$ for $i \leq r$ and $d_i = 0$ for $i > r$. This proves (b). Under condition $r \leq c$ one gets the stronger result (a) since $H^q(Y, \mathcal{O}_Y(t)) = 0$ for $1 \leq q < n$ and for all $t \in \mathbb{Z}$. $\square$

Theorem 4.10.

(c) *If* $r \le c$ *and* $r < n$ *then*

$$H^0(Y, S^r\Omega^1_Y(\log D)(t)) = \dim H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*)) +$$
$$+ \sum_{k=1}^{r-1} (-1)^k \cdot \sum_{1 \le i_1 < \ldots < i_k \le c} \dim H^0(Y, \mathcal{O}_Y(t - \sum_{j=1}^{k} m_{i_j}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^{r-k}\Omega^1_{\mathbb{P}^N}(\log D^*))$$
$$+ (-1)^r \cdot \sum_{1 \le i_1 < \ldots < i_r \le c} \dim H^0(Y, \mathcal{O}_Y(t - m_{i_1} - \ldots - m_{i_r})$$

(d) *If* $c < r$ *and* $c < n-1$ *then*

$$H^0(Y, S^r\Omega^1_Y(\log D)(t)) = \dim H^0(Y, \mathcal{O}_Y(t) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^r\Omega^1_{\mathbb{P}^N}(\log D^*)) +$$
$$+ \sum_{k=1}^{c} (-1)^k \cdot \sum_{1 \le i_1 < \ldots < i_k \le c} \dim H^0(Y, \mathcal{O}_Y(t - \sum_{j=1}^{k} m_{i_j}) \otimes_{\mathcal{O}_{\mathbb{P}^N}} S^{r-k}\Omega^1_{\mathbb{P}^N}(\log D^*))$$

Proof. Statements (c) and (d) follow from the related exact sequences since under these premises by Theorem 4.8 the cohomology groups $H^1(\ldots)$ of all these sheaves vanish (cf. Theorem 4.8 and Theorem 3.7). □

Finally, it is easy to see:

Theorem 4.11.

(e) *If* $t < r \le \min(c, n-1)$ *then* $H^0(Y, S^r\Omega^1_Y(\log D)(t)) = 0$.
(f) *If* $t < r$ *and* $c < \min(r, n-1)$ *then* $H^0(Y, S^r\Omega^1_Y(\log D)(t)) = 0$.
(g) *If* $c < n-1$ *then* $H^0(Y, S^r\Omega^1_Y(\log D)) = 0$ *for all* $r > 0$.
(h) *If* $0 < r < n$ *then* $H^0(Y, S^r\Omega^1_Y(\log D)) = 0$.

Remark 4.12. On the other hand, for regular symmetrical differential forms on complete intersections it is well known:
If $c < n$ then $H^0(Y, S^r\Omega^1_Y) = 0$ for all $r > 0$.

References

[1] P. Brückmann, Zur Kohomologie von projektiven Hyperflächen, *Beiträge zur Algebra und Geometrie*, **2** (1974), 87–101.
[2] P. Brückmann, Zur Kohomologie von vollständigen Durchschnitten mit Koeffizienten in der Garbe der Keime der Differentialformen, *Mathematische Nachrichten*, **71** (1976), 203–210.
[3] P. Brückmann, The Hilbert polynomial of the sheaf Ω^T of germs of $T-$symmetrical tensor differential forms on complete intersections, *Math. Ann.*, **307** (1997), 461–472.
[4] R. Hartshorne, *Algebraic Geometry*, GTM 52, New York, Berlin, Heidelberg (1977).
[5] H. Esnault, E. Viehweg, *Lectures on Vanishing Theorems*, Basel, Boston, Berlin (1992).
[6] I.R. Shafarevich, *Algebraic Geometry II*, Encyclopaedia of Mathematical Sciences Vol. 35, Berlin, Heidelberg, New York (1996).
[7] W. Fulton, J. Harris, *Representation Theory*, GTM 129, New York, Berlin, Heidelberg (1991).
[8] J.-P. Serre, Faisceaux algébriques cohérents, *Ann. of Math.*, **61** (1955), 197–278.

Department of Mathematics, Martin-Luther-University Halle-Wittenberg, 06099 Halle, Germany
E-mail address: brueckmann@mathematik.uni-halle.de

Department of Mathematics, Martin-Luther-University Halle-Wittenberg, 06099 Halle, Germany
E-mail address: patrick.winkert@mathematik.uni-halle.de