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Abstract

We prove existence and uniqueness of weak solutions to certain abstract evolutionary integro-differential equations in Hilbert spaces, including evolution equations of fractional order less than 1. Our results apply, e.g., to parabolic partial integro-differential equations in divergence form with merely bounded and measurable coefficients.

AMS subject classification: 45N05, 45K05

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1 Introduction

Let $\mathcal{V}$ and $\mathcal{H}$ be real separable Hilbert spaces such that $\mathcal{V}$ is densely and continuously embedded into $\mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$, and

$$(h, v)_{\mathcal{H}} = \langle h, v\rangle_{\mathcal{V}' \times \mathcal{V}}, \quad h \in \mathcal{H},\, v \in \mathcal{V}, \tag{1}$$

where $(\cdot,\cdot)_{\mathcal{H}}$ and $\langle \cdot,\cdot\rangle_{\mathcal{V}'\times\mathcal{V}}$ denote the scalar product in $\mathcal{H}$ and the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$, respectively.

In this paper we study the abstract problem

$$\frac{d}{dt}\Big([k*(u-x)](t), v\Big)_{\mathcal{H}} + a(t, u(t), v) = \langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad v \in \mathcal{V},\ \text{a.a.}\, t \in (0,T), \tag{2}$$

where d/dt means the generalized derivative of real functions on $(0,T)$, $k \in L_{1,\,loc}(\mathbb{R}_+)$ is a scalar kernel that belongs to a certain kernel class (k is of type $\mathcal{PC}$, see Definition 2.1 below), $k*u$ stands for the convolution on the positive halfline, i.e. $(k*u)(t) = \int_0^t k(t-\tau)u(\tau)\,d\tau$, $t \geq 0$, and $a : (0,T)\times\mathcal{V}\times\mathcal{V} \to \mathbb{R}$ is a bounded $\mathcal{V}$-coercive bilinear form. Further, $x \in \mathcal{H}$ and $f \in L_2([0,T];\mathcal{V}')$ are given data.

We seek a solution u of (2) in the regularity class

$$W(x,\mathcal{V},\mathcal{H}) := \{u \in L_2([0,T];\mathcal{V}) :\, k*(u-x) \in {}_0H^1_2([0,T];\mathcal{V}')\},$$

where the zero means vanishing trace at $t=0$. The vector x can be regarded as initial data for u, at least in a weak sense. If e.g. u, and $\frac{d}{dt}(k*[u-x])$ belong to $C([0,T];\mathcal{V}')$, then the condition $k*(u-x)(0) = 0$ implies $u(0) = x$, see Section 3.

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An important example for the kernel k we have in mind is given by

$$k(t) \,=\, g_{1-\alpha}(t)e^{-\mu t}, \quad t>0,\ \alpha\in(0,1),\ \mu\ge 0, \tag{3}$$

where g_β denotes the Riemann-Liouville kernel

$$g_\beta(t)=\frac{t^{\beta-1}}{\Gamma(\beta)}\,,\quad t>0,\ \beta>0.$$

In this case, (2) amounts to an abstract differential equation of fractional order $\alpha\in(0,1)$. Recall that for a (sufficiently smooth) function v on $\mathbb{R}_+$, the Riemann-Liouville fractional derivative $D_t^\beta v$ of order $\beta\in(0,1)$ is defined by $D_t^\beta v=\frac{d}{dt}\,(g_{1-\beta}*v)$.

In this paper we prove that the problem (2) possesses exactly one solution in the class $W(x,\mathcal{V},\mathcal{H})$. This result can be regarded as the analogue of the well-known existence and uniqueness result for the corresponding abstract parabolic equation

$$\left\{\begin{array}{c} \frac{d}{dt}\,(u(t),v)_{\mathcal{H}}+a(t,u(t),v)=\langle f(t),v\rangle_{\mathcal{V}'\times\mathcal{V}},\quad v\in\mathcal{V},\ \text{a.a.}\,t\in(0,T),\\ u(0)=x\ \in\mathcal{H},\\ u\in H^1_2([0,T];\mathcal{V}')\cap L_2([0,T];\mathcal{V}), \end{array}\right. \tag{4}$$

see e.g. Theorem 4.1 and Remark 4.3 in Chapter 4 in [7] or [12, Section 23]. We point out that concerning time regularity the bilinear form a is *only* assumed to be *measurable* in t. This allows, e.g., to treat parabolic partial integro-differential equations in divergence form with merely bounded and measurable coefficients, see Section 4.

The proof of the main result is based on the Galerkin method and suitable *a priori* estimates for weak solutions of (2). These estimates are derived by means of the basic identity (17) (see below) for absolutely continuous kernels. It has been known before but does not seem to appear in the literature in the context of problems of the form (2). We remark that recently ([9]) the identity (17) was successfully employed to construct Lyapunov functions for certain nonlinear differential equations of fractional order between 0 and 2.

In order to be able to apply (17), we approximate the kernel k by the sequence (k_n) which is obtained from the Yosida approximation of the operator B defined by $Bv=\frac{d}{dt}(k*v)$, e.g. in $L_2([0,T])$. This method was already used in [9], we also refer to [5], where a more general class of integro-differential operators (in time) is studied.

Note that (2) is equivalent to the equation

$$\frac{d}{dt}[k*(u-x)](t)+A(t)u(t)=f(t),\quad \text{a.a.}\,t\in(0,T), \tag{5}$$

in $\mathcal{V}'$, where the operator $A(t):\mathcal{V}\to\mathcal{V}'$ is defined by

$$\langle A(t)u,v\rangle_{\mathcal{V}'\times\mathcal{V}}=a(t,u,v),\quad u,\,v\in\mathcal{V}. \tag{6}$$

For equations of the form (5) with $A(t)\equiv A$ there exists a vast literature, even in general Banach spaces, see e.g. [5], and [8] and the references given therein. However, in the case of time-dependent A and without smoothness assumption nothing seems to be known in the literature concerning existence and uniqueness.

The paper is organized as follows. In Section 2 we introduce the notion of kernels of type $\mathcal{PC}$, and describe the approximation of such kernels used in this paper. We further prove a trace theorem for functions in the class $W(x,\mathcal{V},\mathcal{H})$, and state the basic identity (17). Section 3 contains the main result on existence and uniqueness as well as some further interpolation results for functions from $W(x,\mathcal{V},\mathcal{H})$. We also look at the special case of fractional evolution equations. In Section 4 we apply the abstract results to second-order parabolic partial integro-differential equations in divergence form.

2 Preliminaries

The following class of kernels is basic to our treatment of (2).

