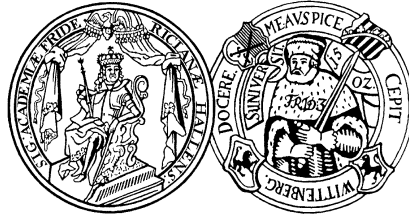


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Report No. 09 (2008)

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# ON CONVERGENCE OF SOLUTIONS TO EQUILIBRIA FOR QUASILINEAR PARABOLIC PROBLEMS

JAN PRÜSS, GIERI SIMONETT, AND RICO ZACHER

ABSTRACT. We show convergence of solutions to equilibria for quasilinear parabolic evolution equations in situations where the set of equilibria is non-discrete, but forms a finite-dimensional  $C^1$ -manifold which is normally hyperbolic. Our results do not depend on the presence of an appropriate Lyapunov functional as in the Lojasiewicz-Simon approach, but are of local nature.

**Keywords:** quasilinear parabolic equations, normally stable, center manifolds, nonlinear boundary conditions, free boundary problems, travelling waves.

**AMS subject classification:** 34G20, 35K55, 35B35, 37D10, 35R35.

## 1. INTRODUCTION

The principle of linearized stability is a well-known and powerful tool for proving stability or instability of equilibria of nonlinear evolution equations. It is known to be true for large classes of nonlinear evolution equations, even for such which are nonlocal. The literature on this subject is large. Since here we are mainly interested in quasilinear parabolic problems, we only refer to the monograph by Lunardi [21], and to [1, 23].

Less known - at least in the PDE-community - seems to be the following fact. Suppose that for a nonlinear evolution equation we have a  $C^1$ -manifold of equilibria  $\mathcal{E}$  such that at a point  $u_* \in \mathcal{E}$ , the kernel  $N(A_0)$  of the linearization  $A_0$  is isomorphic to the tangent space of  $\mathcal{E}$  at  $u_*$ , the eigenvalue 0 of  $A_0$  is semi-simple, and the remaining spectral part of the linearization  $A_0$  is stable. Then solutions starting nearby  $u_*$  exist globally and converge to some point on  $\mathcal{E}$ . This situation occurs frequently in applications. We call it the *generalized principle of linearized stability*, and the equilibrium  $u_*$  is then termed *normally stable*.

A typical example for this situation to occur is the case where the equations under consideration involve symmetries, i.e. are invariant under the action of a Lie-group  $\mathcal{G}$ . If then  $u_*$  is an equilibrium, the manifold  $\mathcal{E}$  includes the action of  $\mathcal{G}$  on  $u_*$  and the manifold  $\mathcal{G}u_*$  is a subset of  $\mathcal{E}$ .

A standard method to handle situations as described above is to refer to *center manifold theory*. In fact in that situation the center manifold of the problem

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in question will be unique, and it coincides with  $\mathcal{E}$  near  $u_*$ . Thus the so-called *shadowing lemma* in center manifold theory implies the result. Center manifolds are well-studied objects in the theory of nonlinear evolution equations. For the parabolic case we refer to the monographs [18, 21], and to the publications [6, 7, 10, 19, 20, 22, 26, 27].

However, the theory of center manifolds in the infinite-dimensional case is a technically difficult matter. It usually involves higher regularity of the involved nonlinearities - in particular concerning the shadowing property - and unfortunately center manifolds need not be unique, not even in the ODE case. Therefore it seems desirable to have a simpler, direct approach to the generalized principle of linearized stability which avoids the technicalities of center manifold theory.

The purpose of this paper is to present such an approach. It turns out that the effort is only slightly larger than that for the proof of the standard linearized stability result - which is simple. We will illustrate by several examples that our result is applicable to a variety of interesting problems in different areas of applied analysis. It is our belief that the approach devised in this manuscript will be fruitful for the stability analysis of equilibria for parabolic evolution equations that involve symmetries in the way described above.

Here we would also like to mention the work in [9], where the action of a Lie group has been used for the stability analysis of equilibrium solutions. However, the approach given here is considerably more general and flexible.

In Section 2 we formulate and prove our main result for abstract autonomous quasilinear parabolic problems. Theorem 2.1 implies, for instance, the main result in [15] on convergence of solutions for the Mullins-Sekerka problem. We also show by means of examples that the conditions of Theorem 2.1 are necessary in order to have convergence to a single equilibrium.

In Section 3, we consider quasilinear parabolic systems with nonlinear boundary conditions and we show that our techniques can also be applied to this situation. Sections 4 and 5 illustrate the scope of our main result, as we show convergence towards equilibria for the Mullins-Sekerka model, and stability of travelling waves for a quasilinear parabolic equation.

In Section 6 we consider the so-called *normally hyperbolic* case, where the remaining part of the spectrum of  $A_0$  also contains an unstable part away from the imaginary axis. In this situation, one cannot expect convergence of all solutions starting near  $u_*$ , but only for those initial values which are on the stable manifold.

Our approach makes use of maximal  $L_p$ -regularity in an essential way to cover the quasilinear case. As general references for this theory we refer to the recent publications [11, 12], to the survey article [23], and also to [2, 3, 4, 8, 21].

In a forthcoming paper these results are extended to the case where the boundary conditions are of relaxation type, i.e. are coupled with an evolution equation on the boundary, as in [13]. Problems of the last kind are important e.g. for the Stefan problem with surface tension, see [14, 25], and for the two-phase Navier-Stokes problem with a free boundary.

Finally, we should like to point out that the generalized principle of linearized stability described in the current paper can also be adapted and applied to other situations where maximal regularity holds, for instance in the case of continuous maximal regularity.

## 2. CONVERGENCE FOR ABSTRACT QUASILINEAR PROBLEMS

Let  $X_0$  and  $X_1$  be two Banach spaces such that  $X_1 \hookrightarrow X_0$ , i.e.  $X_1$  is continuously and densely embedded in  $X_0$ . In this section we consider the autonomous quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \quad u(0) = u_0. \quad (2.1)$$

For  $1 < p < \infty$  we introduce the real interpolation space  $X_\gamma := (X_0, X_1)_{1-1/p, p}$  and we assume that there is an open set  $V \subset X_\gamma$  such that

$$(A, F) \in C^1(V, \mathcal{B}(X_1, X_0) \times X_0). \quad (2.2)$$

Here  $\mathcal{B}(X_1, X_0)$  denotes the space of all bounded linear operators from  $X_1$  into  $X_0$ . In the sequel we use the notation  $|\cdot|_j$  to denote the norm in the respective spaces  $X_j$  for  $j = 0, 1, \gamma$ . Moreover, for any normed space  $X$ ,  $B_X(u, r)$  denotes the open ball in  $X$  with radius  $r > 0$  around  $u \in X$ .

Let  $\mathcal{E} \subset V \cap X_1$  denote the set of equilibrium solutions of (2.1), which means that

$$u \in \mathcal{E} \quad \text{if and only if} \quad u \in V \cap X_1, \quad A(u)u = F(u).$$

Given an element  $u_* \in \mathcal{E}$ , we assume that  $u_*$  is contained in an  $m$ -dimensional manifold of equilibria. This means that there is an open subset  $U \subset \mathbb{R}^m$ ,  $0 \in U$ , and a  $C^1$ -function  $\Psi : U \rightarrow X_1$ , such that

- $\Psi(U) \subset \mathcal{E}$  and  $\Psi(0) = u_*$ ,
  - the rank of  $\Psi'(0)$  equals  $m$ , and
  - $A(\Psi(\zeta))\Psi(\zeta) = F(\Psi(\zeta)), \quad \zeta \in U$ .
- (2.3)

We assume further that near  $u_*$  there are no other equilibria than those given by  $\Psi(U)$ , i.e.  $\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U)$ , for some  $r_1 > 0$ .

We suppose that the operator  $A(u_*)$  has the property of maximal  $L_p$ -regularity. Introducing the deviation  $v = u - u_*$  from the equilibrium  $u_*$ , the equation for  $v$  then reads as

$$\dot{v}(t) + A_0 v(t) = G(v(t)), \quad t > 0, \quad v(0) = v_0, \quad (2.4)$$

where  $v_0 = u_0 - u_*$  and

$$A_0 v = A(u_*)v + (A'(u_*)v)u_* - F'(u_*)v \quad \text{for } v \in X_1. \quad (2.5)$$

The function  $G$  can be written as  $G(v) = G_1(v) + G_2(v, v)$ , where

$$\begin{aligned} G_1(v) &= (F(u_* + v) - F(u_*) - F'(u_*)v) - (A(u_* + v) - A(u_*) - A'(u_*)v)u_*, \\ G_2(v, w) &= -(A(u_* + v) - A(u_*))w, \quad w \in X_1, v \in V_*, \end{aligned}$$

where  $V_* := V - u_*$ . It follows from (2.2) that  $G_1 \in C^1(V_*, X_0)$  and also that  $G_2 \in C^1(V_* \times X_1, X_0)$ . Moreover, we have

$$G_1(0) = G_2(0, 0) = 0, \quad G'_1(0) = G'_2(0, 0) = 0, \quad (2.6)$$

where  $G'_1$  and  $G'_2$  denote the Fréchet derivatives of  $G_1$  and  $G_2$ , respectively.

Setting  $\psi(\zeta) = \Psi(\zeta) - u_*$  results in the following equilibrium equation for problem (2.4)

$$A_0\psi(\zeta) = G(\psi(\zeta)), \quad \text{for all } \zeta \in U. \quad (2.7)$$

Taking the derivative with respect to  $\zeta$  and using the fact that  $G'(0) = 0$  we conclude that  $A_0\psi'(0) = 0$  and this implies that

$$T_{u_*}(\mathcal{E}) \subset N(A_0), \quad (2.8)$$

where  $T_{u_*}(\mathcal{E})$  denotes the tangent space of  $\mathcal{E}$  at  $u_*$ .

After these preparations we can state the following result on convergence of solutions starting near  $u_*$ .

**Theorem 2.1.** *Let  $1 < p < \infty$ . Suppose  $u_* \in V \cap X_1$  is an equilibrium of (2.1), and suppose that the functions  $(A, F)$  satisfy (2.2). Suppose further that  $A(u_*)$  has the property of maximal  $L_p$ -regularity. Let  $A_0$ , defined in (2.5), denote the linearization of (2.1) at  $u_*$ . Suppose that  $u_*$  is normally stable, i.e. assume that*

- (i) *near  $u_*$  the set of equilibria  $\mathcal{E}$  is a  $C^1$ -manifold in  $X_1$  of dimension  $m \in \mathbb{N}$ ,*
- (ii) *the tangent space for  $\mathcal{E}$  at  $u_*$  is given by  $N(A_0)$ ,*
- (iii) *0 is a semi-simple eigenvalue of  $A_0$ , i.e.  $N(A_0) \oplus R(A_0) = X_0$ ,*
- (iv)  *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ .*

*Then  $u_*$  is stable in  $X_\gamma$ , and there exists  $\delta > 0$  such that the unique solution  $u(t)$  of (2.1) with initial value  $u_0 \in X_\gamma$  satisfying  $|u_0 - u_*|_\gamma < \delta$  exists on  $\mathbb{R}_+$  and converges at an exponential rate in  $X_\gamma$  to some  $u_\infty \in \mathcal{E}$  as  $t \rightarrow \infty$ .*

*Proof.* (a) Note first that assumption (iii) implies that 0 is an isolated spectral point of  $\sigma(A_0)$ , the spectrum of  $A_0$ . According to assumption (iv)  $\sigma(A_0)$  admits a decomposition into two disjoint nontrivial parts with

$$\sigma(A_0) = \{0\} \cup \sigma_s, \quad \sigma_s \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

The spectral set  $\sigma_c := \{0\}$  corresponds to the center part, and  $\sigma_s$  to the stable part of the analytic  $C_0$ -semigroup  $e^{-A_0 t}$ , or equivalently of the Cauchy problem  $\dot{w} + A_0 w = f$ .

In the following, we let  $P^l$ ,  $l \in \{c, s\}$ , denote the spectral projections according to the spectral sets  $\sigma_c = \{0\}$  and  $\sigma_s$ , and we set  $X_j^l := P^l X_j$  for  $l \in \{c, s\}$  and  $j \in \{0, 1, \gamma\}$ . The spaces  $X_j^l$  are equipped with the norms  $|\cdot|_j$  for  $j = 0, 1, \gamma$ . We have the topological direct decomposition

$$X_1 = X_1^c \oplus X_1^s, \quad X_0 = X_0^c \oplus X_0^s,$$

and this decomposition reduces  $A_0$  into  $A_0 = A_c \oplus A_s$ , where  $A_l$  is the part of  $A_0$  in  $X_0^l$  for  $l \in \{c, s\}$ . Since  $\sigma_c = \{0\}$  is compact it follows that  $X_0^c \subset X_1$ .

