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Abstract

We study the asymptotic behaviour, as $t \to \infty$, of bounded solutions to a second order integro-differential equation in finite dimensions where the damping term is of memory type and can be of arbitrary fractional order less than 1. We derive appropriate Lyapunov functions for this equation and prove that any global bounded solution converges to an equilibrium of a related equation, if the nonlinear potential $\mathcal{E}$ occurring in the equation satisfies the Łojasiewicz inequality.

AMS subject classification: 45G05, 45M05

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1 Introduction

In this paper we study the asymptotic behaviour, as $t \to \infty$, of bounded solutions to integro-differential equations of the form

$$\ddot{u}(t) + \nabla\mathcal{E}(u(t)) + (k * \dot{u})(t) = f(t),\ t > 0, \quad u(0) = u_0, \quad \dot{u}(0) = u_1. \tag{1}$$

Here $u : \mathbb{R}_+ \to \mathbb{R}^n$, $k \in L_{1,\,loc}(\mathbb{R}_+)$ is a nonnegative kernel, and $k * v$ stands for the convolution on the positive halfline, i.e. $(k * v)(t) = \int_0^t k(t-\tau)v(\tau)\,d\tau$, $t \geq 0$. The scalar nonlinearity $\mathcal{E}$ lies in $C^2(\mathbb{R}^n)$; by $\nabla\mathcal{E}$ we mean the gradient of $\mathcal{E}$. The vectors $u_0,\, u_1 \in \mathbb{R}^n$ as well as the function $f \in L_{1,\,loc}(\mathbb{R}_+;\mathbb{R}^n)$ are given data.

An important example for the kernel k we have in mind is given by

$$k(t) = g_{1-\alpha}(t)e^{-\gamma t}, \quad t > 0,\ \alpha \in (0,1),\ \gamma > 0, \tag{2}$$

where g_β denotes the Riemann-Liouville kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0.$$

In this case, k is a singular kernel and the damping term $k * \dot{u}$ in (1) is of fractional order $\alpha \in (0,1)$.

Concerning applications we primarily regard (1) as a finite-dimensional model problem for more complex evolutionary integral equations in infinite dimensions, which arise in mathematical physics, e.g. in the theory of viscoelasticity. For example, the equation for the viscoelastic Euler-Bernoulli beam with nonlinear load is given by

$$u_{tt} + e_0 u_{txxxx} + e_1 * u_{txxxx} + e_\infty u_{xxxx} + f(x,u) = 0. \tag{3}$$

Here $e_\infty > 0$ denotes the elasticity modulus, $e_0 \geq 0$ a Newtonian viscosity, and $e_1 \in L_{1,loc}(\mathbb{R}_+) \cap C_0((0,\infty))$ is the viscoelastic stress relaxation modulus, see the monograph [24, Section 9.1] and [13]. In this case we view equation (1) as a finite-dimensional model problem for (3) with $e_0 = 0$.

The main objective of this paper is to establish results asserting the convergence to equilibrium as $t \to \infty$ of any global bounded solution of (1). Here by 'equilibrium' we mean a steady state of the related equation $\ddot{u} + \nabla\mathcal{E}(u) + k * \dot{u} = 0$, that is a vector $u_* \in \mathbb{R}^n$ satisfying $\nabla\mathcal{E}(u_*) = 0$. As to the nonlinearity $\mathcal{E}$, the crucial assumption is that given any bounded solution u of (1), $\mathcal{E}$ fulfills the Łojasiewicz inequality near some u_* in the ω-limit set $\omega(u)$ of u. This means there are constants $\theta \in (0, 1/2]$ and σ, $M > 0$ such that

$$|\mathcal{E}(x) - \mathcal{E}(u_*)|^{1-\theta} \leq M|\nabla\mathcal{E}(x)| \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x - u_*| \leq \sigma. \tag{4}$$

In a seminal paper Łojasiewicz was able to prove that any real analytic function $\mathcal{E} : U \to \mathbb{R}$ defined on an open set $U \subset \mathbb{R}^n$ satisfies the Łojasiewicz inequality near each point $u_* \in U$, see [21, Thm. 4]. There exist examples of non-analytic functions satisfying this inequality (see e.g. [5] and [19]). Thus, for the sake of generality, we state the validity of the Łojasiewicz inequality in the formulation of our main result. Note, however, that in general it is very difficult to verify (4) for a non-analytic function.

Using inequality (4), Łojasiewicz ([21], [22]) proved that any bounded solution of the first order ODE system in $\mathbb{R}^n$

$$\dot{u}(t) + \nabla\mathcal{E}(u(t)) = 0,\ t > 0, \tag{5}$$

converges to an equilibrium provided that $\mathcal{E}$ is analytic. It is worth noticing that compared with LaSalle's invariance principle, a significant advantage of the approach based on the Łojasiewicz inequality consists in the fact that this method also works when the set of equilibria is not discrete.