Definition 2.1 *A kernel $k \in L_{1,\,loc}(\mathbb{R}_+)$ is called to be* of type $\mathcal{PC}$ *if it is nonnegative and nonincreasing, and there exists a kernel $l \in L_{1,\,loc}(\mathbb{R}_+)$ such that $k*l = 1$ in $(0,\infty)$. In this case, we say that (k,l) is a $\mathcal{PC}$* pair *and write $(k,l) \in \mathcal{PC}$.*

From $(k,l) \in \mathcal{PC}$ it follows that l is completely positive (see e.g. Theorem 2.2 in [2]), in particular l is nonnegative, cf. [2, Proposition 2.1].

An important example is given by

$$k(t) = g_{1-\alpha}(t)e^{-\mu t} \quad \text{and} \quad l(t) = g_\alpha(t)e^{-\mu t} + \mu(1*[g_\alpha e^{-\mu\cdot}])(t), \quad t>0, \tag{7}$$

with $\alpha \in (0,1)$ and $\mu \geq 0$. Both k and l are strictly positive and decreasing; observe that $l'(t) = g_\alpha'(t)e^{-\mu t} < 0$, $t > 0$. The Laplace transforms are given by

$$\hat{k}(\lambda) = \frac{1}{(\lambda+\mu)^{1-\alpha}}, \quad \hat{l}(\lambda) = \frac{1}{(\lambda+\mu)^{\alpha}}\Big(1+\frac{\mu}{\lambda}\Big), \quad \mathrm{Re}\,\lambda > 0,$$

which shows that $k*l = 1$ on $(0,\infty)$. Hence we have both $(k,l) \in \mathcal{PC}$, and $(l,k) \in \mathcal{PC}$.

$\mathcal{PC}$ pairs enjoy a useful stability property with respect to exponential shifts. Writing $k_\mu(t) = k(t)e^{-\mu t}$, $t > 0$, $\mu \geq 0$, we have

$$(k,l) \in \mathcal{PC} \quad \Rightarrow \quad (k_\mu, l_\mu + \mu(1*l_\mu)) \in \mathcal{PC}, \;\; \mu \geq 0. \tag{8}$$

To prove (8), we first note that for any $\mu \geq 0$, k_μ is evidently nonnegative and nonincreasing, and $l_\mu + \mu(1*l_\mu)$ is nonnegative. Multiplying $k*l = 1$ by $1_\mu(t) = e^{-\mu t}$ gives $k_\mu * l_\mu = 1_\mu$, which in turn implies that $\mu k_\mu * 1 * l_\mu = \mu 1 * 1_\mu = 1 - 1_\mu$. Adding these relations, we see that $k_\mu * [l_\mu + \mu(1*l_\mu)] = 1$.

We next discuss an important method of approximating kernels of type $\mathcal{PC}$. Let $(k,l) \in \mathcal{PC}$, $T > 0$, and $\mathcal{H}$ be a real Hilbert space. Then the operator B defined by

$$Bu = \frac{d}{dt}\,(k*u), \;\; D(B) = \{u \in L_2([0,T];\mathcal{H}) :\; k*u \in {}_0H^1_2([0,T];\mathcal{H})\},$$

is known to be m-accretive in $L_2([0,T];\mathcal{H})$, cf. [1], [3], and [5]. Its Yosida approximations B_n, defined by $B_n = nB(n+B)^{-1}$, $n \in \mathbb{N}$, enjoy the property that for any $u \in D(B)$, one has $B_n u \to Bu$ in $L_2([0,T];\mathcal{H})$ as $n \to \infty$. Furthermore, we have the representation

$$B_n u = \frac{d}{dt}\,(k_n * u), \quad u \in L_2([0,T];\mathcal{H}),\; n \in \mathbb{N}, \tag{9}$$

where $k_n = ns_n$, with s_n being the solution of the scalar-valued Volterra equation

$$s_n(t) + n(l * s_n)(t) = 1, \quad t > 0,\ n \in \mathbb{N},$$

see e.g. [9]. The kernels s_n, $n \in \mathbb{N}$, are nonnegative and nonincreasing in $(0, \infty)$, and $s_n \in H_1^1([0,T])$, cf. [8, Prop. 4.5]. Consequently, the kernels k_n, $n \in \mathbb{N}$, enjoy the same properties. Moreover, $k_n \to k$ in $L_1([0,T])$ as $n \to \infty$.

Let now $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above, that is $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. In the theory of abstract parabolic equations the continuous embedding

$$H_2^1([0,T];\mathcal{V}') \cap L_2([0,T];\mathcal{V}) \hookrightarrow C([0,T];\mathcal{H}) \tag{10}$$

is well-known, see e.g. Proposition 2.1 and Theorem 3.1 in Chapter 1 of [7], or Proposition 23.23 in [12]. The following theorem provides the analogue of (10) in the case of the space $W(x, \mathcal{V}, \mathcal{H})$.

Theorem 2.1 *Let $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above ($\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$). Let further $T > 0$, and $k \in L_{1,\,loc}(\mathbb{R}_+)$ be of type $\mathcal{PC}$. Suppose that $x \in \mathcal{H}$, and $u \in W(x, \mathcal{V}, \mathcal{H})$. Then $k * (u - x)$ and $k * u$ belong to the space $C([0,T];\mathcal{H})$ (after possibly being redefined on a set of measure zero). The mapping $\{t \mapsto |k * u|^2_{\mathcal{H}}(t)\}$ is absolutely continuous on $[0,T]$, with*

$$\frac{d}{dt}\,|k * u|^2_{\mathcal{H}}(t) = 2\Big\langle [k * (u - x)]'(t), [k * u](t)] \Big\rangle_{\mathcal{V}' \times \mathcal{V}} + 2k(t)\Big(x, (k * u)(t)\Big)_{\mathcal{H}} \tag{11}$$

for a.a. $t \in [0,T]$. Furthermore,

$$|k * u|_{C([0,T];\mathcal{H})} \le C\Big(\Big|\frac{d}{dt}[k * (u - x)]\Big|_{L_2([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} + |x|_{\mathcal{H}}\Big), \tag{12}$$

the constant C depending only on T, $|k|_{L_1([0,T])}$, and the constant of the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$.

We remark that in the case $x = 0$, the property $k * u \in C([0,T];\mathcal{H})$ follows immediately from the embedding (10). In fact, $u \in L_2([0,T];\mathcal{V})$ implies $k * u \in L_2([0,T];\mathcal{V})$, by Young's inequality, and so

$$k * u \in H_2^1([0,T];\mathcal{V}') \cap L_2([0,T];\mathcal{V}) \hookrightarrow C([0,T];\mathcal{H}).$$

We point out that for $x \neq 0$ this simple reduction is not feasible.

