MAXIMAL L_p-REGULARITY OF PARABOLIC PROBLEMS WITH BOUNDARY DYNAMICS OF RELAXATION TYPE

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Abstract. In this paper we investigate vector-valued parabolic initial boundary value problems of relaxation type. Typical examples for such boundary conditions are *dynamic* boundary conditions or linearized free boundary value problems like in the Stefan problem. We present a complete L_p-theory for such problems which is based on maximal regularity of certain model problems.

1. Introduction

In the present paper we study the vector-valued parabolic initial boundary value problem of the general form

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x) \quad (t \in J,\ x \in G), \\ \partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x) \quad (t \in J,\ x \in \Gamma), \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x) \quad (t \in J,\ x \in \Gamma,\ j = 1,\dots,m), \\ u(0,x) &= u_0(x) \quad (x \in G), \\ \rho(0,x) &= \rho_0(x) \quad (x \in \Gamma). \end{aligned} \tag{1.1}$$

Here $J = [0,T]$ is a finite interval or $J = \mathbb{R}_+ := [0,\infty)$, and $G \subset \mathbb{R}^n$ is an open connected set with compact smooth boundary $\partial G = \Gamma$. The function u is E-valued and ρ is F-valued, where E and F are Banach spaces of class $\mathcal{HT}$; by definition, a Banach space E is of class $\mathcal{HT}$ if the Hilbert transform is continuous in $L_2(\mathbb{R}; E)$. The coefficients of the differential operators $\mathcal{A}$ and $\mathcal{B}_j$, $j = 1,\dots,m$, are $\mathcal{B}(E)$-valued, while those of $\mathcal{C}_j$ are in $\mathcal{B}(F,E)$, $j = 1,\dots,m$. The coefficients of the differential operator $\mathcal{B}_0$ are in $\mathcal{B}(E,F)$ and that of $\mathcal{C}_0$ in $\mathcal{B}(F)$. A precise formulation of the assumptions on the operators can be found in Section 2.

Problems of this type arise as suitable linearizations in several contexts. So in case of problems with *dynamic boundary conditions* one of the steady boundary conditions would be $\rho = u|_\Gamma$, say $\mathcal{B}_m = 1 = -\mathcal{C}_m$, $g_m = 0$. In *reaction-diffusion problems*, u would be a vector of concentrations, and ρ a vector of surface concentrations which are related by a steady or unsteady adsorption-desorption process. This leads to relations of the form $\rho = Qu|_\Gamma$. In another context arising in the theory of moving boundaries, ρ is the position of the moving boundary while u is the interior variable, like a concentration or the temperature. These examples should give a rough idea of what we have in mind, see Section 3 for other examples and applications. Generally speaking, whenever we encounter a (nonlinear) parabolic problem on a fixed or time-varying domain with dynamics on its boundary, linearization will lead to a problem of type (1.1).

Key words and phrases. Maximal regularity, parabolic initial boundary value problems, dynamic boundary conditions, vector-valued Sobolev spaces.

Here we want to establish a general L_p-theory for problems of this type, which is intimately connected to the concept of *maximal regularity of L_p-type*. This is well-known for classical parabolic initial-boundary value problems, but seems to be new for problems of the form (1.1). Since the boundary conditions do not act instantaneously but involve a coupling with a dynamics on the boundary, we call them *Parabolic Problems with Boundary Dynamics of Relaxation Type*.

We are not aware of any papers dealing with general problems of the form (1.1), although some results are known in special cases. We comment on some of them in Section 3.

The plan for this paper is the following. Section 2 contains the statements of the main results of this paper, namely maximal L_p-regularity of (1.1) and a result on the associated analytic semigroup in the autonomous case. Examples and applications of the main results are presented in Section 3, to explain their scope for concrete problems. The proofs of the main results are given in Section 4, while Sections 5 and 6 deal with the necessity of the relevant Lopatinskii-Shapiro conditions employed in this paper. In particular, it is shown in Section 6 that these conditions are necessary.

2. Statement of the Main Results

Let us consider (1.1) where

$$\mathcal{A}(t,x,D) = \sum_{|\alpha|\le 2m} a_\alpha(t,x)D^\alpha,$$

$$\mathcal{B}_j(t,x,D) = \sum_{|\beta|\le m_j} b_{j\beta}(t,x)D^\beta,$$

$$\mathcal{C}_j(t,x,D_\Gamma) = \sum_{|\gamma|\le k_j} c_{j\gamma}(t,x)D_\Gamma^\gamma$$

are differential operators of order $2m$, $0 \le m_j < 2m$, $0 \le k_j$, respectively, with $m \in \mathbb{N}$ and $m_j, k_j \in \mathbb{N}_0$. The symbols D resp. D_Γ mean $-i\nabla$ resp. $-i\nabla_\Gamma$, where ∇ denotes the gradient in G and ∇_Γ the surface gradient on Γ. We assume that all boundary operators $\mathcal{B}_j$ and at least one $\mathcal{C}_j$ are nontrivial, and we set $k_j = -\infty$ in case $\mathcal{C}_j = 0$. The coefficients of these differential operators will be bounded linear operators, i.e. $a_\alpha(t,x), b_{j\beta}(t,x) \in \mathcal{B}(E)$, $c_{j\gamma}(t,x) \in \mathcal{B}(F,E)$, for $j = 1,\dots,m$, while $b_{0\beta} \in \mathcal{B}(E,F)$, and $c_{0\gamma}(t,x) \in \mathcal{B}(F)$. The initial values u_0 and ρ_0 as well as the right hand sides f and g_j are given functions.

We are interested in L_p-theory, i.e. we are looking for solutions (u,ρ) where $u \in X := L_p(J; L_p(G;E))$ $(1 < p < \infty)$ is such that

$$u \in Z_u := H_p^1(J; L_p(G;E)) \cap L_p(J; H_p^{2m}(G;E)).$$

Here H_p^k stands for the standard (vector-valued) Sobolev space of integer order. Trace theorems then imply that the initial value u_0 of u must belong to

$$\pi Z_u := W_p^{2m(1-1/p)}(G),$$

provided $2m/p \notin \mathbb{N}$, and the traces of the derivatives $D^\beta u$ on Γ belong to the spaces

$$Y_j := W_p^{\kappa_j}(J; L_p(\Gamma;E)) \cap L_p(J; W_p^{2m\kappa_j}(\Gamma;E)),$$

whenever $|\beta| \leq m_j$, with

$$\kappa_j := 1 - m_j/2m - 1/2mp \quad (j = 1, \dots m);$$

here W_p^s denotes the vector-valued Sobolev-Slobodeckii space of non-integer order s. Taking these spaces as the natural spaces for the boundary data g_j, and observing that $\mathcal{C}_j$ is of order k_j, ρ should belong to the spaces

$$\rho \in W_p^{\kappa_j}(J; H_p^{k_j}(\Gamma; F)) \cap L_p(J; W_p^{k_j+2m\kappa_j}(\Gamma; F)) \quad (j = 1, \dots, m)$$

whenever $k_j \neq -\infty$, i.e. whenever ρ is present in the boundary condition j. On the other hand, the boundary space for g_0 this way becomes

$$Y_0 := W_p^{\kappa_0}(J; L_p(\Gamma; F)) \cap L_p(J; W_p^{2m\kappa_0}(\Gamma; F)),$$

where κ_0 is defined analogously. Hence ρ should also satisfy

$$\rho \in W_p^{1+\kappa_0}(J; L_p(\Gamma; F)) \cap H_p^1(J; W_p^{2m\kappa_0}(\Gamma; F)),$$

and

$$\rho \in W_p^{\kappa_0}(J; H_p^{k_0}(\Gamma; F)) \cap L_p(J; W_p^{k_0+2m\kappa_0}(\Gamma; F)),$$

provided $k_0 \neq -\infty$. Setting $l_j = k_j - m_j + m_0$ and $l = \max_{j=0,\dots m} l_j$, this means that we want ρ to belong to the boundary space

$$\begin{aligned}\rho \in Z_\rho :=& W_p^{1+\kappa_0}(J; L_p(\Gamma; F)) \cap H_p^1(J; W_p^{2m\kappa_0}(\Gamma; F)) \\ &\cap \bigcap_{j \in \tilde{\mathcal{J}}} W_p^{\kappa_j}(J; H_p^{k_j}(\Gamma; F)) \cap L_p(J; W_p^{l+2m\kappa_0}(\Gamma; F)),\end{aligned} \tag{2.1}$$

where $\tilde{\mathcal{J}} := \{j \in \{0, \dots, m\} : k_j \neq -\infty\}$. Note that $k_j + 2m\kappa_j = l_j + 2m\kappa_0 \leq l + 2m\kappa_0$, for each $j \in \mathcal{J}_0$. Observe that the points (k_j, κ_j) and $(k_j + 2m\kappa_j, 0)$ are on the parallel lines $2mt + s = 2m\kappa_j + k_j = 2m\kappa_0 + l_j$.

It is not so easy to determine the trace space πZ_ρ where the initial value ρ_0 of ρ should belong to. Moreover, the time derivative of ρ may have a trace as well, we call the corresponding trace space $\pi_1 Z_\rho$.

To find these trace spaces for ρ and $\partial_t \rho$ at time $t = 0$, we proceed as follows. Take the convex hull $\mathcal{NP}$ of $(0, 0)$ and the points corresponding to the indices appearing in the spaces defining Z_ρ, i.e. $(0, 1 + \kappa_0)$, $(2m\kappa_0, 1)$, (k_j, κ_j), and $(k_j + 2m\kappa_j, 0)$, for $j \in \tilde{\mathcal{J}}$. This will be a polygonal set in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1 + \kappa_0)$, $(l + 2m\kappa_0, 0)$, and some of the remaining vertices generating $\mathcal{NP}$. The convex set $\mathcal{NP}$ is called the *Newton polygon* of the problem, and the nontrivial part of the boundary of $\mathcal{NP}$, i.e. the polygon connecting $(0, 1 + \kappa_0)$ to $(l + 2m\kappa_0, 0)$ through the vertices on the boundary of $\mathcal{NP}$ is called the leading part of $\mathcal{NP}$. We then define the set $\mathcal{J}$ as the set of those indices $j \in \{0, \dots, m\}$ such that either $l_j = l$ or (k_j, κ_j) belongs to the leading part of $\mathcal{NP}$ which means that all other such points are in the interior of $\mathcal{NP}$ or on the trivial parts of the boundary of $\mathcal{NP}$. The basic idea to find the time trace spaces is to look at the intersection of the lines $(s, 1/p)$ with the Newton polygon to find πZ_ρ and at $(s, 1 + 1/p)$ to get $\pi_1 Z_\rho$.

Now we have to distinguish three cases.

Case 1: $l = 2m$.
In this case things are simple. Then the points $(0, 1+\kappa_0)$, $(2m\kappa_0, 1)$ and $(l+2m\kappa_0, 0)$ are on the same line, which means that the leading part of the Newton polygon is

the line passing through these points. All other points are below or on this line. In this case we have

$$Z_\rho = W_p^{1+\kappa_0}(J; L_p(\Gamma; F)) \cap L_p(J; W_p^{2m+2m\kappa_0}(\Gamma; F)),$$

and we easily obtain $\pi Z_\rho = W_p^{2m\kappa_0+2m(1-1/p)}(\Gamma; F)$, as well as $\pi_1 Z_\rho = W_p^{2m(\kappa_0-1/p)}(\Gamma; F)$ provided $\kappa_0 > 1/p$.

Case 2: $l < 2m$.
Here the leading part of the Newton polygon is formed by the three points $(0, 1+\kappa_0)$, $(2m\kappa_0, 1)$, and $(l+2m\kappa_0, 0)$, and none of the points (k_j, κ_j) is on the polygon. This implies

$$Z_\rho = W_p^{1+\kappa_0}(J; L_p(\Gamma; F)) \cap H_p^1(J; W_p^{2m\kappa_0}(\Gamma; F)) \cap L_p(J; W_p^{l+2m\kappa_0}(\Gamma; F)).$$

Here we have $\pi Z_\rho = W_p^{2m\kappa_0+l(1-1/p)}(\Gamma; F)$, and $\pi_1 Z_\rho = W_p^{2m(\kappa_0-1/p)}(\Gamma; F)$ provided $\kappa_0 > 1/p$.

Case 3: $l > 2m$.
In this case the point $(2m\kappa_0, 1)$ is interior for $\mathcal{NP}$, so we may concentrate on the points (k_j, κ_j). We may write the space Z_ρ in this case in the form

$$Z_\rho = W_p^{1+\kappa_0}(J; L_p(\Gamma; F)) \cap \bigcap_{j\in\mathcal{J}} W_p^{\kappa_j}(J; H_p^{k_j}(\Gamma; F)) \cap L_p(J; W_p^{l+2m\kappa_0}(\Gamma; F)),$$

a more complicated space than in the previous cases.

Let $\mathcal{J} = \{j_1, \ldots, j_{q_{\max}}\}$ be arranged in such a way that with growing q, the spatial order k_{j_q} increases, hence time order κ_{j_q} decreases, l_{j_q} increases as well, and $l_{j_q} > 2m$ for $q = 1, \ldots, q_{max}$. Thus the vertices of the leading part of the Newton polygon are $P_0 = (0, 1+\kappa_0)$, $P_1 = (k_{j_1}, \kappa_{j_1}), \ldots, P_{q_{\max}} = (k_{j_{q_{\max}}}, \kappa_{j_{q_{\max}}})$, $P_{q_{\max}+1} = (l + 2m\kappa_0, 0)$. It is convenient to define $k_{-1} := 0$ and $\kappa_{-1} := 1 + \kappa_0$, i.e. $m_{-1} := m_0 - 2m$ and $l_{-1} = 2m$.

We denote the edge connecting P_q and P_{q+1} by $\mathcal{NP}_q$, $q = 0, \ldots, q_{\max}$. In the following, we will need the set of indices corresponding to the vertices and edges of the leading part of $\mathcal{NP}$. Therefore, we set

$$\begin{aligned}\mathcal{J}_{2q} &:= \{j \in \mathcal{J} \cup \{-1\} : (k_j, \kappa_j) = P_q\} \quad (q = 0, \ldots, q_{\max}),\\ \mathcal{J}_{2q+1} &:= \{j \in \mathcal{J} \cup \{-1\} : (k_j, \kappa_j) \in \mathcal{NP}_q\} \quad (q = 0, \ldots, q_{\max}).\end{aligned}$$

To determine the trace space for $\partial_t \rho$ choose the lowest spatial order k_{j_1}. The resulting trace space is

$$\pi_1 Z_\rho = W_p^{k_{j_1}(\kappa_0-1/p)/(1+\kappa_0-\kappa_{j_1})}(\Gamma; F),$$

provided $\kappa_0 > 1/p$. In a similar way we determine the trace space of ρ. Find the largest index $i_0 \in \mathcal{J}$ such that $\kappa_{i_0} > 1/p$ and let $i_1 \in \mathcal{J}$ be the smallest one such that $\kappa_{i_1} < 1/p$; we exclude the case $\kappa_i = 1/p$ in the sequel. Interpolating between the points (k_{i_0}, κ_{i_0}) and (k_{i_1}, κ_{i_1}) we obtain

$$\pi Z_\rho = W_p^{k_{i_0} + (\kappa_{i_0} - 1/p)\frac{k_{i_1} - k_{i_0}}{\kappa_{i_0} - \kappa_{i_1}}}(\Gamma; F).$$

This is the generic case, but there are two exceptions. The first one appears when $\kappa_i > 1/p$ for all $i \in \mathcal{J}$. Then we interpolate the points (k_{i_0}, κ_{i_0}) and $(l + 2m\kappa_0, 0)$, where k_{i_0} is the largest k_i of those i with $2m\kappa_i + k_i = l + 2m\kappa_0$, i.e. $i \in \mathcal{J}_{2q_{max}+1}$, to the result

$$\pi Z_\rho = W_p^{l+2m(\kappa_0-1/p)}(\Gamma; F).$$

This situation is encountered for large values of p. The second exception occurs if $\kappa_j < 1/p$ for all $j \in \mathcal{J}$, which corresponds to small values of p. Then we interpolate the points $(0, 1+\kappa_0)$ and (k_{j_1}, κ_{j_1}) and obtain

$$\pi Z_\rho = W_p^{k_{j_1}(1+\kappa_0-1/p)/(1+\kappa_0-\kappa_{j_1})}(\Gamma; F).$$

Actually, here we tacitly assumed that all exponents of the fractional Sobolev spaces appearing are non-integer, otherwise we have to replace them by Besov spaces B^s_{pp}; observe that $B^s_{pp} = W^s_p$ in case $s \notin \mathbb{N}_0$.

One main purpose of this paper is to obtain necessary and sufficient conditions on the data f, g_j, u_0, ρ_0 for the solvability of problem (1.1) in the described class. Obviously, for this, conditions on the coefficients are needed. We begin with the coefficients in the interior of G. Here the subscript # means the principal part of the corresponding differential operator. We set $\mathcal{C}_{j\#} = 0$ if $j \notin \mathcal{J}$. The first condition is normal ellipticity of $\mathcal{A}$ which is known to be necessary for solvability in the L_p-setting as explained above; cf. [4].

(E) (Ellipticity of the interior symbol.) For all $t \in J$, $x \in \overline{G}$, resp. $x \in \overline{G} \cup \{\infty\}$ in case G is unbounded, and for all $\xi \in \mathbb{R}^n$, $|\xi| = 1$, we have

$$\sigma(\mathcal{A}_\#(t,x,\xi)) \subset \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\},$$

i.e. $\mathcal{A}(t,x,D)$ is *normally elliptic*. Here $\sigma(\mathcal{A}_\#(t,x,\xi))$ stands for the spectrum of the bounded operator $\mathcal{A}_\#(t,x,\xi) \in \mathcal{B}(E)$.

Next we turn to smoothness assumptions on the coefficients of $\mathcal{A}$.