Haraux and Jendoubi [18] established the corresponding result for second order ODE systems in $\mathbb{R}^n$ of the prototypical form

$$\ddot{u}(t) + \mu\dot{u}(t) + \nabla\mathcal{E}(u(t)) = 0,\ t > 0,\ \mu > 0. \tag{6}$$

These results have also been extended to the infinite dimensional case ([26], [20]). A seminal contribution was made by Simon [26], who was able to generalize the Łojasiewicz inequality to some analytic functionals defined on Hilbert spaces, and to prove convergence to steady state of bounded solutions of the abstract first order equation $\dot{u} + \mathcal{E}'(u) = 0$ under natural regularity and compactness assumptions on $\mathcal{E}'$ and u. During the last decade the Łojasiewicz inequality and its generalizations to the infinite dimensional case, often called Łojasiewicz-Simon inequality, have been used in many papers to prove convergence to steady state for different evolution equations, see e.g. [5], [7], [14], [17], [19], [25], and the references given therein.

Concerning integro-differential equations, there are only a few papers that derive results on convergence to equilibrium of the type described above. In [1], [2], and [6], first and second order problems with additional memory terms are investigated. Chill and Fašangová [6] study abstract equations in a Hilbert space setting the prototype of which in the finite dimensional case is basically of the form

$$\ddot{u}(t) + \mu\dot{u}(t) + \nabla\mathcal{E}(u(t)) + (k * \dot{u})(t) = 0, \quad t > 0, \tag{7}$$

with $\mu > 0$ and k as in (2). Recently, Vergara and Zacher [27] used the Łojasiewicz inequality to establish convergence to equilibrium for a class of integro-differential equations that includes problems of fractional order both between 1 and 2 and less than 1.

The novelty of the present paper is in proving convergence to equilibrium for a second order system that does not contain a damping term of first order - as in [18] and [6] - , but instead respectively only a *weak damping term of memory type*. This term can be of arbitrary fractional order $\alpha \in (0, 1)$ as in the important example (2). It turns out, somewhat surprisingly, that our result still holds when we send $\alpha \to 0$ in (2), that is for the regular kernel $k(t) = e^{-\gamma t}$, $t \geq 0$, $\gamma > 0$, which leads to a friction term of order 0.

One of the main difficulties in the proof consists in constructing an appropriate Lyapunov function (LF) for (1). We point out that finding an LF for an integro-differential equation, in general, is a highly nontrivial task, see the remarks in Section 14.1 in the monograph [16], the standard reference for such equations in the finite dimensional case. In [1], [2], and [6] Lyapunov functions are constructed by the use of a technique that basically goes back to Dafermos [12]. This technique leads to rather tedious estimates and does not seem to work for the type of problems to be studied in this paper.

In order to succeed we apply and substantially develop further the method used by Vergara and Zacher in [27], the key ingredient being the basic inequality (9) (see below) for nonnegative, nonincreasing kernels. As we will show, this inequality is closely connected to the notion of a *completely positive kernel*, see assumption (K1) and Theorem 3.1 below. This concept also plays a crucial role in the work of Clément and Nohel [10] on nonlinear Volterra equations in Banach spaces with accretive nonlinearity; we further refer to [8], [9], [11], [16] and [24]. Another significant idea in the construction of a suitable LF for (1) is to look at higher energy estimates. Only with the aid of these estimates we are able to obtain an $L_2(\mathbb{R}_+)$ bound for $|\dot{u}|$, which is an important intermediate step in our proof of the convergence result. Note that in the case of equation (6) as well as (7) with $\mu > 0$ such a bound immediately follows from the basic energy estimate. We remark that the method of higher order energies and the techniques from [27] have been recently used in Alabau-Boussouira *et al.* [3] to prove polynomial energy decay for a wave equation with purely boundary memory damping. In Section 4, Step 2 and 3, we make use of some ideas from [3].

We point out that the results of this paper can be generalized to evolutionary equations in a suitable Hilbert space setting, see e.g. the setting in [6]. Assuming sufficiently regular data ensures that the solutions possess the regularity required for the derivation of the higher energy estimates. This way, it is possible to obtain, e.g., results on the convergence to equilibrium for semilinear wave equations with weak damping terms of memory type. Furthermore, the results in [6] can be improved by allowing vanishing first order damping terms.

