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Abstract

In this note we establish the Harnack inequality for the Riemann-Liouville fractional derivation operator ∂_t^α of order $\alpha \in (0,1)$. Here the function under consideration is assumed to be globally nonnegative. We show that the Harnack inequality in general fails if this global positivity assumption is replaced by a local one. A Harnack estimate is also derived for nonnegative solutions of a class of nonhomogeneous fractional differential equations.

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1 Introduction

Harnack inequalities have been proved to be an important tool in the theory of linear and nonlinear partial differential equations. We refer to the recent survey [7] for an introduction into this subject. A variant of the classical Harnack inequality for the Laplace operator can be stated as follows. Denote by $B_\rho(y)$ the open ball in $\mathbb{R}^n$ with radius $\rho > 0$ and center $y \in \mathbb{R}^n$. Suppose that u is a nonnegative harmonic function in $B_{4\rho}(y)$. Then

$$\sup_{B_\rho(y)} u \le 3^n \inf_{B_\rho(y)} u,$$

see e.g. [5, Section 2.3]. The classical parabolic Harnack inequality (i.e. for the heat operator) is due to Hadamard [6] and Pini [12]. The following version was introduced by Moser [11] in a more general context, see also [4]. Letting $\rho > 0$, $\sigma \in (0,1)$, and $y \in \mathbb{R}^n$ we define the boxes

$$Q_- = (-\rho^2, -\sigma\rho^2) \times B_\rho(y), \quad Q_+ = (\sigma\rho^2, \rho^2) \times B_\rho(y).$$

Then there exists a constant $M > 0$ depending only on n and σ such that for any nonnegative and sufficiently smooth function u in $(-4\rho^2, \rho^2) \times B_{4\rho}(y)$ satisfying

$$\partial_t u - \Delta u = 0 \quad \text{in} \quad (-4\rho^2, \rho^2) \times B_{4\rho}(y),$$

there holds the inequality

$$\sup_{Q_-} u \le M \inf_{Q_+} u.$$

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For more general results on Harnack inequalities in the elliptic and parabolic case we refer to [3], [5], [7] [10], and the references given therein.

Concerning *non-local* operators it is known that the Harnack inequality also holds for fractional powers of the negative Laplacian. Let $\alpha \in (0,1)$ and suppose that u is a sufficiently smooth function on $\mathbb{R}^n$ that is nonnegative *everywhere* and satisfies $(-\Delta)^\alpha u = 0$ in $B_{4\rho}(y)$. Then

$$\sup_{B_\rho(y)} u \leq M \inf_{B_\rho(y)} u,$$

where the constant M depends only on α and n, cf. [2, Theorem 5.1]. We point out that here the Harnack inequality fails, if the global positivity assumption is replaced by a local one, cf. [8]. This is due to the non-local nature of $(-\Delta)^\alpha$. More general results on Harnack estimates for integro-differential operators like $(-\Delta)^\alpha$ can be found in [1].

The main objective of this note is to show that a Harnack inequality also holds for the Riemann-Liouville fractional derivation operator ∂_t^α with $\alpha \in (0,1)$ defined by

$$\partial_t^\alpha v(t) = \partial_t \int_0^t g_{1-\alpha}(t-\tau) v(\tau)\, d\tau, \quad t > 0,$$

where ∂_t is the usual derivation operator and g_β stands for the Riemann-Liouville (or standard) kernel given by

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0.$$

To state the main result we need some notation. By $f_1 * f_2$ we denote the convolution defined by $(f_1 * f_2)(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$, $t \geq 0$, of two functions f_1, f_2 supported on the positive half-line. Given $0 \leq t_1 < t_2$ we define the space $Z(t_1, t_2)$ by

$$Z(t_1,t_2) = \{u \in C([0,t_2]) : g_{1-\alpha} * u|_{[t_1,t_2]} \in H_1^1([t_1,t_2])\}.$$

For $t_* \geq 0$, $0 < \sigma_1 < \sigma_2 < \sigma_3$, and $\rho > 0$ we introduce the intervals

$$W_- = (t_* + \sigma_1 \rho, t_* + \sigma_2 \rho), \quad W_+ = (t_* + \sigma_2 \rho, t_* + \sigma_3 \rho).$$

Then the main result is the following.