Proof of Theorem 2.1. Note first that $(k * x)(\cdot) = (1 * k)(\cdot)x \in H_1^1([0,T];\mathcal{H}) \hookrightarrow C([0,T];\mathcal{H})$. Thus $k * (u - x) \in C([0,T];\mathcal{H})$ if and only if $k * u \in C([0,T];\mathcal{H})$.

Let $k_n \in H_1^1([0,T])$, $n \in \mathbb{N}$, be the kernel associated with the Yosida approximation B_n of the operator

$$Bv = \frac{d}{dt}\,(k * v), \;\; D(B) = \{v \in L_2([0,T];\mathcal{V}') :\, k * v \in {}_0H_2^1([0,T];\mathcal{V}')\}. \tag{13}$$

Then $k_n * u \in H_2^1([0,T];\mathcal{V})$, and we have for n, $m \in \mathbb{N}$,

$$\frac{d}{dt}\,\Big|(k_n * u)(t) - (k_m * u)(t)\Big|^2_{\mathcal{H}} = 2\,\Big([k_n * u]'(t) - [k_m * u]'(t), [k_n * u](t) - [k_m * u](t)\Big)_{\mathcal{H}}.$$

Thus, in view of (1) and Young's inequality,

$$\begin{aligned}
&\Big|(k_n*u)(t)-(k_m*u)(t)\Big|_{\mathcal{H}}^2=\Big|(k_n*u)(s)-(k_m*u)(s)\Big|_{\mathcal{H}}^2\\
&+2\int_s^t\Big\langle[k_n*(u-x)]'(\tau)-[k_m*(u-x)]'(\tau),[k_n*u](\tau)-[k_m*u](\tau)\Big\rangle_{\mathcal{V}'\times\mathcal{V}}\,d\tau\\
&+2\int_s^t[k_n(\tau)-k_m(\tau)]\Big(x,[k_n*u](\tau)-[k_m*u](\tau)\Big)_{\mathcal{H}}\,d\tau\\
\le&\Big|(k_n*u)(s)-(k_m*u)(s)\Big|_{\mathcal{H}}^2+\Big|\frac{d}{dt}[k_n*(u-x)]-\frac{d}{dt}[k_m*(u-x)]\Big|_{L_2([0,T];\mathcal{V}')}^2\\
&+\Big|k_n*u-k_m*u\Big|_{L_2([0,T];\mathcal{V})}^2+2|x|_{\mathcal{H}}^2|k_n-k_m|_{L_1([0,T])}^2+\frac{1}{2}\Big|k_n*u-k_m*u\Big|_{C([0,T];\mathcal{H})}^2 \qquad (14)
\end{aligned}$$

for all s, $t\in[0,T]$. Since $k_n\to k$ in $L_1([0,T])$ as $n\to\infty$, we have $k_n*u\to k*u$ in $L_2([0,T];\mathcal{H})$ as well as in $L_2([0,T];\mathcal{V})$. Further, $u-x\in D(B)$ implies that $\frac{d}{dt}[k_n*(u-x)]\to\frac{d}{dt}[k*(u-x)]$ in $L_2([0,T];\mathcal{V}')$.

We fix now a point $s\in(0,T)$ for which

$$(k_n*u)(s)\;\to\;(k*u)(s)\quad\text{in }\mathcal{H}\text{ as }n\to\infty.$$

Taking then in (14) the maximum over all $t\in[0,T]$ and absorbing the last term, it follows that (k_n*u) is a Cauchy sequence in $C([0,T];\mathcal{H})$. Thus k_n*u converges in $C([0,T];\mathcal{H})$ to some $v\in C([0,T];\mathcal{H})$. Since we also know that $k_n*u\to k*u$ in $L_2([0,T];\mathcal{H})$, we deduce $k*u=v$ a.e. in $[0,T]$, proving the first part of the theorem.

Similarly as above we see that

$$\begin{aligned}
|(k_n*u)(t)|_{\mathcal{H}}^2=|(k_n*u)(s)|_{\mathcal{H}}^2+2\int_s^t\Big\langle[k_n*(u-x)]'(\tau),[k_n*u](\tau)\Big\rangle_{\mathcal{V}'\times\mathcal{V}}\,d\tau\\
+2\int_s^t k_n(\tau)\Big(x,(k_n*u)(\tau)\Big)_{\mathcal{H}}\,d\tau
\end{aligned}$$

for all $s,t\in[0,T]$, and $n\in\mathbb{N}$. Taking the limits as $n\to\infty$ we obtain

$$\begin{aligned}
|(k*u)(t)|_{\mathcal{H}}^2=|(k*u)(s)|_{\mathcal{H}}^2+2\int_s^t\Big\langle[k*(u-x)]'(\tau),[k*u](\tau)\Big\rangle_{\mathcal{V}'\times\mathcal{V}}\,d\tau\\
+2\int_s^t k(\tau)\Big(x,(k*u)(\tau)\Big)_{\mathcal{H}}\,d\tau \qquad (15)
\end{aligned}$$

for all $s,t\in[0,T]$. Hence $\{t\mapsto|k*u|_{\mathcal{H}}^2(t)\}$ is absolutely continuous on $[0,T]$, and (11) holds true.

To obtain (12), we estimate the integral terms in (15) similarly as for (14) and integrate with respect to s. This yields

$$\begin{aligned}
|(k*u)(t)|_{\mathcal{H}}^2\le\frac{1}{T}\,|(k*u)|_{L_2([0,T];\mathcal{H})}^2+\Big|\frac{d}{dt}[k*(u-x)]\Big|_{L_2([0,T];\mathcal{V}')}^2\\
+|k*u|_{L_2([0,T];\mathcal{V})}^2+2|x|_{\mathcal{H}}^2|k|_{L_1([0,T])}^2+\frac{1}{2}|k*u|_{C([0,T];\mathcal{H})}^2 \qquad (16)
\end{aligned}$$

for all $t \in [0,T]$. We then take the maximum over all $t \in [0,T]$, absorb the last term, and use Young's inequality for convolutions, to the result

$$\frac{1}{2}|k*u|^2_{C([0,T];\mathcal{H})} \le \frac{1}{T}\,|k|^2_{L_1([0,T])}|u|^2_{L_2([0,T];\mathcal{H})} + \Big|\frac{d}{dt}[k*(u-x)]\Big|^2_{L_2([0,T];\mathcal{V}')} + |k|^2_{L_1([0,T])}|u|^2_{L_2([0,T];\mathcal{V})} + 2|x|^2_{\mathcal{H}}|k|^2_{L_1([0,T])},$$

which implies (12). $\square$

The following lemma is of fundamental importance with regard to *a priori* estimates for (2) and certain interpolation results for functions in the class $W(x,\mathcal{V},\mathcal{H})$.