The paper is organized as follows. In Section 2 we formulate our main result, Theorem 2.1, and give some further remarks. Section 3 provides the basic inequalities which will be repeatedly used in the paper. The key inequality, Theorem 3.1, is formulated in the abstract setting of functions taking values in some Hilbert space. The proof of the main result is given in Sections 4–6.

2 The main result and remarks

We will suppose that the following assumptions are satisfied.

(K1) $k \in L_{1,loc}(\mathbb{R}_+)$ is *completely positive*, that is, there exist $b_0 \geq 0$ and a nonnegative and nonincreasing kernel $b \in L_{1,loc}(\mathbb{R}_+)$ such that

$$b_0 k(t) + (b * k)(t) = 1, \quad t \geq 0;$$

(K2) there are $\gamma > 0$ and $a \in L_1(\mathbb{R}_+)$ strictly positive and nonincreasing such that

$$b(t) = a(t) + \gamma(1 * a)(t), \quad t > 0; \tag{8}$$

(K3) there exists $T_0 > 0$ such that $k \in H_2^1([T_0, \infty))$, and

$$\int_{T_0}^{\infty} (\int_t^{\infty} (k(\tau)^2 + \dot{k}(\tau)^2)\, d\tau)^{\frac{1}{2}}\, dt < \infty;$$

(H$\mathcal{E}$) the function $\mathcal{E}$ belongs to $C^2(\mathbb{R}^n)$;

(Hf) $f \in H_2^1(\mathbb{R}_+; \mathbb{R}^n)$ and there is $T_1 \geq 0$ such that $f \in H_2^2([T_1, \infty); \mathbb{R}^n)$ and

$$\int_{T_1}^{\infty} (\int_t^{\infty} (|f(\tau)|^2 + |\dot{f}(\tau)|^2 + |\ddot{f}(\tau)|^2)\, d\tau)^{1/2}\, dt < \infty.$$

Remarks 2.1 (i) Condition (K1) implies that k is nonnegative, see e.g. [10], [11], and [24]. For equivalent definitions of complete positivity we refer to [9], [10], and [24]. Sufficient conditions which insure that k is completely positive are: $k \in L_{1,loc}(\mathbb{R}_+)$ is nonnegative, nonincreasing, and $\log k$ is convex, see e.g. [10], [23], and [24]. In particular, completely monotonic kernels are completely positive, see [23], [24]. We further remark that assuming (K1) k is locally absolutely continuous on $[0, \infty)$ if and only if $b_0 > 0$; in this case $b_0 = k(0)^{-1}$.

(ii) If we weaken the assumption (K2) by replacing 'strictly positive' with 'nonnegative and not identically 0', then by decreasing γ, we obtain again a decomposition of the form (8) with a strictly positive and nonincreasing, see Remark 3.1(i) in [27].

(iii) Note that (K1) and (K2) imply that $b(t) \geq b_\infty := \lim_{s\to\infty} b(s) = \gamma |a|_{L_1(\mathbb{R}_+)} > 0$, $t > 0$.

(iv) It follows further from (K1) and (K2) that $k \in L_1(\mathbb{R}_+)$. Indeed, since k is nonnegative (cp. (i)), (K1) implies $(b * k)(t) \leq 1$, $t \geq 0$. Using the lower bound for b from (iii) and positivity of k, we see that $|k|_{L_1(\mathbb{R}_+)} \leq 1/b_\infty$.

(v) The kernels k in (2) as well as $k(t) = e^{-\mu t}$, $t \geq 0$, $\mu > 0$, satisfy conditions (K1)–(K3), see Examples 3.1 and 3.2 below.

(vi) (K3) and (Hf) are technical conditions. Note that (K3) entails $k \in H_1^1([T_0, \infty))$, and (Hf) implies $f \in H_1^2([T_1, \infty); \mathbb{R}^n)$, see [19, Chap. 3, Sec. 4].

Definition 2.1 *For $T > 0$ we say that a function $u \in H_1^2([0, T]; \mathbb{R}^n)$ is a* **solution** *of (1) on $[0, T]$ if (1) holds a.e. on $[0, T]$. A function $u \in H_{1,loc}^2([0, \infty); \mathbb{R}^n)$ is called a* **global solution** *of (1) if for any $J = [0, T]$, $T > 0$, the function $u|_J$ is a solution of (1) on J. A global solution u of (1) is called* **global bounded solution** *if $|u|_{L_\infty(\mathbb{R}_+; \mathbb{R}^n)} < \infty$.*

Remarks 2.2 Under the above assumptions local existence and uniqueness for (1) can be shown by means of a simple fixed point argument, cp. the monograph [16]. Observe further that the assumptions on the data imply that any solution u of (1) on $[0, T]$ lies in the space $u \in H_1^3([0, T]; \mathbb{R}^n)$.