Theorem 1.1 *Let $t_* \geq 0$, $0 < \sigma_1 < \sigma_2 < \sigma_3$, and $\rho > 0$. Let further $\alpha \in (0,1)$ and $u_0 \geq 0$. Then for any function $u \in Z(t_*, t_* + \sigma_3\rho)$ that is nonnegative on $(0, t_* + \sigma_3\rho)$ and that satisfies*

$$\partial_t^\alpha (u - u_0)(t) = 0, \quad a.a.\ t \in (t_*, t_* + \sigma_3 \rho), \tag{1}$$

there holds the inequality

$$\sup_{W_-} u \leq \frac{\sigma_3}{\sigma_1} \inf_{W_+} u. \tag{2}$$

Note that in Theorem 1.1 we do not assume that $u(0) = u_0$. So by setting $u_0 = 0$ we obtain the Harnack inequality for the Riemann-Liouville fractional derivative. If we assume in addition that $u(0) = u_0$ then Theorem 1.1 yields the Harnack inequality for the so-called Caputo fractional derivation operator, which is a regularized version of the Riemann-Liouville fractional derivative, cf. the monographs [9] and [13].

We will also show that, similarly to the case of the fractional Laplacian, the Harnack inequality fails if the global positivity assumption is replaced by a local one. Furthermore, we will demonstrate that the above Harnack estimate breaks down if the relation $\partial_t^\alpha(u-u_0)=0$ is only satisfied on the smaller interval $(t_*+\sigma_1\rho, t_*+\sigma_3\rho)$.

In the last section of this note we generalize Theorem 1.1 to nonnegative solutions of the fractional differential equation

$$\partial_t^\alpha(u-u_0)(t)+\mu u(t)=f(t), \quad \text{a.a. } t\in(t_*, t_*+\sigma_3\rho), \tag{3}$$

where $u_0, \mu\geq 0$ and $f\in L_p([t_*, t_*+\sigma_3\rho])$ for some $p>1/\alpha$, see Theorem 4.1 below.

It is highly desirable to have a Harnack inequality also for nonnegative solutions of fractional evolution equations the prototype of which reads

$$\partial_t^\alpha(u-u_0)(t,x)-\Delta u(t,x)=0, \quad t\in(0,T),\ x\in\Omega, \tag{4}$$

where $T>0$, Ω is a domain in $\mathbb{R}^n$, $\alpha\in(0,1)$, and u_0 is a given function depending only on x. This is an open problem. However, our results indicate that a Harnack inequality should hold in this situation, too. In this sense our results can be regarded as an important step towards a better understanding of fractional evolution equations of the type (4). We remark that in the very recent work [15], it is shown that the weak maximum principle is valid for (4), which also supports the conjecture that a Harnack inequality holds in this case, too.

As to literature, hardly anything seems to be known about Harnack estimates for time fractional equations. To the author's knowledge the only paper on this subject is [14], where a *weak* Harnack inequality is established for nonnegative *supersolutions* of (3). Adopting the notation of the present note and assuming for simplicity that $f=0$ and $\mu=0$ it is shown in [14] that for any function $u\in Z(t_*, t_*+\sigma_3\rho)$ that is nonnegative on $(0, t_*+\sigma_3\rho)$ and that satisfies

$$\partial_t^\alpha(u-u_0)(t)\geq 0, \ \text{ a.a. } t\in(t_*, t_*+\sigma_3\rho), \quad u(0)=u_0,$$

we have

$$\rho^{-1/p}|u|_{L_p((t_*, t_*+\sigma_1\rho))}\ \leq\ C\inf_{W_+} u, \tag{5}$$

for all $0<p<\frac{1}{1-\alpha}$, where the constant $C>0$ depends only on $0<\sigma_1<\sigma_2<\sigma_3$, p, and $\alpha\in(0,1)$. The critical exponent $\frac{1}{1-\alpha}$ is optimal. Notice that on the left of (5) we have the interval $(t_*, t_*+\sigma_1\rho)$, not W_- as in (2).