Therefore,  $X_0^c$  and  $X_1^c$  coincide as vector spaces. In the following, we will just write  $X^c = (X^c, |\cdot|_j)$  for either of the spaces  $X_0^c$  and  $X_1^c$ . We note that  $N(A_0)$ , the kernel of  $A_0$ , is contained in  $X^c$ . The operator  $A_s$  inherits the property of  $L_p$ -maximal regularity from  $A_0$ . Since  $\sigma(A_s) = \sigma_s \subset \mathbb{C}_+$  we obtain that the Cauchy problem

$$\dot{w} + A_s w = f, \quad w(0) = 0, \quad (2.9)$$

also enjoys the property of maximal regularity, even on the interval  $J = (0, \infty)$ . In fact the following estimates are true. For any  $a \in (0, \infty]$  let

$$\mathbb{E}_0(a) = L_p((0, a); X_0), \quad \mathbb{E}_1(a) = H_p^1((0, a); X_0) \cap L_p((0, a); X_1). \quad (2.10)$$

The natural norms in  $\mathbb{E}_j(a)$  will be denoted by  $\|\cdot\|_{\mathbb{E}_j(a)}$  for  $j = 0, 1$ . Then the Cauchy problem (2.9) has for each  $f \in L_p((0, a); X_0^s)$  a unique solution

$$w \in H_p^1((0, a); X_0^s) \cap L_p((0, a); X_1^s),$$

and there exists a constant  $M_0$  such that  $\|w\|_{\mathbb{E}_1(a)} \leq M_0 \|f\|_{\mathbb{E}_0(a)}$  for every  $a > 0$ , and every function  $f \in L_p((0, a); X_0^s)$ . In fact, since  $\sigma(A_s - \omega)$  is still contained in  $\mathbb{C}_+$  for  $\omega$  small enough, we see that the operator  $A_s - \omega$  enjoys the same properties as  $A_s$ . Therefore, every solution of the Cauchy problem (2.9) satisfies the estimate

$$\|e^{\sigma t} w\|_{\mathbb{E}_1(a)} \leq M_0 \|e^{\sigma t} f\|_{\mathbb{E}_0(a)}, \quad \sigma \in [0, \omega], \quad a > 0, \quad (2.11)$$

for  $f \in L_p((0, a); X_0^s)$ , where  $M_0 = M_0(\omega)$  for  $\omega > 0$  fixed; cf. [23, Sec. 6]. Furthermore, there exists a constant  $M_1 > 0$  such that

$$\|e^{\omega t} e^{-A_s t} P^s u\|_{\mathbb{E}_1(a)} + \sup_{t \in [0, a]} |e^{\omega t} e^{-A_s t} P^s u|_\gamma \leq M_1 |P^s u|_\gamma \quad (2.12)$$

for every  $u \in X_\gamma$  and  $a \in (0, \infty]$ . For future use we note that

$$\sup_{t \in [0, a]} |w(t)|_\gamma \leq c_0 \|w\|_{\mathbb{E}_1(a)} \quad \text{for all } w \in \mathbb{E}_1(a) \text{ with } w(0) = 0 \quad (2.13)$$

with a constant  $c_0$  that is independent of  $a \in (0, \infty]$ , see for instance the proof of [24, Proposition 6.2]. We remind that  $N(A_0)$  is contained in  $X^c$ .

(b) It follows from the considerations above and assumptions (i)-(iii) that in fact

$$N(A_0) = X^c \quad \text{and} \quad \dim(X^c) = m.$$

As  $X^c$  has finite dimension, the norms  $|\cdot|_j$  for  $j = 0, 1, \gamma$  are equivalent, and we equip  $X^c$  with one of these equivalent norms, say with  $|\cdot|_0$ . Let us now consider the mapping

$$g : U \subset \mathbb{R}^m \rightarrow X^c, \quad g(\zeta) := P^c \psi(\zeta), \quad \zeta \in U.$$

It follows from our assumptions that  $g'(0) = P^c \psi'(0) : \mathbb{R}^m \rightarrow X^c$  is an isomorphism (between the finite dimensional spaces  $\mathbb{R}^m$  and  $X^c$ ). By the inverse function theorem,  $g$  is a  $C^1$ -diffeomorphism of a neighborhood of 0 in  $\mathbb{R}^m$  into a neighborhood, say  $B_{X^c}(0, \rho_0)$ , of 0 in  $X^c$ . Let  $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$  be the inverse mapping. Then  $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$  is  $C^1$  and  $g^{-1}(0) = 0$ . Next we set  $\Phi(x) := \psi(g^{-1}(x))$  for  $x \in B_{X^c}(0, \rho_0)$  and we note that

$$\Phi \in C^1(B_{X^c}(0, \rho_0), X_1^s), \quad \Phi(0) = 0, \quad \{\Phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W,$$



where  $W$  is an appropriate neighborhood of  $u_*$  in  $X_1$ . One readily verifies that

$$P^c\Phi(x) = ((P^c \circ \psi) \circ g^{-1})(x) = (g \circ g^{-1})(x) = x, \quad x \in B_{X^c}(0, \rho_0),$$

and this yields  $\Phi(x) = P^c\Phi(x) + P^s\Phi(x) = x + P^s\Phi(x)$  for  $x \in B_{X^c}(0, \rho_0)$ . Setting  $\phi(x) := P^s\Phi(x)$  we conclude that

$$\phi \in C^1(B_{X^c}(0, \rho_0), X_1^s), \quad \phi(0) = \phi'(0) = 0, \quad (2.14)$$

and that

$$\{x + \phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W,$$

where  $W$  is a neighborhood of  $u_*$  in  $X_1$ . This shows that the manifold  $\mathcal{E}$  can be represented as the (translated) graph of the function  $\phi$  in a neighborhood of  $u_*$ . Moreover, the tangent space of  $\mathcal{E}$  at  $u_*$  coincides with  $N(A_0) = X^c$ . By applying the projections  $P^l$ ,  $l \in \{c, s\}$ , to equation (2.7) and using that  $x + \phi(x) = \psi(g^{-1}(x))$  for  $x \in B_{X^c}(0, \rho_0)$ , and that  $A_c \equiv 0$ , we obtain the following equivalent system of equations for the equilibria of (2.4)

$$P^cG(x + \phi(x)) = 0, \quad P^sG(x + \phi(x)) = A_s\phi(x), \quad x \in B_{X^c}(0, \rho_0). \quad (2.15)$$

Finally, let us also agree that  $\rho_0$  has already been chosen small enough so that

$$|\phi'(x)|_{\mathcal{B}(X^c, X_1^s)} \leq 1, \quad |\phi(x)|_1 \leq |x|, \quad x \in B_{X^c}(0, \rho_0). \quad (2.16)$$

This can always be achieved, thanks to (2.14).

(c) Introducing the new variables

$$\begin{aligned} x &= P^c v = P^c(u - u_*), \\ y &= P^s v - \phi(P^c v) = P^s(u - u_*) - \phi(P^c(u - u_*)) \end{aligned}$$

we then obtain the following system of evolution equations in  $X^c \times X_0^s$

$$\begin{cases} \dot{x} = T(x, y), & x(0) = x_0, \\ \dot{y} + A_s y = R(x, y), & y(0) = y_0, \end{cases} \quad (2.17)$$

with  $x_0 = P^c v_0$  and  $y_0 = P^s v_0 - \phi(P^c v_0)$ , where the functions  $T$  and  $R$  are given by

$$\begin{aligned} T(x, y) &= P^cG(x + \phi(x) + y), \\ R(x, y) &= P^sG(x + \phi(x) + y) - A_s\phi(x) - \phi'(x)T(x, y). \end{aligned}$$

Using the equilibrium equations (2.15), the expressions for  $R$  and  $T$  can be rewritten as

$$\begin{aligned} T(x, y) &= P^c(G(x + \phi(x) + y) - G(x + \phi(x))), \\ R(x, y) &= P^s(G(x + \phi(x) + y) - G(x + \phi(x))) - \phi'(x)T(x, y). \end{aligned} \quad (2.18)$$

Although the term  $P^cG(x + \phi(x))$  in  $T$  is zero, see (2.15), we include it here for reasons of symmetry, and for justifying the estimates for  $T$  below. Equation (2.18) immediately yields

$$T(x, 0) = R(x, 0) = 0 \quad \text{for all } x \in B_{X^c}(0, \rho_0),$$

showing that the equilibrium set  $\mathcal{E}$  of (2.1) near  $u_*$  has been reduced to the set  $B_{X^c}(0, \rho_0) \times \{0\} \subset X^c \times X_1^s$ .

Observe also that there is a unique correspondence between the solutions of (2.1) close to  $u_*$  in  $X_\gamma$  and those of (2.17) close to 0. We call system (2.17) the *normal form* of (2.1) near its normally stable equilibrium  $u_*$ .

(d) From the representation of  $G$  and (2.6) we obtain the following estimates for  $G_1$  and  $G_2$ : for given  $\eta > 0$  we may choose  $r = r(\eta) > 0$  small enough such that

$$|G_1(v_1) - G_1(v_2)|_0 \leq \eta |v_1 - v_2|_\gamma, \quad v_1, v_2 \in B_{X_\gamma}(0, r).$$

Moreover, there is a constant  $L > 0$  such that

$$\begin{aligned} |G_2(v_1, w) - G_2(v_2, w)|_0 &\leq L |w|_1 |v_1 - v_2|_\gamma, \quad w \in X_1, \quad v_1, v_2 \in B_{X_\gamma}(0, r), \\ |G_2(v, w_1) - G_2(v, w_2)|_0 &\leq L r |w_1 - w_2|_1, \quad w_1, w_2 \in X_1, \quad v \in B_{X_\gamma}(0, r). \end{aligned}$$

We remark that  $L$  does not depend on  $r \in (0, r_0]$  with  $r_0$  appropriately chosen. Combining these estimates we have

$$\begin{aligned} |G(v_1) - G(v_2)|_0 &\leq (\eta + L |v_2|_1) |v_1 - v_2|_\gamma + L r |v_1 - v_2|_1 \\ &\leq C_0 (\eta + r + |v_2|_1) |v_1 - v_2|_1 \end{aligned} \quad (2.19)$$

for all  $v_1, v_2 \in B_{X_\gamma}(0, r) \cap X_1$ , where  $C_0$  is independent of  $r \in (0, r_0]$ .

In the following, we will always assume that  $r \in (0, r_0]$  and  $r_0 \leq 3\rho_0$ . Taking  $v_1 = x + \phi(x) + y$  and  $v_2 = x + \phi(x)$  in (2.19) we infer from (2.16) and (2.18) that

$$|T(x, y)|, |R(x, y)|_0 \leq C_1 (\eta + r + |x + \phi(x)|_1) |y|_1 \leq C_2 (\eta + r) |y|_1, \quad (2.20)$$

for all  $x \in \bar{B}_{X^c}(0, \rho)$ ,  $y \in \bar{B}_{X_1^s}(0, \rho) \cap X_1$  and all  $\rho \in (0, r/3)$ , where  $C_1$  and  $C_2$  are uniform constants. Suppose that  $\eta$  and, accordingly,  $r$  were already chosen small enough so that

$$\beta := M_0 C_2 (\eta + r) \leq 1/2. \quad (2.21)$$

(e) By [23, Theorem 3.1], problem (2.4) admits for each  $v_0 \in B_{X_\gamma}(0, r)$  a unique local strong solution

$$v \in \mathbb{E}_1(a) \cap C([0, a]; X_\gamma) \quad (2.22)$$

for some number  $a > 0$ . This solution can be extended to a maximal interval of existence  $[0, t_*)$ . If  $t_*$  is finite, then either  $v(t)$  leaves the ball  $B_{X_\gamma}(0, r)$  at time  $t_*$ , or the limit  $\lim_{t \rightarrow t_*} v(t)$  does not exist in  $X_\gamma$ . We show that this cannot happen for initial values  $v_0 \in B_{X_\gamma}(0, \delta)$ , with  $\delta \leq r$  to be chosen later.

Suppose that  $x_0 \in B_{X^c}(0, N\delta)$  and  $y_0 \in B_{X_1^s}(0, N\delta)$  are given, where the number  $\delta$  will be determined later and  $N := \|P^c\|_{\mathcal{B}(X_0)} + \|P^s\|_{\mathcal{B}(X_\gamma)}$ . Let  $t_*$  denote the existence time for the solution  $(x(t), y(t))$  of system (2.17) with initial values  $(x_0, y_0)$ , or equivalently, for the solution  $v(t)$  of (2.4) with initial value  $v_0 = x_0 + \phi(x_0) + y_0$ . Let  $\rho$  be fixed so that the estimates in (2.20) hold. Set

$$t_1 := t_1(x_0, y_0) := \sup\{t \in (0, t_*) : |x(\tau)|, |y(\tau)|_\gamma \leq \rho, \tau \in [0, t]\}$$

and suppose that  $t_1 < t_*$ . Due to (2.11)–(2.12) and (2.20) we obtain

$$\begin{aligned} \|e^{\omega t}y\|_{\mathbb{E}_1(t_1)} &\leq M_1|y_0|_\gamma + M_0\|e^{\omega t}R(x, y)\|_{\mathbb{E}_0(t_1)} \\ &\leq M_1|y_0|_\gamma + M_0C_2(\eta + r)\|e^{\omega t}y\|_{\mathbb{E}_1(t_1)}. \end{aligned}$$

This yields with (2.21)

$$\|e^{\sigma t}y\|_{\mathbb{E}_1(t_1)} \leq 2M_1|y_0|_\gamma, \quad \sigma \in [0, \omega]. \quad (2.23)$$

Using this estimate as well as (2.12)–(2.13) we further have for  $t \in [0, t_1]$

$$\begin{aligned} |e^{\sigma t}y(t)|_\gamma &\leq |e^{\sigma t}y(t) - e^{\sigma t}e^{-A_s t}y_0|_\gamma + |e^{\sigma t}e^{-A_s t}y_0|_\gamma \\ &\leq c_0\|e^{\sigma t}y(t) - e^{\sigma t}e^{-A_s t}y_0\|_{\mathbb{E}_1(t_1)} + M_1|y_0|_\gamma \\ &\leq (3c_0M_1 + M_1)|y_0|_\gamma, \end{aligned}$$

which yields with  $M_2 = (3c_0 + 1)M_1$ ,

$$|y(t)|_\gamma \leq M_2e^{-\sigma t}|y_0|_\gamma, \quad t \in [0, t_1], \quad \sigma \in [0, \omega]. \quad (2.24)$$

We deduce from the equation for  $x$ , the estimate for  $T$  in (2.20), and Hölder's inequality that

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |T(x(s), y(s))| ds \\ &\leq |x_0| + C_2(\eta + r) \int_0^t |y(s)|_1 ds \\ &\leq |x_0| + C_2(\eta + r) \left( \int_0^\infty e^{-\omega s p'} ds \right)^{1/p'} \|e^{\omega t}y\|_{\mathbb{E}_1(t_1)} \\ &= |x_0| + C_2(\eta + r)c_1\|e^{\omega t}y\|_{\mathbb{E}_1(t_1)} \leq |x_0| + M_3|y_0|_\gamma, \quad t \in [0, t_1], \end{aligned}$$

where  $M_3 = M_1c_1/M_0$  and  $c_1 = (1/[\omega p'])^{1/p'}$ . Summarizing, we have shown that  $|x(t)| + |y(t)|_\gamma \leq |x_0| + (M_2 + M_3)|y_0|_\gamma$  for all  $t \in [0, t_1]$ . By continuity and the assumption  $t_1 < t_*$  this inequality also holds for  $t = t_1$ . Hence

$$|x(t_1)| + |y(t_1)|_\gamma \leq |x_0| + (M_2 + M_3)|y_0|_\gamma \leq (1 + M_2 + M_3)N\delta < \rho/2,$$

provided  $\delta \leq \rho/[2N(1 + M_2 + M_3)]$ . This contradicts the definition of  $t_1$  and we conclude that  $t_1 = t_*$ .