We are now in position to state our main result.

Theorem 2.1 *Suppose (K1), (K2), (K3), (H$\mathcal{E}$), and (Hf) are satisfied. Let u_0, $u_1 \in \mathbb{R}^n$ and u be a global bounded solution of (1). Assume further that there exists some $u_* \in \omega(u)$ such that $\mathcal{E}$ fulfills the Lojasiewicz inequality near u_*, i.e. there are constants $\theta \in (0, 1/2]$ and σ, $M > 0$ such that*

$$|\mathcal{E}(x) - \mathcal{E}(u_*)|^{1-\theta} \leq M |\nabla \mathcal{E}(x)| \quad \textit{for all } x \in \mathbb{R}^n \textit{ with } |x - u_*| \leq \sigma.$$

Then $\lim_{t\to\infty} u(t) = u_*$, and $\nabla \mathcal{E}(u_*) = 0$.

Remarks 2.3 It is instructive to have a look at the case $\mathcal{E} \equiv 0$. Suppose u is a global solution of (1) with $k \geq 0$ not identically 0 and that $\lim_{t\to\infty} u(t) = u_\infty$. Then, by a well-known Abelian theorem, see e.g. [4, Theorem 4.1.2], $u(t)$ converges to u_∞ in the sense of Abel as $t \to \infty$, that is the Laplace transform $\hat{u}(\lambda)$ exists for $\operatorname{Re}\lambda > 0$ and $\lim_{\lambda\downarrow 0} \lambda\hat{u}(\lambda) = u_\infty$. Taking the Laplace transform we infer from (1) that

$$\big(\lambda + \hat{k}(\lambda)\big)\, \lambda\hat{u}(\lambda) = \hat{f}(\lambda) + \lambda u_0 + u_1 + \hat{k}(\lambda)u_0,$$

which shows that it is natural to assume $k \in L_1(\mathbb{R}_+)$ and $f \in L_1(\mathbb{R}_+;\mathbb{R}^n)$. We then obtain

$$u_\infty = u_0 + \frac{u_1 + \int_0^\infty f(t)\,dt}{\int_0^\infty k(t)\,dt}.$$

3 Preliminaries

Let $\mathcal{H}$ denote a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. For an interval $J \subset \mathbb{R}$, $s \geq 0$ an integer, and $1 \leq p < \infty$, by $H^s_p(J;\mathcal{H})$ we mean the Sobolev space of $\mathcal{H}$-valued functions on J. We write $H^s_p(J) = H^s_p(J;\mathbb{R})$ for short.

The following result is basic to deriving suitable energy estimates for solutions of (1). In the special case $b_0 = 0$ it is due to Vergara and Zacher [27, Theorem 2.1 and Remark 2.1].

Theorem 3.1 *Let $\mathcal{H}$ be a real Hilbert space, $T > 0$, and $b \in L_{1,\,loc}(\mathbb{R}_+)$ be nonnegative and nonincreasing such that $b_0 k + b * k = 1$ in $(0,\infty)$ for some $b_0 \geq 0$ and some nonnegative kernel $k \in L_{1,\,loc}(\mathbb{R}_+)$. Then for any function $v \in H^1_1([0,T];\mathcal{H})$,*

$$\Big\langle v(t), \frac{d}{dt}\,(b * v)(t)\Big\rangle_{\mathcal{H}} \geq \frac{1}{2}\,\frac{d}{dt}\,(b * |v|^2_{\mathcal{H}})(t) + \frac{1}{2}\,b(t)|v(t)|^2_{\mathcal{H}}, \quad a.a.\ t \in (0,T). \tag{9}$$

Proof: It remains to prove the statement in the case $b_0 > 0$. We proceed similarly as in [27]. Let $v \in H^1_1([0,T];\mathcal{H})$ and $x = v(0) \in \mathcal{H}$. The computations in Step 1 of the proof of Theorem 2.1 in [27] show that v satisfies the inequality (9) whenever $v - x$ does so, thus we may restrict ourselves to the case $v \in {}_0H^1_1([0,T];\mathcal{H})$, where the 0 means vanishing trace at $t = 0$.