2 Proof of the Harnack inequality

Suppose $u\in Z(t_*, t_*+\sigma_3\rho)$ is nonnegative on $(0, t_*+\sigma_3\rho)$ and satisfies (1). We introduce the shifted time $s=t-t_*$ and define the function $\tilde{u}$ by means of $\tilde{u}(s)=u(s+t_*)$, $s\in(0,\sigma_3\rho)$. Then (1) implies that

$$\partial_s^\alpha\tilde{u}(s)=g_{1-\alpha}(t_*+s)u_0+h(s), \quad s\in(0,\sigma_3\rho), \tag{6}$$

where the history term $h(s)$ is given by

$$h(s)=\int_0^{t_*}[-\dot{g}_{1-\alpha}(t_*+s-\tau)]u(\tau)\,d\tau, \quad s\in(0,\sigma_3\rho). \tag{7}$$

Since $(g_{1-\alpha} * \tilde{u})(0) = 0$ and $g_\alpha * g_{1-\alpha} = 1$, we have

$$g_\alpha * \partial_s^\alpha \tilde{u} = g_\alpha * \partial_s (g_{1-\alpha} * \tilde{u}) = \partial_s (g_\alpha * g_{1-\alpha} * \tilde{u}) = \tilde{u}.$$

Therefore convolving (6) with g_α yields

$$\tilde{u}(s) = u_0 \big(g_\alpha * g_{1-\alpha}(\cdot + t_*)\big)(s) + (g_\alpha * h)(s), \quad s \in (0, \sigma_3 \rho). \tag{8}$$

The first term on the right-hand side of (8) can be rewritten by the use of the identity

$$\begin{aligned} \big(g_\alpha * g_{1-\alpha}(\cdot + t_*)\big)(s) &= \int_0^s g_\alpha(s-\sigma) g_{1-\alpha}(t_* + \sigma)\, d\sigma \\ &= s \int_0^1 g_\alpha(s - rs) g_{1-\alpha}(t_* + rs)\, dr \\ &= \int_0^1 g_\alpha(1-r) g_{1-\alpha}(r + \frac{t_*}{s})\, dr \qquad (9) \\ &=: \varphi(s), \quad s \in (0, \sigma_3 \rho). \end{aligned}$$

Similarly, we have for the second term

$$\begin{aligned} (g_\alpha * h)(s) &= \int_0^s g_\alpha(s-\sigma) \int_0^{t_*} [-\dot{g}_{1-\alpha}(t_* + \sigma - \tau)] u(\tau)\, d\tau\, d\sigma \\ &= \frac{1}{s} \int_0^1 g_\alpha(1-r) \int_0^{t_*} [-\dot{g}_{1-\alpha}(r + \frac{t_* - \tau}{s})] u(\tau)\, d\tau\, dr \qquad (10) \\ &=: \psi(s), \quad s \in (0, \sigma_3 \rho). \end{aligned}$$

Consequently, (8) is equivalent to

$$\tilde{u}(s) = u_0 \varphi(s) + \psi(s), \quad s \in (0, \sigma_3 \rho).$$

Let now $s \in (\sigma_1 \rho, \sigma_2 \rho)$ and $\bar{s} \in (\sigma_2 \rho, \sigma_3 \rho)$. Since $g_{1-\alpha}$ is nonincreasing, we evidently have $\varphi(s) \le \varphi(\bar{s})$. As to ψ, we use the positivity of u on $(0, t_*)$ and the monotonicity of $-\dot{g}_{1-\alpha}$ to estimate as follows.