(SD) For $|\alpha| = k < 2m$ there are $r_\alpha, s_\alpha \geq p$, $s_\alpha < \infty$, with $\frac{1}{s_\alpha} + \frac{n}{2mr_\alpha} \leq 1 - \frac{k}{2m}$ such that

$$a_\alpha \in L_{s_\alpha}(J; (L_{r_\alpha} + L_\infty)(G; \mathcal{B}(E))).$$

For $|\alpha| = 2m$ assume

$$a_\alpha \in C(J \times \overline{G}; \mathcal{B}(E)).$$

If G is unbounded, the limits $a_\alpha(t,\infty) := \lim_{|x|\to\infty,\, x\in G} a_\alpha(t,x)$ exist uniformly with respect to $t \in J$ for all $|\alpha| = 2m$.

Smoothness of the boundary coefficients should be such that they are pointwise multipliers for the boundary spaces Y_j. Hence we require

(SB) Let $\mathcal{E}_0 = \mathcal{B}(E,F)$, $\mathcal{E}_j = \mathcal{B}(E)$ for $j = 1, \ldots, m$. For each $j = 0, \ldots, m$ and each β with $|\beta| = k \leq m_j$ there are $s_{j\beta}, r_{j\beta} \geq p$, $s_{j\beta} < \infty$, with $\frac{1}{s_{j\beta}} + \frac{n-1}{2mr_{j\beta}} \leq \kappa_j + \frac{m_j - k}{2m}$ such that

$$b_{j\beta} \in B^{\kappa_j}_{s_{j\beta},p}(J; L_{r_{j\beta}}(\Gamma; \mathcal{E}_j)) \cap L_{s_{j\beta}}(J; B^{2m\kappa_j}_{r_{j\beta},p}(\Gamma; \mathcal{E}_j)),$$

and in addition

$$b_{j\beta} \in C(J \times \Gamma; \mathcal{E}_j) \;\text{ for }\; |\beta| = m_j.$$

The assumptions on the coefficients of the boundary operators $\mathcal{C}_j$ are of the same nature.

(SC) Let $\mathcal{F}_0 = \mathcal{B}(F)$, $\mathcal{F}_j = \mathcal{B}(F, E)$ for $j = 1, \ldots, m$. For each $j = 0, \ldots, m$ and each γ with $|\gamma| = k \le k_j$, there are $s^c_{j\gamma}, r^c_{j\gamma} \ge p$, $s^c_{j\gamma} < \infty$, with $\frac{1}{s^c_{j\gamma}} + \frac{n-1}{2mr^c_{j\gamma}} \le \kappa_j + \frac{m_j - k}{2m}$ such that

$$c_{j\gamma} \in B^{\kappa_j}_{s^c_{j\gamma},p}(J; L_{r^c_{j\gamma}}(\Gamma; \mathcal{F}_j)) \cap L_{s^c_{j\gamma}}(J; B^{2m\kappa_j}_{r^c_{j\gamma},p}(\Gamma; \mathcal{F}_j)),$$

and in addition

$$c_{j\gamma} \in C(J \times \Gamma; \mathcal{F}_j) \quad \text{for} \quad |\gamma| = k_j, j \in \mathcal{J}.$$

Of course ellipticity conditions on the boundary operators are also needed. These are conditions of Lopatinskii-Shapiro type.

(LS) For each fixed $t \in J$ and $x \in \Gamma$ we rewrite the boundary value problem (1.1) in coordinates associated to x. They are obtained from the original coordinates by a translation and a rotation after which the positive x_n-axis has the direction of the inner normal to Γ at x. Then for all $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \overline{\mathbb{C}}_+$ with $|\xi'| + |\lambda| \neq 0$, all $h_j \in E$ and all $h_0 \in F$ the ordinary differential equation in $\mathbb{R}_+ = [0, \infty)$ given by

$$\begin{aligned} \big(\lambda + \mathcal{A}_\#(t, x, \xi', D_y)\big)v(y) &= 0 \quad (y > 0), \\ \mathcal{B}_{0\#}(t, x, \xi', D_y)v(0) + \big(\lambda + \mathcal{C}_{0\#}(t, x, \xi')\big)\sigma &= h_0, \\ \mathcal{B}_{j\#}(t, x, \xi', D_y)v(0) + \mathcal{C}_{j\#}(t, x, \xi')\sigma &= h_j \quad (j = 1, \ldots, m), \end{aligned} \tag{2.2}$$

has a unique solution $(v, \sigma) \in C_0(\mathbb{R}_+; E) \times F$.

This condition is a natural one and in Case 1 this is it. However, it is not sufficient in Cases 2 and 3, due to the inherent nonisotropy of the differential operators. Another condition is needed, which we call the *asymptotic Lopatinskii-Shapiro condition.* We have to distinguish these cases.

($\mathbf{LS}^-_\infty$) Let $l < 2m$.
For all fixed $t \in J$ and $x \in \Gamma$ rewrite (1.1) in coordinates associated to x. Then for all $h_j \in E$, all $h_0 \in F$, all $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \overline{\mathbb{C}}_+$ with $|\xi'| + |\lambda| \neq 0$, the equations

$$\begin{aligned} \lambda v(y) + \mathcal{A}_\#(t, x, \xi', D_y)v(y) &= 0 \quad (y > 0), \\ \mathcal{B}_{j\#}(t, x, \xi', D_y)v(0) &= h_j \quad (j = 1, \ldots, m), \end{aligned} \tag{2.3}$$

and for $|\xi'| = 1$ and $\lambda \in \overline{\mathbb{C}}_+$,

$$\begin{aligned} \mathcal{A}_\#(t, x, \xi', D_y)v(y) &= 0 \quad (y > 0), \\ \mathcal{B}_{0\#}(t, x, \xi', D_y)v(0) + (\lambda + \mathcal{C}_{0\#}(t, x, \xi'))\sigma &= h_0, \\ \mathcal{B}_{j\#}(t, x, \xi', D_y)v(0) + \mathcal{C}_{j\#}(t, x, \xi')\sigma &= h_j \quad (j = 1, \ldots, m), \end{aligned} \tag{2.4}$$

admit unique solutions $(v, \sigma) \in C_0(\mathbb{R}_+; E) \times F$.

Note that the first condition in **($\mathbf{LS}^-_\infty$)** means that the standard problem with $\sigma = 0$ and without the equation for σ satisfies the Lopatinskii-Shapiro condition, while the second one means that the quasi-steady problem is also subject to this condition.

In Case 3 things are more involved.

(LS^+_∞) Let $l > 2m$.
For all $t \in J$, $x \in \Gamma$, all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, all $h_j \in E$, $h_0 \in F$ and all $\lambda \in \overline{\mathbb{C}}_+$, the ordinary differential equation systems in $\mathbb{R}_+$

$$\begin{aligned} \lambda v(y) + \mathcal{A}_\#(t,x,\xi',D_y)v(y) &= 0 \quad (y>0), \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) + \delta_{j,\mathcal{J}_{2q_{\max}+1}}\mathcal{C}_{j\#}(t,x,\xi')\sigma &= h_j \quad (j=0,\dots,m), \end{aligned} \tag{2.5}$$

and for $|\xi'| = 1$ and $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$, $q = 1,2,\dots,2q_{\max}$,

$$\begin{aligned} \lambda v(y) + \mathcal{A}_\#(t,x,0,D_y)v(y) &= 0 \quad (y>0), \\ \mathcal{B}_{0\#}(t,x,0,D_y)v(0) + \delta_{-1,\mathcal{J}_q}\lambda\sigma + \delta_{0,\mathcal{J}_q}\mathcal{C}_{0\#}(t,x,\xi')\sigma &= h_0, \\ \mathcal{B}_{j\#}(t,x,0,D_y)v(0) + \delta_{j,\mathcal{J}_q}\mathcal{C}_{j\#}(t,x,\xi')\sigma &= h_j \quad (j=1,\dots,m), \end{aligned} \tag{2.6}$$

admit unique solutions $(v,\sigma) \in C_0(\mathbb{R}_+;E) \times F$. Here $\delta_{j,\mathcal{J}_q} = 1$ if $j \in \mathcal{J}_q$ and zero otherwise.

For another equivalent description of these asymptotic Lopatinskii-Shapiro conditions, see Section 5. We remark that in the case of finite-dimensional E and F, the LS conditions (LS) and ($\mathrm{LS}^\pm_\infty$) are satisfied if the ODE system with $h_j = 0$ has only the trivial solution.

The asymptotic Lopatinskii-Shapiro conditions look quite complicated. However, we show in Section 6 that they are necessary for maximal L_p-regularity of (1.1), hence are unavoidable. Fortunately, in explicit examples it is not so difficult to verify them, see Section 3.

After these preparations we can state our first main result of this paper which shows that under the assumptions made so far the problem (1.1) admits maximal L_p-regularity.

Theorem 2.1. *Let $J = [0,T]$, $G \subset \mathbb{R}^n$ a domain with compact boundary $\Gamma = \partial G$ of class C^{2m+l-m_0}. Suppose the Banach spaces E and F are of class $\mathcal{HT}$, let assumptions* (E), (SD), (SB), (SC), (LS) *and for $l < 2m$ condition* (LS^-_∞)*, for $l > 2m$ accordingly* (LS^+_∞) *be satisfied, and let $1 < p < \infty$ be such that $2m/p \notin \mathbb{N}$, $\kappa_j \neq 1/p$, $j = 0,\dots,m$, where κ_j, the spaces X, Z_u, Z_ρ, Y_j, as well as the trace spaces πZ_u, πZ_ρ and $\pi_1 Z_\rho$ are defined as above.*

Then problem (1.1) *admits a unique solution $(u,\rho) \in Z_u \times Z_\rho$ if and only if the data are subject to the conditions*

$$f \in X,\ u_0 \in \pi Z_u,\ \rho_0 \in \pi Z_\rho,\ g_j \in Y_j,\ j = 0,\dots,m,$$

and the compatibility conditions

$$\mathcal{B}_j(0,x)u_0(x) + \mathcal{C}_j(0,x)\rho_0(x) = g_j(0,x), \quad x \in \Gamma,\ \text{if } \kappa_j > 1/p,\ j = 1,\dots,m$$

and

$$g_0(0,\cdot) - \mathcal{B}_0(0,\cdot)u_0 - \mathcal{C}_0(0,\cdot)\rho_0 \in \pi_1 Z_\rho, \qquad \text{if } \kappa_0 > 1/p,$$

are satisfied.

There is also a semigroup formulation of problem (1.1) in the autonomous case which works as follows. As a base space we choose $X_0 := L_p(G;E) \times W^s_p(\Gamma;F)$, we define an operator A in X_0 by means of

$$A(u,\rho) = (\mathcal{A}(x,D)u, \mathcal{B}_0(x,D)u + \mathcal{C}_0(x,D_\Gamma)\rho), \quad (u,\rho) \in D(A), \tag{2.7}$$

with domain

(2.8)

$$D(A) = \{(u,\rho) \in H_p^{2m}(G;E) \times W_p^{2m\kappa_0 + l}(\Gamma;F) :$$
$$\mathcal{B}_j(x,D)u + \mathcal{C}_j(x,D_\Gamma)\rho = 0, j = 1,\dots,m,\ \mathcal{B}_0(x,D)u + \mathcal{C}_0(x,D_\Gamma)\rho \in W_p^s(\Gamma;F)\}.$$

The number s is determined by the intersection of the line $(\sigma, 1)$ with the Newton polygon $\mathcal{NP}$ of the problem. In Cases 1 and 2 this obviously leads to $s = 2m\kappa_0$, while in Case 3 we obtain

$$s = k_{j_1}\kappa_0/(1+\kappa_0-\kappa_{j_1}) = k_{j_1}2m\kappa_0/(2m+m_{j_1}-m_0),$$

where j_1 is defined as before.

In the autonomous case, where the coefficients are time-independent, one would like also to consider the halfline $J = \mathbb{R}_+$ instead of a finite interval. The regularity conditions on the coefficients **(SD), (SB), (SC)** then should be read according to $s_\alpha = s_{j\beta} = s^c_{j\gamma} = \infty$, with strict inequalities $\frac{n}{2mr_\alpha} < 1 - \frac{k}{2m}$, $\frac{n-1}{2mr_{j\beta}} < \kappa_j + \frac{m_j - k}{2m}$ and similarly for $r_{j\gamma^c}$.

We can state now the following result.

Theorem 2.2. *Let $G \subset \mathbb{R}^n$ a domain with compact boundary $\Gamma := \partial G$ of class C^{2m+l-m_0}, and assume that the coefficients of the operators do not depend on time. Suppose the Banach spaces E and F are of class $\mathcal{HT}$, let assumptions* (E), (SD), (SB), (SC), (LS) *and for $l < 2m$ condition* (LS$^-_\infty$)*, for $l > 2m$ accordingly* (LS$^+_\infty$) *be satisfied, and let $1 < p < \infty$ be such that $\kappa_j \neq 1/p$, $j = 0,\dots,m$. Define $s = 2m\kappa_0$ in case $l \le 2m$, and $s = k_{j_1}2m\kappa_0/(2m+m_{j_1}-m_0)$ for $l > 2m$. Assume that $s \notin \mathbb{N}$ in case $l > 2m$.*

Then the operator $-A$ defined by (2.7) and (2.8) generates an analytic C_0-semigroup in $X_0 = L_p(G;E) \times W_p^s(\Gamma;F)$ which has the property of maximal L_p-regularity on each finite interval $J = [0,T]$. Consequently, there is $\omega \ge 0$ such that $-(A+\omega)$ has maximal L_p-regularity on the halfline $J = \mathbb{R}_+$.

The maximal regularity result in the semigroup setting can be stated as follows.

Corollary 2.3. *Let the assumptions of Theorem 2.1 be valid, $J = [0,T]$. There is a unique solution (u,ρ) of (1.1) such that $u \in Z_u$,*

$$\rho \in H^1_p(J;W^s_p(\Gamma;F)) \cap \bigcap_{j\in\mathcal{J}} W_p^{\kappa_j}(J;H_p^{k_j}(\Gamma;F)) \cap L_p(J;W_p^{l+2m\kappa_0}(\Gamma;F)),$$

and

$$\mathcal{B}_0(\cdot,D)u + \mathcal{C}_0(\cdot,D_\Gamma)\rho \in L_p(J;W_p^s(\Gamma;F))$$

if and only if $u_0 \in \pi Z_u$, $\rho_0 \in \pi Z_\rho$, $f \in L_p(J\times G;E)$, $g_j \in Y_j$, $j = 1,\dots,m$, the compatibility conditions

$$\mathcal{B}_j(0,x,D)u_0 + \mathcal{C}_j(0,x,D_\Gamma)\rho_0 = g_j(0,x), \quad x \in \Gamma,\ \text{if } \kappa_j > 1/p,\ j = 1,\dots,m,$$

hold, as well as $g_0 \in L_p(J;W^s_p(\Gamma;F))$, and

$$\mathcal{B}_0(0,x,D)u_0 + \mathcal{C}_0(0,x,D_\Gamma)\rho_0 \in W_p^{(s/\kappa_0)(\kappa_0-1/p)}(\Gamma;F) \quad \text{if } \kappa_0 > \frac{1}{p}.$$

In the autonomous case this result is also true for $J = \mathbb{R}_+$, in case ∂_t is replaced by $\partial_t + \omega$, with some sufficiently large $\omega > 0$.

Here the last compatibility condition for $v(0)$ with $v(t) := \mathcal{B}_0(t,x,D)u(t) + \mathcal{C}_0(t,x,D_\Gamma)\rho(t)$ comes from the regularity

$$v \in W_p^{\kappa_0}(J; L_p(\Gamma; F)) \cap L_p(J; W_p^s(\Gamma; F)).$$

In particular, v has a time trace at $t = 0$ provided $\kappa_0 > 1/p$ which belongs to $W_p^{(s/\kappa_0)(\kappa_0 - 1/p)}(\Gamma; F)$. Note that this space coincides with $\pi_1 Z_\rho$.

3. Examples and Applications

In this section we want to present a number of prominent examples which can be treated by our theory. This shows that the approach taken in this paper is general enough to unify prior theory designed for special situations. We give also examples which are not covered by known results.

In the analysis of problem (1.1), we will see that the symbol

$$s(\xi', \lambda) := \begin{cases} \lambda + |\xi'|^l & \text{in Cases 1 and 2,} \\ \lambda + \sum_{j \in \mathcal{J}} |\xi'|^{k_j} (\lambda + |\xi'|^{2m})^{(m_0 - m_j)/2m} & \text{in Case 3} \end{cases}$$

plays a crucial role. We will call this the boundary symbol of the problem.

Our first example deals with dynamic boundary conditions for the diffusion equations, which has been studied e.g. in Escher [6].

Example 3.1. *Dynamic boundary conditions for the diffusion equation*

$$\begin{aligned} \partial_t u - \Delta u &= f \quad (t \in J,\ x \in G), \\ \partial_t u + \partial_\nu u &= g \quad (t \in J,\ x \in \Gamma), \\ u(0,x) &= u_0(x) \quad (x \in G). \end{aligned}$$

Here ν denotes the outer unit normal on Γ. This problem fits into our setting by taking $E = F = \mathbb{C}$, $\mathcal{A} = -\Delta$, $\mathcal{C}_0 = 0$, $\mathcal{B}_0 = \partial_\nu$, $\mathcal{B}_1 = -\mathcal{C}_1 = 1$, $g_1 = 0$. Here we have $2m = 2$, $m_0 = 1$, $k_0 = -\infty$, $m_1 = k_1 = 0$, $l = l_1 = 1$, hence this example is in Case 2. We have $\mathcal{J} = \{1\}$, and the system in (LS) with trivial right-hand sides is given by

$$\begin{aligned} (\lambda + |\xi'|^2 - \partial_y^2)v(y) &= 0 \quad (y > 0), \\ -\partial_y v(0) + \lambda\sigma &= 0, \\ v(0) - \sigma &= 0. \end{aligned} \tag{3.1}$$

Every stable solution of the first line in (3.1) is of the form $v(y) = e^{-\mu y} v(0)$ with $\mu := \sqrt{\lambda + |\xi'|^2}$. The boundary conditions yield $(\mu + \lambda)v(0) = 0$, and consequently $v = \sigma = 0$. The first problem in (LS_∞^-) is given by

$$\begin{aligned} (\lambda + |\xi'|^2 - \partial_y^2)v(y) &= 0, \\ v(0) &= 0 \end{aligned}$$

and has only the trivial solution for $|\xi'| + |\lambda| \neq 0$. The second problem in (LS_∞^-) is obtained by taking $\lambda = 0$ in the first line of (3.1). In this case we get $(|\xi'| + \lambda)v(0) = 0$, and again $v = \sigma = 0$. Note that the boundary symbol is given by $s(\xi', \lambda) = \lambda + |\xi'|$.