In the following, we assume that  $\delta \leq \rho/[2N(1 + M_2 + M_3)]$ . Then the estimates derived above and (2.16) yield the uniform bounds

$$\|v\|_{\mathbb{E}_1(a)} + \sup_{t \in [0, a]} |v(t)|_\gamma \leq M, \quad (2.25)$$

for every initial value  $v_0 \in B_{X_\gamma}(0, \delta)$  and every  $a < t_*$ . It follows from Corollary 3.2 in [23] that the solution  $v(t)$  of (2.4) exists on  $\mathbb{R}_+$ .

(f) By repeating the above estimates on the interval  $(0, \infty)$  we obtain the estimates

$$|x(t)| \leq |x_0| + M_3|y_0|_\gamma, \quad |y(t)|_\gamma \leq M_2e^{-\omega t}|y_0|_\gamma, \quad t \in [0, \infty), \quad (2.26)$$

for all  $x_0 \in B_{X^c}(0, N\delta)$  and  $y_0 \in B_{X_\gamma^s}(0, N\delta)$ . Moreover,

$$\lim_{t \rightarrow \infty} x(t) = x_0 + \int_0^\infty T(x(s), y(s)) ds =: x_\infty$$

exists since the integral is absolutely convergent. Next observe that we in fact obtain exponential convergence of  $x(t)$  towards  $x_\infty$ , as

$$\begin{aligned} |x(t) - x_\infty| &= \left| \int_t^\infty T(x(s), y(s)) ds \right| \\ &\leq C_2(\eta + r) \int_t^\infty |y(s)|_1 ds \\ &\leq C_2(\eta + r) \left( \int_t^\infty e^{-\omega s p'} ds \right)^{1/p'} \|e^{\omega s} y\|_{\mathbb{E}_1(\infty)} \\ &\leq M_4 e^{-\omega t} |y_0|_\gamma, \quad t \geq 0. \end{aligned}$$

This yields existence of

$$v_\infty := \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} x(t) + \phi(x(t)) + y(t) = x_\infty + \phi(x_\infty).$$

Clearly,  $v_\infty$  is an equilibrium for equation (2.4), and  $v_\infty + u_* \in \mathcal{E}$  is an equilibrium for (2.2). Due to (2.16), (2.26) and the exponential estimate for  $|x(t) - x_\infty|$  we get

$$\begin{aligned} |v(t) - v_\infty|_\gamma &= |x(t) + \phi(x(t)) + y(t) - v_\infty|_\gamma \\ &\leq |x(t) - x_\infty|_\gamma + |\phi(x(t)) - \phi(x_\infty)|_\gamma + |y(t)|_\gamma \\ &\leq (CM_4 + M_2)e^{-\omega t} |y_0|_\gamma \\ &\leq Me^{-\omega t} |P^s v_0 - \phi(P^c v_0)|_\gamma, \end{aligned} \tag{2.27}$$

thereby completing the proof of the second part of Theorem 2.1. Concerning stability, note that given  $r > 0$  small enough we may choose  $0 < \delta \leq r$  such that the solution starting in  $B_{X_\gamma}(u_*, \delta)$  exists on  $\mathbb{R}_+$  and stays within  $B_{X_\gamma}(u_*, r)$ .  $\square$

**Remarks 2.2.** (a) Theorem 2.1 shows, given that situation, that near  $u_*$  the set of equilibria constitutes the (unique) center manifold for (2.1).

(b) It is worthwhile to point out a slightly different way to obtain the function  $\phi$  used in the proof of Theorem 2.1. Applying the projections  $P^s$  and  $P^c$  to the equilibrium equation (2.7) yields the following equivalent system of equations near  $v = 0$

$$A_s z = P^s G(x + z), \quad A_c x = P^c G(x + z), \tag{2.28}$$

with  $z = P^s \psi(\zeta)$  and  $x = P^c \psi(\zeta)$ . Since  $G(0) = G'(0) = 0$  and  $A_s$  is invertible, by the implicit function theorem we may solve the first equation for  $z$  in terms of  $x$ , i.e. there is a  $C^1$ -function  $\phi : B_{X^c}(0, \rho_0) \rightarrow X_1^s$  such that

$$\phi(0) = 0 \quad \text{and} \quad A_s \phi(x) = P^s G(x + \phi(x)), \quad x \in B_{X^c}(0, \rho_0).$$

As  $x + \phi(x)$  is the unique solution of the first equation in (2.28) we additionally have  $A_c x = P^c G(x + \phi(x))$ , as well as  $P^s \psi(\zeta) = \phi(P^c \psi(\zeta))$  for all  $\zeta \in U$ . Since

$G'(0) = 0$  we obtain  $A_s \phi'(0) = P^s G'(0) = 0$  and this implies  $\phi'(0) = 0$ . This shows that  $\mathcal{E} \subset \mathcal{M}$  with  $\mathcal{M} = \{x + \phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\}$  in a neighborhood of  $u_*$  in  $X_1$ .

$\mathcal{M}$  is a  $C^1$ -manifold of dimension  $\ell := \dim(X^c)$  with tangent space  $T_{u_*}(\mathcal{M}) = X^c$  and  $\mathcal{E}$  is a submanifold in  $\mathcal{M}$ . In general,  $\mathcal{E}$  has lower dimension than  $\mathcal{M}$ . Our assumptions in Theorem 2.1 do in fact exactly amount to asserting that  $\mathcal{E}$  and  $\mathcal{M}$  are of equal dimension. Since  $\mathcal{E} \subset \mathcal{M}$  we can then conclude that they coincide in a neighborhood of  $u_*$ .

(c) An inspection of the argument given above shows that in fact all equilibria of equation (2.1) that are close to the equilibrium  $u_*$  are contained in a manifold  $\mathcal{M} = \{x + \phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\}$  such that  $\phi(0) = \phi'(0) = 0$ , with no additional assumptions on the structure of the equilibria. To see this, let us once more consider the equation

$$A_s z = P^s G(x + z), \quad x \in X^c, \quad z \in X_1^s. \quad (2.29)$$

Clearly,  $x = z = 0$  is a solution. Exactly as in the remark above, we can solve (2.29) by the implicit function theorem for  $z$  in terms of  $x$ , obtaining a  $C^1$ -function  $\phi : B_{X^c}(0, \rho_0) \rightarrow X_1^s$  with  $\phi(0) = \phi'(0) = 0$ . If  $v \in X_1$  is an equilibrium for the evolution equation (2.4) close to 0, then the pair  $x = P^c v$ ,  $z = P^s v$  necessarily satisfies equation (2.29), and therefore lies on the graph of  $\phi$ .

(d) We illustrate by means of examples that convergence to equilibria fails if one of the conditions (i)-(iii) in Theorem 2.1 does not hold.

**Example 1.** Consider in  $G := \mathbb{R}^2 \setminus \{0\}$  the ODE system

$$\begin{aligned} \dot{x} &= (x + y)(1 - \sqrt{x^2 + y^2}), \\ \dot{y} &= (y - x)(1 - \sqrt{x^2 + y^2}). \end{aligned} \quad (2.30)$$

In polar coordinates (2.30) reads as

$$\begin{aligned} \dot{r} &= -r(r - 1), \\ \dot{\theta} &= r - 1, \end{aligned}$$

thus the set of equilibria  $\mathcal{E}$  of (2.30) in  $G$  is the unit circle, and for any initial value  $(x_0, y_0) \in G$  we have  $r(t) \rightarrow 1$  as  $t \rightarrow \infty$ . Since the phase portrait is rotationally invariant we may restrict the stability analysis for  $\mathcal{E}$  to one equilibrium, say  $u_* = (0, 1)$ . Denoting the right-hand side of (2.30) by  $F(x, y)$ , we have  $F \in C^1(G)$ , and

$$A_0 = -F'(u_*) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of  $A_0$  are 0 and 1 with eigenvectors  $(1, 0)$  and  $(1, 1)$ , respectively. Thus 0 is semi-simple, and  $N(A_0)$  coincides with the tangent space  $T_{u_*}(\mathcal{E})$ . Consequently,  $u_*$  is normally stable, and hence we can apply Theorem 2.1 to conclude that each trajectory converges to some point on the unit circle as  $t \rightarrow \infty$ . It is readily seen that the trajectories satisfy the relation  $\theta(r) = c_0 - \ln r$  for some appropriate constant  $c_0$ , and this confirms that  $\theta(r)$  converges as  $r \rightarrow 1$ .

**Example 2.** In this example, we consider in  $G := \mathbb{R}^2 \setminus \{0\}$  the ODE system

$$\begin{aligned}\dot{x} &= -x(\sqrt{x^2 + y^2} - 1)^3 - y(\sqrt{x^2 + y^2} - 1)^m, \\ \dot{y} &= -y(\sqrt{x^2 + y^2} - 1)^3 + x(\sqrt{x^2 + y^2} - 1)^m,\end{aligned}\tag{2.31}$$

with  $m = 1$ . In polar coordinates (2.31) reads as

$$\begin{aligned}\dot{r} &= -r(r - 1)^3, \\ \dot{\theta} &= (r - 1)^m.\end{aligned}$$

Again, the set of equilibria  $\mathcal{E}$  of (2.31) in  $G$  is the unit circle. As above, we may restrict the stability analysis for  $\mathcal{E}$  to one equilibrium, say  $u_* = (0, 1)$ . Denoting the right side of (2.31) by  $F(x, y)$  we obtain (in case  $m = 1$ )

$$A_0 = -F'(u_*) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly,  $\{0\}$  is an eigenvalue of  $A_0$  with algebraic multiplicity 2, and  $N(A_0) = \text{span}\{(1, 0)\}$ . Therefore, the eigenvalue  $\{0\}$  has geometric multiplicity 1 and algebraic multiplicity 2. So we have the following situation:

- (i)  $\mathcal{E} = \{(x, y) : x^2 + y^2 = 1\}$  is a smooth manifold of dimension 1 in  $\mathbb{R}^2$ ,
- (ii) the tangent space of  $\mathcal{E}$  at  $u_*$  is given by  $N(A_0)$ ,
- (iii)  $\{0\}$  is *not* semi-simple,

and hence condition (iii) of Theorem 2.1 is not satisfied. We will show that the trajectories of system (2.31) still converge towards the unit circle, but will spiral around the circle at increasing speed as  $r \rightarrow 1$ . This can be seen as follows. First we observe that  $V(x, y) := (r - 1)^2$  with  $r = \sqrt{x^2 + y^2}$  is a Lyapunov function for system (2.31), since for every solution  $(x, y)$  of (2.31) we have

$$\frac{d}{dt}V(x, y) = 2\dot{r}(r - 1) = -2r(r - 1)^4 \leq 0.$$

So  $r(t) \rightarrow 1$  as  $t \rightarrow \infty$  for every solution. On the other hand one verifies that the trajectories satisfy the relation  $\theta(r) = c_0 + \ln(|r - 1|/r) + 1/(r - 1)$ . This shows that all trajectories spiral around  $\mathcal{E}$  with increasing speed, in clockwise direction as  $r \nearrow 1$ , and in counter-clockwise direction as  $r \searrow 1$ .

**Example 3:** Here we consider system (2.31) with  $m = 2$ . This example is similar to the one in [5, p. 4]. In this case we have

$$A_0 = -F'(u_*) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly,  $\{0\}$  is now an eigenvalue with geometric multiplicity 2, and  $N(A_0) = \mathbb{R}^2$ . So condition (ii) of Theorem 2.1 is not satisfied. The function  $V$  from the previous example is again a Lyapunov function, and this yields  $r(t) \rightarrow 1$  as  $t \rightarrow \infty$ . The trajectories satisfy  $\theta(r) = c_0 - \ln(|r - 1|/r)$ , showing that they spiral counter-clockwise with increasing speed around the unit circle as  $r \rightarrow 1$ .