$$\begin{aligned} \psi(s) &\le \frac{1}{\sigma_1 \rho} \int_0^1 g_\alpha(1-r) \int_0^{t_*} [-\dot{g}_{1-\alpha}(r + \frac{t_* - \tau}{\sigma_2 \rho})] u(\tau)\, d\tau\, dr \\ &\le \frac{\sigma_3}{\sigma_1 \bar{s}} \int_0^1 g_\alpha(1-r) \int_0^{t_*} [-\dot{g}_{1-\alpha}(r + \frac{t_* - \tau}{\bar{s}})] u(\tau)\, d\tau\, dr \\ &= \frac{\sigma_3}{\sigma_1} \psi(\bar{s}). \end{aligned}$$

By positivity of u_0, we thus obtain

$$\tilde{u}(s) \le \frac{\sigma_3}{\sigma_1} \tilde{u}(\bar{s}),$$

which immediately implies inequality (2). This completes the proof of Theorem 1.1.

Remark 2.1 Note that in case $t_* = 0$ relation (1) implies $u(t) = u_0$ for all $t \in [0, \sigma_3 \rho]$, thus the Harnack inequality (2) trivially holds with the constant $\frac{\sigma_3}{\sigma_1} > 1$ being replaced by 1.

3 Counterexamples

Example 3.1 We show first that the Harnack inequality fails for nonnegative functions $u \in Z(t_* + \sigma_1\rho, t_* + \sigma_3\rho)$ satisfying the relation $\partial_t^\alpha(u - u_0) = 0$ only on the smaller interval $(t_* + \sigma_1\rho, t_* + \sigma_3\rho)$.

To this purpose fix $W_- = (1,2)$ and $W_+ = (2,3)$ and consider the family of functions u_ε, $\varepsilon \in (0,1]$, defined by

$$u_\varepsilon(t) = \begin{cases} 0 & : 0 \le t \le 1-\varepsilon \\ \frac{1}{\varepsilon}(t - 1 + \varepsilon) & : 1-\varepsilon \le t \le 1, \end{cases} \tag{11}$$

and

$$\partial_t^\alpha u_\varepsilon = 0, \quad \text{a.a. } t \in (1,3). \tag{12}$$

Apparently $u_\varepsilon|_{[0,1]} \in H_1^1([0,1])$ so that (12) means that with $s = t-1$ and $\tilde{u}_\varepsilon(s) = u_\varepsilon(s+1)$ we have

$$\tilde{u}_\varepsilon(s) = (g_\alpha * h_\varepsilon)(s), \quad s \in (0,2), \tag{13}$$

where

$$h_\varepsilon(s) = \int_0^1 [-\dot{g}_{1-\alpha}(1+s-\tau)] u_\varepsilon(\tau)\, d\tau, \quad s \in (0,2).$$

Observe that u_ε is nonnegative on $(0,3)$ and that $u_\varepsilon \in Z(1,3)$ for all $\varepsilon \in (0,1]$. From $u_\varepsilon = 0$ in $[0, 1-\varepsilon]$ and $u_\varepsilon \le 1$ in $[1-\varepsilon, 1]$ we infer the estimate

$$h_\varepsilon(s) \le \int_{1-\varepsilon}^1 [-\dot{g}_{1-\alpha}(1+s-\tau)]\, d\tau = g_{1-\alpha}(s) - g_{1-\alpha}(s+\varepsilon), \quad s \in (0,2).$$

In view of (13) this gives for $s \in (1,2)$

$$\begin{aligned} \tilde{u}(s) &\le \int_0^s g_\alpha(s-\sigma)[g_{1-\alpha}(\sigma) - g_{1-\alpha}(\sigma+\varepsilon)]\, d\sigma \\ &= \int_0^1 g_\alpha(1-r)[g_{1-\alpha}(r) - g_{1-\alpha}(r + \frac{\varepsilon}{s})]\, dr \\ &\le \int_0^1 g_\alpha(1-r)[g_{1-\alpha}(r) - g_{1-\alpha}(r+\varepsilon)]\, dr =: \delta(\varepsilon). \end{aligned}$$