Lemma 2.1 *Let $\mathcal{H}$ be a real Hilbert space and $T>0$. Then for any $k \in H_1^1([0,T])$ and any $v \in L_2([0,T];\mathcal{H})$ there holds*

$$\Big(\frac{d}{dt}\,(k*v)(t), v(t)\Big)_{\mathcal{H}} = \frac{1}{2}\,\frac{d}{dt}\,(k*|v(\cdot)|^2_{\mathcal{H}})(t) + \frac{1}{2}\,k(t)|v(t)|^2_{\mathcal{H}} + \frac{1}{2}\int_0^t [-\dot{k}(s)]\,|v(t)-v(t-s)|^2_{\mathcal{H}}\,ds, \quad a.a.\ t \in (0,T). \tag{17}$$

The assertion of Lemma 2.1 follows from a straightforward computation. We remark that a more general version of (17) (with $\mathcal{H} = \mathbb{R}^n$) in integrated form can be found in [6, Lemma 18.4.1].

3 The main existence and uniqueness result

In this section we are concerned with existence and uniqueness for the abstract problem (2). Recall that $\mathcal{V}$ and $\mathcal{H}$ are real separable Hilbert spaces such that $\mathcal{V}$ is densely and continuously embedded into $\mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we have $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$, and the relation (1) holds. It will be assumed that $\dim \mathcal{V} = \infty$.

We will suppose that the following assumptions are satisfied.

(Hk) $(k,l) \in \mathcal{PC}$ for some $l \in L_{1,\,loc}(\mathbb{R}_+)$.

(Hd) $x \in \mathcal{H}$, $f \in L_2([0,T];\mathcal{V}')$.

(Ha) For a.a. $t \in (0,T)$, the mapping $a(t,\cdot,\cdot): \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ is bilinear, and there exist constants $M>0$, $c>0$, and $d \ge 0$, which are independent of t, such that

$$|a(t,u,v)| \le M|u|_{\mathcal{V}}|v|_{\mathcal{V}}, \tag{18}$$

$$a(t,u,u) \ge c|u|^2_{\mathcal{V}} - d|u|^2_{\mathcal{H}}, \tag{19}$$

for all $u,\, v \in \mathcal{V}$ and a.a. $t \in (0,T)$. Moreover, the function $\{t \mapsto a(t,u,v)\}$ is measurable on $(0,T)$ for all $u,\, v \in \mathcal{V}$.

We seek a solution of (2) in the space

$$W(x,\mathcal{V},\mathcal{H}) = \{u \in L_2([0,T];\mathcal{V}):\ k*(u-x) \in {}_0H_2^1([0,T];\mathcal{V}')\}.$$

Note that the vector x plays the role of the initial data for u, at least in a weak sense. If e.g. u, and $\frac{d}{dt}(k*[u-x]) =: \tilde{f}$ belong to $C([0,T];\mathcal{V}')$, then the assumption (Hk) and the condition $k*(u-x)(0)=0$ entail that

$$u - x = \frac{d}{dt}\,(l*k*[u-x]) = l*\tilde{f}$$

in $C([0,T];\mathcal{V}')$, and therefore $u(0)=x$.

In order to construct a solution in the desired class, we will use the Galerkin method. We will assume that

(Hb) $\{w_1, w_2, \ldots\}$ is a basis in $\mathcal{V}$, and (x_m) is a sequence in $\mathcal{H}$ such that $x_m \in \mathrm{span}\{w_1,\ldots,w_m\}$, $m\in\mathbb{N}$, and $x_m \to x$ in $\mathcal{H}$ as $m\to\infty$.

Setting

$$u_m(t) = \sum_{j=1}^m c_{jm}(t) w_j, \qquad x_m = \sum_{j=1}^m \beta_{jm} w_j,$$

and replacing u, x, and v in (2) by u_m, x_m, and w_i, respectively, we formally obtain for every $m\in\mathbb{N}$, the system of Galerkin equations

$$\sum_{j=1}^m \frac{d}{dt}\,[k*(c_{jm}-\beta_{jm})](t)(w_j,w_i)_{\mathcal{H}} + \sum_{j=1}^m c_{jm}(t) a(t,w_j,w_i) = \langle f(t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}}, \tag{20}$$

for a.a. $t\in(0,T)$, where i runs through the set $\{1,\ldots,m\}$.

The main result in this section is the following.

Theorem 3.1 *Let $T>0$, and $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above. Suppose the assumptions (Hk), (Hd), (Ha), and (Hb) hold. Then the problem (2) has exactly one solution u in the space $W(x,\mathcal{V},\mathcal{H})$. The mapping $(x,f)\mapsto u$ is linear, and there exists a constant $M_0>0$ such that*

$$|k*(u-x)|_{H_2^1([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} \le M_0\Big(|x|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big) \tag{21}$$

for all $x\in\mathcal{H}$ and $f\in L_2([0,T];\mathcal{V}')$. Moreover, for every $m\in\mathbb{N}$, the Galerkin equation (20) possesses precisely one solution $u_m\in W(x_m,\mathcal{V},\mathcal{H})$. The sequence (u_m) converges weakly to u in $L_2([0,T];\mathcal{V})$ as $m\to\infty$.

Proof. **Uniqueness.** Suppose that u_1, $u_2\in W(x,\mathcal{V},\mathcal{H})$ are solutions of (2). The difference $u=u_1-u_2$ then belongs to the space $W(0,\mathcal{V},\mathcal{H})$ and satisfies the equation

$$\langle (k*u)'(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} + a(t,u(t),v) = 0, \quad v\in V,\ \text{a.a.}\, t\in(0,T).$$

We may take $v=u(t)$, thereby getting

$$\langle (k*u)'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + a(t,u(t),u(t)) = 0, \quad \text{a.a.}\, t\in(0,T). \tag{22}$$

Let $k_n\in H_1^1([0,T])$, $n\in\mathbb{N}$, be the kernel associated with the Yosida approximation B_n of the operator B defined in (13). Then (22) is equivalent to

$$\langle (k_n*u)'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + a(t,u(t),u(t)) = h_n(t), \quad \text{a.a.}\, t\in(0,T), \tag{23}$$

where

$$h_n(t) = \langle (k_n * u)'(t) - (k * u)'(t), u(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad \text{a.a.}\, t \in (0,T).$$

Since $k_n * u \in H^1_2([0,T];\mathcal{H})$, we may apply (1) to the first term in (23), to the result

$$\Big(\frac{d}{dt}\,(k_n * u)(t), u(t)\Big)_{\mathcal{H}} + a(t, u(t), u(t)) = h_n(t), \quad \text{a.a.}\, t \in (0,T), \tag{24}$$

for all $n \in \mathbb{N}$.