Define the operators

$$B_1 u = b_0\,\frac{d}{dt}\,u + \frac{d}{dt}\,(b * u),\ \ D(B_1) = {}_0H^1_1([0,T]),$$
$$B_2 u = b_0\,\frac{d}{dt}\,u + \frac{d}{dt}\,(b * u),\ \ D(B_2) = {}_0H^1_1([0,T];\mathcal{H}).$$

Then B_1 and B_2 are known to be m-accretive in $X_1 := L_1([0,T])$ and $X_2 := L_1([0,T];\mathcal{H})$, respectively, see [8], [11], and [15]. Their Yosida approximations $B_{i,\,n}$, $n \in \mathbb{N}$, $i = 1,2$, defined by

$$B_{i,\,n} = nB_i(n + B_i)^{-1}, \quad n \in \mathbb{N},\ i = 1,\,2,$$

enjoy the property that for any $u \in D(B_i)$, we have $B_{i,\,n}u \to B_i u$ in X_i as $n \to \infty$. We will show that

$$B_{i,n}u = \frac{d}{dt}\,(ns_n * u), \quad u \in X_i,\ \ i = 1,2, \tag{10}$$

where the kernels $s_n \in H^1_1([0,T])$ are defined as solutions of the scalar Volterra equations

$$s_n(t) + n(k * s_n)(t) = 1, \quad t > 0,\ n \in \mathbb{N}. \tag{11}$$

Since k is completely positive, we know that s_n is nonnegative and nonincreasing, cf. [10, Section 2] or [24, Prop. 4.5]. Let $i \in \{1,2\}$, $n \in \mathbb{N}$, and suppose that $u \in D(B_i)$ satisfying

$$nu + b_0 \, \frac{d}{dt} \, u + \frac{d}{dt} \, (b * u) = f \quad \text{on } (0,T). \tag{12}$$

Convolving (12) with k and employing the identity $b_0 k + b * k = 1$ as well as $u(0) = 0$ and $(b * u)(0) = 0$ results into

$$n(k * u) + u = k * f \quad \text{on } (0,T). \tag{13}$$

We then convolve (13) with s_n and use (11), thereby obtaining $1 * u = s_n * k * f$, that is

$$u = (n + B_i)^{-1} f = \frac{d}{dt} \, (s_n * k * f) \quad \text{on } (0,T). \tag{14}$$

In fact, it is not difficult to see that the relation (14) with $f \in X_i$ also implies (12). From (14) and $b_0 k + b * k = 1$ we then deduce that

$$\begin{aligned} B_{i,n} f \; &= n b_0 \, \frac{d^2}{dt^2} \, (s_n * k * f) + n \, \frac{d}{dt} \Big(b * \, \frac{d}{dt} \, (s_n * k * f) \Big) \\ &= n b_0 \, \frac{d^2}{dt^2} \, (s_n * k * f) + n \, \frac{d^2}{dt^2} \, (s_n * [1 - b_0 k] * f), \end{aligned}$$

which yields (10) with u replaced by f.

Putting $b_n = n s_n$ we have $b_n \in H_1^1([0,T])$, and b_n is nonnegative and nonincreasing. Consequently, by [27, Lemma 2.2],

$$\Big\langle v(t), \frac{d}{dt} \, (b_n * v)(t) \Big\rangle_{\mathcal{H}} \geq \frac{1}{2} \, \frac{d}{dt} \, (b_n * |v|^2_{\mathcal{H}})(t) + \frac{1}{2} \, b_n(t) |v(t)|^2_{\mathcal{H}}, \quad \text{a.a. } t \in (0,T), \tag{15}$$

for every $n \in \mathbb{N}$. Further, by definition of b_n,

$$\begin{aligned} b_n \; &= \; \frac{d}{dt} \, (b_n * 1) = \frac{d}{dt} \Big(b_n * (b_0 k + b * k) \Big) \\ &= \; b_0 \, \frac{d}{dt} \, (b_n * k) + \frac{d}{dt} \, (b_n * k * b) \\ &= \; b_0(-\dot{s}_n) + \frac{d}{dt} \, (b_n * k * b). \end{aligned}$$

Since s_n is nonincreasing, this together with (15) yields

$$\Big\langle v(t), \frac{d}{dt} \, (b_n * v)(t) \Big\rangle_{\mathcal{H}} \geq \frac{1}{2} \, \frac{d}{dt} \, (b_n * |v|^2_{\mathcal{H}})(t) + \frac{1}{2} \, |v(t)|^2_{\mathcal{H}} \, \frac{d}{dt} \, (b_n * k * b)(t), \quad \text{a.a. } t \in (0,T). \tag{16}$$

Recall that $v \in {}_0H_1^1([0,T];\mathcal{H})$, that is $v \in D(B_2)$ as well as $|v(\cdot)|^2_{\mathcal{H}} \in D(B_1)$. Therefore, we have

$$\frac{d}{dt} \, (b_n * v) \; \to b_0 \, \frac{d}{dt} \, v + \frac{d}{dt} \, (b * v) \; \text{ in } \; L_1([0,T];\mathcal{H}) \; \text{ as } \; n \to \infty, \tag{17}$$

$$\frac{d}{dt} \, (b_n * |v|^2_{\mathcal{H}}) \; \to b_0 \, \frac{d}{dt} \, |v|^2_{\mathcal{H}} + \frac{d}{dt} \, (b * |v|^2_{\mathcal{H}}) \; \text{ in } \; L_1([0,T]) \; \text{ as } \; n \to \infty. \tag{18}$$