### 3. QUASILINEAR PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

The analysis in the previous section applies in particular to quasilinear parabolic systems of partial differential equations with *linear autonomous* boundary conditions. In this section we show how this can be extended to the case where also the boundary conditions are nonlinear. For this purpose, let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain with boundary  $\partial\Omega \in C^{2m}$ . The outer normal at a point  $x \in \partial\Omega$  will be denoted by  $\nu(x)$ . Consider the problem

$$\begin{cases} \partial_t u(t) + A(u(t))u(t) = F(u(t)) & \text{in } \Omega, \\ B_j(u(t)) = 0 & \text{on } \partial\Omega, \quad j \in \{1, \dots, m\}, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (3.1)$$

Here we employ the maps

$$\begin{aligned} [A(u)v](x) &= \sum_{|\alpha|=2m} a_\alpha(x, u(x), \nabla u(x), \dots, \nabla^{2m-1}u(x)) D^\alpha v(x), \quad x \in \Omega, \\ [F(u)](x) &= f(x, u(x), \nabla u(x), \dots, \nabla^{2m-1}u(x)), \quad x \in \Omega, \\ [B_j(u)](x) &= b_j(x, u(x), \nabla u(x), \dots, \nabla^{m_j}u(x)), \quad x \in \partial\Omega, \end{aligned} \quad (3.2)$$

for functions  $u \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$ , and  $v \in W_p^{2m}(\Omega; \mathbb{C}^N)$ . The numbers  $m_j$  are integers strictly smaller than  $2m$ , and with  $E = \mathbb{C}^N$ , the coefficients are subject to the following regularity assumptions

$$\begin{aligned} \text{(R)} \quad a_\alpha &\in C^1(E \times E^N \times \dots \times E^{N^{2m-1}}; BC(\overline{\Omega}; \mathcal{B}(E))) \text{ for } |\alpha| = 2m, \\ f &\in C^1(E \times E^N \times \dots \times E^{N^{2m-1}}; BC(\overline{\Omega}; E)), \\ b_j &\in C^{2m+1-m_j}(\partial\Omega \times E \times E^N \times \dots \times E^{N^{m_j}}; E) \text{ for } j \in \{1, \dots, m\}. \end{aligned}$$

We set  $B = (B_1, \dots, B_m)$ . We point out that, for a fixed  $u_0 \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$ ,  $A(u_0)$  is a linear differential operator of order  $2m$  with bounded coefficients; whereas  $F$  contains all terms involving derivatives of order  $|\alpha| < 2m$ .

We will employ the  $L_p$ -setting for this problem as in [19], hence we fix  $p > n+2m$  and the basic spaces

$$X_0 = L_p(\Omega; E), \quad X_1 = W_p^{2m}(\Omega; E), \quad X_\gamma = (X_0, X_1)_{1-1/p} = W_p^{2m(1-1/p)}(\Omega; E).$$

As in Section 2 we denote the norm in  $X_j$  by  $|\cdot|_j$  and open balls in  $X_j$  by  $B_{X_j}(u, r)$ ,  $j = 0, 1, \gamma$ . Note that by the Sobolev embedding theorem we have  $X_\gamma \hookrightarrow BC^{2m-1}(\overline{\Omega}; E)$ , which allows us to plug in functions  $u \in X_\gamma$  into the coefficients of  $A$ , into  $f$  and into the functions  $b_j$  pointwise, without any growth restrictions on these nonlinearities.

Assume we have a  $C^1$ -manifold of equilibria  $\Psi : U \rightarrow X_1$  where  $U \subset \mathbb{R}^k$  is an open neighborhood of 0,

$$\begin{aligned} A(\Psi(\zeta))\Psi(\zeta) &= F(\Psi(\zeta)) & \text{in } \Omega, \quad \zeta \in U, \\ B(\Psi(\zeta)) &= 0, & \text{on } \partial\Omega, \quad \zeta \in U, \end{aligned} \quad (3.3)$$

and set  $u_* = \Psi(0)$ . Assume that the rank of  $\Psi'(0)$  is  $k$  and that there are no other equilibria near  $u_*$  in  $X_1$ , i.e.  $\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U)$ , for some  $r_1 > 0$ , where as in Section 2,  $\mathcal{E}$  denotes the set of equilibria of (3.1).

The linearization of (3.1) at  $u_*$  is given by the operator  $A_0$  defined as follows

$$\begin{aligned} A_*v &= A(u_*)v + (A'(u_*)v)u_* - F'(u_*)v && \text{in } \Omega, \\ B_*v &= B'(u_*)v && \text{on } \partial\Omega, \\ \text{with } v &\in D(A_*) = D(B_*) = W_p^{2m}(\Omega; E), \\ A_0 &= A_*|_{N(B_*)}. \end{aligned} \tag{3.4}$$

Next we consider the property of maximal  $L_p$ -regularity for the pair  $(A_*, B_*)$ , and in particular for the operator  $A_0$ . For this we only need to consider the principal parts of the corresponding differential operator and of the boundary operators, i.e.

$$\begin{aligned} A_{\#}(x, D) &= \sum_{|\alpha|=2m} a_{\alpha}(x, u_*(x), \dots, \nabla^{2m-1}u_*(x))D^{\alpha}, \\ B_{j\#}(x, D) &= \sum_{|\beta|=m_j} i^{m_j} [\partial b_j / \partial (\partial_x^{\beta} u)](x, \dots, \nabla^{2m-1}u_*(x))D^{\beta}, \end{aligned}$$

for  $j = 1, \dots, m$ . Note that we use the notation  $D = -i\nabla$ , hence  $\nabla^{\beta} = i^{|\beta|}D^{\beta}$ . It is shown in [12] that normal ellipticity of  $A_{\#}$  and the Lopatinskii-Shapiro condition for  $(A_{\#}, B_{\#})$  are necessary, and in [11] that they are also sufficient for  $L_p$ -maximal regularity of  $A_0$ . These conditions read as follows.

**(E)** For all  $x \in \bar{\Omega}$ ,  $\xi \in \mathbb{R}^n$ ,  $|\xi| = 1$ ,  $\sigma(A_{\#}(x, \xi)) \subset \mathbb{C}_+$ ,  
i.e.  $\mathcal{A}(x, D)$  is **normally elliptic**.

**(LS)** For all  $x \in \partial\Omega$ ,  $\xi \in \mathbb{R}^n$ , with  $\xi \cdot \nu(x) = 0$ ,  $\lambda \in \overline{\mathbb{C}_+}$ ,  $\lambda \neq 0$ , and  $h \in E^m$ ,  
the system of ordinary differential equations on the half-line

$$\begin{aligned} \lambda v(y) + \mathcal{A}_{\#}(x, \xi + i\nu(x)\partial_y)v(y) &= 0, & y > 0, \\ \mathcal{B}_{j\#}(x, \xi + i\nu(x)\partial_y)v(0) &= h_j, & j = 1, \dots, m, \end{aligned}$$

admits a unique solution  $v \in C_0(\mathbb{R}_+; E)$ .

**(Lopatinskii-Shapiro condition.)**

Now assume that  $u_* \in X_1$  is an equilibrium of (3.1), and let conditions (R), (E), and (LS) be satisfied. It was shown in [19] that (3.1) then admits a local strong solution in the  $L_p$ -sense for each initial value  $u_0 \in X_{\gamma}$ , provided the compatibility condition  $B(u_0) = 0$  holds and  $|u_0 - u_*|_{\gamma}$  is sufficiently small. The solution map  $[u_0 \mapsto u(t, u_0)]$  defines a local semi-flow in  $X_{\gamma}$  near  $u_*$  on the nonlinear phase-manifold

$$\mathcal{M} = \{u \in X_{\gamma} : B(u) = 0 \text{ on } \partial\Omega\}.$$

In case the equilibrium  $u_*$  is *hyperbolic*, i.e.  $\sigma(A_0) \cap i\mathbb{R} = \emptyset$ , it was moreover shown in [19] that it is isolated and that it has the so-called *saddle point property*, which means that the local semi-flow in  $\mathcal{M}$  admits a unique stable and unstable manifold



near  $u_*$ . We refer to [19] for details as well as to [23] in the case of linear boundary conditions.

Returning to our situation, differentiating (3.3) w.r.t.  $\zeta$  we obtain for  $\zeta = 0$

$$\begin{aligned} A(u_*)\Psi'(0) + [A'(u_*)\Psi'(0)]u_* - F'(u_*)\Psi'(0) &= 0 \quad \text{in } \Omega, \\ B'_j(u_*)\Psi'(0) &= 0 \quad \text{on } \partial\Omega, \quad j = 1, \dots, m. \end{aligned}$$

This shows that the image of  $\Psi'(0)$  is contained in the kernel  $N(A_0)$  of  $A_0$ , and also that  $T_{u_*}(\mathcal{E})$ , the tangential space of  $\mathcal{E}$  at  $u_*$ , is contained in  $N(A_0)$ . As in Section 2 we assume now that  $R(\Psi'(0)) = N(A_0)$ , that the eigenvalue 0 of  $A_0$  is semi-simple, and that the remaining spectrum of  $A_0$  is contained in the open right half-plane  $\mathbb{C}_+$ . Note that by boundedness of  $\Omega$  and compact embedding, the spectrum of  $A_0$  consists only of isolated eigenvalues of finite algebraic multiplicity, anyway. We can now state the main result of this section.

**Theorem 3.1.** *Let  $2m + n < p < \infty$ , let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain with boundary of class  $C^{2m}$ , and let the spaces  $X_j$ ,  $j = 0, 1, \gamma$ , be defined as above. Suppose  $u_* \in X_1$  is an equilibrium of (3.1), and assume that conditions (R), (E), and (LS) are satisfied. Let  $A_0$  defined in (3.4) denote the linearization of (3.1) at  $u_*$ , and suppose that  $u_*$  is normally stable, i.e. assume that*

- (i) *near  $u_*$  the set of equilibria  $\mathcal{E}$  is a  $C^1$ -manifold in  $X_1$  of dimension  $k \in \mathbb{N}$ ,*
- (ii) *the tangent space for  $\mathcal{E}$  at  $u_*$  is given by  $N(A_0)$ ,*
- (iii) *0 is a semi-simple eigenvalue of  $A_0$ , i.e.  $R(A_0) \oplus N(A_0) = X_0$ ,*
- (iv)  *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ .*

*Then  $u_*$  is stable in  $X_\gamma$ , and there exists  $\delta > 0$  such that the unique solution  $u(t)$  of (3.1) with initial value  $u_0 \in X_\gamma$ , satisfying  $|u_0 - u_*|_\gamma < \delta$  and the compatibility condition  $B(u_0) = 0$  on  $\partial\Omega$ , exists on  $\mathbb{R}_+$  and converges exponentially fast in  $X_\gamma$  to some  $u_\infty \in \mathcal{E}$  as  $t \rightarrow \infty$ .*

*Proof.* (a) The proof is similar to that of Theorem 2.1. It is based again on the reduction to normal form. We use the notation introduced above and denote as in Section 2 by  $P^s$  and  $P^c$  the projections onto  $X_0^s = R(A_0)$  resp.  $X^c = N(A_0)$ . We first center (3.1) around  $u_*$  by setting  $\bar{u} = u - u_*$ , and obtain the following problem for  $\bar{u}$ .

$$\begin{cases} \partial_t \bar{u} + A_* \bar{u} = G(\bar{u}) & \text{in } \Omega, \\ B_* \bar{u} = H(\bar{u}) & \text{on } \partial\Omega, \\ \bar{u}(0) = \bar{u}_0 := u_0 - u_* & \text{in } \Omega. \end{cases} \quad (3.5)$$

Here  $G$  is defined as in Section 2, and

$$H(\bar{u}) = B_* \bar{u} - B(u_* + \bar{u}) = -[B(u_* + \bar{u}) - B(u_*) - B'(u_*)\bar{u}].$$

Exactly as in the proof of Theorem 2.1 we obtain a function  $\phi \in C^1(B_{X^c}(0, \rho_0), X_1^s)$  with  $\phi(0) = \phi'(0) = 0$  such that the equilibrium equation

$$\begin{aligned} A_* \psi(\zeta) &= G(\psi(\zeta)) \quad \text{in } \Omega, \quad \zeta \in U, \\ B_* \psi(\zeta) &= H(\psi(\zeta)) \quad \text{on } \partial\Omega, \quad \zeta \in U, \end{aligned}$$

for (3.5) can equivalently be expressed by

$$\begin{aligned} P^c A_* \phi(v) &= P^c G(v + \phi(v)), \\ P^s A_* \phi(v) &= P^s G(v + \phi(v)), \quad B_* \phi(v) = H(v + \phi(v)), \end{aligned} \quad (3.6)$$

for every  $v \in B_{X^c}(0, \rho_0)$ . We can now introduce the normal form of (3.1) for the variables

$$v := P^c(u - u_*) = P^c \bar{u}, \quad w := P^s(u - u_*) - \phi(P^c(u - u_*)) = P^s \bar{u} - \phi(P^c \bar{u}),$$

which reads as

$$\begin{cases} \partial_t v = T(v, w) & \text{in } \Omega, \\ \partial_t w + P^s A_* P^s w = R(v, w) & \text{in } \Omega, \\ B_* w = S(v, w) & \text{on } \partial\Omega, \\ v(0) = v_0, w(0) = w_0 & \text{in } \Omega. \end{cases} \quad (3.7)$$

Using the equilibrium equations in (3.6) we can derive, similarly as in Section 2, the following expressions for  $T$ ,  $R$  and  $S$ :

$$\begin{aligned} T(v, w) &= P^c(G(v + \phi(v) + w) - G(v + \phi(v))) - P^c A_* w, \\ R(v, w) &= P^s(G(v + \phi(v) + w) - G(v + \phi(v))) - \phi'(v)T(v, w), \\ S(v, w) &= H(v + \phi(v) + w) - B_* \phi(v) \\ &= H(v + \phi(v) + w) - H(v + \phi(v)). \end{aligned}$$

Clearly,

$$R(v, 0) = T(v, 0) = S(v, 0) = 0, \quad v \in B_{X^c}(0, \rho_0).$$

Therefore we are in the same situation as in Section 2, except that here the infinite dimensional part, i.e. the equation for  $w$ , has a nonlinear boundary condition in case  $S(v, w) \not\equiv 0$ .