The kernels k_n are nonnegative and nonincreasing. Thus, by Lemma 2.1,

$$\frac{1}{2}\,\frac{d}{dt}\,(k_n * |u(\cdot)|^2_{\mathcal{H}})(t) \;\le\; \Big(\frac{d}{dt}\,(k_n * u)(t), u(t)\Big)_{\mathcal{H}}, \quad \text{a.a.}\, t \in (0,T).$$

The second term in (24) is estimated by means of the abstract Gårding inequality (19) in (Ha). Proceeding this way, it follows from (24) that

$$\frac{d}{dt}\,(k_n * |u(\cdot)|^2_{\mathcal{H}})(t) \le 2d|u(t)|^2_{\mathcal{H}} + 2h_n(t), \quad \text{a.a.}\, t \in (0,T). \tag{25}$$

Observe that all terms in (25) viewed as functions of t belong to $L_1([0,T])$. Therefore we may convolve (25) with the kernel l from assumption (Hk). Letting then n go to ∞ and selecting an appropriate subsequence, if necessary, we arrive at

$$|u(t)|^2_{\mathcal{H}} \le 2d\,(l * |u(\cdot)|^2_{\mathcal{H}})(t), \quad \text{a.a.}\, t \in (0,T). \tag{26}$$

Here we use the fact that $h_n \to 0$ in $L_1([0,T])$, which entails $l * h_n \to 0$ in $L_1([0,T])$, and that

$$l * \frac{d}{dt}\,(k_n * |u(\cdot)|^2_{\mathcal{H}}) = \frac{d}{dt}\,(k_n * l * |u(\cdot)|^2_{\mathcal{H}}) \to \frac{d}{dt}\,(k * l * |u(\cdot)|^2_{\mathcal{H}}) = |u(\cdot)|^2_{\mathcal{H}}$$

in $L_1([0,T])$ as $n \to \infty$.

Since l is nonnegative, (26) implies that $|u(t)|^2_{\mathcal{H}} = 0$ a.e. in $(0,T)$, by the abstract Gronwall lemma [11, Prop. 7.15], i.e. $u = 0$.

Existence. 1. We show first that for every $m \in \mathbb{N}$, the system of Galerkin equations (20) admits a unique solution $\psi := \psi_m := (c_{1m}, \ldots, c_{mm})^T$ on $[0,T]$ in the class $W(\xi, \mathbb{R}^m, \mathbb{R}^m)$, where $\xi := \xi_m := (\beta_{1m}, \ldots, \beta_{mm})^T$.

Since the vectors $w_1, \ldots, w_m$ are linearly independent, the matrix $((w_j, w_i)_{\mathcal{H}}) \in \mathbb{R}^{m\times m}$ is invertible. Hence (20) can be solved for $\frac{d}{dt}\,[k * (c_{jm} - \beta_{jm})]$, which leads to an equivalent system of the form

$$\frac{d}{dt}\,[k * (\psi - \xi)](t) = B(t)\psi(t) + g(t), \quad \text{a.a.}\, t \in (0,T), \tag{27}$$

where $B \in L_\infty([0,T];\mathbb{R}^{m\times m})$, and $g \in L_2([0,T];\mathbb{R}^m)$, by the assumptions (Ha) and (Hd). In order to solve (27), we transform it into the system of Volterra equations

$$\psi(t) = \xi + l * [B(\cdot)\psi(\cdot)](t) + (l * g)(t), \quad \text{a.a.}\, t \in (0,T),$$

which has a unique solution $\psi \in L_2([0,T];\mathbb{R}^m)$, see e.g. [6, Chapter 9]. But then $\psi \in W(\xi, \mathbb{R}^m, \mathbb{R}^m)$, and hence it is also a solution of (27). This shows that for every $m \in \mathbb{N}$, the Galerkin equation (20) has exactly one solution $u_m \in W(x_m, \mathcal{V}, \mathcal{H})$.

2. We next derive *a priori* estimates for the Galerkin solutions. The Galerkin equations (20) are equivalent to

$$\Big(\frac{d}{dt}\,[k*(u_m-x_m)](t),w_i\Big)_{\mathcal{H}}+a(t,u_m(t),w_i)=\langle f(t),w_i\rangle_{\mathcal{V}'\times\mathcal{V}},\quad \text{a.a.}\,t\in(0,T),\tag{28}$$

$i=1,\ldots,m$. Multiplying (28) by c_{im} and summing over i, we obtain

$$\Big(\frac{d}{dt}\,[(k*(u_m-x_m)](t),u_m(t)\Big)_{\mathcal{H}}+a(t,u_m(t),u_m(t))=\langle f(t),u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}}.\tag{29}$$

Let $k_n\in H^1_1([0,T])$, $n\in\mathbb{N}$, be as in the uniqueness part above. Then (29) can be written as

$$\begin{aligned}\Big(\frac{d}{dt}\,(k_n*u_m)(t),u_m(t)\Big)_{\mathcal{H}}&+a(t,u_m(t),u_m(t))\\&=k_n(t)(x_m,u_m(t))_{\mathcal{H}}+\langle f(t),u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}}+h_{mn}(t),\quad \text{a.a.}\,t\in(0,T),\end{aligned}\tag{30}$$

with

$$h_{mn}(t)=\langle[k_n*(u_m-x_m)]'(t)-[k*(u_m-x_m)]'(t),u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}}.$$

Using Lemma 2.1 and inequality (19), we find that

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,(k_n*|u_m(\cdot)|^2_{\mathcal{H}})(t)&+\frac{1}{2}\,k_n(t)|u_m(t)|^2_{\mathcal{H}}+c|u_m(t)|^2_{\mathcal{V}}\\&\le d|u_m(t)|^2_{\mathcal{H}}+k_n(t)(x_m,u_m(t))_{\mathcal{H}}+\langle f(t),u_m(t)\rangle_{\mathcal{V}'\times\mathcal{V}}+h_{mn}(t),\end{aligned}$$

which, by Young's inequality, yields the estimate

$$\frac{d}{dt}\,(k_n*|u_m(\cdot)|^2_{\mathcal{H}})(t)+c|u_m(t)|^2_{\mathcal{V}}\le 2d|u_m(t)|^2_{\mathcal{H}}+k_n(t)|x_m|^2_{\mathcal{H}}+\frac{1}{c}\,|f(t)|^2_{\mathcal{V}'}+2h_{mn}(t).\tag{31}$$

Similarly as in the uniqueness part, we see that $l*h_{mn}\to 0$ and

$$l*\frac{d}{dt}\,(k_n*|u_m(\cdot)|^2_{\mathcal{H}})\to|u_m(\cdot)|^2_{\mathcal{H}}$$

in $L_1([0,T])$ as $n\to\infty$. Consequently, if we convolve (31) with l, and let n tend to ∞, selecting an appropriate subsequence, if necessary, we obtain the estimate

$$|u_m(t)|^2_{\mathcal{H}}\le 2d\,(l*|u_m(\cdot)|^2_{\mathcal{H}})(t)+|x_m|^2_{\mathcal{H}}+\frac{1}{c}\,(l*|f(\cdot)|^2_{\mathcal{V}'})(t)\tag{32}$$

for a.a. $t\in(0,T)$, and all $m\in\mathbb{N}$. By positivity of l, it follows from (32) that

$$|u_m|_{L_2([0,T];\mathcal{H})}\le C\Big(|x_m|_{\mathcal{H}}+|f|_{L_2([0,T];\mathcal{V}')}\Big),\tag{33}$$

where the constant C depends only on c, d, l, T.