Notice as well that $k \in H_1^1([0,T])$ implies $k * b \in D(B_1)$, and thus

$$\frac{d}{dt} \, (b_n * k * b) \to b_0 \, \frac{d}{dt} \, (k * b) + \frac{d}{dt} \, (b * k * b) = b \; \text{ in } \; L_1([0,T]) \; \text{ as } \; n \to \infty. \tag{19}$$

By choosing an appropriate subsequence of (b_n), again denoted by (b_n), we may assume that the sequences in (17), (18), and (19) converge also pointwise a.e. in $(0,T)$. Using these properties we may send $n \to \infty$ in (16), to the result

$$\Big\langle v(t), b_0 \frac{d}{dt}\, v + \frac{d}{dt}\,(b * v)(t)\Big\rangle_{\mathcal{H}} \geq \frac{b_0}{2}\, \frac{d}{dt}\, |v(t)|^2_{\mathcal{H}} + \frac{1}{2}\, \frac{d}{dt}\,(b * |v|^2_{\mathcal{H}})(t) + \frac{1}{2}\, b(t)|v(t)|^2_{\mathcal{H}},$$

for a.a. $t \in (0,T)$, which in turn implies the desired inequality (9). $\square$

The subsequent simple lemma (cf. [27, Lemma 2.1]) will be frequently used in the estimates below.

Lemma 3.1 *Let $\mathcal{H}$ be a real Hilbert space and $T > 0$. Suppose that $l \in L_{1,\,loc}(\mathbb{R}_+)$ is nonnegative. Then for any $v \in L_2([0,T];\mathcal{H})$ there holds*

$$|(l * v)(t)|^2_{\mathcal{H}} \leq (l * |v|^2_{\mathcal{H}})(t)\,(1 * l)(t), \quad a.a.\ t \in (0,T).$$

We next describe two important examples of kernels k satisfying the conditions (K1)–(K3). We first consider the case of singular kernels.

Example 3.1 Let $\alpha \in (0,1)$ and $\gamma > 0$. Set

$$k(t) = g_{1-\alpha}(t)e^{-\gamma t} \quad \text{and} \quad b(t) = g_\alpha(t)e^{-\gamma t} + \gamma(1 * [g_\alpha(\cdot)e^{-\gamma\cdot}])(t), \quad t > 0. \tag{20}$$

Then both kernels are strictly positive and decreasing; observe that $\dot{b}(t) = \dot{g}_\alpha(t)e^{-\gamma t} < 0$, $t > 0$. Their Laplace transforms are given by

$$\hat{k}(\lambda) = \frac{1}{(\lambda+\gamma)^{1-\alpha}}, \quad \hat{b}(\lambda) = \frac{1}{(\lambda+\gamma)^{\alpha}}\Big(1 + \frac{\gamma}{\lambda}\Big), \quad \mathrm{Re}\,\lambda > 0,$$

which shows that $k * b = 1$ on $(0,\infty)$. It is readily seen that k satisfies all of the conditions (K1)–(K3).

The next example describes a smooth kernel which is admissible.

Example 3.2 Let $\mu > 0$. Set

$$k(t) = e^{-\mu t}, \quad b_0 = 1, \ \text{ and } \quad b(t) = \mu, \quad t \geq 0. \tag{21}$$

Then $b_0 k + b * k = 1$ on $[0,\infty)$. Letting $\gamma > 0$ and setting

$$a(t) = \mu e^{-\gamma t}, \quad t \geq 0,$$

the kernel b decomposes as $b = a + \gamma(1 * a)$ on $[0,\infty)$. It is then easy to see that k satisfies the conditions (K1)–(K3).

4 Lyapunov functions

We commence by deriving suitable energy estimates. We proceed in several steps.