By the dominated convergence theorem, $\delta(\varepsilon)$ vanishes as $\varepsilon \to 0+$. Hence

$$\lim_{\varepsilon \to 0+} \inf_{W_+} u_\varepsilon = 0.$$

On the other hand we have $u_\varepsilon(1) = \tilde{u}(0) = 1$ for all $\varepsilon \in (0,1]$, and therefore

$$\sup_{W_-} u_\varepsilon \ge 1, \quad \varepsilon \in (0,1].$$

This shows that an estimate of the form

$$\sup_{W_-} u \le M \inf_{W_+} u$$

with M independent of u cannot hold.

Example 3.2 We next show that the Harnack inequality fails if the positivity assumptions $u_0 \geq 0$ and $u \geq 0$ in $(0, t_*)$ are dropped.

Fix $t_* > 0$ and consider the family of functions u_ε, $\varepsilon > 0$, defined by

$$u_\varepsilon(t) = \frac{1}{\varepsilon}\,(t - t_* + \varepsilon), \quad 0 \leq t \leq t_*,$$

and

$$\partial_t^\alpha (u_\varepsilon - u_{0,\varepsilon}) = 0, \quad \text{a.a. } t > t_*, \tag{14}$$

where

$$u_{0,\varepsilon} = u_\varepsilon(0) = 1 - \frac{t_*}{\varepsilon}.$$

Observe that u_ε has negative values in $[0, t_*]$ if and only if $\varepsilon \in (0, t_*)$. Setting $s = t - t_*$ and $\tilde{u}_\varepsilon(s) = u_\varepsilon(s + t_*)$, $s \geq 0$, (14) is equivalent to

$$\tilde{u}_\varepsilon(s) = u_{0,\varepsilon}\big(g_\alpha * g_{1-\alpha}(\cdot + t_*)\big)(s) + (g_\alpha * h_\varepsilon)(s), \quad s > 0, \tag{15}$$

where

$$\begin{aligned} h_\varepsilon(s) &= \int_0^{t_*} [-\dot{g}_{1-\alpha}(t_* + s - \tau)] u_\varepsilon(\tau)\, d\tau \\ &= \Big[g_{1-\alpha}(t_* + s - \tau) u_\varepsilon(\tau) \Big]_{\tau=0}^{\tau=t_*} - \int_0^{t_*} g_{1-\alpha}(t_* + s - \tau) \dot{u}_\varepsilon(\tau)\, d\tau \\ &= g_{1-\alpha}(s) - g_{1-\alpha}(t_* + s) u_{0,\varepsilon} + \frac{1}{\varepsilon}\big(g_{2-\alpha}(s) - g_{2-\alpha}(s + t_*) \big), \quad s > 0. \end{aligned}$$

Inserting the last identity into (15) yields

$$\tilde{u}_\varepsilon(s) = 1 + \frac{1}{\varepsilon}\,(g_\alpha * [g_{2-\alpha} - g_{2-\alpha}(\cdot + t_*)])(s), \quad s \geq 0.$$

In particular $\tilde{u}_\varepsilon$ is differentiable in $(0, \infty)$ and we have

$$\begin{aligned} \dot{\tilde{u}}_\varepsilon(s) &= \frac{1}{\varepsilon}\,(g_\alpha * [g_{1-\alpha} - g_{1-\alpha}(\cdot + t_*)])(s) - \frac{1}{\varepsilon}\, g_\alpha(s) g_{2-\alpha}(t_*) \\ &< \frac{1}{\varepsilon}\big(1 - g_\alpha(s) g_{2-\alpha}(t_*)\big), \quad s > 0. \end{aligned}$$