Returning to (31), we may integrate from 0 to T - note that $(k_n*|u_m(\cdot)|^2_{\mathcal{H}})(0)=0$ - and then let n go to ∞ to find that

$$c\int_0^T|u_m(t)|^2_{\mathcal{V}}\,dt\le 2d\int_0^T|u_m(t)|^2_{\mathcal{H}}\,dt+|k|_{L_1([0,T])}|x_m|^2_{\mathcal{H}}+\frac{1}{c}\int_0^T|f(t)|^2_{\mathcal{V}'}\,dt.$$

This, together with (33) and the assumption $x_m \to x$ in $\mathcal{H}$, yields the *a priori* bound

$$|u_m|_{L_2([0,T];\mathcal{V})} \le C_1\Big(|x|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big), \quad m \in \mathbb{N}, \tag{34}$$

with some $C_1 > 0$ being independent of $m \in \mathbb{N}$.

3. By (34) there exists a subsequence of (u_m), which we will again denote by (u_m), such that

$$u_m \rightharpoonup u \quad \text{in } L_2([0,T];\mathcal{V}) \quad \text{as } m \to \infty, \tag{35}$$

for some $u \in L_2([0,T];\mathcal{V})$. We will show that $u \in W(x,\mathcal{V},\mathcal{H})$, and that u is a solution of (2).

Let $\varphi \in C^1([0,T];\mathbb{R})$ with $\varphi(T) = 0$. Multiplying (28) by φ and using integration by parts, we obtain

$$\begin{aligned} -\int_0^T \varphi'(t)([k*(u_m - x_m)](t), w_i)_{\mathcal{H}}\, dt + \int_0^T \varphi(t) a(t, u_m(t), w_i)\, dt \\ = \int_0^T \varphi(t)\langle f(t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}} \end{aligned} \tag{36}$$

for all $m \ge i$, because $[k*(u_m - x_m)](0) = 0$. We apply then the limits (35), and $x_m \to x$ in $\mathcal{H}$ to equation (36). By means of (18), the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$, and Young's and Hölder's inequality, one easily verifies that this leads to

$$-\int_0^T \varphi'(t)([k*(u-x)](t), w_i)_{\mathcal{H}}\, dt + \int_0^T \varphi(t) a(t, u(t), w_i)\, dt = \int_0^T \varphi(t)\langle f(t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}} \tag{37}$$

for all $i \in \mathbb{N}$. Observe that $([k*(u-x)](t), w_i)_{\mathcal{H}} = \langle [k*(u-x)](t), w_i\rangle_{\mathcal{V}'\times\mathcal{V}}$, by (1). It is not difficult to see that the terms in (37) represent linear continuous functionals on the space $\mathcal{V}$, with respect to w_i. Consequently, in light of (Hb), (37) implies

$$-\int_0^T \varphi'(t)\langle [k*(u-x)](t), v\rangle_{\mathcal{V}'\times\mathcal{V}}\, dt + \int_0^T \varphi(t) a(t, u(t), v)\, dt = \int_0^T \varphi(t)\langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \tag{38}$$

for all $v \in \mathcal{V}$.

Since (38) holds in particular for all $\varphi \in C_0^\infty(0,T)$, we infer that $k*(u-x)$ has a generalized derivative on $(0,T)$ with

$$\frac{d}{dt}[k*(u-x)](t) + A(t)u(t) = f(t), \quad \text{a.a. } t \in (0,T), \tag{39}$$

where the operator $A(t) : \mathcal{V} \to \mathcal{V}'$ is defined as in (6). From $u \in L_2([0,T];\mathcal{V})$ and $|A(t)u(t)|_{\mathcal{V}'} \le M|u(t)|_{\mathcal{V}}$ for a.a. $t \in (0,T)$, we deduce that $A(\cdot)u \in L_2([0,T];\mathcal{V}')$. Since $f \in L_2([0,T];\mathcal{V}')$, too, it follows that $[k*(u-x)]' \in L_2([0,T];\mathcal{V}')$.

To see that $u \in W(x,\mathcal{V},\mathcal{H})$, it remains to show that $[k*(u-x)](0) = 0$. We set $z := k*(u-x)$. Then $z \in H_2^1([0,T];\mathcal{V}') \hookrightarrow C([0,T];\mathcal{V}')$, and by (38) and (39), there holds

$$-\int_0^T \varphi'(t)\langle z(t), v\rangle_{\mathcal{V}'\times\mathcal{V}}\, dt = \int_0^T \varphi(t)\langle z'(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} \tag{40}$$

for all $v \in \mathcal{V}$, and all $\varphi \in C^1([0,T];\mathbb{R})$ with $\varphi(T) = 0$. Choosing φ such that $\varphi(0) = 1$, and approximating z in $H_2^1([0,T];\mathcal{V}')$ by a sequence of functions $z_n \in C^1([0,T];\mathcal{V}')$, it follows from (40) and the formula of integration by parts that $\langle z(0), v\rangle_{\mathcal{V}'\times\mathcal{V}} = 0$ for all $v \in \mathcal{V}$. Hence $z(0) = 0$.

Summarizing, we have found a function $u \in W(x,\mathcal{V},\mathcal{H})$ that solves the operator equation (39). Since (39) is equivalent to (2), the existence proof is complete.

Moreover, (39) has exactly one solution in the class $W(x,\mathcal{V},\mathcal{H})$. Consequently, all subsequences of the *original* sequence (u_m) that are weakly convergent in $L_2([0,T];\mathcal{V})$ have the same limit u. Hence, the original sequence (u_m) converges weakly to u in $L_2([0,T];\mathcal{V})$.