The next example is related to diffusion problems with surface diffusion as they appear in the chemistry of surface active agents, so-called *surfactants*, like tensides; cf. Bothe, Prüss and Simonett [1].

Example 3.2. *Dynamic boundary condition and surface diffusion for the diffusion equation*

$$\begin{aligned} \partial_t u - \Delta u &= f \quad (t \in J,\ x \in G),\\ \partial_t u + \partial_\nu u - \Delta_\Gamma u &= g \quad (t \in J,\ x \in \Gamma),\\ u(0,x) &= u_0(x) \quad (x \in G). \end{aligned}$$

Taking $E = F = \mathbb{C}$, $\mathcal{A} = -\Delta$, $\mathcal{C}_0 = -\Delta_\Gamma$, the Laplace-Beltrami operator on the manifold Γ, $\mathcal{B}_0 = \partial_\nu$, $\mathcal{B}_1 = -\mathcal{C}_1 = 1$, $g_1 = 0$, this problem is of the form (1.1). Here we have $2m = k_0 = l_0 = l = 2$, $m_0 = 1$, $m_1 = k_1 = 0$, $l_1 = 1$, this example is in Case 1. It is easy to verify (LS). Here the boundary symbol equals $\lambda + |\xi'|^2$ which is the symbol of the heat equation.

Our third example is a problem from the theory of phase transitions; cf. e.g. Racke and Zheng [11].

Example 3.3. *Dynamic boundary condition and surface diffusion for the linear Cahn-Hilliard equation*

$$\begin{aligned} \partial_t u + \Delta^2 u &= f \quad (t \in J,\ x \in G),\\ \partial_\nu \Delta u &= g \quad (t \in J,\ x \in \Gamma),\\ \partial_t u + \partial_\nu u - \Delta_\Gamma u &= h \quad (t \in J,\ x \in \Gamma),\\ u(0,x) &= u_0(x) \quad (x \in G). \end{aligned}$$

This problem is of the form (1.1) by taking $E = F = \mathbb{C}$, $\mathcal{A} = \Delta^2$, $\mathcal{C}_0 = -\Delta_\Gamma$, $\mathcal{B}_0 = \partial_\nu$, $\mathcal{B}_1 = -\mathcal{C}_1 = 1$, $g_1 = 0$, $\mathcal{B}_2 = \partial_\nu\Delta$, $\mathcal{C}_2 = 0$. In this example $2m = 4$, $k_0 = l_0 = l = 2$, $m_0 = 1$, $m_1 = k_1 = 0$, $l_1 = 1$, $m_2 = 3$, $k_2 = -\infty$, hence it belongs to Case 2. Here we have $\mathcal{J} = \{0\}$ and $\mathcal{C}_{1\#} = 0$, and the system in (LS) is given by

$$\big(\lambda + (|\xi'|^2 - \partial_y^2)^2\big) v(y) = 0 \quad (y > 0), \tag{3.2}$$

$$-\partial_y v(0) + (\lambda + |\xi'|^2)\sigma = 0, \tag{3.3}$$

$$v(0) = 0, \tag{3.4}$$

$$(|\xi'|^2 - \partial_y^2)\partial_y v(0) = 0. \tag{3.5}$$

Every stable solution of (3.2) is of the form $v(y) = c_1 e^{z_1 y} + c_2 e^{z_2 y}$ with $c_{1,2} \in \mathbb{C}$ where $z_{1,2} := -\sqrt{|\xi'|^2 \pm \sqrt{-\lambda}}$. From (3.4) we get $c_1 = -c_2$, and by (3.5) we see

$$\begin{aligned} 0 &= c_1\big[|\xi'|^2(z_1 - z_2) - (z_1^3 - z_2^3)\big] = c_1(z_1 - z_2)\big[|\xi'|^2 - (z_1^2 + z_2^2 + z_1 z_2)\big]\\ &= -c_1(z_1 - z_2)(|\xi'|^2 + \sqrt{\lambda + |\xi'|^4}). \end{aligned}$$

For $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\mathbb{C}}_+ \setminus \{(0,0)\}$ this yields $c_1 = c_2 = 0$, i.e. $v = 0$ and, by (3.3), $\sigma = 0$. The first problem in (LS_∞^-) is given by (3.2), (3.4), (3.5) and also has only the trivial solution. To verify the second condition in (LS_∞^-) we have to study the quasi-steady problem that is

$$(|\xi'|^2 - \partial_y^2)^2 v(y) = 0 \quad (y > 0), \tag{3.6}$$

together with (3.3), (3.4), and (3.5). Every stable solution of (3.6) is of the form $v(y) = c_1 e^{-|\xi'|y} + c_2 y e^{-|\xi'|y}$, where $c_{1,2} \in \mathbb{C}$. From (3.4) we infer $c_1 = 0$. Condition (3.5) then implies that

$$0 = (|\xi'|^2 - \partial_y^2)\partial_y v(0) = -2c_2|\xi'|^2,$$

hence $c_2 = 0$ for $|\xi'| > 0$. Finally (3.3) entails that $\sigma = 0$. The boundary symbol is now given by $s(\xi', \lambda) = \lambda + |\xi'|^2$.

An interesting example occurs in connection with the Stefan problem with surface tension; cf. Escher, Prüss and Simonett [7].

Example 3.4. *Linearized Stefan problem with surface tension*

$$\begin{aligned}
\partial_t u - \Delta u &= f \quad (t \in J,\ x \in G),\\
u &= -\Delta_\Gamma \rho \quad (t \in J,\ x \in \Gamma),\\
\partial_t \rho + \partial_\nu u &= g \quad (t \in J,\ x \in \Gamma),\\
u(0,x) &= u_0(x) \quad (x \in G),\\
\rho(0,x) &= \rho_0(x) \quad (x \in \Gamma).
\end{aligned}$$

This problem fits into our setting by taking $E = F = \mathbb{C}$, $\mathcal{A} = -\Delta$, $\mathcal{C}_0 = 0$, $\mathcal{B}_0 = \partial_\nu$, $\mathcal{B}_1 = 1$, $\mathcal{C}_1 = \Delta_\Gamma$. Here $2m = k_1 = 2$, $m_1 = 0$, $k_0 = -\infty$, $m_0 = 1$, $l = l_1 = 3$, this is a prominent example for Case 3. The problem in the (LS) condition is given by

$$\begin{aligned}
(\lambda + |\xi'|^2 - \partial_y^2)v(y) &= 0 \quad (y > 0),\\
-\partial_y v(0) + \lambda\sigma &= 0,\\
v(0) - |\xi'|^2\sigma &= 0.
\end{aligned}$$

It is easily seen that there exists only the trivial stable solution. Now we have $q_{\max} = 1$, $\mathcal{J}_0 = \{-1\}$, $\mathcal{J}_1 = \{-1, 1\}$ and $\mathcal{J}_2 = \mathcal{J}_3 = \{1\}$, and for $\xi' \neq 0$ and $\lambda \neq 0$ all asymptotic (LS_∞^+)-conditions are satisfied. The boundary symbol is $\lambda + |\xi'|^2\sqrt{\lambda + |\xi'|^2}$, and here we have $s = 2 - 2/p$, while $2m\kappa_0 = 1 - 1/p$.

Linearization of the Stefan problem with surface tension and kinetic undercooling leads to the following example.

Example 3.5. *Linearized Stefan problem with surface tension and kinetic undercooling*

$$\begin{aligned}
\partial_t u - \Delta u &= f \quad (t \in J,\ x \in G),\\
u &= \partial_t \rho - \Delta_\Gamma \rho + g \quad (t \in J,\ x \in \Gamma),\\
\partial_t \rho + \partial_\nu u &= h \quad (t \in J,\ x \in \Gamma),\\
u(0,x) &= u_0(x) \quad (x \in G),\\
\rho(0,x) &= \rho_0(x) \quad (x \in \Gamma).
\end{aligned}$$

Inserting the second boundary conditions into the first, this problem is of the form (1.1) with $E = F = \mathbb{C}$, $\mathcal{A} = -\Delta$, $\mathcal{C}_0 = 0$, $\mathcal{B}_0 = \partial_\nu$, $\mathcal{B}_1 = \partial_\nu + 1$, $\mathcal{C}_1 = \Delta_\Gamma$. In this example we have $2m = k_1 = 2$, $m_0 = m_1 = 1$, $k_0 = -\infty$, hence $l_1 = l = 2$ and so this is another example for Case 1. It is easily verified that condition (LS) is satisfied. The boundary symbol reads $\lambda + |\xi'|^2$, which is much simpler than that of the previous example.

The following example cannot be treated by the operator sum method.

Example 3.6. *Dynamic boundary conditions and surface convection for the diffusion equation*

$$\begin{aligned}\partial_t u - \Delta u &= f \quad (t \in J,\ x \in G),\\ \partial_t u + \partial_\nu u + a(x)\cdot\nabla_\Gamma u &= g \quad (t \in J,\ x \in \Gamma),\\ u(0,x) &= u_0(x) \quad (x \in G).\end{aligned}$$

Here $a \in C^1(\Gamma;\mathbb{R}^{n-1})$ is a tangent vector field on the surface Γ. This problem fits into our setting by taking $E = F = \mathbb{C}$, $\mathcal{A} = -\Delta$, $\mathcal{C}_0 = a(x)\cdot\nabla_\Gamma$, $\mathcal{B}_0 = \partial_\nu$, $\mathcal{B}_1 = -\mathcal{C}_1 = 1$, $g_1 = 0$. Here we have $2m = 2$, $m_0 = 1$, $k_0 = 1$, $m_1 = k_1 = 0$, $l = l_1 = l_0 = 1$, hence this example is in Case 2. We have $\mathcal{J} = \{0,1\}$, and the system in (LS) is given by

$$\begin{aligned}(\lambda + |\xi'|^2 - \partial_y^2)v(y) &= 0 \quad (y > 0),\\ -\partial_y v(0) + (\lambda + ia\cdot\xi')\sigma &= 0,\\ v(0) - \sigma &= 0.\end{aligned}$$

Setting $v_0 = v(0)$, the only stable solution of the ODE is $v(y) = e^{-\mu y}v_0$ with $\mu := \sqrt{|\xi'|^2+\lambda}$. The boundary conditions yield $(\lambda + ia\cdot\xi' + \mu)\sigma = 0$, hence $\sigma = v_0 = 0$. Similarly one verifies that (LS^-_∞) is satisfied. Note that the symbol appearing in the second boundary condition cannot be treated by the operator-sum method.

The next example is related to the free boundary value problem for the Navier-Stokes equation; cf. Prüss and Simonett [9].

Example 3.7. Consider

$$\begin{aligned}\partial_t u - \Delta u &= f \quad (t \in J,\ x \in G),\\ \partial_\nu u &= -\Delta_\Gamma \rho + g \quad (t \in J,\ x \in \Gamma),\\ \partial_t \rho - u &= h \quad (t \in J,\ x \in \Gamma),\\ u(0,x) &= u_0(x) \quad (x \in G),\\ \rho(0,x) &= \rho_0(x) \quad (x \in \Gamma).\end{aligned}$$

Here we take $E = F = \mathbb{C}$, $m = 1$, $m_0 = 0$, $k_0 = -\infty$, $m_1 = 1$, $k_1 = 2$, hence $l = l_1 = 1$, i.e. this is another example for Case 2. The boundary symbol in this example becomes $s(\xi',\lambda) = \lambda + |\xi'|$.

Our final example is more of academic nature. It shows that even for $m = 1$ the maximum number $m+3$ of corners on the leading part of the Newton polygon may occur. It is not difficult to extend this example to arbitrary $m \geq 1$.

Example 3.8.

$$\begin{aligned}\partial_t u - \Delta u &= f \quad (t \in J,\ x \in G),\\ u &= \Delta_\Gamma^2 \rho + g \quad (t \in J,\ x \in \Gamma),\\ \partial_t \rho - \partial_\nu u - \Delta_\Gamma^3 \rho &= h \quad (t \in J,\ x \in \Gamma),\\ u(0,x) &= u_0(x) \quad (x \in G),\\ \rho(0,x) &= \rho_0(x) \quad (x \in \Gamma).\end{aligned}$$

Here we take $E = F = \mathbb{C}$, $m = 1$, $m_0 = 1$, $k_0 = 6$, $m_1 = 0$, $k_1 = 4$, hence $l = l_0 = 6$, $l_1 = 5$, i.e. we are in Case 3. The boundary symbol in this example becomes $s(\xi',\lambda) = \lambda + |\xi'|^6 + |\xi'|^4\sqrt{\lambda + |\xi'|^2}$. In this example the nontrivial points on the Newton polygon are $(4, 1-\frac{1}{2p})$ and $(6, \frac{1}{2} - \frac{1}{2p})$, we have $2m\kappa_0 = 1 - \frac{1}{p}$ and $s = 4 - \frac{4}{p}$.

4. Proof of Theorems 2.1 and 2.2.

Following a standard approach in parabolic theory, we will first prove Theorems 2.1 and 2.2 for the model problem. So we assume that $G = \mathbb{R}^n_+$ and that all differential operators in question have constant coefficients and coincide with their principal parts. In particular, $\mathcal{C}_j = 0$ if $j \notin \mathcal{J}$. The proof is carried out in several steps.

4.1. **Reduction to Time Trace 0.** We first reduce the problem to $u_0 = \rho_0 = 0 = f$ and $\rho_1 := \partial_t\rho(0) = 0$. For this purpose let $\omega \geq 0$, extend u_0 to all of $\mathbb{R}^n$ in the class $W_p^{2m(1-1/p)}(\mathbb{R}^n)$ and f trivially by zero. Then by [3] there is a unique solution $u_* \in Z_u$ of the problem

$$\partial_t u + \omega u + \mathcal{A}u = f, \quad u(0) = u_0.$$

Restricting u_* to $\mathbb{R}^n_+$ and subtracting u_* from u shows that we may assume $u_0 = f = 0$ if we choose $\omega = 0$. In addition, in case $\omega > 0$, we obtain the estimate

$$|u_*|_{L_p(\mathbb{R}_+\times G)} \leq C(|u_0|_{W_p^{2m(1-1/p)}(G)} + |f|_{L_p(\mathbb{R}_+\times G)})/\omega.$$

It is more involved to remove the traces of ρ at $t = 0$. For this purpose we introduce the operator B on $X := L_p(\mathbb{R}^{n-1};F)$ by means of

$$Bv(x) := (\omega^2 - \Delta)^{s/2}v(x) \quad (x \in \mathbb{R}^{n-1}),$$

with domain $D(B) = H_p^s(\mathbb{R}^{n-1};F)$, $s > 0$, and $\omega > 0$. Here H_p^s denotes the vector-valued Bessel potential space of order s. It is well-known that B is sectorial with angle 0 and invertible. The C_0-semigroup generated by $-B$ is analytic and exponentially stable. The real interpolation spaces of B are given by

$$D_B(\alpha,p) = W^{s\alpha}_{pp}(\mathbb{R}^{n-1};F),$$

for each $\alpha \in (0,1)$ whenever $s\alpha \notin \mathbb{N}$.

Now consider an initial value $\varphi \in X$, and let the function $\sigma(t)$ be defined by

$$\sigma(t) = e^{-Bt}\varphi, \quad t \geq 0.$$

Then elementary semigroup theory shows for $\alpha > 1/p$, that $\sigma \in W_p^\alpha(\mathbb{R}_+;X)$ if and only if $\varphi \in D_B(\alpha - 1/p, p)$ and then $\sigma \in L_p(\mathbb{R}_+; D_B(\alpha,p))$ as well. This implies for $\alpha > 1/p$ with $s\alpha, s(\alpha - 1/p) \notin \mathbb{N}$ the equivalence

$$\sigma \in W_p^\alpha(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};F)) \cap L_p(\mathbb{R}_+; W_p^{s\alpha}(\mathbb{R}^{n-1};F)) \iff \varphi \in W_p^{\alpha s - s/p}(\mathbb{R}^{n-1}).$$

Let ρ_0 and ρ_1 be given where $\rho_1 = 0$ in case $\kappa_0 < \frac{1}{p}$ and $\rho_1 = g_0|_{t=0} - \mathcal{B}_0(0)u_0 - \mathcal{C}_0(0)\rho_0$ in case $\kappa_0 > \frac{1}{p}$. We set

$$\rho_*(t) = (2e^{-B_0t} - e^{-2B_0t})\rho_0 + (e^{-B_1t} - e^{-2B_1t})B_1^{-1}\rho_1 = \rho_*^0(t) + \rho_*^1(t).$$

Observe that $\rho_*^0(0) = \rho_0$, $\frac{d}{dt}\rho_*^0(0) = 0$ and $\rho_*^1(0) = 0$, $\frac{d}{dt}\rho_*^1(0) = \rho_1$. Here $\rho_1 = 0$ in case $\kappa_0 < 1/p$, and $\rho_1 = g_0|_{t=0} - \mathcal{B}_0(0)u_0 - \mathcal{C}_0(0)\rho_0$ in case $\kappa_0 > 1/p$.