1. The basic energy estimate. Letting u be a global solution of (1), we take the inner product of (1) and $\dot{u}$ to find that

$$\frac{d}{dt}\Big(\frac{1}{2}\,|\dot{u}|^2 + \mathcal{E}(u)\Big) + \langle \dot{u}, k * \dot{u}\rangle = \langle f, \dot{u}\rangle, \quad t > 0. \tag{22}$$

Introducing the function

$$v = k * \dot{u}$$

we use property (K1) to write

$$\dot{u} = \frac{d}{dt}\,([b_0 k + b * k] * \dot{u}) = b_0\,\frac{d}{dt}\,v + \frac{d}{dt}\,(b * v), \tag{23}$$

which yields

$$\langle \dot{u}, k * \dot{u} \rangle = \frac{b_0}{2}\,\frac{d}{dt}\,|v|^2 + \langle v, \frac{d}{dt}\,(b * v) \rangle.$$

By Theorem 3.1,

$$\langle v(t), \frac{d}{dt}\,(b * v)(t) \rangle \geq \frac{1}{2}\,\frac{d}{dt}\,(b * |v|^2)(t) + \frac{1}{2}\,b(t)|v(t)|^2, \quad t > 0.$$

Combining (22) and the preceding relations, and using the decomposition $b = a + \gamma(1 * a)$ from (K2) it follows that

$$\frac{d}{dt}\Big(\frac{1}{2}\,|\dot{u}|^2 + \mathcal{E}(u) + \frac{b_0}{2}\,|v|^2 + \frac{1}{2}\,a * |v|^2\Big) \leq -\frac{b_\infty}{2}\,|v|^2 - \frac{\gamma}{2}\,a * |v|^2 + \langle f, \dot{u} \rangle, \quad t > 0. \tag{24}$$

Here $b_\infty := \lim_{t\to\infty} b(t) = \gamma |a|_1$ with $|a|_1 := |a|_{L_1(\mathbb{R}_+)}$. As to the term $\langle f, \dot{u} \rangle$ we have, by (23), (K2), Lemma 3.1, and Young's inequality,

$$\begin{aligned}
\langle f, \dot{u} \rangle &= \langle f, b_0 \dot{v} + \frac{d}{dt}\,(b * v) \rangle = \langle f, b_0 \dot{v} + \frac{d}{dt}\,(a * v) \rangle + \gamma \langle f, a * v \rangle \\
&= \frac{d}{dt}\,\langle f, b_0 v + a * v \rangle - \langle \dot{f}, b_0 v + a * v \rangle + \gamma \langle f, a * v \rangle \\
&\leq \frac{d}{dt}\,\langle f, b_0 v + a * v \rangle + 2|a|_1 \gamma |f|^2 + \Big(2|a|_1 \gamma^{-1} + \frac{b_0^2}{b_\infty}\Big)|\dot{f}|^2 + \frac{b_\infty}{4}\,|v|^2 + \frac{\gamma}{4|a|_1}\,|a * v|^2 \\
&\leq \frac{d}{dt}\,\langle f, b_0 v + a * v \rangle + M\Big(|f|^2 + |\dot{f}|^2\Big) + \frac{b_\infty}{4}\,|v|^2 + \frac{\gamma}{4}\,a * |v|^2, \quad t > 0,
\end{aligned} \tag{25}$$

where $M = \max\{2|a|_1 \gamma, 2|a|_1 \gamma^{-1} + b_0^2 b_\infty^{-1}\}$. Setting

$$\begin{aligned}
F(t) =& \frac{1}{2}\,|\dot{u}(t)|^2 + \mathcal{E}(u(t)) + \frac{b_0}{2}\,|v(t)|^2 + \frac{1}{2}\,(a * |v|^2)(t) - \langle f(t), b_0 v(t) + (a * v)(t) \rangle \\
&+ M \int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2)\,d\tau, \quad t \geq 0,
\end{aligned} \tag{26}$$

we infer from (24) and (25) that

$$\dot{F} \leq -\frac{b_\infty}{4}\,|v|^2 - \frac{\gamma}{4}\,a * |v|^2, \quad t > 0. \tag{27}$$

2. Higher energy estimates. Let now u be a global bounded solution of (1). Differentiating (1) we find that $w := \dot{u}$ satisfies

$$\ddot{w}(t) + \nabla^2 \mathcal{E}(u(t))\,w(t) + (k * \dot{w})(t) + k(t)u_1 = \dot{f}(t),\ t > 0, \quad w(0) = u_1, \tag{28}$$

recall Remark 2.2. Let

$$z = k * \ddot{u} = k * \dot{w}$$

and

$$\begin{aligned} G_1(t) = & \frac{1}{2}\,|\dot w(t)|^2 + \frac{b_0}{2}\,|z(t)|^2 + \frac{1}{2}\,(a*|z|^2)(t) - \langle \dot f(t), b_0 z(t) + (a*z)(t)\rangle \\ & + M\int_t^\infty (|\dot f(\tau)|^2 + |\ddot f(\tau)|^2)\,d\tau, \quad t \ge T_1. \end{aligned}$$