(b) Let  $0 < a \leq \infty$  and define the following function spaces on  $(0, a) \times \Omega$ :

$$\mathbb{E}_1(a) = H_p^1((0, a); X_0) \cap L_p((0, a); X_1), \quad \mathbb{E}_0(a) = L_p((0, a); X_0).$$

We also need spaces for the boundary values. For this purpose, we set with  $\kappa_j = 1 - m_j/2m - 1/2mp$

$$Y_0 = L_p(\partial\Omega; \mathbb{C}^N), \quad Y_j = W_p^{2m\kappa_j}(\partial\Omega; \mathbb{C}^N),$$

and

$$\mathbb{F}(a) = \prod_{j=1}^m \mathbb{F}_j(a), \quad \mathbb{F}_j(a) = W_p^{\kappa_j}((0, a); Y_0) \cap L_p((0, a); Y_j), \quad j = 1, \dots, m.$$

Note that by trace theory we have

$$\begin{aligned} \mathbb{F}_j(a) &\hookrightarrow BC([0, a]; W_p^{2m\kappa_j - 2m/p}(\partial\Omega; \mathbb{C}^N)), \\ W_p^{2m\kappa_j - 2m/p}(\partial\Omega; \mathbb{C}^N) &\hookrightarrow BC(\partial\Omega; \mathbb{C}^N) \end{aligned}$$

by the condition  $p > 2m + n$  and since  $m_j < 2m$ . The spaces  $\mathbb{F}_j(a)$  are the trace spaces on the lateral boundary  $(0, a) \times \partial\Omega$  of  $(0, a) \times \Omega$  for the derivatives  $D^\beta u$  of

order  $|\beta| = m_j$  for  $u \in \mathbb{E}_1(a)$ .

The basic solvability theorem for the fully inhomogeneous linear problem

$$\begin{cases} \partial_t u + A_* u = f(t) & \text{in } \Omega, \quad t > 0, \\ B_* u = g(t) & \text{on } \partial\Omega, \quad t > 0, \\ u(0) = u_0 & \text{in } \Omega \end{cases} \quad (3.8)$$

in the  $L_p$ -setting reads as follows, see [12].

**Proposition 3.2.** *Let  $a < \infty$ . The linear problem (3.8) admits a unique solution  $u \in \mathbb{E}_1(a)$  if and only if  $f \in \mathbb{E}_0(a)$ ,  $g \in \mathbb{F}(a)$ ,  $u_0 \in X_\gamma$ , and the compatibility condition  $B_* u_0 = g(0)$  holds. There is a constant  $C = C(a) > 0$  such that the estimate*

$$\|u\|_{\mathbb{E}_1(a)} \leq C(|u_0|_\gamma + \|f\|_{\mathbb{E}_0(a)} + \|g\|_{\mathbb{F}(a)})$$

holds for the solution  $u$  of (3.8).

We shall also need a variant of Proposition 3.2 for the problem

$$\begin{cases} \partial_t w + P^s A_* P^s w = f(t) & \text{in } \Omega, \quad t > 0, \\ B_* w = g(t) & \text{on } \partial\Omega, \quad t > 0, \\ w(0) = w_0 & \text{in } \Omega \end{cases} \quad (3.9)$$

on the half-line, where we assume  $w_0 \in X_\gamma^s$  and  $f \in L_p(\mathbb{R}_+; X_0^s)$ . For this purpose we proceed as follows. Suppose first that  $u$  solves (3.8) with  $u_0 = w_0$ . Since  $A_* P^c u = B_* P^c u = 0$  we then conclude that  $w = P^s u$  solves problem (3.9). Let  $u_1$  denote the solution of (3.8) with  $A_*$  replaced by  $A_* + 1$ . The spectrum of  $A_0 + 1$  is contained in  $\mathbb{C}_+$ , hence we may apply Proposition 3.1 of [19] to obtain a uniform estimate for  $u_1$  in  $\mathbb{E}_1(\infty)$ . Then  $u_2 = u - u_1$  solves the problem

$$\partial_t u_2 + A_* u_2 = u_1, \quad B_* u_2 = 0, \quad u_2(0) = 0.$$

As  $\sigma(A_s) \subset \mathbb{C}_+$ ,  $A_s$  has maximal  $L_p$ -regularity on the half-line, hence we obtain also a uniform estimate for  $P^s u_2$  in  $\mathbb{E}_1(\infty)$ . These arguments yield the following result.

**Proposition 3.3.** *Let  $a \leq \infty$ . The linear problem (3.9) admits a unique solution  $w \in \mathbb{E}_1(a) \cap L_p((0, a); X_0^s)$  if and only if  $f \in L_p((0, a); X_0^s)$ ,  $g \in \mathbb{F}(a)$ ,  $w_0 \in X_\gamma^s$ , and the compatibility condition  $B_* w_0 = g(0)$  holds. There is a constant  $C_0 > 0$ , independent of  $a$ , such that the estimate*

$$\|w\|_{\mathbb{E}_1(a)} \leq C_0(|w_0|_\gamma + \|f\|_{\mathbb{E}_0(a)} + \|g\|_{\mathbb{F}(a)})$$

holds for the solution  $w$  of (3.9), for all functions  $f \in L_p((0, a); X_0^s)$ ,  $g \in \mathbb{F}(a)$  and all initial values  $w_0 \in X_\gamma^s$ .

Proposition 3.3 remains valid when we replace  $w(t)$  by  $e^{\sigma t} w(t)$ ,  $f(t)$  by  $e^{\sigma t} f(t)$  and  $g(t)$  by  $e^{\sigma t} g(t)$  where  $0 < \sigma \leq \omega$ ,  $\omega < \inf\{\operatorname{Re} \lambda : \lambda \in \sigma(A_s)\}$ .

(c) Next we consider the nonlinearities  $R$ ,  $T$ , and  $S$ . Since by assumption the functions  $a_\alpha$  and  $f$  are in  $C^1$  and  $p > n + 2m$ , it follows easily via the embedding

$X_\gamma \hookrightarrow BC(\bar{\Omega})$  that  $A$  and  $F$  are as in Section 2. Hence we obtain as there the estimates

$$|T(v, w)|_0 \leq C_2(\eta + r)|w|_1 + C_3|w|_1, \quad v \in B_{X^c}(0, \rho), \quad w \in B_{X_\gamma^s}(0, \rho) \cap X_1,$$

where  $r = 3\rho$  and  $C_3 := \|P^c A_* P^c\|_{\mathcal{B}(X_1, X^c)}$ . Since  $\phi'(0) = 0$  we can assume that  $\rho_0$  was chosen so small that  $|\phi'(w)|_{\mathcal{B}(X^c, X_1^s)} \leq \eta$  for all  $w \in B_{X^c}(0, \rho_0)$ . With this we obtain

$$|R(v, w)|_0 \leq C_4(\eta + r)|w|_1, \quad v \in \bar{B}_{X^c}(0, \rho), \quad w \in \bar{B}_{X_\gamma^s}(0, \rho) \cap X_1.$$

Observe that in contrast to the previous section the constant  $C_3$  is no longer small since  $P^l$ ,  $l \in \{c, s\}$ , and  $A_*$  do not commute. However, this does not alter our conclusions. It is more involved to derive the estimates on  $S$  needed for Proposition 3.3. Fortunately, we can refer to [19, Proposition 3.3]. This result implies

$$\|e^{\omega t}(H(\bar{u}_1) - H(\bar{u}_2))\|_{\mathbb{F}(a)} \leq \eta \|e^{\omega t}(\bar{u}_1 - \bar{u}_2)\|_{\mathbb{E}_1(a)},$$

for all  $e^{\omega t}\bar{u}_1, e^{\omega t}\bar{u}_2 \in \mathbb{E}_1(a)$  such that  $|\bar{u}_1(t)|_\gamma, |\bar{u}_2(t)|_\gamma \leq r$ ,  $t \in [0, a]$ . Therefore, by possibly decreasing  $r > 0$ , with  $\bar{u}_1 = v + \phi(v) + w$  and  $\bar{u}_2 = v + \phi(v)$  this yields

$$\|e^{\omega t}S(v, w)\|_{\mathbb{F}(a)} \leq \eta \|e^{\omega t}w\|_{\mathbb{E}_1(a)},$$

for all  $e^{\omega t}v, e^{\omega t}w \in \mathbb{E}_1(a)$  such that  $v([0, a]) \subset B_{X^c}(0, \rho)$  and  $w([0, a]) \subset B_{X_\gamma^s}(0, \rho)$ . These are the estimates we need for applying Proposition 3.3.

(d) We may now follow part (c)-(f) of the proof of Theorem 2.1 to complete the proof of Theorem 3.1, the needed local well-posedness result being Proposition 4.1 in [19].  $\square$

#### 4. CONVERGENCE OF SOLUTIONS FOR THE MULLINS-SEKERKA PROBLEM

We consider the two-phase quasi-stationary Stefan problem with surface tension, which has also been termed Mullins-Sekerka model (or Hele-Shaw model with surface tension) and is a model for phase transitions in liquid-solid systems. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , with smooth boundary  $\partial\Omega$ . Let  $\Gamma_0 \subset \Omega$  be a compact connected hypersurface in  $\Omega$  which is the boundary of an open set  $\Omega_0 \subset \Omega$ , and let  $\Gamma(t)$  be its position at time  $t \geq 0$ . Denote by  $V(t, \cdot)$  and  $\kappa(t, \cdot)$  the normal velocity and the mean curvature of  $\Gamma(t)$ , and let  $\Omega_1(t)$  (liquid phase) and  $\Omega_2(t)$  (solid phase) be the two regions in  $\Omega$  separated by  $\Gamma(t)$ , with  $\Omega_1(t)$  being the interior region. Let further  $\nu(t, \cdot)$  be the outer unit normal field on  $\Gamma(t)$  with respect to  $\Omega_1(t)$ . We shall use the convention that  $V$  is positive if  $\Omega_1(t)$  is expanding, and that the mean curvature is positive for uniformly convex hypersurfaces. The *two-phase Mullins-Sekerka problem* consists in finding a family  $\Gamma(t)$ ,  $t \geq 0$ , of hypersurfaces satisfying

$$V = [\partial_\nu u_\kappa], \quad t > 0, \quad \Gamma(0) = \Gamma_0, \quad (4.1)$$

where  $u_\kappa = u_\kappa(t, \cdot)$  is, for each  $t \geq 0$ , the solution of the elliptic boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_1(t) \cup \Omega_2(t), \\ u = \kappa & \text{on } \Gamma(t), \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

Here  $[\partial_\nu u_\kappa] := \partial_\nu u_\kappa^2 - \partial_\nu u_\kappa^1$  stands for the jump of the normal derivative of  $u_\kappa$  across the interface  $\Gamma(t)$ , and  $\partial_\nu u$  denotes the normal derivative of  $u$  on  $\partial\Omega$ .

Assuming connected phases and that the interface does not touch the fixed boundary  $\partial\Omega$ , the set of equilibrium states of (4.1), (4.2) consists precisely of all spheres  $S_R(x_0) \subset \Omega$ , where  $R$  denotes the radius and  $x_0$  the center. Thus there is an  $(n+1)$ -parameter family of equilibria, the parameters being the  $n$  coordinates of the center  $x_0$  and the radius  $R$ .

Let now  $\Sigma \subset \Omega$  be some fixed sphere without boundary contact. We are interested in the asymptotic properties of solutions of the Mullins-Sekerka problem that start in a neighbourhood of  $\Sigma$ , that is  $\Gamma_0$  is close to  $\Sigma$ . Following [15] we first use Hanzawa's method to transform the original problem to a system of equations on a fixed domain. Here the basic idea is to represent the moving interface  $\Gamma(t)$  as the graph of a function in normal direction of a fixed reference surface, which will be  $\Sigma$  in our case. Denoting the parameterizing function by  $\rho(t, \cdot)$  this leads to a problem on  $\Sigma$  of the form

$$\dot{\rho} + B(\rho)S(\rho) = 0, \quad t > 0, \quad \rho(0) = \rho_0, \quad (4.3)$$

where  $S(\rho)$  is the solution of the transformed elliptic boundary value problem

$$\begin{cases} A(\rho)v = 0 & \text{in } \Omega_1 \cup \Omega_2, \\ v = K(\rho) & \text{on } \Sigma, \\ \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4)$$

Here  $\Omega_1$  and  $\Omega_2$  are the two regions in  $\Omega$  separated by  $\Sigma$ , with  $\Omega_1$  being enclosed by  $\Sigma$ . By construction, the solution  $\Gamma(\cdot) \equiv \Sigma$  of the original problem (4.1) corresponds to the solution  $\rho \equiv 0$  of (4.3). The operator  $F(\cdot) := B(\cdot)S(\cdot)$  in (4.3) is a nonlocal pseudo-differential operator of third order and renders (4.3) a quasilinear parabolic problem, see [15] for its precise definition and more details.