To obtain $\rho_* \in Z_\rho$ for $\rho_0 \in \pi Z_\rho$, $\rho_1 \in \pi_1 Z_\rho$ we set

Case 1: $B_0 = B_1 = (\omega^2 - \Delta)^m$,

Case 2: $B_0 = (\omega^2 - \Delta)^{l/2}$, $B_1 = (\omega^2 - \Delta)^m$,

Case 3 (i): $B_0 = (\omega^2 - \Delta)^m$, $B_1 = (\omega^2 - \Delta)^{k_{j_1} m/(2m+m_{j_1}-m_0)}$,
for $\min_{j\in\mathcal{J}} \kappa_j > 1/p$,

Case 3 (ii): $B_0 = (\omega^2 - \Delta)^{\frac{k_{i_1}-k_{i_0}}{2(\kappa_{i_0}-\kappa_{i_1})}}$, $B_1 = (\omega^2 - \Delta)^{k_{j_1} m/(2m+m_{j_1}-m_0)}$,
for $\kappa_{i_0} > 1/p > \kappa_{i_1}$ as in Section 2;

Case 3 (iii): $B_0 = (\omega^2 - \Delta)^{\frac{k_{j_1}}{2(1+\kappa_0-\kappa_{j_1})}}$, for $\max_{j\in\mathcal{J}} \kappa_j < 1/p$.

Then in Cases 1 and 3 (i) we have $\rho_0 \in D_{B_0}(\frac{l}{2m} + \kappa_0 - \frac{1}{p}, p)$, hence

$$\rho_*^0 \in W_p^{\kappa_0+l/2m}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; F)) \cap L_p(\mathbb{R}_+; W_p^{l+2m\kappa_0}(\mathbb{R}^{n-1}; F)),$$

which embeds into Z_ρ since $l \geq 2m$. Similarly, in Cases 1 and 2 we have $B_1^{-1}\rho_1 \in D_{B_1}(1 + \kappa_0 - \frac{1}{p}, p)$, hence

$$\rho_*^1 \in W_p^{1+\kappa_0}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; F)) \cap L_p(\mathbb{R}_+; W_p^{2m+2m\kappa_0}(\mathbb{R}^{n-1}; F)),$$

which also embeds into Z_ρ since now $2m \geq l$. In Case 2 we have $\rho_0 \in D_{B_0}(\frac{2m\kappa_0}{l} + 1 - \frac{1}{p}, p)$ hence

$$\rho_*^0 \in W_p^{1+2m\kappa_0/l}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; F)) \cap L_p(\mathbb{R}_+; W_p^{l+2m\kappa_0}(\mathbb{R}^{n-1}; F)),$$

which again embeds into Z_ρ since $2m > l$. Next, consider Case 3 (i), (ii); here we have $B_1^{-1}\rho_1 \in D_{B_1}(1 + \kappa_0 - \frac{1}{p}, p)$, hence with $s = k_{j_1}\kappa_0/(1 + \kappa_0 - \kappa_{j_1})$,

$$\rho_*^1 \in W_p^{1+\kappa_0}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; F)) \cap L_p(\mathbb{R}_+; W_p^{(s/\kappa_0)(1+\kappa_0)}(\mathbb{R}^{n-1}; F)),$$

which embeds into Z_ρ since by construction $\mathcal{NP}$ is left from the line passing through the points $(0, 1 + \kappa_0)$ and (k_{j_1}, κ_{j_1}). In Case 3 (ii) we have $(1 - \Delta)^{k_{i_r}/2}\rho_0 \in D_{B_0}(\kappa_{i_r} - 1/p, p)$ for $r = 0, 1$, hence

$$\begin{aligned}\rho_*^0 \in\, & W_p^{\kappa_{i_0} + k_{i_0}\frac{\kappa_{i_0}-\kappa_{i_1}}{k_{i_1}-k_{i_0}}}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; F)) \cap W_p^{\kappa_{i_r}}(\mathbb{R}_+; H_p^{k_{i_r}}(\mathbb{R}^{n-1}; F)) \\ & \cap L_p(\mathbb{R}_+; W_p^{k_{i_0} + \kappa_{i_0}\frac{k_{i_1}-k_{i_0}}{\kappa_{i_0}-\kappa_{i_1}}}(\mathbb{R}^{n-1}; F)),\end{aligned}$$

$r = 0, 1$, which implies $\rho_*^0 \in Z_\rho$ since $\mathcal{NP}$ is left from the line passing through the points (k_{i_r}, κ_{i_r}), $r = 0, 1$. Finally, Case 3 (iii) is treated in a similar way.

This shows that ρ_* belongs to Z_ρ in all three cases. Moreover, the dependence on $\omega > 0$ implies

$$|\rho_*|_{Y_0(\mathbb{R}_+)} \leq C(|\rho_0|_{\pi Z_\rho} + |\rho_1|_{\pi_1 Z_\rho})/\omega^\tau,$$

for some $\tau > 0$, depending only on the orders $2m$, m_j, k_j. Here $Y_0(\mathbb{R}_+)$ stands for the space Y_0 with $J = \mathbb{R}_+$.

4.2. **The Boundary Symbol.** We concentrate here on Case 3, Cases 1 and 2 can be treated in a similar but simpler way. Denoting by ξ' the Fourier variable in the tangential direction x', by λ the Laplace variable in t, and with $\mu = (\lambda + |\xi'|^{2m})^{1/2m}$, the boundary symbol, i.e. the symbol corresponding to the space Z_ρ reads

$$s(\xi', \lambda) = \lambda + \sum_{j\in\mathcal{J}} |\xi'|^{k_j} \mu^{m_0-m_j} \quad (\xi' \in \mathbb{R}^{n-1},\ \lambda \in \overline{\mathbb{C}}_+).$$

The corresponding operator S given by

$$S = \frac{d}{dt} + \sum_{j\in\mathcal{J}} (-\Delta')^{k_j/2} L^{(m_0-m_j)/2m},\ L = \frac{d}{dt} + (-\Delta')^m,$$

maps the space

$$ {}_0Z_\rho := {}_0W_p^{1+\kappa_0}(J;L_p(\Gamma;F)) \cap \bigcap_{j\in\mathcal{J}} W_p^{\kappa_j}(J;H_p^{k_j}(\Gamma,F)) \cap L_p(J;W_p^{l+2m\kappa_0}(\Gamma;F)) $$

boundedly into the boundary space

$$ {}_0Y_0 := {}_0W_p^{\kappa_0}(J;L_p(\mathbb{R}^{n-1};F)) \cap L_p(J;W_p^{2m\kappa_0}(\mathbb{R}^{n-1};F)). $$

Here the zero means that the traces at $t=0$ of the function and its derivative w.r.t. t vanish whenever they exist. We show that in fact S is an isomorphism.

For this purpose we employ the Dore-Venni theorem in ${}_0Y_0$, which belongs to the class $\mathcal{HT}$. Let $G = d/dt$ with natural domain

$$ D(G) = {}_0W_p^{1+\kappa_0}(J;L_p(\mathbb{R}^{n-1};F)) \cap H_p^1(J;W_p^{2m\kappa_0}(\mathbb{R}^{n-1};F)); $$

G is sectorial, invertible and admits an $\mathcal{H}^\infty$-calculus in ${}_0Y_0$ of angle $\pi/2$. Similarly we let $D_{n-1} = -\Delta'$ with domain

$$ D(D_{n-1}) = {}_0W_p^{\kappa_0}(J;H_p^2(\mathbb{R}^{n-1};F)) \cap L_p(J;W_p^{2+2m\kappa_0}(\mathbb{R}^{n-1};F)). $$

Then D_n is sectorial and admits an $\mathcal{H}^\infty$-calculus in ${}_0Y_0$ of angle zero, hence each of its fractional powers $D_n^{k_j/2}$ has the same property. Further, L is also sectorial and admits an $\mathcal{H}^\infty$-calculus of angle $\frac{\pi}{2}$, L is invertible and in fact, as an operator in $L_p(J\times\mathbb{R}^{n-1};F)$ we have ${}_0Y_0 = D_L(\kappa_0,p)$. The domain of L in ${}_0Y_0$ is therefore the space

$$ D(L) = {}_0W_p^{1+\kappa_0}(J;L_p(\mathbb{R}^{n-1};F)) \cap L_p(J;W_p^{2m+2m\kappa_0}(\mathbb{R}^{n-1};F)). $$

The fractional powers $L^{(m_0-m_j)/2m}$ of L have the same properties, with angles $\frac{|m_0-m_j|}{2m}\,\frac{\pi}{2} < \frac{\pi}{2}$. Thus the Dore-Venni theorem for products and iterated sums implies that S is an isomorphism between $D(S) = {}_0Z_\rho$ and ${}_0Y_0$, since all operators commute and the parabolicity condition is valid; cf. [10].

If $J=\mathbb{R}_+$ is the halfline we obtain the same result in case $G=d/dt$ is replaced by $G+\omega$ where $\omega>0$, since $G+\omega$ is invertible in ${}_0Y_0$. Then we obtain in addition the estimate

$$ |S^{-1}|_{\mathcal{B}({}_0Y_0)} \le \frac{|(G+\omega)S^{-1}|_{\mathcal{B}({}_0Y_0)}}{\omega} \le \frac{C}{\omega}, \quad \omega>0, $$

with some constant $C>0$ which is independent of ω.

For the proofs of Corollary 2.3 and Theorem 2.2 we also need the regularity of the solution of the equation $S\rho = h$ when $h \in L_p(J;W_p^s(\mathbb{R}^{n-1};F))$, where $s = k_{j_1}\kappa_0/(1+\kappa_0-\kappa_{j_1})$ as in Theorem 2.2. For such function h we obviously get

$$ \rho \in {}_0H_p^1(J;W_p^s(\mathbb{R}^{n-1};F)) \cap L_p(J;W_p^{s+l}(\mathbb{R}^{n-1};F)), $$

but also the mixed time-space regularity

$$ \rho \in \bigcap_{j\in\mathcal{J}} {}_0H_p^{\kappa_j-\kappa_0}(J;W_p^{s+k_j}(\mathbb{R}^{n-1};F). $$

By the definition of s we then obtain via the mixed derivative theorem

$$ \begin{aligned} &{}_0H_p^1(J;W_p^s(\mathbb{R}^{n-1};F)) \cap {}_0H_p^{\kappa_{j_1}-\kappa_0}(J;W_p^{s+k_{j_1}}(\mathbb{R}^{n-1};F)) \\ &\qquad\qquad \hookrightarrow {}_0W_p^{\kappa_{j_1}}(J;H_p^{k_{j_1}}(\mathbb{R}^{n-1};F)). \end{aligned} $$

Since the slopes

$$\frac{k_{j_{r+1}} - k_{j_r}}{\kappa_{j_r} - \kappa_{j_{r+1}}}$$

are positive and nonincreasing in r we obtain inductively the embeddings

$$\begin{aligned} {}_0H_p^{\kappa_{j_r}}(J; H_p^{k_{j_r}}(\mathbb{R}^{n-1}; F)) \cap {}_0H_p^{\kappa_{j_{r+1}}-\kappa_0}(J; W_p^{s+k_{j_{r+1}}}(\mathbb{R}^{n-1}; F)) \\ \hookrightarrow {}_0W_p^{\kappa_{j_{r+1}}}(J; H_p^{k_{j_{r+1}}}(\mathbb{R}^{n-1}; F)). \end{aligned}$$

Thus, for $h \in L_p(J; W_p^s(\mathbb{R}^{n-1}; F))$, the solution ρ of $S\rho = h$ belongs to the space

$$\rho \in {}_0H_p^1(J; W_p^s(\mathbb{R}^{n-1}; F)) \cap \bigcap_{j\in\mathcal{J}} W_p^{\kappa_j}(J; H_p^{k_j}(\mathbb{R}^{n-1}; F)) \cap L_p(J; W_p^{s+l}(\mathbb{R}^{n-1}; F)).$$

In Cases 1 and 2 the corresponding boundary symbol will be

$$s(\xi', \lambda) = \lambda + |\xi'|^l \quad (\xi' \in \mathbb{R}^{n-1},\ \lambda \in \overline{\mathbb{C}}_+),$$

which is much simpler than that in Case 3.

4.3. **Partial Fourier Transform.** We follow here the presentation in [3], Section 6. Taking Fourier transform in the spatial variables x' and Laplace transform in t we obtain the following ordinary differential equations.

$$\begin{aligned} \lambda v + \mathcal{A}(\xi', D_y)v = 0 \quad (y > 0,\ \xi' \in \mathbb{R}^{n-1},\ \lambda \in \overline{\mathbb{C}}_+), \\ \mathcal{B}_j(\xi', D_y)v(0) + \mathcal{C}_j(\xi')\sigma = h_j \quad (\xi' \in \mathbb{R}^{n-1},\ j = 1, \dots, m), \\ \mathcal{B}_0(\xi', D_y)v(0) + (\lambda + \mathcal{C}_0(\xi'))\sigma = h_0 \quad (y > 0,\ \xi' \in \mathbb{R}^{n-1},\ \lambda \in \overline{\mathbb{C}}_+). \end{aligned} \tag{4.1}$$

Here we assumed $f = u_0 = \rho_0 = \rho_1 = 0$, h_j, v and σ denote the transforms of g_j, u and ρ, respectively. The Lopatinskii-Shapiro condition means that for each $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\mathbb{C}}_+ \setminus \{(0, 0)\}$ and for any given vectors h_j $(j = 0, \dots, m)$ there is a unique solution $v \in C_0^{2m}(\mathbb{R}_+; E)$, $\sigma \in F$ of (4.1). We obtain a suitable representation of this solution as follows.

We have

$$\mathcal{A}(\xi', D_y) = \sum_{k=0}^{2m} a_k(\xi')D_y^{2m-k}, \quad \mathcal{B}_j(\xi', D_y) = \sum_{k=0}^{m_j} b_{jk}(\xi')D_y^{m_j-k},$$

where $a_k(\xi')$ and $b_{jk}(\xi')$ are homogeneous of degree k. Rewrite the ordinary differential equation of order $2m$ as a first order system by introducing the matrix operator

$$A_0(\xi', \lambda) = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I \\ c_{2m} & c_{2m-1} & \dots & c_2 & c_1 \end{pmatrix},$$

where

$$c_j = c_j(\xi') = -a_0^{-1}a_j(\xi') \quad (j = 1, \dots, 2m-1)$$

and

$$c_{2m} = c_{2m}(\xi', \lambda) = -a_0^{-1}(a_{2m}(\xi') + \lambda I).$$

Note that a_0 does not depend on ξ' and is invertible by ellipticity. It has been shown in [3], Section 6, that the spectrum $\sigma(iA_0(\xi', \lambda))$ as an operator in E^{2m} does not intersect the imaginary axis, hence splits into two parts $S_\pm(\xi', \lambda)$ located in the right resp. in the left half-plane. The associated spectral projections will be denoted by $P_\pm(\xi',\lambda)$.

It is convenient to introduce the scaling $a=\lambda/\mu^{2m}$, $b=|\xi'|/\mu$, $\zeta=\xi'/|\xi'|$ with $\mu=(\lambda+|\xi'|^{2m})^{1/2m}$. Then (a,b,ζ) runs through a compact set. Note that $a+b^{2m}=1$, hence a is even redundant. Introducing the vector

$$w(y)=\Big(v(y),\frac{1}{\mu}D_yv(y),\dots,\frac{1}{\mu^{2m-1}}D_y^{2m-1}v(y)\Big)^T,$$

we obtain the following differential equation of first order for w.

$$\partial_y w=i\mu A_0(b\zeta,a)w,$$

The solutions of this equation are given by

$$w(y)=e^{\mu iA_0(b\zeta,a)y}w_0\quad(y\geq 0)$$

where $w_0:=w(0)$ still has to be determined in such a way that $w(y)$ is decaying at infinity, which means $P_+(b\zeta,a)w_0=0$.

Thus the pair $(w_0,\sigma)\in E^{2m}\times F$ has to satisfy the following system of equations.

$$\begin{aligned}B_0(b\zeta)w_0+|\xi'|^{k_0}\mu^{-m_0}\mathcal{C}_0(\zeta)\sigma+a\mu^{2m-m_0}\sigma&=\mu^{-m_0}h_0,\\ B_j(b\zeta)w_0+|\xi'|^{k_j}\mu^{-m_j}\mathcal{C}_j(\zeta)\sigma&=\mu^{-m_j}h_j\quad(j=1,\dots,m),\\ P_+(b\zeta,a)w_0&=0.\end{aligned}\tag{4.2}$$

Here $B_j(\xi')=(b_{jm_j}(\xi'),\dots,b_{j0},0\dots,0)^T$ for $j=0,\dots,m$. Note that by the ellipticity assumption (E), the Lopatinskii-Shapiro condition (LS) is equivalent to unique solvability of (4.2) for all parameter values of $\xi'\in\mathbb{R}^{n-1}$, $\operatorname{Re}\lambda\geq 0$, such that $\lambda+|\xi'|^{2m}=\mu^{2m}\neq 0$, and for all given right hand sides $h=(\mu^{-m_0}h_0,\dots,\mu^{-m_m}h_m)^T\in F\times E^m$. We denote the unique solution (w_0,σ) for a fixed vector $h\in F\times E^m$ by

$$w_0=M_w(b,\zeta,\mu)h,\quad \sigma=M_\sigma(b,\zeta,\mu)h.$$

Next let h_j be the Fourier-Laplace transform of the function $g_j\in {}_0Y_j$. Define the space ${}_0Y_E$ as

$${}_0Y_E={}_0W_p^{1-1/2mp}(J;L_p(\mathbb{R}^{n-1};E))\cap L_p(J;W_p^{2m-1/p}(\mathbb{R}^{n-1};E)),$$

and similarly we define ${}_0Y_F$. Since μ is the symbol of the operator $L^{1/2m}$ where $L=G+D_n^m$ has been introduced above, and ${}_0Y_j=D_L(\kappa_j,p)$ for L considered as an operator in $X=L_p(J\times\mathbb{R}^{n-1};E)$, we see that μ^{-m_j} is the symbol of $L^{-m_j/2m}$ which maps ${}_0Y_j$ onto ${}_0Y_E$. Thus h is the Fourier-Laplace transform of a function $g\in{}_0Y_F\times{}_0Y_E^m$.