Continuous dependence on the data. From $u_m \rightharpoonup u$ in $L_2([0,T];\mathcal{V})$ and the estimate (34), it follows by means of the theorem of Banach and Steinhaus, that

$$|u|_{L_2([0,T];\mathcal{V})} \leq \liminf_{m\to\infty} |u_m|_{L_2([0,T];\mathcal{V})} \leq C_1\Big(|x|_{\mathcal{H}} + |f|_{L_2([0,T];\mathcal{V}')}\Big).$$

Using this estimate, together with $|A(\cdot)u|_{L_2([0,T];\mathcal{V}')} \leq M|u|_{L_2([0,T];\mathcal{V})}$, (Hd), and (39), we obtain the desired estimate (21). $\square$

If not only the kernel l in (Hk) but also some p-th power of it with $p > 1$ belongs to $L_1([0,T])$, then one can get an additional estimate for solutions of (2). This is the consequence of the first part of the subsequent interpolation result for functions in the space $W(x,\mathcal{V},\mathcal{H})$. It also contains the analogue of

$$\int_0^T t^{-1}|u(t)|_{\mathcal{H}}^2\,dt < \infty \quad \text{for all } u \in {}_0H_2^1([0,T];\mathcal{V}') \cap L_2([0,T];\mathcal{V}),$$

see [7, Chap. 3, Prop. 5.3 and Prop. 5.4], in the case of the space $W(x,\mathcal{V},\mathcal{H})$.

By $L_{p,w}([0,T])$, $p \in [1,\infty)$ we mean the weak L_p space of Lebesgue measurable functions on $(0,T)$.

Theorem 3.2 *Let $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above ($\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$). Let further $T > 0$, $(k,l) \in \mathcal{PC}$, and suppose that $x \in \mathcal{H}$, and $u \in W(x,\mathcal{V},\mathcal{H})$. Then the following statements hold.*

(i) If $l \in L_{p,w}([0,T])$ for some $p > 1$ then $u \in L_{2p,w}([0,T];\mathcal{H})$, and there holds

$$|u|_{L_{2p,w}([0,T];\mathcal{H})} \leq C\Big(\Big|\frac{d}{dt}[k*(u-x)]\Big|_{L_2([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} + |x|_{\mathcal{H}}\Big), \tag{41}$$

the constant C depending only on T, and $|l|_{L_{p,w}([0,T])}$. If $l \in L_p([0,T])$ for some $p > 1$ then $u \in L_{2p}([0,T];\mathcal{H})$, and the estimate corresponding to (41) holds.

(ii) There holds the estimate

$$\Big(\int_0^T k(t)|u(t)|_{\mathcal{H}}^2\,dt\Big)^{1/2} \leq C_1\Big(\Big|\frac{d}{dt}[k*(u-x)]\Big|_{L_2([0,T];\mathcal{V}')} + |u|_{L_2([0,T];\mathcal{V})} + |x|_{\mathcal{H}}\Big),$$

where the constant C_1 only depends on $|k|_{L_1([0,T])}$.

Proof. We proceed similarly as in the proof of the previous result. The key idea again is to apply the identity (17) from Lemma 2.1.

Let (k_n) be the sequence of approximating kernels used above, and put

$$g(t) = \langle (k * [u - x])'(t), u(t) \rangle_{\mathcal{V}' \times \mathcal{V}}, \quad t \in (0, T),$$

and

$$h_n(t) = \langle (k_n * [u - x])'(t) - (k * [u - x])'(t), u(t) \rangle_{\mathcal{V}' \times \mathcal{V}}, \quad t \in (0, T).$$

Then

$$\Big(\frac{d}{dt} (k_n * u)(t), u(t) \Big)_{\mathcal{H}} = k_n(t)(x, u(t))_{\mathcal{H}} + g(t) + h_n(t), \quad \text{a.a. } t \in (0, T),$$

with each term being in $L_1([0, T])$. Using (17) and the inequality $ab \leq \frac{1}{4}a^2 + b^2$ it follows that

$$\frac{1}{2} \frac{d}{dt} (k_n * |u(\cdot)|^2_{\mathcal{H}})(t) + \frac{1}{4} k_n(t) |u(t)|^2_{\mathcal{H}} \leq k_n(t) |x|^2_{\mathcal{H}} + g(t) + h_n(t), \quad \text{a.a. } t \in (0, T). \tag{42}$$

To prove (i), we drop the second term on the left, which is nonnegative, convolve the resulting inequality with l, and send n to ∞. Arguing as in the proof of Theorem 3.1 we obtain

$$|u(t)|^2_{\mathcal{H}} \leq 2 \Big(|x|^2_{\mathcal{H}} + (l * g)(t) \Big), \quad \text{a.a. } t \in (0, T).$$

Young's inequality for weak type L_p spaces (see e.g. [4, Theorem 1.2.13]) then gives

$$|u|^2_{L_{2p,w}([0,T];\mathcal{H})} = \Big| |u(\cdot)|^2_{\mathcal{H}} \Big|_{L_{p,w}([0,T])} \leq 2 \Big(|l|_{L_{p,w}([0,T])} |g|_{L_1([0,T])} + T^{1/p} |x|^2_{\mathcal{H}} \Big),$$

which together with

$$2|g|_{L_1([0,T])} \leq \Big| \frac{d}{dt} [k * (u - x)] \Big|^2_{L_2([0,T];\mathcal{V}')} + |u|^2_{L_2([0,T];\mathcal{V})}$$

implies the first desired bound in (i). If $l \in L_p([0, T])$ one may apply Young's classical inequality for convolutions to establish the asserted estimate in (i).

As to (ii), we integrate (42) from 0 to T, and drop the term $k_n * |u(\cdot)|^2_{\mathcal{H}})(T)$, to the result

$$\int_0^T k_n(t) |u(t)|^2_{\mathcal{H}} \, dt \leq 4 \Big((1 * k_n)(T) |x|^2_{\mathcal{H}} + |g|_{L_1([0,T])} + |h_n|_{L_1([0,T])} \Big).$$

Observe that $(1 * k_n)(T) \to (1 * k)(T)$, and $|h_n|_{L_1([0,T])} \to 0$ as $n \to \infty$. Therefore, for sufficiently large n we have

$$\int_0^T k_n(t) |u(t)|^2_{\mathcal{H}} \, dt \leq 8 \Big((1 * k)(T) |x|^2_{\mathcal{H}} + |g|_{L_1([0,T])} \Big).$$

Since $k_n \to k$ in $L_1([0, 1])$, the assertion then follows from Fatou's lemma. □

In the case of fractional evolution equations we have the following corollary. Here we set

$${}_0H^\alpha_2([0, T]; \mathcal{V}') := \{ v|_{[0,T]} : v \in H^\alpha_2(\mathbb{R}; \mathcal{V}') \text{ and } \operatorname{supp} v \subseteq \mathbb{R}_+ \},$$

where $H^\alpha_2(\mathbb{R}; \mathcal{V}')$ stands for the Bessel potential space of order α of $\mathcal{V}'$-valued functions on the line.