We want to study (4.3) in an  $L_p$  setting. Let  $p > n+2$  and define

$$X_0 = W_p^{1-1/p}(\Sigma), \quad X_1 = W_p^{4-1/p}(\Sigma).$$

Given  $J = (0, a)$ ,  $a > 0$ , we view (4.3) as an evolution equation in the space  $\mathbb{E}_0(J) = L_p(J; X_0)$ , that is we are interested in solutions of (4.3) in the class  $\mathbb{E}_1(J) = H_p^1(J; X_0) \cap L_p(J; X_1)$ . For the corresponding trace space we have

$$X_\gamma := (X_0, X_1)_{1-1/p, p} = W_p^{4-4/p}(\Sigma).$$

Note that, by Sobolev embedding, we have  $X_1 \hookrightarrow X_\gamma \hookrightarrow C^2(\Sigma)$ .

Let

$$\mathfrak{A} := \{\rho \in C^2(\Sigma) : |\rho|_{C(\Sigma)} < \eta\}$$

with  $\eta > 0$  sufficiently small denote the set of admissible parameterizations. Setting  $U := X_1 \cap \mathfrak{A}$ , one has  $F \in C^\infty(U; X_0)$  and the linearization  $L := F'(0)$  is given by

$$L\rho = -[\partial_\nu T A_\Sigma \rho],$$

where  $Tg$  denotes the solution of the elliptic problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_1 \cup \Omega_2, \\ v = g & \text{on } \Sigma, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.5)$$

and

$$A_\Sigma = -\frac{1}{n-1} \left( \frac{n-1}{R^2} + \Delta_\Sigma \right),$$

with  $R$  being the radius of  $\Sigma$  and  $\Delta_\Sigma$  the Laplace-Beltrami operator on  $\Sigma$ . The operator  $A_\Sigma$  is the linearization  $K'(0)$  of the transformed mean curvature operator  $K(\rho)$  at  $\rho = 0$ . Concerning the linearization  $L = F'(0)$  we refer to [15, 25].

One can show (cf. [25, Theorem 2.1]) that the spectrum of  $L$  consists of countably many real nonnegative eigenvalues of finite algebraic multiplicity, and that 0 is a semi-simple eigenvalue of  $L$  with multiplicity  $n+1$ , see also [15, Proposition 5.4 and Lemma 6.1]. Moreover, the kernel of  $L$  is given by  $N(L) = \text{span}\{Y_0, Y_1, \dots, Y_n\}$ , where  $Y_0 \equiv 1$ , and where  $Y_j$ ,  $1 \leq j \leq n$ , are the spherical harmonics of degree 1. We may assume that  $Y_j = R^{-1}p_j|_\Sigma$ ,  $1 \leq j \leq n$ , with  $p_j$  being the harmonic polynomial of degree 1 given by  $p_j(x) = x_j$  for  $x \in \mathbb{R}^n$ ; by  $p_j|_\Sigma$  we mean the restriction of  $p_j$  to  $\Sigma$ .

Let us assume that  $\Sigma$  is centered at the origin of  $\mathbb{R}^n$ . Suppose  $\mathcal{S} \subset \Omega$  is a sphere that is sufficiently close to  $\Sigma$ . Denote by  $(z_1, \dots, z_n)$  the coordinates of its center and let  $z_0$  be such that  $R + z_0$  corresponds to its radius. Then, by [15, Section 6], the sphere  $\mathcal{S}$  can be parameterized over  $\Sigma$  by the distance function

$$\rho(z) = \sum_{j=1}^n z_j Y_j - R + \sqrt{\left(\sum_{j=1}^n z_j Y_j\right)^2 + (R + z_0)^2 - \sum_{j=1}^n z_j^2}.$$

Denoting by  $O$  a sufficiently small neighbourhood of 0 in  $\mathbb{R}^{n+1}$ , the mapping  $[z \mapsto \rho(z)] : O \rightarrow W_p^{4-1/p}(\Sigma)$  is smooth and the derivative at 0 is given by

$$\rho'(0)h = \sum_{j=0}^n h_j Y_j, \quad h \in \mathbb{R}^{n+1}.$$

So we see that near  $\Sigma$  the set  $\mathcal{E}$  of equilibria of (4.3) is a smooth manifold in  $X_1$  of dimension  $n+1$ , and that the tangent space  $T_\Sigma(\mathcal{E})$  coincides with  $N(L)$ .

In order to be able to apply Theorem 2.1 from Section 2 it remains to verify that the operator  $L$  has the property of maximal  $L_p$ -regularity. This means we have to show that for any  $J = (0, a)$ ,  $a > 0$ , and any  $g \in \mathbb{E}_0(J)$  the problem

$$\dot{\rho} + L\rho = g, \quad t \in J, \quad \rho(0) = 0, \quad (4.6)$$

has a unique solution in the space  $\mathbb{E}_1(J)$ . By means of the standard localization method, perturbation arguments, and by solving certain elliptic auxiliary problems, (4.6) can be reduced to the following two-phase problem on  $\mathbb{R}^n \times \dot{\mathbb{R}}$  with  $\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$ :

$$-\Delta_x w - \partial_y^2 w = 0, \quad t \in J, x \in \mathbb{R}^n, y \in \dot{\mathbb{R}}, \quad (4.7)$$

$$w|_{y=0} + \Delta_x \sigma = 0, \quad t \in J, x \in \mathbb{R}^n, \quad (4.8)$$

$$\partial_t \sigma - [\partial_y w] = h, \quad t \in J, x \in \mathbb{R}^n, \quad (4.9)$$

$$\sigma(0) = 0, \quad x \in \mathbb{R}^n.$$

Here  $[\partial_y w] = \partial_y w|_{y=0^+} - \partial_y w|_{y=0^-}$ , and  $h \in L_p(J; W_p^{1-1/p}(\mathbb{R}^n))$  is a given function. We take the Fourier transform w.r.t.  $x$  and denote the transformed functions by  $\tilde{w}$  and  $\tilde{\sigma}$ . Then (4.7) and (4.8) imply that  $\tilde{w} = e^{-|\xi||y|}|\xi|^2 \tilde{\sigma}$ . Inserting this into (4.9) leads to the subsequent problem for  $\tilde{\sigma}$  on  $\mathbb{R}^n$ :

$$\partial_t \tilde{\sigma} + 2|\xi|^3 \tilde{\sigma} = \tilde{h}, \quad t \in J, \quad \tilde{\sigma}(0) = 0.$$

Set  $Y_0 = W_p^{1-1/p}(\mathbb{R}^n)$  and  $Y_1 = W_p^{4-1/p}(\mathbb{R}^n)$  and let  $G$  be defined by  $G = d/dt$  with domain  $D(G) = {}_0H_p^1(J; Y_0)$ , here the zero means vanishing trace at  $t = 0$ . Then  $G$  is sectorial, invertible and admits an  $\mathcal{H}^\infty$ -calculus in  $L_p(J; Y_0)$  of angle  $\pi/2$ . Let further  $D$  be the operator in  $L_p(J; Y_0)$  with symbol  $2|\xi|^3$  and domain  $D(D) = L_p(J; Y_1)$ . Then  $D$  is sectorial and admits an  $\mathcal{H}^\infty$ -calculus in  $L_p(J; Y_0)$  of angle 0. Thus by the Dore-Venni theorem, the equation  $G\sigma + D\sigma = h$  possesses a unique solution  $\sigma \in D(G) \cap D(D)$ . Hence  $L$  has the property of maximal  $L_p$ -regularity.

So all assumptions of Theorem 2.1 are satisfied, hence we obtain the following result, which is the main result in [15] except for the different functional analytic setting.

**Theorem 4.1.** *Let  $p > n + 2$  and  $\Omega \subset \mathbb{R}^n$  be a bounded domain with boundary of class  $C^2$ . Suppose  $\Sigma$  is an arbitrary sphere in  $\Omega$  of radius  $R$  without boundary contact. Then  $\rho \equiv 0$  is a stable equilibrium of (4.3) in  $X_\gamma = W_p^{4-4/p}(\Sigma)$ , and there exists  $\delta > 0$  such that if  $|\rho_0|_\gamma < \delta$ , then the corresponding solution of (4.3) exists globally and converges at an exponential rate in  $X_\gamma$  to some equilibrium  $\rho_\infty$  as  $t \rightarrow \infty$ . In this sense, the sphere  $\Sigma$  is a stable equilibrium of the Mullins-Sekerka problem, and any solution  $\Gamma(\cdot)$  of (4.1) that starts sufficiently close to  $\Sigma$  exists globally and converges to some sphere at an exponential rate as  $t \rightarrow \infty$ .*

This approach can also be used to show the stability of spheres for the two-phase quasi-stationary Stokes flow in a bounded domain, see [17, 16] for alternate approaches in the one-phase case. Moreover, it can be applied to models in tumor growth, see [9] for a discussion of existing work.

## 5. STABILITY OF TRAVELLING WAVE SOLUTIONS TO A QUASILINEAR PARABOLIC EQUATION

The situation of the generalized principle of linearized stability may occur when studying the stability of travelling wave solutions of parabolic equations, see e.g. [18, Section 5.4] for the semilinear case. In what follows we want to consider a quasilinear variant of the Huxley equation:

$$u_t - (\sigma(u_x))_x = f(u), \quad t > 0, x \in \mathbb{R}. \quad (5.1)$$

Here  $f(r) = r(1-r)(r-a)$ ,  $r \in \mathbb{R}$ , where  $a \in (0, 1/2)$  is a constant, and  $\sigma$  is a  $C^2$  smooth function on  $\mathbb{R}$  satisfying

$$0 < c_1 \leq \sigma'(r) \leq c_2, \quad r \in \mathbb{R}. \quad (5.2)$$

A travelling wave  $u(t, x) = w(x + Vt)$  with speed  $V$  satisfies

$$(\sigma(w'(s)))' - Vw'(s) + f(w(s)) = 0, \quad s \in \mathbb{R}. \quad (5.3)$$

Similarly to the special case  $\sigma(r) = r$  (cf. [18]), one can show, by means of a phase plane analysis, that for some  $V > 0$  (5.3) admits a solution  $w$  with  $w(s) \rightarrow 0$  as  $s \rightarrow -\infty$  and  $w(s) \rightarrow 1$  as  $s \rightarrow \infty$ . For this purpose we introduce the variable  $z := w'$ . Then (5.3) is equivalent to the system

$$\begin{cases} \dot{w} = z, \\ \dot{z} = \frac{1}{\sigma'(z)} (Vz - f(w)). \end{cases}$$

Denoting its right-hand side by  $H(w, z)$ , we find that

$$H'(i, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{f'(i)}{\sigma'(0)} & \frac{V}{\sigma'(0)} \end{bmatrix}, \quad i \in \{0, 1\}.$$

We have  $f'(0) = -a$  and  $f'(1) = a - 1$ , thus the equilibria  $(0, 0)$  and  $(1, 0)$  are both saddle points. The eigenvalues of  $H'(0, 0)$  are given by

$$\lambda_{1,2} = \frac{1}{2\sigma'(0)} \left( V \pm \sqrt{V^2 + 4a\sigma'(0)} \right),$$

$(1/\lambda_1, 1)$  is an eigenvector to  $\lambda_1 > 0$ , thus the unstable manifold points into the first and third quadrant, with steeper slopes for higher values of  $V \geq 0$ .

Define the functions

$$F(y) = \int_0^y f(r) dr, \quad G(y) = \int_0^y \sigma'(r)r dr, \quad y \in \mathbb{R}.$$

Then (5.3) implies that

$$\frac{d}{ds} \left( G(z(s)) + F(w(s)) \right) = Vz(s)^2, \quad s \in \mathbb{R}. \quad (5.4)$$

In particular  $G(z) + F(w)$  is a first integral if  $V = 0$ . By (5.2) we further have  $C_1 y^2 \leq 2G(y) \leq C_2 y^2$ ,  $y \in \mathbb{R}$ .



We now consider the trajectory  $\gamma$  that (near the origin) lies on the unstable manifold to  $(0, 0)$  in the first quadrant. For  $V = 0$ ,  $\gamma$  cannot reach the line  $w = 1$ , since  $F(w) \leq 0$  on  $\gamma$ , and  $F(1) = \frac{1}{6}(\frac{1}{2} - a) > 0$ . In case  $V > 0$ , (5.4) shows that  $\gamma$ , as long as it remains in the first quadrant, moves through increasing values  $c$  of the level curves  $G(z) + F(w) = c$ . For  $V$  sufficiently large,  $\gamma$  will reach the level curve to  $c = F(1)$  at some point with  $z > 0$ . For continuity reasons, there exists then  $V > 0$  for which  $\gamma$  becomes a heteroclinic orbit, connecting  $(0, 0)$  and  $(1, 0)$ ; observe that  $\gamma$  is the only such orbit. Hence there is a smooth solution  $w$  to (5.3) satisfying  $w'(s) > 0$  for all  $s \in \mathbb{R}$  and

$$(w(s), w'(s)) \rightarrow (0, 0) \text{ as } s \rightarrow -\infty, \quad (w(s), w'(s)) \rightarrow (1, 0) \text{ as } s \rightarrow \infty, \quad (5.5)$$

both exponentially fast. Clearly each translate  $w(\cdot + \alpha)$  with  $\alpha \in \mathbb{R}$  enjoys the same properties.

In order to investigate the stability of the travelling wave we change to moving coordinates with  $y := x + Vt$  and  $\tilde{u}(t, y) := u(t, x)$ , thereby transforming (5.1) into

$$\tilde{u}_t - (\sigma(\tilde{u}_y))_y + V\tilde{u}_y = f(\tilde{u}), \quad t > 0, \quad y \in \mathbb{R}. \quad (5.6)$$

Evidently,  $\tilde{u} = w$  and all translates of it are equilibria of (5.6). Letting  $v := \tilde{u} - w$  be the deviation from  $w$ , the equation for  $v$  reads as

$$v_t - (\sigma(v_y + w_y))_y + V(v_y + w_y) = f(v + w), \quad t > 0, \quad y \in \mathbb{R}. \quad (5.7)$$

For this equation, the set

$$\mathcal{E} = \{w(\cdot + \alpha) - w(\cdot) : \alpha \in \mathbb{R}\} \quad (5.8)$$

forms a one-dimensional smooth manifold of equilibria. Observe that  $\mathcal{E}$  contains all equilibria  $w(y)$  of (5.7) satisfying  $w(y) \rightarrow 0$  as  $|y| \rightarrow \infty$ . By definition, the travelling wave under study is stable, if  $v = 0$  is stable for (5.7). Of course, this has to be understood in the sense of a suitable functional analytic setting.