Assume that M_w is a Fourier-Laplace multiplier from this space into ${}_0Y_E^{2m}$, which means that w_0 belongs to this space, for each given g_j. Then to obtain the right regularity for the solution v we only need to know that the extension operator defined by the symbol

$$\tau(\xi',\lambda,y)w_0=\mu^{2m}e^{\mu iA_0(b\zeta,a)y}(I-P_+(b\zeta,a))w_0$$

maps ${}_0Y_E^{2m}$ into $L_p(J\times\mathbb{R}^n;E^{2m})$. This has been proved in [4]. Therefore it remains to study the symbols M_w and M_σ.

It is now convenient to introduce a scaling of σ by $\sigma_0=s(\xi',\lambda)\mu^{-m_0}\sigma$, where the boundary symbol $s(\xi',\lambda)$ has been studied in the previous subsection. According to the results obtained there, the operator S^{-1} with symbol $1/s(\xi',\lambda)$ maps ${}_0Y_0$

isomorphically onto ${}_0Z_\rho$. Since on the other hand the operator $L^{m_0/2m}$ with symbol μ^{m_0} maps ${}_0Y_F$ isomorphically onto ${}_0Y_0$, we see that we need to obtain $\sigma_0 \in {}_0Y_F$. The problem for (w_0, σ_0) reads now as follows.

$$
\begin{aligned}
B_0(b\zeta)w_0 + \frac{\mu^{l_0}\mathcal{C}_0(b\zeta) + a\mu^{2m}}{s(\xi',\lambda)}\sigma_0 &= h_0^0, \\
B_j(b\zeta)w_0 + \frac{\mu^{l_j}}{s(\xi',\lambda)}\mathcal{C}_j(b\zeta)\sigma_0 &= h_j^0 \quad (j = 1, \dots, m), \\
P_+(b\zeta, a)w_0 &= 0.
\end{aligned} \tag{4.3}
$$

Here we have set $h_j^0 = \mu^{-m_j}h_j$. Note that $s(\xi',\lambda)$ can be rewritten as

$$
s(\xi',\lambda) = a\mu^{2m} + \sum_{r\in\mathcal{J}} b^{k_r}\mu^{l_r},
$$

for $l > 2m$ and $s(\xi',\lambda) = \mu^{2m}$ if $l = 2m$ and $s(\xi',\lambda) = (1 - b^{2m})\mu^{2m} + b^l\mu^l$ for $l < 2m$. Since $\mathcal{B}_j$ and $\mathcal{C}_j$ are polynomial and according to [3], Section 6, P_+ is holomorphic, we see that $M_w^0(b,\zeta,\mu)$ and $M_\sigma^0(b,\zeta,\mu)$ are holomorphic, where

$$
w_0 = M_w^0(b,\zeta,\mu)h^0, \quad \sigma_0 = M_\sigma^0(b,\zeta,\mu)h^0
$$

denote the rescaled solutions. Now b and ζ run through a compact set, $\zeta \in \mathbb{R}^{n-1}$, $|\zeta| = 1$ and $b \in (\overline{B}_{1/2}(1/2))^{1/2m}$. However μ is not bounded, therefore we have to study the asymptotic properties of M_w^0 and M_ρ^0 as $|\mu| \to \infty$.

(i) Let us begin with the simplest case $l = 2m$. Then $b^{k_j}\mu^{l_j}/s(\xi',\lambda) = b^{k_j}$ for all $j \in \mathcal{J}$, and $\lambda/s(\xi',\lambda) = 1 - b^{2m}$, since in this case we have $l_j = l = 2m$ for all $j \in \mathcal{J}$. Thus in Case 1 there is no dependence of M_w^0 and M_σ^0 on μ, this is the homogeneous case. To complete the proof in Case 1, we first invert the Fourier transform w.r.t. $\zeta = \xi'/|\xi'|$. The function $\xi' \mapsto M^0(b,\zeta) = (M_w^0(b,\zeta), M_\sigma^0(b,\zeta))$ is homogeneous of degree 0, and by the proof of Proposition 6.2 in [3], M^0 is holomorphic on $D_b \times D_\zeta$, with some open sets $D_b \supset (\overline{B}_{1/2}(1/2))^{1/2m}$, $D_\zeta \supset \{\zeta \in \mathbb{R}^{n-1} : |\zeta| = 1\}$. Therefore

$$
\{|\xi'|^{|\alpha'|}D_{\xi'}^{\alpha'}M^0(b,\zeta) : \xi' \in \mathbb{R}^{n-1},\ \xi' \neq 0,\ b \in D_b\}
$$

is $\mathcal{R}$-bounded in $\mathcal{B}(F \times E^m; F \times E^{2m})$, by [3], Proposition 3.10. Hence, there is a family of linear operators $\{T(b) : b \in D_b\} \subset \mathcal{B}(L_p(\mathbb{R}^{n-1}; F \times E^m); L_p(\mathbb{R}^{n-1}; F \times E^{2m}))$ such that $\mathcal{F}T(b)(\xi') = M^0(b,\zeta)$ for all $\xi' \in \mathbb{R}^{n-1}$ and $b \in D_b$. Moreover, the map $b \mapsto T(b)$ is holomorphic and uniformly bounded on D_b. This result also holds with L_p replaced by H_p^s, and then by real interpolation also for W_p^s. By canonical extension we therefore obtain

$$
T \in \mathcal{H}^\infty(D_b; \mathcal{B}({}_0Y_F \times {}_0Y_E^m; {}_0Y_F \times {}_0Y_E^{2m})).
$$

Define an operator B by means of its symbol $b = |\xi'|/\mu$, i.e. $B = (-\Delta')^{1/2}L^{-1/2m}$ which is bounded and has spectrum contained in the set $\overline{B}_{1/2}(1/2) \subset \mathbb{C}$. Then the operator-valued Dunford calculus implies that $\mathcal{T} := T(B)$ is a bounded linear operator from ${}_0Y_F \times {}_0Y_E^m$ to ${}_0Y_F \times {}_0Y_E^{2m}$. This completes the proof in Case 1.

(ii) Next in Case 2 we have for $b = 1$, i.e. $\lambda = 0$ the relation $b^{k_j}\mu^{l_j}/s(\xi',\lambda) = 1$ for all $j \in \mathcal{J}$, since $j \in \mathcal{J}$ if and only if $l_j = l$. The corresponding problem is uniquely solvable by (LS) by setting $\lambda = 0$ and $|\xi'| = 1$.

On the other hand, if in this case $b \neq 1$ then $b^{k_j}\mu^{l_j}/s(\xi',\lambda) \to 0$ for all $j \in \mathcal{J}$, i.e. for $\mu \to \infty$ we obtain the limiting problem

$$\begin{aligned} B_0(b\zeta)w_0 + \sigma_0 &= h_0^0, \\ B_j(b\zeta)w_0 &= h_j^0 \quad (j = 1, \ldots, m), \\ P_+(b\zeta, a)w_0 &= 0. \end{aligned} \tag{4.4}$$

This problem arises if we perform the same scaling for the first problem appearing in the asymptotic Lopatinskii-Shapiro condition (LS_∞^-).

If we let $\mu \to \infty$ and at the same time $b \to 1$, the corresponding limiting problem is not unique, which shows that there is a discontinuity at $(b, \mu) = (1, \infty)$. In fact, choose $a = c/\mu^s$ with some fixed exponent $s > 0$ and some constant c. Then for $\mu \to \infty$ we obtain

$$\frac{\mu^{l_j}}{s(\xi',\lambda)} = \frac{\mu^l}{c\mu^{2m-s} + b^l\mu^l} \to \begin{cases} 1 & \text{for } s > 2m - l \\ \frac{1}{1+c} & \text{for } s = 2m - l \\ 0 & \text{for } s < 2m - l. \end{cases}$$

Thus the limiting problem for $s > 2m - l$ is the same as (LS) with $\lambda = 0$, $|\xi'| = 1$, and for $s < 2m - l$ it becomes again (4.4). However, for $s = 2m - l$ we obtain the problem

$$\begin{aligned} B_0(\zeta)w_0 + \mathcal{C}_0(\zeta)\frac{\sigma_0}{1+c} + c\frac{\sigma_0}{1+c} &= h_0^0, \\ B_j(\zeta)w_0 + \mathcal{C}_j(\zeta)\frac{\sigma_0}{1+c} &= h_j^0 \quad (j = 1, \ldots, m), \\ P_+(\zeta, 0)w_0 &= 0, \end{aligned} \tag{4.5}$$

where $c \in \overline{\mathbb{C}}_+$ is arbitrary. These are all possible limit problems since as $\mu \to \infty$ and $a \to 0$ we see that $b^l\mu^l/(a\mu^{2m} + b^l\mu^l)$ stays bounded and belongs to $\overline{\mathbb{C}}_+$, hence admits a convergent subsequence. Problem (4.5) is uniquely solvable by the second condition in (LS_∞^-), note that it corresponds to the corresponding quasisteady problem.

Since the limiting problems are uniquely solvable and holomorphic in ζ, b and $\eta = 1/(1+c)$, applying [3] again, we obtain holomorphy of $M(b, \zeta, \infty)$ for $b \neq 1$ and also of $M(1, \zeta, \infty, \eta)$, which implies as in (i) $\mathcal{R}$-boundedness of the family $\{M(b, \zeta, \mu) : |\zeta| = 1, b \in D_b, \mu \in \Sigma_\theta\}$, for some open set $D_b \supset \overline{B}_{1/2}(1/2)$ and some $\theta > (\pi/2)/2m$. Then as in (i) we may invert the Fourier transform w.r.t. ζ and apply the Dunford calculus for B to obtain a family of operators $\mathcal{T}(\mu)$ uniformly bounded and holomorphic from ${}_0Y_F \times {}_0Y_E^m$ to ${}_0Y_F \times {}_0Y_E^{2m}$. To finish this case, we employ a variant of the $\mathcal{H}^\infty$ calculus for L in the interpolation spaces $D_L(1 - 1/p, p)$, cf. [2], Corollary 1. This yields an operator $T(L)$ linear and bounded from ${}_0Y_F \times {}_0Y_E^m$ to ${}_0Y_F \times {}_0Y_E^{2m}$.

(iii) Similarly, in Case 3 we have for $b = 0$, i.e. $\xi' = 0$ the relation $b^{k_j}\mu^{l_j}/s(\xi',\lambda) = 0$. The corresponding problem is uniquely solvable by (LS) with $\xi' = 0$. If in Case 3 we consider $b \neq 0$ then

$$\beta_j(b,\mu) := b^{k_j}\mu^{l_j}/s(\xi',\lambda) \to b^{k_j}\delta_j,$$

where $\delta_j = 1$ if $l_j = l$, and $\delta_j = 0$ in case $l_j < l$, and $\lambda/s(\xi',\lambda) \to 0$. The corresponding problem is uniquely solvable thanks to the first condition in (LS_∞^+), by the same scaling as above.

As in Case 2 there is a discontinuity in the asymptotic problems, this time at $(b,\mu) = (0,\infty)$. Suppose that $\mu \to \infty$, $b \to 0$; then β_j are bounded, and $\sum_{j\in\mathcal{J}\cup\{-1\}} \beta_j = 1$ in the limit sense. Hence we may assume $\beta_j \to \beta_j^\infty$. If $\beta_j^\infty = 0$ for all $j \in \mathcal{J}$ then the corresponding limiting problem is that in (LS) with $\xi' = 0$. If $\beta_i^\infty \neq 0$ and $\beta_j^\infty = 0$ for all $j \neq i \in \mathcal{J}$ then

$$\varepsilon_j := b^{k_j}\mu^{l_j}/b^{k_i}\mu^{l_i} \to 0 \quad \text{ for all } j \neq i.$$

We will show that (k_i, κ_i) must be the left endpoint of some edge $\mathcal{NP}_r$. Suppose the contrary, and consider first the case $l_i = l$. Choose $j \in \mathcal{J}_{2q_{max}}$, that is (k_j, κ_j) is the left endpoint of the edge through (k_i, κ_i). Then $l_j = l_i = l$ and $k_j < k_i$, and thus $\varepsilon_j = b^{k_j - k_i}$ tends to ∞ as $b \to 0$, a contradiction. Suppose now that $l_i < 2m$. Let $i-1$ and $i+1$ denote the indices corresponding to the left and right endpoint of the edge through (k_i, κ_i), respectively. Then $\frac{l_{i+1}-l_i}{k_{i+1}-k_i} = \frac{l_{i-1}-l_i}{k_{i-1}-k_i}$, and since

$$b = \varepsilon_j^{\frac{1}{k_j - k_i}} \Big(\frac{1}{\mu}\Big)^{\frac{l_j - l_i}{k_j - k_i}} \quad \text{ for all } j \neq i,$$

it follows that $\varepsilon_{i+1}^{\frac{1}{k_{i+1}-k_i}} = \varepsilon_{i-1}^{\frac{1}{k_{i-1}-k_i}}$. In view of $k_{i-1} < k_i < k_{i+1}$ and $\varepsilon_j \to 0$, $j = i-1, i+1$, the left-hand side of the last equation tends to 0 while the right-hand side goes to ∞, a contradiction. Hence (k_i, κ_i) is the left endpoint of some edge $\mathcal{NP}_r$.

Suppose on the other hand that $\beta_i^\infty \neq 0$ and $\beta_j^\infty \neq 0$ for at least two indices $i \neq j \in \mathcal{J} \cup \{-1\}$. Then we have

$$b^{k_j - k_i}\mu^{l_j - l_i} \to \beta_j^\infty/\beta_i^\infty.$$

Assuming $k_j > k_i$ this means that

$$b \sim (\beta_j^\infty/\beta_i^\infty)^{\frac{1}{k_j-k_i}} \Big(\frac{1}{\mu}\Big)^{\frac{l_j - l_i}{k_j - k_i}} =: c\mu^{-s}.$$

Observe that this situation cannot occur for $i, j \in \mathcal{J}_{2q_{max}+1}$, because in this case $l_i = l_j = l$. We conclude that $l_i - k_i s = l_j - k_j s = \max\{l_r - k_r s : r \in \mathcal{J} \cup \{-1\}\}$. Now consider the function

$$\varphi(s) = \max\{l_j - k_j s : \, j \in \mathcal{J} \cup \{-1\}\}, \quad s \geq 0.$$

The function φ defines also a polygon, it is strictly decreasing and convex for $0 \leq s \leq (l_{j_1} - 2m)/k_{j_1}$, we have $\varphi(0) = l$ and $\varphi(s) = 2m$ for $s \geq (l_{j_1} - 2m)/k_{j_1}$. Observe that $l_j - k_j s = l_i - k_i s$ if and only if

$$s = \frac{l_j - l_i}{k_j - k_i} = 1 + 2m\frac{\kappa_j - \kappa_i}{k_j - k_i},$$

hence the slopes of the Newton polygon correspond to the vertices of φ.

Now if $s \in (0, (l_{j_1} - 2m)/k_{j_1})$ is not a vertex of φ, then there exists $q(s) \in \{1, \ldots, q_{max}\}$ such that the maximum defining $\varphi(s)$ is taken at precisely those i for which $i \in \mathcal{J}_{2q(s)}$. Denoting the number of these indices by $|\mathcal{J}_{2q(s)}|$, it follows that $\beta_j^\infty = 1/|\mathcal{J}_{2q(s)}|$ for all $j \in \mathcal{J}_{2q(s)}$, and $\beta_j^\infty = 0$ for all $j \notin \mathcal{J}_{2q(s)}$. This yields the limiting problems

$$\begin{aligned} B_j(0)w_0 + \delta_{j,\mathcal{J}_{2q}}\mathcal{C}_j(\zeta)\frac{\sigma_0}{|\mathcal{J}_{2q}|} &= h_j^0 \quad (j = 0, \ldots, m), \\ P_+(0,1)w_0 &= 0, \end{aligned} \tag{4.6}$$

where q runs through the set $\{1,\dots,q_{max}\}$, and $\zeta\in\mathbb{R}^{n-1}$, $|\zeta|=1$. For $s>(l_{j_1}-2m)/k_{j_1}$ the limiting problem is that in (LS) with $\xi'=0$.

On the other hand, if $s>0$ is a vertex of φ then s corresponds to the slope of one of the edges $\mathcal{NP}_q$, $q\in\{0,\dots,q_{max-1}\}$, of the Newton polygon, say $(\kappa_{j_q}-\kappa_{j_{q+1}})/(k_{j_q}-k_{j_{q+1}})$, and then $\beta_j^\infty=0$ for all $j\notin\mathcal{J}_{2q+1}$. Thus we obtain the limiting problems

$$B_j(0)w_0+\delta_{j,\mathcal{J}_{2q+1}}\mathcal{C}_j(c\zeta)\sigma_0/\sum_i\delta_{i,\mathcal{J}_{2q+1}}c^{k_i}=h_j^0\quad(j=0,\dots,m),$$
$$P_+(0,1)w_0=0,$$

where q runs through the set $\{1,\dots,q_{max}-1\}$. Here $\delta_{j,\mathcal{J}_{2q+1}}=1$ if $j\in\mathcal{J}_{2q+1}$ and zero otherwise, $\zeta\in\mathbb{R}^{n-1}$, $|\zeta|=1$, and $c\in\overline{\Sigma}_{\phi_q}$, where $\phi_q=\frac{\pi(m_{j_q}-m_{j_{q+1}})}{4m(k_{j_q}-m_{j_{q+1}})}$.