This shows that $\tilde{u}_\varepsilon$ is strictly decreasing in the interval $[0, s_*]$ with

$$s_* = \frac{t_*}{[\Gamma(\alpha)\Gamma(2-\alpha)]^{1/(1-\alpha)}}.$$

Selecting

$$\varepsilon = (g_\alpha * [g_{2-\alpha}(\cdot + t_*) - g_{2-\alpha}])(s_*),$$

we have

$$\tilde{u}_\varepsilon(s_*) = 0 \quad \text{and} \quad \tilde{u}_\varepsilon(s) > 0,\ s \in [0, s_*). \tag{16}$$

Note that $\varepsilon < t_*$, for otherwise we would have $u_{0,\varepsilon} \geq 0$ and $u_\varepsilon > 0$ in $(0, t_*]$, which by (15), entails strict positivity of $\tilde{u}_\varepsilon$, a contradiction.

Choosing the parameters in such a way that $s_* = t_* + \sigma_3\rho$, (16) shows that an estimate of the form

$$\sup_{W_-} u_\varepsilon \le M \inf_{W_+} u_\varepsilon$$

cannot hold.

4 Nonhomogeneous fractional differential equations

In this section we derive a Harnack estimate for nonnegative solutions of the more general equation

$$\partial_t^\alpha (u - u_0)(t) + \mu u(t) = f(t), \quad \text{a.a. } t \in (t_*, t_* + \sigma_3\rho), \tag{17}$$

here $\mu \ge 0$ is another parameter and we assume that $f \in L_p([t_*, t_* + \sigma_3\rho])$ for some $p > 1/\alpha$. The other parameters are as before.

Suppose $u \in Z(t_*, t_* + \sigma_3\rho)$ is nonnegative on $(0, t_* + \sigma_3\rho)$ and satisfies (17). Setting $s = t - t_*$ and $\tilde{u}(s) = u(s + t_*)$, $\tilde{f}(s) = f(s + t_*)$, $\tilde{g}_{1-\alpha}(s) = g_{1-\alpha}(s + t_*)$, $s \in (0, \sigma_3\rho)$, we infer from (17) that

$$\partial_s^\alpha \tilde{u}(s) + \mu \tilde{u}(s) = \tilde{g}_{1-\alpha}(s) u_0 + h(s) + \tilde{f}(s), \quad s \in (0, \sigma_3\rho), \tag{18}$$

where $h(s)$ is given by (7). Let $r_{\alpha,\mu}$ denote the resolvent kernel corresponding to (17), that is

$$r_{\alpha,\mu}(s) + \mu(r_{\alpha,\mu} * g_\alpha)(s) = g_\alpha(s), \quad s > 0.$$

Equation (18) then implies

$$\tilde{u}(s) = (r_{\alpha,\mu} * [\tilde{g}_{1-\alpha} u_0 + h + \tilde{f}])(s), \quad s \in (0, \sigma_3\rho). \tag{19}$$

It is well-known (see e.g. [14, Section 2.1]) that

$$r_{\alpha,\mu}(s) = \Gamma(\alpha) g_\alpha(s)\, E_{\alpha,\alpha}(-\mu s^\alpha), \quad s > 0,$$

where $E_{\alpha,\beta}$ denotes the generalized Mittag-Leffler-function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}\,, \quad z \in \mathbb{C}.$$

Let now $\omega > 0$ be a fixed constant and assume that

$$\mu\rho^\alpha \le \omega.$$

By continuity and strict positivity of $E_{\alpha,\alpha}$ in $(-\infty, 0]$ we then have

$$0 < c_1 := \min_{z \in [0, \omega\sigma_3^\alpha]} E_{\alpha,\alpha}(-z) \le E_{\alpha,\alpha}(-\mu s^\alpha) \le \max_{z \in [0, \omega\sigma_3^\alpha]} E_{\alpha,\alpha}(-z) =: c_2, \quad s \in (0, \sigma_3\rho).$$