Let us choose again the  $L_p$  setting to study (5.7). Let  $1 < p < \infty$ ,  $X_0 = L_p(\mathbb{R})$ , and  $X_1 = H_p^2(\mathbb{R})$ . Define  $M : X_1 \rightarrow X_0$  by

$$M(v) = -(\sigma(v_y + w_y))_y + V(v_y + w_y) - f(v + w), \quad v \in X_1,$$

This definition makes sense since by (5.2)–(5.3) and (5.5)  $w_y$ ,  $w_{yy}$ , and  $f(w)$  belong to  $L_p(\mathbb{R})$ . The fact that  $f(v + w)$  belongs to the space  $L_p(\mathbb{R})$  can be justified by observing that

$$f(v(y) + w(y)) = f(w(y)) + \int_0^1 f'(w(y) + \tau v(y)) d\tau v(y), \quad y \in \mathbb{R}.$$

Since  $v$  and  $w$  belong to  $C_0(\mathbb{R})$  one readily verifies that  $f'(v + w)$  is continuous and bounded, and this yields the statement for  $f(v + w)$ . The linearization  $A_0 := M'(0)$  of  $M$  at  $v = 0$  is given by

$$A_0 v = -(\sigma'(w_y)v_y)_y + Vv_y - f'(w)v, \quad v \in D(A_0) := X_1.$$

By (5.2),  $A_0$  is a uniformly elliptic operator with smooth coefficients whose leading coefficient satisfies  $-\sigma'(w'(y)) \rightarrow -\sigma'(0)$  as  $|y| \rightarrow \infty$ . Thus, by [11, Theorem 5.7],  $A_0$  enjoys the property of maximal  $L_p$ -regularity.

Next observe that  $\mathcal{E} \subset H_p^3(\mathbb{R}) \hookrightarrow X_1$  and that the tangent space for the manifold  $\mathcal{E}$  at  $v = 0$  coincides with  $\text{span}\{w'\}$ , the span of  $w' \in X_1$ . On the other hand, by differentiating (5.3) we see that  $A_0 w' = 0$ . So to show normal stability of  $v = 0$ , it remains to prove that 0 is a simple eigenvalue of  $A_0$  and that the remainder of the spectrum of  $A_0$  lies in  $\{z \in \mathbb{C} : \text{Re } z > 0\}$ . We proceed similarly as in [18, p. 131], generalizing the proof given there to the quasilinear case. Suppose  $\lambda$  with  $\text{Re } \lambda \leq 0$  is an eigenvalue of  $A_0$  with eigenfunction  $v \in X_1$ , that is

$$v'' + 2b(y)v' + \frac{f'(w(y))}{\sigma'(w'(y))}v + \frac{\lambda v}{\sigma'(w'(y))} = 0, \quad y \in \mathbb{R},$$

where

$$b(y) = \frac{\sigma''(w'(y))w''(y) - V}{2\sigma'(w'(y))}, \quad y \in \mathbb{R}.$$

By studying the characteristic equation for the limits  $y \rightarrow \pm\infty$ , one sees that there exist constants  $\delta, C > 0$  such that

$$|v(y)| \leq Ce^{(\delta+V/\sigma'(0))y}, \quad y \leq 0, \quad |v(y)| \leq Ce^{-\delta y}, \quad y \geq 0.$$

Letting  $\varphi(y) = v(y) \exp(\int_0^y b(r) dr)$ , we have at least  $\varphi(y) = O(e^{-V|y|/(2\sigma'(0))})$  for  $|y| \rightarrow \infty$ , and

$$\varphi'' + \left( \frac{\lambda}{\sigma'(w'(y))} + \frac{f'(w(y))}{\sigma'(w'(y))} - b'(y) - b(y)^2 \right) \varphi = 0, \quad y \in \mathbb{R}. \quad (5.9)$$

The function  $\psi(y) := w'(y) \exp(\int_0^y b(r) dr)$  is strictly positive, and since  $A_0 w' = 0$ , it satisfies

$$\psi'' + \left( \frac{f'(w(y))}{\sigma'(w'(y))} - b'(y) - b(y)^2 \right) \psi = 0, \quad y \in \mathbb{R}. \quad (5.10)$$

Combining (5.9), (5.10) yields

$$\varphi'' + \frac{\lambda}{\sigma'(w'(y))} \varphi - \frac{\psi''}{\psi} \varphi = 0.$$

Multiplying this equation by  $\bar{\varphi}$ , integrating over  $\mathbb{R}$ , and integrating by parts gives

$$\begin{aligned} \lambda \int_{\mathbb{R}} \frac{|\varphi(y)|^2}{\sigma'(w'(y))} dy &= \int_{\mathbb{R}} -\frac{\bar{\varphi}(y)}{\psi(y)} (\varphi''(y)\psi(y) - \varphi(y)\psi''(y)) dy \\ &= \int_{\mathbb{R}} \psi(y)^2 \left\{ [(\text{Re } \varphi(y)/\psi(y))']^2 + [(\text{Im } \varphi(y)/\psi(y))']^2 \right\} dy, \end{aligned}$$

hence  $\lambda = 0$ . In view of (5.9) we may then suppose that  $\varphi$  is real and nonnegative, and so the last formula above implies that  $\varphi/\psi = v/w'$  is constant on  $\mathbb{R}$ . Hence  $N(A_0) = \text{span}\{w'\}$ .

We now show that the essential spectrum  $\sigma_e(A_0)$  of  $A_0$ , that is the set of all spectral points except isolated eigenvalues of finite multiplicity, is contained in  $[\text{Re } z > 0]$ . This can be seen as follows.

By the asymptotics of  $w$ , see (5.5), we know that  $f'(w(y)) \rightarrow f'(0) = -a$  as  $y \rightarrow -\infty$ , and that  $f'(w(y)) \rightarrow f'(1) = -(1-a)$  as  $y \rightarrow \infty$  at an exponential rate. Due to  $a \in (0, 1/2)$ ,  $a$  is the smaller of the two numbers  $\{a, (1-a)\}$ . Fix  $\varepsilon > 0$

so that  $a - \varepsilon > 0$ . We can then find a number  $R > 0$  so that  $-f'(w(y)) \geq a - \varepsilon$  whenever  $|y| \geq R$ . Let  $c$  be a bounded continuous function on  $\mathbb{R}$  that agrees with  $-f'(w)$  on  $[|y| \geq R]$  and satisfies  $c(y) \geq a - \varepsilon$  for all  $y \in \mathbb{R}$ . Let  $B : X_1 \rightarrow X_0$  be the operator defined by

$$Bv := -(\sigma'(w_y)v_y)_y + Vv_y + cv.$$

One readily shows that  $B$  is accretive on  $L_p(\mathbb{R})$  and also that  $\sigma(B)$ , the spectrum of  $B$ , is contained in  $[\operatorname{Re} z \geq (a - \varepsilon)]$ . Next, note that  $A_0$  can be written as  $A_0 = B + S$ , where  $S$  is a perturbation which is relatively compact with respect to  $B$ . For this we note that  $S$  can be written as  $Sv = -(f'(w) + c)\chi v$ , where  $\chi$  is a smooth cut-off function for the interval  $[-R, R]$ , with support contained, say, in  $\Omega = (-2R, 2R)$ . Since  $H_p^s(\Omega)$  is compactly embedded in  $L_p(\Omega)$  for  $s > 0$  we conclude that  $S$  is a compact operator from  $H_p^s(\mathbb{R}) \rightarrow L_p(\mathbb{R})$ . A well-known perturbation result now shows that  $\sigma_\varepsilon(A_0)$  must also be contained in  $[\operatorname{Re} z \geq (a - \varepsilon)]$ . Since this is true for every  $\varepsilon$ , we have proved that  $\sigma_e(A_0) \subset [\operatorname{Re} z \geq a]$ .  $A_0$  might still have isolated eigenvalues outside of this set. As  $A_0$  generates an analytic semigroup, they must be contained in an appropriate sector with opening angle  $\theta < \pi/2$ . Since we have already shown that there are no eigenvalues in  $[\operatorname{Re} z < 0]$  we conclude that there is a number  $\alpha > 0$  so that  $\sigma(A_0) \setminus \{0\} \subset [\operatorname{Re} z \geq \alpha]$ . Finally, since the operator  $A_1$  defined by

$$A_1v := v_{yy} + \left( \frac{f'(w)}{\sigma'(w')} - b'(y) - b(y)^2 \right)v, \quad v \in D(A_1) = X_1,$$

is self-adjoint in  $L_2(\mathbb{R})$ , it follows in view of (5.9) with  $\lambda = 0$  that the eigenvalue 0 of  $A_0$  is semi-simple.

Summarizing we have shown that

- the set  $\mathcal{E}$  consists of all equilibria of (5.7) in  $X_1$  and forms a smooth 1-dimensional manifold,
- $T_0(\mathcal{E}) = \operatorname{span} \{w'\} = N(A_0)$ ,
- the eigenvalue 0 is semi-simple,
- $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ ,

and we can now formulate our main result for this section.

**Theorem 5.1.** *Let  $p \in (1, \infty)$ ,  $f(r) = r(1 - r)(r - a)$ ,  $r \in \mathbb{R}$ , with  $a \in (0, 1/2)$ , and  $\sigma \in C^2(\mathbb{R})$  such that (5.2) holds.*

*Then (5.1) possesses a travelling wave solution  $u(t, x) = w(x + Vt)$  with speed  $V > 0$  and profile  $w \in C^3(\mathbb{R})$  satisfying  $w'(r) > 0$ ,  $r \in \mathbb{R}$ , and  $(w(r), w'(r)) \rightarrow (0, 0)$  as  $r \rightarrow -\infty$ , as well as  $(w(r), w'(r)) \rightarrow (1, 0)$  as  $r \rightarrow \infty$ .*

*This travelling wave is stable in the sense that  $v \equiv 0$  is a stable equilibrium of (5.7) in  $X_\gamma = W_p^{2-2/p}(\mathbb{R})$ . Moreover there exists  $\delta > 0$  such that if  $|v_0|_\gamma < \delta$ , then the solution  $v$  of (5.7) with  $v(0) = v_0$  exists globally and converges at an exponential rate in  $X_\gamma$  to some equilibrium  $v_\infty$ , i.e. to some element of the set  $\mathcal{E}$  defined in (5.8). In this sense, any solution  $u$  of (5.1) that starts sufficiently*

close to  $w$  exists globally and converges at an exponential rate as  $t \rightarrow \infty$  to some translate  $w(x + Vt + \alpha)$ ,  $\alpha \in \mathbb{R}$ , of the travelling wave solution.

This approach applies to many other travelling wave solutions of quasilinear parabolic systems, as soon as the condition of normal stability is satisfied.

## 6. CONVERGENCE FOR ABSTRACT QUASILINEAR PROBLEMS II

We return to the setting of Section 2 for the case that  $\sigma(A_0)$  also contains an unstable part, i.e. we now assume that

$$\sigma(A_0) = \{0\} \cup \sigma_s \cup \sigma_u, \quad \text{with } \sigma_s \subset \mathbb{C}_+, \sigma_u \subset \mathbb{C}_-, \quad (6.1)$$

such that  $\sigma_u \neq \emptyset$ . In this situation we can prove the following result.

**Theorem 6.1.** *Let  $1 < p < \infty$ . Suppose  $u_* \in V \cap X_1$  is an equilibrium of (2.1), and suppose that the functions  $(A, F)$  satisfy (2.2). Suppose further that  $A(u_*)$  has the property of maximal  $L_p$ -regularity. Let  $A_0$  be the linearization of (2.1) at  $u_*$ . Suppose that  $u_*$  is normally hyperbolic, i.e. assume that*

- (i) *near  $u_*$  the set of equilibria  $\mathcal{E}$  is a  $C^1$ -manifold in  $X_1$  of dimension  $m \in \mathbb{N}$ ,*
- (ii) *the tangent space for  $\mathcal{E}$  at  $u_*$  is given by  $N(A_0)$ ,*
- (iii) *0 is a semi-simple eigenvalue of  $A_0$ , i.e.  $N(A_0) \oplus R(A_0) = X_0$ ,*
- (iv)  *$\sigma(A_0) \cap i\mathbb{R} = \{0\}$ ,  $\sigma_u := \sigma(A_0) \cap \mathbb{C}_- \neq \emptyset$ .*

*Then there exists  $\delta > 0$  such that if the unique solution  $u(t)$  of (2.1) exists globally and satisfies  $u(t) \in B_{X_\gamma}(u_*, \delta)$  for all  $t \geq 0$ , then  $u(t)$  converges at an exponential rate to some  $u_\infty \in \mathcal{E}$  in  $X_\gamma$  as  $t \rightarrow \infty$ . The equilibrium  $u_*$  is unstable in  $X_\gamma$  and even in  $X_0$ .*

*Proof.* (a) Let  $P^l$  denote the spectral projections corresponding to the spectral sets  $\sigma_l$ , where  $\sigma_c = \{0\}$  and  $\sigma_s, \sigma_u$  are as in (6.1). Let  $X_j^l = P^l(X_j)$ ,  $l \in \{c, s, u\}$ , where these spaces are equipped with the norms of  $X_j$  for  $j \in \{0, 1, \gamma\}$ . We may assume that  $X_1$  is equipped with the graph norm of  $A_0$ , i.e.  $|v|_1 := |v|_0 + |A_0 v|_0$  for  $v \in X_1$ . Since the operator  $-A_0$  generates an analytic  $C_0$ -semigroup on  $X_0$ ,  $\sigma_u$  is a compact spectral set for  $A_0$ . This implies that  $P^u(X_0) \subset X_1$ . Consequently,  $X_0^u$  and  $X_1^u$  coincide as vector spaces. In addition, since  $A_u$ , the part of  $A_0$  in  $X_0^u$ , is invertible, we conclude that the spaces  $X_j^u$  carry equivalent norms. We set  $X^u := X_0^u = X_1^u$  and equip  $X^u$  with the norm of  $X_0$ , that is,  $X^u = (X^u, |\cdot|_0)$ . As in the proof of Theorem 2.1 we obtain the decomposition

$$X_1 = X^c \oplus X_1^s \oplus X^u, \quad X_0 = X^c \oplus X_0^s \oplus X^u,$$

and this decomposition reduces  $A_0$  into  $A_0 = A_c \oplus A_s \oplus A_u$ , where  $A_l$  is the part of  $A_0$  in  $X_0^l$  for  $l \in \{c, s, u\}$ . It follows that  $\sigma(A_l) = \sigma_l$  for  $l \in \{c, s, u\}$ . Moreover, due to assumption (iii),  $A_c \equiv 0$ .