This covers all segments on the Newton polygon except for the first one which connects the points $(0,1+\kappa_0)$ and (k_{j_1},κ_{j_1}). For this segment we have the following limiting problem.

$$\begin{aligned}B_0(0)w_0+\frac{(1+\delta_{0,\mathcal{J}_1}\mathcal{C}_0(c\zeta))\sigma_0}{\sum_i\delta_{0,\mathcal{J}_1}c^{k_i}}&=h_0^0,\\ B_j(0)w_0+\frac{\delta_{j,\mathcal{J}_1}\mathcal{C}_j(c\zeta)\sigma_0}{\sum_i\delta_{i,\mathcal{J}_1}c^{k_i}}&=h_j^0\quad(j=1,\dots,m),\\ P_+(0,1)w_0&=0,\end{aligned}\tag{4.7}$$

where $\delta_{j,\mathcal{J}_1}=1$ if $j\in\mathcal{J}_1$ and zero otherwise, $\zeta\in\mathbb{R}^{n-1}$, $|\zeta|=1$, and $c\in\overline{\Sigma}_{\phi_1}$, with $\phi_1=\frac{\pi(l_{j_1}-2m)}{4mk_{j_1}}$.

By the asymptotic Lopatinskii-Shapiro conditions (LS$_\infty^+$) these problems are uniquely solvable, and the solution operators $M(b,\zeta,\infty,c,r)$ are holomorphic for each r. These limiting problems resolve the discontinuity at $(b,\mu)=(0,\infty)$. We may then continue as in Case 2 to obtain $\mathcal{R}$-boundedness of the family $\{M(b,\zeta,\mu)\}$ and then to derive a bounded operator $T(L)$ with symbol $M(b,\zeta,\mu)$. This completes the proof in Case 3.

If we consider the halfline $J=\mathbb{R}_+$, the results remain valid if $G=d/dt$ is replaced by $G+\omega$, i.e. λ is replaced by $\lambda+\omega$ on the symbolic level. For the solution (u,ρ) we then obtain the estimate

$$|u|_{L_p(\mathbb{R}_+;X)}+|\rho|_{{}_0Y_0}\le\frac{C|g|_{{}_0Y}}{\omega},\quad\omega>0,$$

with a constant independent of $\omega>0$.

4.4. **Proof of Theorem 2.2 and Corollary 2.3.** In a first step, we set $g_0=0$. Since $W_p^{(s/\kappa_0)(\kappa_0-1/p)}(\Gamma;F)$ coincides with the space π_1Z_ρ we know from Theorem 2.1 that there exists a unique solution $(u,\rho)\in Z_u\times Z_\rho$, so we may assume that $u_0=\rho_0=f=g_j=0$ for $j=1,\dots,m$ but $g_0\in L_p(W_p^s(\mathbb{R}^{n-1};F))$.

Proceeding as in subsection 4.3, this yields $h_0^0\in\mathcal{LF}L_p(J;W_p^{s+m_0}(\mathbb{R}^{n-1};F)$, hence w_0 and σ_0 belong to the same class, by the arguments given in subsection 4.3. Here the notation $\mathcal{LF}$ refers to Laplace-Fourier-transform. But since $s\ge2m\kappa_0$, this implies $u\in L_p(J;H_p^{2m}(\mathbb{R}^n;E))$, and then by the equation for u we obtain

$u \in Z_u$. Similarly, we get

$$\rho \in H_p^1(J; W^s(\mathbb{R}^{n-1}; F)) \bigcap_{j\in\mathcal{J}} W_p^{\kappa_j}(J; H_p^{k_j}(\mathbb{R}^{n-1}; F)) \cap L_p(J; W_p^{l+2m\kappa_0}(\mathbb{R}^{n-1}; F)),$$

by the results of subsection 4.2. This proves the maximal regularity assertion in the semigroup case, i.e. Corollary 2.3.

Finally, in virtue of maximal L_p-regularity, Proposition 1.2 in [8] shows that the operator $-A$ is the generator of an analytic C_0-semigroup in $L_p(\mathbb{R}^n_+; E) \times W_p^s(\mathbb{R}^{n-1}; F)$ where s is defined in Theorem 2.2. □

4.5. **General Domains.** The general case will be proved by the result on the model problem via localization coordinate transform and perturbation. Since this method is well-known and worked in detail in [3] we shall be concise here, indicating only the important steps and arguments.

Firstly, observe that the ellipticty condition (E) as well as the Lopatinskii-Shapiro conditions (LS), (LS^-_∞), and (LS^+_∞) hold uniformly for $t \in J$ and $x \in \overline{G}$ or $x \in \overline{G} \cup \{\infty\}$ in case G is unbounded, and for $x \in \Gamma$, respectively, in the sense that the maximal regularity constants, i.e. the norm of the solution maps for the model problems, are uniform in (t, x). Since maximal regularity is invariant under small perturbations, the coefficients of the model problem a_α, $b_{j\beta}$ and $c_{j\gamma}$ can be perturbed by nonconstant a_α^{small}, $b_{j\beta}^{small}$ and $c_{j\gamma}^{small}$ which are subject to (SD), (SB) and (SC), respectively, and which satisfy in addition $|a_\alpha^{small}|_\infty, |b_{j\beta}^{small}|_\infty, |c_{j\gamma}^{small}|_\infty \le \eta$, where $\eta > 0$ is a small but positive and uniform constant.

Secondly, note that the ellipticity condition (E) as well as the Lopatinskii-Shapiro conditions (LS), (LS^-_∞), and (LS^+_∞) are invariant w.r.t. coordinate transformations. Together with perturbation we thus obtain maximal regularity also for so called bended half spaces which come from transformations of the form $(x, y) \mapsto (x, y + \phi(x))$ where $|\phi|_\infty + |\phi'|_\infty$ is small. Note that due to the assumed smoothness of the boundary $\Gamma \in C^{2m+l-m_0}$ all of the relevant Sobolev spaces are invariant w.r.t. such coordinate transformations, and the compatibility conditions are also preserved.

Now we employ the usual localization procedure. Let $h > 0$ be sufficiently small and divide the time interval J into intervals $J_k = [kh, (k+1)h]$, $k = 0, \dots, N_1$. In virtue of causality, it is enough to consider the problem on each of these intervals, w.l.o.g. we consider only J_0. By arguments similar to those given in subsection 4.1, we may assume w.l.o.g. initial values $u_0 = \rho_0 = 0$. We let L denote the operator defined by the left hand side of (1.1). L is a linear bounded operator from the solution space $Z = {}_0Z_u \times {}_0Z_\rho$ into the data space $Y = X \times \Pi_{j=0}^m {}_0Y_j$. Accordingly we define its principal part $L_\#$.

Next, in view of the compactness of Γ given $r > 0$ cover the boundary Γ of the underlying domain by finitely many balls $U^k := B_r(x_k)$, with $x_k \in \Gamma$, $k = 1, \dots, N_2$, and set $U^0 := \{x \in G : \mathrm{dist}\,(x, \Gamma) > r_0\}$. If $r_0 > 0$ is small enough, then $\{U^k\}_{k=0}^{N_2}$ covers $\overline{G}$. Choose a partition of unity of class C^∞ subordinate to this covering $\{\varphi^k\}_{k=0}^{N_2}$ such that that each φ^k with $k \ge 1$ has compact support. We further choose C^∞-functions ψ^k such that $\mathrm{supp}\,\psi^k \subset U^k$ and $\psi^k = 1$ on $\mathrm{supp}\,\varphi^k$. Then we form local differential operators $\mathcal{A}^k$ by extending its coefficients from U^i to all of $\mathbb{R}^n$ such that $|a_\alpha^i - a_\alpha(0, x_i)|_\infty \le \eta$. This is possible by continuity of a_α provided $r > 0$ is small enough. At the boundary we proceed in a similar way; here we flatten the boundary near x_k by a transformation to local coordinates, extend the transformed coefficients to all of $\mathbb{R}^{n-1}$ and invert the transformation, to obtain local boundary operators $\mathcal{B}_j^k$ and $\mathcal{C}_j^k$ with coefficients subject to (SB) and (SC), and $|b_{j\beta}^k - b_{j\beta}(0,x_k)|_\infty, |c_{j\gamma}^k - c_{j\gamma}(0,x_k)|_\infty \leq \eta$. Then the local problems with operators $\mathcal{A}^k$, $\mathcal{B}_j^k$ and $\mathcal{C}_j^k$ have maximal regularity. We denote by $S^k = (L^k)^{-1}$ the corresponding solution operators.

Now suppose we have a solution (u,ρ) of (1.1). Multiply each equation with the cutoffs φ^k and define $u^k = \varphi^k u$, $\rho^k = \varphi^k \rho$, and similarly $f^k = \varphi^k f$, $g_j^k = \varphi^k g_j$. For $k \geq 1$ this gives the problems on bended half spaces

$$\begin{aligned} \partial_t u^k + \mathcal{A}^k u^k &= f^k - \varphi^k \mathcal{A}^{low} u + [\mathcal{A}_\#, \varphi^k]u, \\ \partial_t \rho^k + \mathcal{B}_0^k u^k + \mathcal{C}_0^k \rho^k &= g_0^k - \varphi^k(\mathcal{B}_0^{low} u + \mathcal{C}_0^{low}\rho) + [\mathcal{B}_{0\#}, \varphi^k]u + [\mathcal{C}_{0\#}, \varphi^k]\rho, \\ \mathcal{B}_j^k u^k + \mathcal{C}_j^k \rho^k &= g_j^k - \varphi^k(\mathcal{B}_j^{low} u + \mathcal{C}_j^{low}\rho) + [\mathcal{B}_{j\#}, \varphi^k]u + [\mathcal{C}_{j\#}, \varphi^k]\rho, \\ u^k(0) = 0, & \qquad \rho^k(0) = 0. \end{aligned} \tag{4.8}$$

Here $\mathcal{A}^{low}$, $\mathcal{B}_j^{low}$ and $\mathcal{C}_j^{low}$ designate lower order terms, and $[\mathcal{A}_\#, \varphi^k]u = \mathcal{A}_\# \varphi^k u - \varphi^k \mathcal{A}_\# u$ means a commutator. Observe that such terms are all of lower order or zero. For $k = 0$ we have the parabolic problem on $\mathbb{R}^n$

$$\partial_t u^0 + \mathcal{A}^0 u^0 = f^0 - \varphi^k \mathcal{A}^{low} u + [\mathcal{A}_\#, \varphi^k]u, \; u^0 = 0.$$

Denote the lower order terms on the right hand sides of (4.8) by T^k resp. T_j^k. Then by maximal regularity we have

$$(u^k, \rho^k) = S^k(f^k - T^k u, g_j^k - T_j^k(u,\rho)), \quad k = 0, \ldots, N_2,$$

here the ρ component for $k = 0$ is void. From this we obtain the following representation of the solution.

$$\begin{aligned} (u,\rho) &= \sum_{k=0}^{N_2} \psi^k (u^k, \rho^k) = \sum_{k=0}^{N_2} \psi^k S^k (f^k - T^k u, g_j^k - T_j^k(u,\rho)) \\ &= \sum_{k=0}^{N_2} \psi^k S^k (f^k, g_j^k) - \sum_{k=0}^{N_2} \psi^k S^k (T^k u, T_j^k(u,\rho)) \\ &= (u^{data}, \rho^{data}) - R(u,\rho) = S^{data}(f, g_j) - R(u,\rho), \end{aligned} \tag{4.9}$$

where (u^{data}, ρ^{data}) belongs to $Z(J_0)$ and is determined by the data f and g_j alone, and R is the remainder, a linear operator bounded from $Z(J_0) := {}_0Z_u(J_0) \times {}_0Z_\rho(J_0)$ into itself. Due to the fact that T^k and T_j^k are of lower order we obtain an estimate of the form

$$|R|_{\mathcal{B}(Z,Z)} \leq MCh^\tau,$$

where M denotes the uniform maximal regularity constant, and C a constant depending on the partition of unity and on the coefficients. Here $\tau > 0$ is determined by the orders $2m$, m_j, k_j and $p \in (1,\infty)$, only. Therefore choosing h small enough we see that $|R|_{\mathcal{B}(Z,Z)} \leq 1/2$, say, hence the solution is unique and satisfies the maximal regularity estimate on J_0. Therefore, L admits a left-inverse which we call $S = (S_u, S_\rho)$. Thus we have the identity

$$S = S^{data} - RS.$$

On the other hand, if data f and g_j are given, we may use (4.9) as a definition of a function $(u,\rho) \in Z$, inverting $I + R$ by a Neumannn series. Applying $L_\#$ to (4.9), we obtain

$$LS = I + \sum_k [L_\#, \psi^k] S^k(\varphi^k - T^k S_u, \varphi^k - T_j^k(S_u, S_\rho)) = I + R_0.$$

This is an equation in the data space $Y = Y(J_0)$, where due to the fact that the commutators $[L_\#, \psi^k]$ are of lower order, the operator R_0 has norm less than $1/2$, say, provided $h > 0$ is small enough. Then another Neumann series shows that L has a right inverse. As operators with left and right inverse are invertible, these arguments prove Theorem 2.1, as well as Corollary 2.3 for finite intervals.

Theorem 2.2 follows by abstract theory; cf. Prüss [8], Proposition 1.2.

Finally, we comment on the autonomous case where $J = \mathbb{R}_+$ is the halfline, i.e. the last assertion of Corollary 2.3. In this case we do not localize in time. Instead we use ω and the estimate $|S_\omega^k|_{\mathcal{B}(Y(\mathbb{R}_+))} \le C/\omega$ of the solution operators S_ω^k in the data space. By means of interpolation this allows to control the lower order terms by C/ω^τ, where $C > 0$ and $\tau > 0$ are uniform. If we choose $\omega > 0$ large enough this way the lower order terms can be made small so that the same arguments as above show invertibility of L also on the halfline.

5. Remarks on the asymptotic Lopatinskii-Shapiro conditions

As mentioned above, the main difficulty in treating the boundary value problem (1.1) lies in the inherent inhomogeneity of the symbol. More precisely, the co-variable λ corresponding to the time variable, has no definite weight compared to the space co-variables ξ.

To analyze boundary value problems with inhomogeneous symbol, the Newton polygon approach was developed, see, e.g., [5] and the references therein. One way of describing the inhomogeneity uses the r-principal part of the symbol where $r > 0$ denotes the weight of λ with respect to ξ. In the homogeneous case, the weight r is given by the symbol, in the inhomogeneous case we have to take any $r > 0$. In the following we will develop the notion of the r-principal part of a Lopatinskii-Shapiro condition. This will allow us to find a unified description of the conditions (LS_∞^-) and (LS_∞^+) formulated above.

We fix $t \in J$ and $x \in \Gamma$ and rewrite the boundary value problem (1.1) in coordinates associated to x. We assume that the operators $\mathcal{B}_j$ and $\mathcal{C}_j$ have no lower-order terms. In matrix form the Lopatinskii-Shapiro condition (LS) can be written as

$$L(\xi', D_y, \lambda) \begin{pmatrix} v(y) \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ g_0 \\ \vdots \\ g_m \end{pmatrix}$$

with

$$L(\xi', \tau, \lambda) := \big(L_{ij}(\xi', \tau, \lambda)\big)_{\substack{i=-1,0,1,\dots,m \\ j=1,2}} := \begin{pmatrix} \lambda + \mathcal{A}(\xi', \tau) & 0 \\ \mathcal{B}_0(\xi', \tau) & \lambda + \mathcal{C}_0(\xi') \\ \mathcal{B}_1(\xi', \tau) & \mathcal{C}_1(\xi') \\ \vdots & \vdots \\ \mathcal{B}_m(\xi', \tau) & \mathcal{C}_m(\xi') \end{pmatrix}.$$

In the matrix $L(\xi', D_y, \lambda)$ the differential operators $\mathcal{B}_j = \mathcal{B}_{j\#}$ have to be understood as boundary operators where taking the trace at $y = 0$ is included. The coefficients of $L(\xi', \tau, \lambda)$ are symbols of pseudo-differential operators with different weights of λ with respect to ξ'. Therefore, there is no natural order of these symbols, and we have to consider their r-order for arbitrary $r > 0$.

For $r > 0$, we define $\operatorname{ord}_r \lambda := r$ and $\operatorname{ord}_r \xi' := 1$. Now we consider the ordinary differential equation

$$\big(\lambda + \mathcal{A}_\#(\xi', D_y)\big)v(y) = 0 \quad (y > 0).$$

Its solutions are determined by the roots $\tau = \tau(\xi', \lambda)$ of the equation $\lambda + \mathcal{A}_\#(\xi', \tau) = 0$. As $\tau(\xi', \lambda)$ is homogeneous in $\mu = (\lambda + |\xi'|^{2m})^{1/2m}$ in the sense that $\tau(\xi', \lambda) = \mu\tau(\frac{\xi'}{\mu}, \frac{\lambda}{\mu^{2m}})$, it makes sense to define

$$\operatorname{ord}_r \tau := \tilde{r} := \max\big\{1, \frac{r}{2m}\big\}.$$

For the symbols of the differential operators appearing in (1.1) we obtain

$$\begin{aligned}
\operatorname{ord}_r \mathcal{A}_\#(\xi', \tau) &= 2m\tilde{r},\\
\operatorname{ord}_r \mathcal{B}_{j\#}(\xi', \tau) &= m_j\tilde{r},\\
\operatorname{ord}_r \mathcal{C}_{j\#}(\xi') &= k_j.
\end{aligned}$$

The r-principal part of a scalar operator with symbol $\mathcal{P}(\xi', \tau, \lambda)$ is defined as

$$\mathcal{P}^{(r)}(\xi', \tau, \lambda) := \lim_{R\to\infty} R^{-\operatorname{ord}_r \mathcal{P}}\, \mathcal{P}(R\xi', R^{\tilde{r}}\tau, R^r\lambda).$$

For the operators in (1.1) we get

$$\begin{aligned}
\big(\lambda + \mathcal{A}(\xi', \tau)\big)^{(r)} &= \begin{cases} \mathcal{A}(\xi', \tau) & \text{if } 0 < r < 2m,\\ \lambda + \mathcal{A}(\xi', \tau) & \text{if } r = 2m,\\ \lambda + \mathcal{A}(0, \tau) & \text{if } r > 2m,\end{cases}\\
\mathcal{B}_j^{(r)}(\xi', \tau) &= \begin{cases} \mathcal{B}_j(\xi', \tau) & \text{if } 0 < r \le 2m,\\ \mathcal{B}_j(0, \tau) & \text{if } r > 2m,\end{cases}\\
\mathcal{C}_j^{(r)}(\xi') &= \mathcal{C}_j(\xi').
\end{aligned}$$

The matrix L is an example of a mixed-order system (Douglis-Nirenberg system). For every $r > 0$ we have

$$\operatorname{ord}_r L_{ij}(\xi', \tau, \lambda) \le s_i(r) + t_j(r)$$

with $(s_{-1}, s_0, \ldots, s_m) = (2m\tilde{r}, m_0\tilde{r}, \ldots, m_m\tilde{r})$ and $(t_1, t_2) = (0, \tilde{t})$. Here

$$\tilde{t} = \tilde{t}(r) := \max\{r - m_0\tilde{r}, k_0 - m_0\tilde{r}, \ldots, k_m - m_m\tilde{r}\}.$$

Following the general idea of mixed-order systems, the r-principal part of L is given by

$$\big(L^{(r)}(\xi', \tau, \lambda)\big)_{ij} := \begin{cases} L_{ij}^{(r)}(\xi', \tau, \lambda) & \text{if } \operatorname{ord}_r L_{ij} = s_i + t_j,\\ 0 & \text{if } \operatorname{ord}_r L_{ij} < s_i + t_j.\end{cases}$$

The asymptotic Lopatinskii-Shapiro condition now means that for *every* $r > 0$ the following r-principal Lopatinskii-Shapiro condition $(\mathrm{LS}_\infty^{(r)})$ is satisfied.