Setting $C_i = C_i(\alpha, \omega, \sigma_3) = c_i \Gamma(\alpha)$, $i = 1, 2$, we thus have

$$C_1 g_\alpha(s) \le r_{\alpha,\mu}(s) \le C_2 g_\alpha(s), \quad s \in (0, \sigma_3\rho). \tag{20}$$

Further,

$$\max_{s\in[0,\sigma_3\rho]}(g_\alpha * |\tilde{f}|)(s) \le |g_\alpha|_{L_{p'}([0,\sigma_3\rho])}|\tilde{f}|_{L_p([0,\sigma_3\rho])} = C_3\rho^{\alpha-\frac{1}{p}}|\tilde{f}|_{L_p([0,\sigma_3\rho])}, \tag{21}$$

with

$$C_3 = \frac{\sigma_3^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)[(\alpha-1)p'+1]^{1/p'}}.$$

Using the functions φ and ψ from Section 2, we infer from (19), (20), and (21) that

$$\tilde{u}(s) \le C_2\big(\varphi(s)u_0 + \psi(s) + C_3\rho^{\alpha-\frac{1}{p}}|\tilde{f}|_{L_p([0,\sigma_3\rho])}\big), \quad s\in(0,\sigma_3\rho), \tag{22}$$

as well as

$$\tilde{u}(s) \ge C_1\big(\varphi(s)u_0 + \psi(s)\big) - C_2C_3\rho^{\alpha-\frac{1}{p}}|\tilde{f}|_{L_p([0,\sigma_3\rho])}, \quad s\in(0,\sigma_3\rho). \tag{23}$$

Suppose now that $s\in(\sigma_1\rho,\sigma_2\rho)$ and $\bar{s}\in(\sigma_2\rho,\sigma_3\rho)$. Employing (22), (23), and the estimates for φ and ψ from Section 2, we have

$$\begin{aligned}
\tilde{u}(s) &\le C_2\big(\varphi(\bar{s})u_0 + \frac{\sigma_3}{\sigma_1}\,\psi(\bar{s}) + C_3\rho^{\alpha-\frac{1}{p}}|\tilde{f}|_{L_p([0,\sigma_3\rho])}\big)\\
&\le \frac{C_2\sigma_3}{C_1\sigma_1}\,\big(C_1[\varphi(\bar{s})u_0 + \psi(s)] - C_2C_3\rho^{\alpha-\frac{1}{p}}|\tilde{f}|_{L_p([0,\sigma_3\rho])}\big)\\
&\quad + C_2C_3\big(1+\frac{C_2\sigma_3}{C_1\sigma_1}\big)\rho^{\alpha-\frac{1}{p}}|\tilde{f}|_{L_p([0,\sigma_3\rho])}\\
&\le \frac{C_2\sigma_3}{C_1\sigma_1}\,\tilde{u}(\bar{s}) + C_2C_3\big(1+\frac{C_2\sigma_3}{C_1\sigma_1}\big)\rho^{\alpha-\frac{1}{p}}|\tilde{f}|_{L_p([0,\sigma_3\rho])}.
\end{aligned}$$

We have thus proved the following result.

Theorem 4.1 *Let $\omega>0$ be fixed. Let $t_*,\mu\ge 0$, $0<\sigma_1<\sigma_2<\sigma_3$, and $\rho>0$. Let further $\alpha\in(0,1)$, $u_0\ge 0$, and $f\in L_p([t_*,t_*+\sigma_3\rho])$ for some $p>1/\alpha$. Assume that $\mu\rho^\alpha\le\omega$. Then there exists a positive constant $M=M(\alpha,p,\sigma_1,\sigma_3,\omega)$ such that for any function $u\in Z(t_*,t_*+\sigma_3\rho)$ that is nonnegative on $(0,t_*+\sigma_3\rho)$ and that satisfies (17) there holds the inequality*

$$\sup_{W_-} u \le M\left(\inf_{W_+} u + \rho^{\alpha-\frac{1}{p}}|f|_{L_p([t_*,t_*+\sigma_3\rho])}\right). \tag{24}$$

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