(b) Let  $\Phi$  be the mapping obtained in step (b) of the proof of Theorem 2.1, and set  $\phi_l(x) := P^l \Phi(x)$  for  $l \in \{s, u\}$  and for  $x \in B_{X^c}(0, \rho_0)$ . Then

$$\phi_l \in C^1(B_{X^c}(0, \rho_0), X_1^l), \quad \phi_l(0) = \phi_l'(0) = 0 \quad \text{for } l \in \{s, u\}. \quad (6.2)$$

These mappings parametrize the manifold  $\mathcal{E}$  of equilibria near  $u_*$  via

$$[x \mapsto (x + \phi_s(x) + \phi_u(x) + u_*)], \quad x \in B_{X^c}(0, \rho_0).$$

We may assume that  $\rho_0$  has been chosen small enough so that

$$\|\phi'_l(x)\|_{\mathcal{B}(X^c, X_l^1)} \leq 1, \quad x \in B_{X^c}(0, \rho_0), \quad l \in \{s, u\}. \quad (6.3)$$

(c) The equilibrium equation (2.7) now corresponds to the system

$$\begin{aligned} P^c G(x + \phi_s(x) + \phi_u(x)) &= 0, \\ P^l G(x + \phi_s(x) + \phi_u(x)) &= A_l \phi_l(x), \quad x \in B_{X^c}(0, \rho_0), \quad l \in \{s, u\}. \end{aligned} \quad (6.4)$$

The canonical variables are

$$x = P^c v, \quad y = P^s v - \phi_s(x), \quad z = P^u v - \phi_u(x)$$

and the canonical form of the system is given by

$$\begin{cases} \dot{x} = T(x, y, z), & x(0) = x_0, \\ \dot{y} + A_s y = R_s(x, y, z), & y(0) = y_0, \\ \dot{z} + A_u z = R_u(x, y, z), & z(0) = z_0. \end{cases} \quad (6.5)$$

Here the functions  $T$ ,  $R_s$ , and  $R_u$  are given by

$$\begin{aligned} T(x, y, z) &= P^c(G(x + y + z + \phi_s(x) + \phi_u(x)) - G(x + \phi_s(x) + \phi_u(x))), \\ R_l(x, y, z) &= P^l(G(x + y + z + \phi_s(x) + \phi_u(x)) - G(x + \phi_s(x) + \phi_u(x))) \\ &\quad - \phi'_l(x)T(x, y, z), \quad l \in \{s, u\}, \end{aligned} \quad (6.6)$$

where we have used the equilibrium equations (6.4). Clearly,

$$R_l(x, 0, 0) = T(x, 0, 0) = 0, \quad x \in B_{X^c}(0, \rho_0), \quad l \in \{s, u\},$$

showing that the equilibrium set  $\mathcal{E}$  of (2.1) near  $u_*$  has been reduced to the set  $B_{X^c}(0, \rho_0) \times \{0\} \times \{0\} \subset X^c \times X^s \times X^u$ .

There is a unique correspondence between the solutions of (2.1) close to  $u_*$  in  $X_\gamma$  and those of (6.5) close to 0. We again call (6.5) the *normal form* of (2.1) near its *normally hyperbolic* equilibrium  $u_*$ .

(d) The estimates for  $R_l$  and  $T$  are similar to those derived in Section 2, and we have

$$|T(x, y, z)|, |R_l(x, y, z)|_0 \leq C_2(\eta + r)(|y|_1 + |z|), \quad l \in \{s, u\}, \quad (6.7)$$

for all  $x, z \in \bar{B}_{X^c}(0, \rho)$ ,  $\tilde{l} \in \{c, u\}$ , and  $y \in \bar{B}_{X_\gamma^s}(0, \rho) \cap X_1$ , where  $r = 5\rho$ . We remind that the spaces  $X_j^l$  have been given the norm of  $X_0^l$  for  $l \in \{c, u\}$  and  $j \in \{0, 1, \gamma\}$ .

(e) Let  $(x(t), y(t), z(t))$  be a global solution of (6.5) that satisfies

$$|x(t)|, |y(t)|_\gamma, |z(t)| \leq \rho, \quad \text{for all } t \geq 0.$$

Similarly to the proof of Theorem 2.1, part (e), we obtain from (6.7)

$$\|e^{\omega t} y\|_{\mathbb{E}_1(\infty)} \leq 2M_1 |y_0|_\gamma + 2\beta \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)}, \quad (6.8)$$

where  $\omega \in (0, \inf\{\operatorname{Re} \lambda : \lambda \in \sigma(A_s)\})$  is a fixed number and  $\beta$  is given in (2.21). Repeating the estimates leading up to (2.24) we now get

$$|e^{\omega t} y(t)|_\gamma \leq M_2 |y_0|_\gamma + 2c_0 \beta \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)}, \quad t \geq 0. \quad (6.9)$$

From the equation for  $z$  we infer that

$$\begin{aligned} z(t) &= e^{-A_u t} z_0 + \int_0^t e^{-A_u(t-s)} R_u(x(s), y(s), z(s)) ds \\ &= e^{-A_u t} [z_0 + \int_0^\infty e^{A_u s} R_u(x(s), y(s), z(s)) ds] \\ &\quad - \int_t^\infty e^{-A_u(t-s)} R_u(x(s), y(s), z(s)) ds \\ &= - \int_t^\infty e^{-A_u(t-s)} R_u(x(s), y(s), z(s)) ds, \quad t \geq 0, \end{aligned} \quad (6.10)$$

since the last integral is bounded, and  $e^{-A_u t}$  is an exponentially growing group for  $t \rightarrow \infty$ . Using (6.10) and the estimate for  $R_u$  from (6.7) and proceeding as in the proof of Young's inequality for convolution integrals one shows that

$$\|e^{\omega t} z\|_{\mathbb{E}_0(\infty)} \leq C_3 \beta (\|e^{\omega t} y\|_{\mathbb{E}_1(\infty)} + \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)}). \quad (6.11)$$

Making  $\beta$  sufficiently small (by decreasing  $\eta$  and, accordingly,  $r$ ) it follows from (6.8) and (6.11) that

$$\|e^{\omega t} y\|_{\mathbb{E}_1(\infty)} + \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)} \leq C_4 |y_0|_\gamma.$$

This estimate in turn, together with (6.9), implies  $|y(t)|_\gamma \rightarrow 0$  and  $|z(t)| \rightarrow 0$  exponentially fast as  $t \rightarrow \infty$ . As in the proof of Theorem 2.1 part (f) we get

$$x(t) \rightarrow x_\infty := x_0 + \int_0^\infty T(x(s), y(s), z(s)) ds.$$

This yields existence of the limit

$$v_\infty := \lim_{t \rightarrow \infty} v(t) = x_\infty + \phi_s(x_\infty) + \phi_u(x_\infty) \in \mathcal{E}.$$

Similar arguments as in Section 2 yield exponential convergence of  $v(t)$  to  $v_\infty$ . The first assertion of Theorem 6.1 then follows by setting

$$\delta = \rho / (\|P^c\|_{\mathcal{B}(X_0)} + \|P^s\|_{\mathcal{B}(X_\gamma)} + \|P^u\|_{\mathcal{B}(X_0)}).$$

Finally, the instability assertion is a consequence of [23, Theorem 6.2].  $\square$

A result similar to Theorem 6.1 is also valid in the setting of Section 3. We leave the details to the interested reader.

## REFERENCES

- [1] H. Amann, Dynamic theory of quasilinear parabolic equations. II. Reaction–diffusion systems. *Differential Integral Equations* **3** (1990), 13–75.
- [2] H. Amann, *Linear and Quasilinear Parabolic Problems, Vol. I. Abstract Linear Theory*. Monographs in Mathematics **89**, Birkhäuser, Boston (MA), 1995.
- [3] H. Amann, Maximal regularity for nonautonomous evolution equations. *Adv. Nonlinear Stud.* **4** (2004), 417–430.
- [4] H. Amann, Quasilinear parabolic problems via maximal regularity. *Adv. Differential Equations* **10** (2005), 1081–1110.
- [5] B. Aulbach, *Continuous and discrete dynamics near manifolds of equilibria*, Lecture Notes in Math. **1058**, Springer, Berlin, 1984.
- [6] P. Bates and C. Jones, Invariant manifolds for semilinear partial differential equations. In: *Dynamics Reported* **2**, Wiley, Chichester, 1989, 1–38.
- [7] C.-M. Brauner, J. Hulshof and A. Lunardi, A general approach to stability in free boundary problems. *J. Differential Equations* **164** (2000), 16–48.
- [8] Ph. Clément and S. Li, Abstract parabolic quasilinear equations and applications to a groundwater flow problem. *Adv. Math. Sci. Appl.* **3** (1994), 17–32.
- [9] Shangbin Cui, Lie group action and stability analysis of stationary solutions for a free boundary problem modeling tumor growth, *ArXiv:0712.2483v1*, preprint.
- [10] G. Da Prato and A. Lunardi, Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in Banach space. *Arch. Rational Mech. Anal.* **101** (1988), 115–141.
- [11] R. Denk, M. Hieber, and J. Prüss,  $\mathcal{R}$ -boundedness and problems of elliptic and parabolic type. *Memoirs of the AMS* vol. **166**, No. 788 (2003).
- [12] R. Denk, M. Hieber, and J. Prüss, Optimal  $L_p - L_q$ -regularity for parabolic problems with inhomogeneous boundary data. *Math. Z.* **257** (2007), 193–224.
- [13] R. Denk, J. Prüss, and R. Zacher, Maximal  $L_p$ -regularity of parabolic problems with boundary dynamics of relaxation type. Submitted (2008).
- [14] J. Escher, J. Prüss, and G. Simonett, Analytic solutions for a Stefan problem with Gibbs-Thomson correction. *J. Reine Angew. Math.* **563** (2003), 1–52.
- [15] J. Escher and G. Simonett, A center manifold analysis for the Mullins-Sekerka model. *J. Differential Equations* **143** (1998), 267–292.
- [16] A. Friedman and F. Reitich, Quasi-static motion of a capillary drop. II. The three-dimensional case. *J. Differential Equations* **186** (2002), 509–557.
- [17] M. Günther and G. Prokert, Existence results for the quasistationary motion of a free capillary liquid drop. *Z. Anal. Anwendungen* **16** (1997), 311–348.
- [18] D. Henry, *Geometric theory of semilinear parabolic equations*. Lecture Notes in Math. **840**, Springer, Berlin, 1981.
- [19] Y. Latushkin, J. Prüss, and R. Schnaubelt, Stable and unstable manifolds for quasilinear parabolic systems with fully nonlinear boundary conditions. *J. Evol. Equ.* **6** (2006), 537–576.
- [20] Y. Latushkin, J. Prüss, and R. Schnaubelt, Center manifolds and dynamics near equilibria for quasilinear parabolic systems with fully nonlinear boundary conditions. *Discrete Contin. Dyn. Syst. Ser. B* **9** (2008), 595–633.
- [21] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Progress in Nonlinear Differential Equations and their Applications **16**, Birkhäuser, Basel, 1995.
- [22] A. Mielke, Locally invariant manifolds for quasilinear parabolic equations. *Rocky Mountain J. Math.* **21** (1991), 707–714.
- [23] J. Prüss, Maximal regularity for evolution equations in  $L_p$ -spaces. *Conf. Semin. Mat. Univ. Bari* **285** (2003), 1–39.
- [24] J. Prüss, J. Saal, and G. Simonett, Existence of analytic solutions for the classical Stefan problem. *Math. Ann.* **338** (2007), 703–755.

- [25] J. Prüss and G. Simonett, Stability of equilibria for the Stefan problem with surface tension. *SIAM J. Math. Anal.* To appear.
- [26] G. Simonett, Center manifolds for quasilinear reaction-diffusion systems. *Differential Integral Equations* **8** (1995), 753–796.
- [27] A. Vanderbauwhede, G. Iooss, Center manifold theory in infinite dimensions. *Dynamics reported: expositions in dynamical systems*. Dynam. Report. Expositions Dynam. Systems (N.S.) **1**, Springer, Berlin, 1992, 125–163.

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