($\mathbf{LS}_\infty^{(r)}$) For all $h_j \in E$, all $h_0 \in F$, $t \in J$, $x \in \Gamma$, all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$ the initial-value problem

$$L^{(r)}(t, x, \xi', D_y, \lambda) \begin{pmatrix} v(y) \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ h_0 \\ \vdots \\ h_m \end{pmatrix}$$

has a unique solution $\begin{pmatrix} v \\ \rho \end{pmatrix} \in C_0(\mathbb{R}_+; E) \times F$.

Remark 5.1. By definition of the r-principal part of the scalar operators and of the mixed-order system L, we have

$$L^{(r)}(\xi', \tau, \lambda) = \lim_{R\to\infty} \begin{pmatrix} R^{-2m\tilde{r}} & & & \\ & R^{-m_0\tilde{r}} & & \\ & & \ddots & \\ & & & R^{-m_m\tilde{r}} \end{pmatrix} L(R\xi', R^{\tilde{r}}\tau, R^r\lambda) \begin{pmatrix} 1 & 0 \\ 0 & R^{-\tilde{t}} \end{pmatrix}$$

with $\tilde{r}$ and $\tilde{t}$ being defined above.

It will turn out that in all cases the validity of ($\mathrm{LS}_\infty^{(r)}$) for every $r > 0$ is equivalent to the asymptotic LS-conditions formulated in Section 2. We start with some elementary observations.

Remark 5.2. a) In Case 1 we have $j \in \mathcal{J}$ if and only if $l_j = l$. This is equivalent to the condition $k_j - m_j = \max_{i=0,\dots,m}(k_i - m_i)$. The points (k_j, κ_j) with $j \in \mathcal{J}$ are lying on the nontrivial edge of $\mathcal{NP}$.

b) In Case 2 again we have $j \in \mathcal{J}$ if and only if $l_j = l$, but all points (k_j, κ_j), $j \in \mathcal{J}$, are lying in the interior of $\mathcal{NP}$.

c) In Case 3 there are two groups of indices in $\mathcal{J}$. The first group consists of all points (k_j, κ_j) lying on the edge $\mathcal{NP}_{q_{\max}}$, this is equivalent to $l_j = l$ and to $k_j - m_j = \max_i(k_i - m_i)$. The second group consists of all j for which the points (k_j, κ_j) are lying on another edge of $\mathcal{NP}$. This is the case if and only if there exists an $r > 2m$ such that

$$k_j - m_j\tilde{r} = \max_{i=0,\dots,m}(k_i - m_i\tilde{r}). \tag{5.1}$$

For every fixed $r \geq 2m$ the set of all $j \in \mathcal{J}$ satisfying (5.1) coincides with one of the index sets $J_1, \dots, J_{2q_{\max}+1}$.

Theorem 5.3. *In the situation of Theorem 2.1, let assumptions* (E), (SD), (SB) *and* (LS) *be satisfied. Then the following statements are equivalent.*

(i) *In case $l < 2m$ condition* (LS_∞^-) *and in case $l > 2m$ condition* (LS_∞^+) *holds.*

(ii) *For every $r > 0$ condition* ($\mathrm{LS}_\infty^{(r)}$) *holds.*

Proof. a) We start with Case 1 where $l = 2m$. For $r < 2m$ we get $\tilde{r} = 1$ and $\tilde{t} = 2m - m_0$. As $k_j - m_j = \tilde{t}$ iff $j \in \mathcal{J}$ and because of $\mathcal{C}_{j\#} = 0$ for $j \notin \mathcal{J}$, we obtain

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \mathcal{A}_\#(\xi', \tau) & 0 \\ \mathcal{B}_{0\#}(\xi', \tau) & \mathcal{C}_{0\#}(\xi') \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(\xi', \tau) & \mathcal{C}_{m\#}(\xi') \end{pmatrix}, \tag{5.2}$$

i.e. condition ($\mathrm{LS}_\infty^{(r)}$) is equivalent to (LS) with $\lambda = 0$. In the same way, for $r = 2m$ we get

$$L^{(2m)}(\xi', \tau, \lambda) = \begin{pmatrix} \lambda + \mathcal{A}_\#(\xi', \tau) & 0 \\ \mathcal{B}_{0\#}(\xi', \tau) & \lambda + \mathcal{C}_{0\#}(\xi') \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(\xi', \tau) & \mathcal{C}_{m\#}(\xi') \end{pmatrix}$$

which equals (LS).

For $r > 2m$ we have $\tilde{r} = \frac{r}{2m}$, $\tilde{t}(r) = r - m_0 \tilde{r} > k_j - m_j \tilde{r}$, $j = 0, \dots, m$, and

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \lambda + \mathcal{A}_\#(0, \tau) & 0 \\ \mathcal{B}_{0\#}(0, \tau) & \lambda \\ \mathcal{B}_{1\#}(0, \tau) & 0 \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(0, \tau) & 0 \end{pmatrix}. \tag{5.3}$$

The corresponding condition is equivalent to (LS) with $\xi' = 0$.

b) In Case 2 we have $l < 2m$, i.e. $2m - m_0 > \max_j (k_j - m_j)$. For $r < 2m$ we get $\tilde{r} = 1$ and $\tilde{t} = \max\{\max_j(k_j - m_j), r - m_0\}$. If $r - m_0 < \max_j(k_j - m_j)$, the asymptotic LS condition is given by (5.2) which again equals (LS) with $\lambda = 0$.

For $r < 2m$ and $r - m_0 = \max_j(k_j - m_j)$ we have

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \mathcal{A}_\#(\xi', \tau) & 0 \\ \mathcal{B}_{0\#}(\xi', \tau) & \lambda + \mathcal{C}_{0\#}(\xi') \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(\xi', \tau) & \mathcal{C}_{m\#}(\xi') \end{pmatrix}$$

which corresponds to the asymptotic (LS_∞^-)-condition (2.4). For $r < 2m$ and $r - m_0 > \max_j(k_j - m_j)$ we get

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \mathcal{A}_\#(\xi', \tau) & 0 \\ \mathcal{B}_{0\#}(\xi', \tau) & \lambda \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(\xi', \tau) & 0 \end{pmatrix}$$

which equals (2.3) with $\lambda = 0$.

For $r = 2m$ we have $r - m_0 = 2m - m_0 > \max_j(k_j - m_j)$ and

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \lambda + \mathcal{A}_\#(\xi', \tau) & 0 \\ \mathcal{B}_{0\#}(\xi', \tau) & \lambda \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(\xi', \tau) & 0 \end{pmatrix}.$$

This coincides with (LS_∞^-). For $r > 2m$ we get (5.3) again.

c) Finally, in Case 3 we have $l > 2m$, i.e. $\max_j(k_j - m_j) > 2m - m_0$. For $r < 2m$ we have $r - m_0 < 2m - m_0 < \max_j(k_j - m_j) = l - m_0$. Hence $\tilde{t}(r) = l - m_0$, and so we obtain

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \mathcal{A}_\#(\xi', \tau) & 0 \\ \mathcal{B}_{0\#}(\xi', \tau) & \mathcal{C}_{0\#}(\xi')\delta_{0, J_{2q_{\max}+1}} \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(\xi', \tau) & \mathcal{C}_{m\#}(\xi')\delta_{m, J_{2q_{\max}+1}} \end{pmatrix}$$

which is the first part of (LS^+_∞), equation (2.5), with $\lambda = 0$.

For $r \geq 2m$ the second column of $L^{(r)}$ contains all $\mathcal{C}_{j\#}$ for which j satisfies (5.1). From Remark 5.2 we see that the sets $J_1, \dots, J_{2q_{\max}+1}$ appear as index sets. For $r = 2m$ we have

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \lambda + \mathcal{A}_\#(\xi', \tau) & 0 \\ \mathcal{B}_{0\#}(\xi', \tau) & \mathcal{C}_{0\#}(\xi')\delta_{0,J_{2q_{\max}+1}} \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(\xi', \tau) & \mathcal{C}_{m\#}(\xi')\delta_{m,J_{2q_{\max}+1}} \end{pmatrix}$$

which is the first part of (LS^+_∞), equation (2.5). In the case $r > 2m$ and $r(1 - \frac{m_0}{2m}) \leq \max_j(k_j - \frac{m_j r}{2m})$ we have

$$L^{(r)}(\xi', \tau, \lambda) = \begin{pmatrix} \lambda + \mathcal{A}_\#(0, \tau) & 0 \\ \mathcal{B}_{0\#}(0, \tau) & \lambda\delta_{-1,J_p} + \mathcal{C}_{0\#}(\xi')\delta_{0,J_q} \\ \vdots & \vdots \\ \mathcal{B}_{m\#}(0, \tau) & \mathcal{C}_{m\#}(\xi')\delta_{m,J_q} \end{pmatrix}$$

where q runs through $\{1, \dots, 2q_{\max}\}$. Therefore, the corresponding condition equals the second part of (LS^+_∞), equation (2.6). For $r(1 - \frac{m_0}{2m}) > \max_j(k_j - \frac{m_j r}{2m})$, which implies $r > 2m$, the corresponding condition coincides with (LS) with $\xi' = 0$. $\square$

In fact, the asymptotic LS conditions are in some sense more important than (LS). This can be seen from the following result.

Theorem 5.4. *In the situation of Theorem 2.1 let assumptions* (E), (SD), (SB) and (SC) *be satisfied. If the asymptotic Lopatinskii-Shapiro condition* $(\mathrm{LS}^{(r)}_\infty)$ *holds for all* $r > 0$ *then there exists a* $\lambda_0 > 0$ *such that* (LS) *is satisfied for all* $\lambda \in \overline{\mathbb{C}}_+$ *with* $|\lambda| \geq \lambda_0$.

Proof. In Case 1 we have seen in the proof of Theorem 5.3 that (LS) and $(\mathrm{LS}^{(r)}_\infty)_{r>0}$ are equivalent conditions. Therefore, we will restrict ourselves to Case 2, the proof in Case 3 follows by the same arguments.

We first fix $(t, x) \in J \times \Gamma$. If there is no $\lambda_0 > 0$ such that (LS) holds for $|\lambda| \geq \lambda_0$, there exist sequences $(\lambda_n)_{n\in\mathbb{N}} \subset \overline{\mathbb{C}}_+$ and $(\xi'_n)_{n\in\mathbb{N}} \subset \mathbb{R}^{n-1}$ with $|\lambda_n| \to \infty$ such that (LS) with $(\lambda, \xi') = (\lambda_n, \xi'_n)$ is violated for all $n \in \mathbb{N}$. We employ the scaling from the proof of Theorem 2.1 and consider the corresponding sequence $b_n = \frac{|\xi'_n|}{\mu_n} \in \overline{B_{1/2}(1/2)}$. By compactness, we may assume $b_n \to b_0 \in \overline{B_{1/2}(1/2)}$.

If $b_0 \neq 1$ we have seen in the proof of Theorem 2.1 that (LS) is equivalent to (4.3) and that the limiting problem for $\lambda \to \infty$ (and, consequently, $\mu \to \infty$) is given by (4.4). By assumption, $(\mathrm{LS}^{(r)}_\infty)_{r>0}$ and therefore (LS^-_∞) holds, so the limiting problem is uniquely solvable. But this implies that (4.3) is uniquely solvable for $(\lambda, \xi') = (\lambda_n, \xi'_n)$ with sufficiently large n which yields a contradiction.

In the same way, for $b_0 = 1$ we obtain either (LS) with $\lambda = 0$ or (LS^-_∞) as limiting problems. But we have seen in the proof of Theorem 5.3 that each of these problems coincides with $(\mathrm{LS}^{(r)}_\infty)$ for suitable $r > 0$. Again the unique solvability of the limiting problems yields a contradiction for sufficiently large n.

Finally, as the coefficients in (LS) depend continuously on $(t, x) \in J \times \Gamma$, a compactness argument shows that λ_0 may be chosen independently of t and x. $\square$

6. Necessity of the ellipticity conditions

In this section, we will prove that the ellipticity condition (E) as well as the Lopatinskii-Shapiro conditions (LS) and $(\mathrm{LS}_\infty^{(r)})$ are necessary. Condition (E), i.e. normal ellipticity of the interior symbol, is known to be necessary from [4], so the essential point is the asymptotic LS condition. The precise formulation reads as follows.

Theorem 6.1. *Let $G \subset \mathbb{R}^n$ be a domain with compact boundary Γ of class C^{2m+l-m_0}. Suppose the Banach spaces E and F are of class $\mathcal{HT}$, and let assumptions* (SD), (SB) *and* (SC) *be satisfied. Let $1 < p < \infty$ be such that $\kappa_j \neq \frac{1}{p}$ $(j = 0, \dots, m)$. For an interval $J_a = [0, a] \subset J$ define the space $X_a := L_p(J_a; L_p(G; E))$, i.e. J in the definition of X is replaced by J_a, and in an analogous way the spaces $Z_{u,a}$, $Z_{\rho,a}$ and $Y_{j,a}$.*

Assume that there exists a constant $C > 0$ such that for all $(u, \rho) \in Z_u \times Z_\rho$ satisfying $u(0, \cdot) = 0$ and $\rho(0, \cdot) = 0$ the inequality

$$\begin{aligned} |u|_{Z_{u,a}} + |\rho|_{Z_{\rho,a}} \leq C\Big(& |\big(\partial_t u + \mathcal{A}(t, x, D)\big)u|_{X_a} \\ & + |\big(\partial_t + \mathcal{C}_0(t, x, D_\Gamma)\big)\rho + \mathcal{B}_0(t, x, D)u|_{Y_{0,a}} \\ & + \sum_{j=1}^{m} |\mathcal{B}_j(t, x, D)u + \mathcal{C}_j(t, x, D_\Gamma)\rho|_{Y_{j,a}}\Big) \end{aligned}$$

holds for every $J_a \subset J$. Then the ellipticity conditions (E) *and* $(\mathrm{LS}_\infty^{(r)})$ *hold for any $r > 0$. Consequently, for sufficiently large $\lambda \in \overline{\mathbb{C}}_+$ condition* (LS) *is satisfied.*

Proof. The last statement follows from Theorem 5.4 and the necessity of (E) was already shown in [4], so we have to prove $(\mathrm{LS}_\infty^{(r)})$. The proof is done in several steps.

(i) Reduction to the model problem. Assume that there exists an $r > 0$, $x_0 \in \Gamma$, $t_0 \in J$ and $\xi_0' \in \mathbb{R}^{n-1} \setminus \{0\}$ such that the ordinary differential equation in $(\mathrm{LS}_\infty^{(r)})$ is not uniquely solvable in $C_0(\mathbb{R}_+; E) \times F$. We write $(\mathrm{LS}_\infty^{(r)})$ in the coordinate system associated to x_0.