Corollary 3.1 *Let $T>0$, and $\mathcal{V}$ and $\mathcal{H}$ be real Hilbert spaces as described above. Suppose (Hd), (Ha), and (Hb), and assume that $k(t)=g_{1-\alpha}(t)e^{-\mu t}$, $t>0$, with $\alpha\in(0,1)$, and $\mu\geq 0$. Then the problem (2) admits exactly one solution u in the space*

$$W(\alpha;x,\mathcal{V},\mathcal{H}):=\{u\in L_2([0,T];\mathcal{V}):\, u-x\in {}_0H_2^\alpha([0,T];\mathcal{V}')\}.$$

Furthermore, we have

$$(g_{1-\alpha}e^{-\mu\cdot})*u\in C([0,T];\mathcal{H}),\ \ u\in L_{\frac{2}{1-\alpha},\,w}([0,T];\mathcal{H}),\ \ \text{and}\ \int_0^T t^{-\alpha}|u(t)|^2_{\mathcal{H}}\,dt<\infty.$$

There exists a constant $M=M(\alpha,\mu,T)>0$ such that

$$|u-x|_{{}_0H_2^\alpha([0,T];\mathcal{V}')}+|u|_{L_2([0,T];\mathcal{V})}+|(g_{1-\alpha}e^{-\mu\cdot})*u|_{C([0,T];\mathcal{H})}+|u|_{L_{\frac{2}{1-\alpha},\,w}([0,T];\mathcal{H})}$$
$$+\Big(\int_0^T t^{-\alpha}|u(t)|^2_{\mathcal{H}}\,dt\Big)^{1/2}\leq M\Big(|x|_{\mathcal{H}}+|f|_{L_2([0,T];\mathcal{V}')}\Big),$$

for all $x\in\mathcal{H}$ and $f\in L_2([0,T];\mathcal{V}')$.

Proof. Let l as in (7). Then

$$\begin{aligned}{}_0H_2^\alpha([0,T];\mathcal{V}')&=\{l*v:\,v\in L_2([0,T];\mathcal{V}')\}\\&=\{v\in L_2([0,T];\mathcal{V}'):\,(g_{1-\alpha}e^{-\mu\cdot})*v\in {}_0H_2^1([0,T];\mathcal{V}'),\end{aligned}$$

for the first equals sign see e.g. [10, Corollary 2.1]; the second one follows from $k*l=1$. Note further that $l\in L_{1/(1-\alpha),w}([0,T])$. So the assertions of the corollary follow immediately from the previous results. ☐

Note that taking *formally* the limit $\alpha\to 1$ in the above estimates (with $\mu=0$) we recover the well-known estimates for solutions of the abstract parabolic equation (4).

4 Example

Let $T>0$, and Ω be a bounded domain in $\mathbb{R}^N$ with $N\geq 3$. We consider the problem

$$\left\{\begin{aligned}\partial_t(k*(u-u_0))-\operatorname{div}(A\,Du)+b\cdot Du+cu&=g, && t\in(0,T),\,x\in\Omega,\\ u(t,x)&=0, && t\in(0,T),\,x\in\partial\Omega,\\ u(0,x)&=u_0(x), && x\in\Omega.\end{aligned}\right.\tag{43}$$

Here Du denotes the gradient of u w.r.t. the spatial variables, and $z_1\cdot z_2$ means the scalar product of $z_1,z_2\in\mathbb{R}^N$. We make the following assumptions on the kernel k, the coefficients, and the data.

(Hk) $(k,l)\in\mathcal{PC}$ for some $l\in L_{1,\,loc}(\mathbb{R}_+)$.

(Hd) $u_0\in L_2(\Omega)$, $g\in L_2([0,T];L_{\frac{2N}{N+2}}(\Omega))$.

(HA) $A \in L_\infty((0,T)\times\Omega;\mathbb{R}^{N\times N})$, and $\exists \nu > 0$ such that

$$A(t,x)\xi\cdot\xi \geq \nu|\xi|^2, \quad \text{for a.a. } t\in(0,T),\ x\in\Omega, \text{ and all } \xi\in\mathbb{R}^n.$$

(Hc) $b \in L_\infty((0,T)\times\Omega;\mathbb{R}^N)$, $c \in L_\infty((0,T)\times\Omega)$.

We set $\mathcal{V} = \mathring{H}_2^1(\Omega)$, and $\mathcal{H} = L_2(\Omega)$, endowed with the inner product $(u,v)_{\mathcal{H}} = \int_\Omega uv\,dx$. Define

$$a(t,u,v) = \int_\Omega \Big(A(t,x)Du(x)\cdot Dv(x) + (b(t,x)\cdot Du(x))v(x) + c(t,x)u(x)v(x)\Big)\,dx$$

and

$$\langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \int_\Omega g(t,x)v(x)\,dx, \quad \text{a.a. } t\in(0,T).$$

Then the weak formulation of (43) reads

$$\frac{d}{dt}\Big([k*(u-u_0)](t), v\Big)_{\mathcal{H}} + a(t,u(t),v) = \langle f(t), v\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad v\in\mathcal{V}, \text{ a.a. } t\in(0,T), \tag{44}$$

and we seek a solution in the class

$$W(u_0, \mathring{H}_2^1(\Omega), L_2(\Omega)) = \{u\in L_2([0,T];\mathring{H}_2^1(\Omega)) : k*(u-u_0)\in {}_0H_2^1([0,T];H_2^{-1}(\Omega))\}.$$

It is folklore that in the described setting the assumptions (Hd), and (Ha) in Theorem 3.1 are satisfied. Concerning (Hb), we could take $\{w_1, w_2, \ldots\}$ to be the complete set of eigenfunctions for $-\Delta$ in $\mathring{H}_2^1(\Omega)$. Consequently, we obtain

Corollary 4.1 *Suppose the assumptions (Hk), (Hd), (HA), and (Hc) hold. Then the problem (43) has a unique weak solution $u \in W(u_0, \mathring{H}_2^1(\Omega), L_2(\Omega))$ in the sense that (44) is satisfied. Further, $k*u \in C([0,T];L_2(\Omega))$. In the case $k(t) = g_{1-\alpha}(t)e^{-\mu t}$, $t>0$, with $\alpha\in(0,1)$, and $\mu \geq 0$, we have*

$$u \in L_{\frac{2}{1-\alpha},w}([0,T];L_2(\Omega)) \cap L_2([0,T];\mathring{H}_2^1(\Omega)), \text{ and } u-u_0 \in {}_0H_2^\alpha([0,T];H_2^{-1}(\Omega))\}.$$

Of course, a corresponding result also holds in the case $N \leq 2$ with the assumption on f appropriately modified.

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