We can see in exactly the same way as in [4] that there exists an $a \in (0, T)$ and $\delta > 0$, $B_\delta(a) \subset (0, T)$, with the following property: For all $(u, \rho) \in Z_u \times Z_\rho$ with $\operatorname{supp} u \subset \overline{B_\delta(a)} \times (\overline{B_\delta(x_0)} \cap \overline{\mathbb{R}^n_+})$ and $\operatorname{supp} \rho \subset \overline{B_\delta(a)} \times (\overline{B_\delta(x_0)} \cap \overline{\mathbb{R}^{n-1}})$ the inequality

$$\begin{aligned} & |\partial_t u|_{X_a} + |(-\Delta_{n-1})^m u|_{X_a} + |\rho|_{Z_{\rho,a}} \\ & \leq C\Big(|f|_{X_a} + \sum_{j=0}^{m} |g_j|_{Y_{j,a}} + |u|_{X_a} + |\rho|_{L_p(J_a; L_p(\mathbb{R}^{n-1}; F))}\Big) \end{aligned} \tag{6.1}$$

holds. Here the spaces $X_a, Z_{\rho,a}$ and $Y_{j,a}$ refer to the model problem in the half-space, i.e. $X_a = L_p(J_a; L_p(\mathbb{R}^n_+; E))$ etc., and

$$\begin{aligned} f &:= \big(\partial_t + \mathcal{A}_\#(t_0, x_0, D)\big)u, \\ g_0 &:= \mathcal{B}_{0\#}(t_0, x_0, D)u + \big(\partial_t + \mathcal{C}_{0\#}(t_0, x_0, D')\big)\rho, \\ g_j &:= \mathcal{B}_{j\#}(t_0, x_0, D)u + \mathcal{C}_{j\#}(t_0, x_0, D')\rho \quad (j = 1, \dots, m). \end{aligned}$$

(ii) *Choice of u and ρ.* Consider the operator pencil $T(\lambda)\colon L_p(\mathbb{R}_+;E)\times F\to L_p(\mathbb{R}_+;E)\times F\times E^m$ being defined by $D(T(\lambda))=H_p^{2m}(\mathbb{R}_+;E)\times F$ and

$$T(\lambda)\begin{pmatrix} v\\ \sigma\end{pmatrix}=L^{(r)}(t_0,x_0,\xi_0',D_y,\lambda)\begin{pmatrix} v\\ \sigma\end{pmatrix}.$$

By assumption, the spectrum of T has a nontrivial intersection with $\overline{\mathbb{C}}_+\setminus\{0\}$. From [4], Lemma 5.1, we obtain that there exists a $\lambda_0\in\overline{\mathbb{C}}_+\setminus\{0\}$ such that for every $\eta>0$ there exists $\binom{v_\eta}{\sigma_\eta}\in H_p^{2m}(\mathbb{R}_+;E)\times F$ with

$$\left|\begin{pmatrix} v_\eta\\ \sigma_\eta\end{pmatrix}\right|=1\quad\text{ and }\left|T(\lambda_0)\begin{pmatrix} v_\eta\\ \sigma_\eta\end{pmatrix}\right|<\eta.$$

Following the idea in [4], we fix $\chi\in C^\infty([0,a])$, $\psi\in C^\infty(\mathbb{R}^{n-1})$ and $\varphi\in C^\infty([0,\infty))$ with $0\le\chi,\psi,\varphi\le 1$ and $\operatorname{supp}\chi\subset(a-\delta,a]$, $\operatorname{supp}\psi\subset\overline{B_\delta(x_0)}\cap\mathbb{R}^{n-1}$ and $\operatorname{supp}\varphi\subset[0,\delta)$. Additionally we assume $\chi=0$ in $[a-\delta,a-\frac{2}{3}\delta]$, $\chi=1$ in $[a-\frac{1}{3}\delta,a]$, $\varphi=1$ in $[0,\frac{1}{2}\delta]$, and $\psi=1$ in $\overline{B_{\delta/2}(x_0)}$.

For $R>0$ we define

$$\begin{aligned} u(t,x',y)&:=R^{(-2m+1/p)\tilde r}\chi(t)e^{R^r\lambda_0 t}\psi(x')e^{iR\xi_0'x'}\varphi(y)v_\eta(R^{\tilde r}y),\\ \rho(t,x')&:=R^{(-2m+1/p)\tilde r-\tilde t}\chi(t)e^{R^r\lambda_0 t}\psi(x')e^{iR\xi_0'x'}\sigma_\eta.\end{aligned}$$

Here $\tilde r=\max\{1,\frac{r}{2m}\}$ and $\tilde t=\tilde t(r)$ are defined as in Section 5.

(iii) *Estimate of $|u|_{Z_{u,a}}$.* We start with some remarks. For $k\in\mathbb{N}$ and $j=1,\dots,n-1$ we have

$$\partial_j^k\big(\psi(x')e^{iR\xi_0'x'}\big)=(iR\xi_{0j}')^k\psi(x')e^{iR\xi_0'x'}+O(R^{k-1}),$$

and therefore

$$C_1R^k|\psi|_{L^p(\mathbb{R}^{n-1})}\le|\psi e^{iR\xi_0'x'}|_{H_p^k(\mathbb{R}^{n-1})}\le C_2R^k|\psi|_{L^p(\mathbb{R}^{n-1})}$$

for sufficiently large R with constants $C_{1,2}$ independent of R. From the interpolation inequality we obtain

$$|\psi e^{iR\xi_0'x'}|_{W_p^{k\kappa}(\mathbb{R}^{n-1})}\le CR^{k\kappa}|\psi|_{L^p(\mathbb{R}^{n-1})}$$

for $\kappa\in(0,1)$ and large R. We will show that the last inequality is two-sided, too.

Let $k\kappa = m+s$ with $m\in\mathbb{N}_0$, $s\in(0,1)$. Then

$$\begin{aligned}
&|\psi e^{iR\xi_0' x'}|^p_{W_p^{k\kappa}(\mathbb{R}^{n-1})}\\
&\geq C\sum_{j=1}^{n-1}\int_{B_\delta(0)}\int_{B_\delta(0)}\frac{|\partial_j^m\big(\psi(x'+h')e^{iR\xi_0'(x'+h')}-\psi(x')e^{iR\xi_0'x'}\big)|^p}{|h'|^{sp+n-1}}\,dh'dx'\\
&\geq C\sum_{j=1}^{n-1}\int_{B_{\delta/4}(0)}\int_{B_{\delta/4}(0)}\frac{R^{mp}|\xi_{0j}'|^{mp}|e^{iR\xi_0'h'}-1|^p}{|h'|^{sp+n-1}}\,dh'dx'\\
&\geq CR^{mp}\sum_{j=1}^{n-1}|\xi_{0j}'|^{mp}\int_{B_{\delta/4}(0)}1dx'\int_{B_{\delta/4}(0)}\frac{|e^{iR\xi_0'h'}-1|^p}{|h'|^{sp+n-1}}\,dh'\\
&= CR^{mp+sp}\int_{B_{R\delta/4}(0)}\frac{|e^{i\xi_0'h'}-1|^p}{|h'|^{sp+n-1}}\,dh'\\
&\geq CR^{k\kappa p}\int_{B_1(0)}\frac{|e^{i\xi_0'h'}-1|^p}{|h'|^{sp+n-1}}\,dh'
\end{aligned}$$

for large R. As the last integral is a positive constant depending on ξ_0' but not on R, we obtain

$$\begin{aligned}
C_1R^{k\kappa}|\psi(x')e^{iR\xi_0'x'}|_{L_p(\mathbb{R}^{n-1})} &\leq |\psi(x')e^{iR\xi_0'x'}|_{W_p^{k\kappa}(\mathbb{R}^{n-1})}\\
&\leq C_2R^{k\kappa}|\psi(x')e^{iR\xi_0'x'}|_{L_p(\mathbb{R}^{n-1})}.
\end{aligned}\tag{6.2}$$

In a similar way, one can estimate $|\chi e^{R^r\lambda_0 t}|_{W_p^\kappa([0,a])}$, starting from the obvious inequality

$$|\dot\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}\leq|\dot\chi|_\infty e^{R^r\lambda_0(a-\delta/3)}\leq|\dot\chi|_\infty|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}.$$

We get $|\chi e^{R^r\lambda_0 t}|_{H_p^1([0,a])}\leq CR^r|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}$ for sufficiently large R. By interpolation,

$$|\chi e^{R^r\lambda_0 t}|_{W_p^\kappa([0,a])}\leq CR^{r\kappa}|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}.$$

Using similar scaling arguments as above, we can see that this inequality is two-sided again, i.e. we have

$$C_1R^{r\kappa}|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}\leq|\chi e^{R^r\lambda_0 t}|_{W_p^\kappa([0,a])}\leq C_2R^{r\kappa}|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}\tag{6.3}$$

for $0<\kappa<1$ and sufficiently large R.

Finally, to deal with v_η, we note that $\varphi=1$ near 0 which implies

$$\begin{aligned}
|v_\eta(R^{\tilde r}y)|_{L_p(\mathbb{R}_+;E)} &\geq|\varphi v_\eta(R^{\tilde r}y)|_{L_p(\mathbb{R}_+;E)}\geq\frac12|v_\eta(R^{\tilde r}y)|_{L_p(\mathbb{R}_+;E)}\\
&=\frac12R^{-\tilde r/p}|v_\eta|_{L_p(\mathbb{R}_+;E)}
\end{aligned}\tag{6.4}$$

for large R.

After these remarks, we first estimate $\partial_t u$ and write

$$\begin{aligned}
|\partial_t u|_{X_a} &= R^{(-2m+1/p)\tilde r}|\partial_t(\chi e^{R^r\lambda_0 t})|_{L_p([0,a])}|\psi|_{L_p(\mathbb{R}^{n-1})}|\varphi v_\eta(R^{\tilde r}y)|_{L_p(\mathbb{R}_+;E)}\\
&\geq CR^{r-2m\tilde r}|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}|\psi|_{L_p(\mathbb{R}^{n-1})}|v_\eta|_{L_p(\mathbb{R}_+;E)}.
\end{aligned}$$

In the same way,

$$|u|_{L_p([0,a];H_p^{2m}(\mathbb{R}_+^n;E))} \geq \sum_{j=1}^{n-1} |\partial_j^{2m} u|_{X_a} \geq CR^{2m-2m\tilde r} |\chi e^{R^r\lambda_0 t}|_{L_p([0,a])} |\psi|_{L_p(\mathbb{R}^{n-1})} |v_\eta|_{L_p(\mathbb{R}_+;E)}.$$

To combine these two estimates, we remark that $\max\{r-2m\tilde r, 2m-2m\tilde r\} = 0$ by definition of $\tilde r$. Consequently,

$$|u|_{Z_{u,a}} \geq C|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])} |\psi|_{L_p(\mathbb{R}^{n-1})} |v_\eta|_{L_p(\mathbb{R}_+;E)}. \tag{6.5}$$

(iv) *Estimate of* $|\rho|_{Z_{\rho,a}}$. We consider all three cases simultaneously and write

$$|\rho|_{Z_{\rho,a}} = \sum_{(k,\kappa)} |\rho|_{W_p^\kappa([0,a];W_p^k(\mathbb{R}^{n-1}))}$$

where (k,κ) runs through all vertices of the Newton polygon except $(0,0)$. We get

$$|\rho|_{Z_{\rho,a}} = R^{(-2m+1/p)\tilde r - \tilde t} \sum_{(k,\kappa)} |\chi e^{R^r\lambda_0 t}|_{W_p^\kappa([0,a])} |\psi e^{iR\xi_0' x'}|_{W_p^k(\mathbb{R}^{n-1})} |\sigma_\eta|_F.$$

Using (6.3) and (6.2), we can estimate

$$|\rho|_{Z_{\rho,a}} \geq C|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])} |\psi|_{L_p(\mathbb{R}^{n-1})} |\sigma_\eta|_F \; R^{(-2m+1/p)\tilde r-\tilde t} \sum_{(k,\kappa)} R^{r\kappa+k} \tag{6.6}$$

for sufficiently large R.

We will see that

$$R^{(-2m+1/p)\tilde r-\tilde t} \sum_{(k,\kappa)} R^{r\kappa+k} \geq 1. \tag{6.7}$$

In fact, in all cases there exists a vertex (k,κ) of the Newton polygon with

$$(2m-1/p)\tilde r + \tilde t = r\kappa + k. \tag{6.8}$$

To see this, we first consider $r \leq 2m$, i.e. $\tilde r = 1$. In Case 1 we have $l = 2m$ and $\tilde t = 2m - m_0$, and for the vertex $(k,\kappa) = (l + 2m\kappa_0, 0)$ the equality (6.8) holds because of

$$(2m-1/p)\tilde r + \tilde t = 4m - m_0 - 1/p = l + 2m\kappa_0.$$

In Case 2 we have $l < 2m$. If $r - m_0 \geq \max_{j=0,\dots,m}(k_j - m_j)$ then $\tilde t = r - m_0$. Now we take the vertex $(2m\kappa_0, 1)$ and obtain

$$(2m-1/p)\tilde r + \tilde t = 2m - 1/p + r - m_0 = r + 2m\kappa_0.$$

If $r - m_0 < \max(k_j - m_j)$ then $\tilde t = \max_j(k_j - m_j) = l - m_0$ and we take the vertex $(l + 2m\kappa_0, 0)$. For this the equality

$$(2m-1/p)\tilde r + \tilde t = 2m - 1/p + l - m_0 = l + 2m\kappa_0$$

holds, and (6.8) is satisfied again. In Case 3 we have $\tilde t = l - m_0$, and so (6.8) holds for the vertex $(l + 2m\kappa_0, 0)$.

This shows that for $r \leq 2m$ in all three cases (6.8) holds for at least one vertex of the Newton polygon. Consequently, (6.7) is satisfied. For $r > 2m$ the proof of (6.7) is similar. Inserting (6.7) into (6.6), we see that

$$|\rho|_{Z_{\rho,a}} \geq C|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])} |\psi|_{L_p(\mathbb{R}^{n-1})} |\sigma_\eta|_F.$$

From this and (6.5) one obtains that there exist constants $R_0 > 0$ and $C_0 > 0$ such that for all $R \geq R_0$ the inequality

$$\begin{aligned}\Big|\begin{pmatrix}u\\ \rho\end{pmatrix}\Big|_{Z_{u,a}\times Z_{\rho,a}} &\geq C_0|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}|\psi|_{L_p(\mathbb{R}^{n-1})}\Big|\begin{pmatrix}v_\eta\\ \sigma_\eta\end{pmatrix}\Big|_{L_p(\mathbb{R}_+;E)\times F}\\ &= C_0|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])}|\psi|_{L_p(\mathbb{R}^{n-1})}\end{aligned} \tag{6.9}$$

holds.

(v) *Estimate of the right-hand side in* (6.1). By the product rule,

$$\begin{aligned}\begin{pmatrix}f\\ g_0\\ \vdots\\ g_m\end{pmatrix} &= L(t_0,x_0,D',D_y,\partial_t)\begin{pmatrix}u\\ \rho\end{pmatrix}\\ &= R^{(-2m+1/p)\tilde r}\chi(t)e^{R^r\lambda_0 t}\psi(x')e^{iR\xi_0'x'}\\ &\qquad L(t_0,x_0,R\xi_0',R^{\tilde r}D_y,R^r\lambda_0)\begin{pmatrix}1 & 0\\ 0 & R^{-\tilde t}\end{pmatrix}\begin{pmatrix}v_\eta(R^{\tilde r}y)\\ \sigma_\eta\end{pmatrix}+O\Big(\frac{1}{R}\Big)\end{aligned}$$

for $R \to \infty$. We have to estimate

$$|g_j|_{Y_{j,a}} = |g_j|_{W_p^{\kappa_j}([0,a];L_p(\mathbb{R}^{n-1};E))} + |g_j|_{L_p([0,a];W_p^{2m\kappa_j}(\mathbb{R}^{n-1};E))}.$$

For this we use the right inequalities in (6.3) and (6.2) and get

$$\begin{aligned}|\chi e^{R^r\lambda_0 t}|_{W_p^{\kappa_j}([0,a])} &\leq CR^{\kappa_j r}|\chi e^{R^r\lambda_0 t}|_{L_p([0,a])},\\ |\psi e^{iR\xi_0'x'}|_{W_p^{2m\kappa_j}(\mathbb{R}^{n-1})} &\leq CR^{2m\kappa_j}|\psi|_{L_p(\mathbb{R}^{n-1})}.\end{aligned}$$

Note that $\max\{\kappa_j r, 2m\kappa_j\} = 2m\tilde r\kappa_j$. Therefore,

$$\left|\begin{pmatrix}f\\ g_0\\ \vdots\\ g_m\end{pmatrix}\right|_{X_a\times Y_{0,a}\times\cdots\times Y_{m,a}} \leq C|\chi(t)e^{R^r\lambda_0 t}|_{L_p([0,a])}|\psi|_{L_p(\mathbb{R}^{n-1})}|N_\eta|_{L_p(\mathbb{R}_+;E)\times F}$$

with

$$\begin{aligned}N_\eta := R^{(-2m+1/p)\tilde r}&\begin{pmatrix}1 & & & \\ & R^{2m\tilde r\kappa_0} & & \\ & & \ddots & \\ & & & R^{2m\tilde r\kappa_m}\end{pmatrix}\\ &\cdot L(t_0,x_0,R\xi_0',R^{\tilde r}D_y,R^r\lambda_0)\begin{pmatrix}1 & 0\\ 0 & R^{-\tilde t}\end{pmatrix}\begin{pmatrix}v_\eta(R^{\tilde r}y)\\ \sigma_\eta\end{pmatrix}\end{aligned}$$

We remark that $(-2m+1/p)\tilde r + 2m\tilde r\kappa_j = -m_j\tilde r$ and that

$$|[L(\ldots,R^{\tilde r}D_y,R^r\lambda_0)v_\eta](R^{\tilde r}y)|_{L_p(\mathbb{R}_+;E)} = R^{-\tilde r/p}|L(\ldots,R^{\tilde r}D_y,R^r\lambda_0)v_\eta|_{L_p(\mathbb{R}_+)}.$$

Thus we can apply Remark 5.1 to see that the operator acting on $\binom{v_\eta(R^{\tilde r}y)}{\sigma_\eta}$ is a differential operator in y with coefficients which tend uniformly to the coefficients of

$L^{(r)}(t_0, x_0, \xi_0', D_y, \lambda_0)$ as $R \to \infty$. Consequently, there exists an $R_1 > 0$ depending on η such that

$$|N_\eta|_{L_p(\mathbb{R}_+;E)\times F} \leq \eta + \left| L^{(r)}(t_0, x_0, \xi_0', D_y, \lambda_0) \begin{pmatrix} v_\eta \\ \sigma_\eta \end{pmatrix} \right| \leq 2\eta$$

for all $R \geq R_1$. Altogether, we have seen that there exists a constant $C_1 > 0$ and for every $\eta > 0$ a constant $R_1 > 0$ such that

$$\left| \begin{pmatrix} f \\ g_0 \\ \vdots \\ g_m \end{pmatrix} \right|_{X_a \times Y_{0,a} \times \cdots \times Y_{m,a}} \leq C_1 \eta |\chi(t) e^{R^r \lambda_0 t}|_{L_p([0,a])} |\psi|_{L_p(\mathbb{R}^{n-1})} \tag{6.10}$$

holds for all $R \geq R_1$. But from (6.9) and (6.10) with sufficiently small η we obtain a contradiction to (6.1). □

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