Taking the inner product of (28) and $\dot w$, and arguing as in Step 1 we obtain

$$\dot G_1(t) + \langle \nabla^2\mathcal{E}(u(t))\,w(t), \dot w(t)\rangle + \langle k(t)u_1, \dot w(t)\rangle \le -\frac{b_\infty}{4}\,|z(t)|^2 - \frac{\gamma}{4}\,(a*|z|^2)(t), \quad t > T_1. \tag{29}$$

The second term can be rewritten as follows.

$$\begin{aligned} \langle \nabla^2\mathcal{E}(u)\,w, \dot w\rangle = & \frac{d}{dt}\,\langle \nabla\mathcal{E}(u), \dot w\rangle - \langle \nabla\mathcal{E}(u), \ddot w\rangle \\ = & \frac{d}{dt}\,\langle \nabla\mathcal{E}(u), \dot w\rangle + \langle \nabla\mathcal{E}(u), \nabla^2\mathcal{E}(u)\,w + z + k(t)u_1 - \dot f\rangle \\ = & \frac{d}{dt}\,\langle \nabla\mathcal{E}(u), \dot w\rangle + \frac{1}{2}\frac{d}{dt}\,|\nabla\mathcal{E}(u)|^2 + \langle \nabla\mathcal{E}(u), \dot v\rangle - \langle \nabla\mathcal{E}(u), \dot f\rangle \\ = & \frac{d}{dt}\,\langle \nabla\mathcal{E}(u), \dot w\rangle + \frac{1}{2}\frac{d}{dt}\,|\nabla\mathcal{E}(u)|^2 + \frac{d}{dt}\,\langle \nabla\mathcal{E}(u), v\rangle \\ & - \langle \nabla^2\mathcal{E}(u)\dot u, v\rangle - \langle \nabla\mathcal{E}(u), \dot f\rangle. \end{aligned} \tag{30}$$

Turning to the third term in (29), observe that

$$\begin{aligned} w = & 1*\ddot u + w(0) = (b_0 k + b*k)*\ddot u + w(0) \\ = & b_0 z + b*z + u_1 = b_0 z + a*z + \gamma(1*a*z) + u_1, \end{aligned} \tag{31}$$

and thus

$$\frac{d}{dt}\,\langle k(t)u_1, b_0 z + a*z\rangle = \langle k(t)u_1, \dot w(t)\rangle - \gamma\langle k(t)u_1, a*z\rangle + \langle \dot k(t)u_1, b_0 z + a*z\rangle, \quad t > T_0. \tag{32}$$

Since

$$\dot w + \nabla\mathcal{E}(u) + v = f,$$

by (1), it follows from (29), (30), and (32) that the function G_2 defined by

$$\begin{aligned} G_2 = & \frac{1}{2}\,|v - f|^2 + \frac{b_0}{2}\,|z|^2 + \frac{1}{2}\,(a*|z|^2) - \langle \dot f, b_0 z + a*z\rangle + \langle \nabla\mathcal{E}(u), v\rangle \\ & + \langle k(t)u_1, b_0 z + a*z\rangle + M\int_t^\infty (|\dot f(\tau)|^2 + |\ddot f(\tau)|^2)\,d\tau, \quad t \ge T_2 := \max\{T_0, T_1\}, \end{aligned}$$

satisfies

$$\begin{aligned} \dot G_2 \le & -\frac{b_\infty}{4}\,|z|^2 - \frac{\gamma}{4}\,a*|z|^2 + \langle \nabla^2\mathcal{E}(u)\dot u, v\rangle \\ & + \langle \nabla\mathcal{E}(u), \dot f\rangle - \gamma\langle k(t)u_1, a*z\rangle + \langle \dot k(t)u_1, b_0 z + a*z\rangle, \quad t > T_2. \end{aligned} \tag{33}$$

Using Young's inequality and Lemma 3.1, we have

$$|\gamma\langle k(t)u_1, a*z\rangle| \le \frac{\gamma}{16}\,a*|z|^2 + 4\gamma|a|_1 k(t)^2|u_1|^2,$$

and similarly

$$|\langle \dot{k}(t)u_1, b_0 z + a * z\rangle| \le \frac{b_\infty}{8}\,|z|^2 \;+\; \frac{\gamma}{16}\,a * |z|^2 + \Big(\frac{4|a|_1}{\gamma} \;+\; \frac{2b_0^2}{b_\infty}\Big)\dot{k}(t)^2|u_1|^2.$$

By (H$\mathcal{E}$) and global boundedness, we evidently have $|\nabla^2\mathcal{E}(u)|_{L_\infty(\mathbb{R}_+;\mathbb{R}^{n\times n})} \le C_1$ for some constant $C_1 > 0$. Therefore

$$|\langle \nabla^2\mathcal{E}(u)\dot{u}, v\rangle| \le C_1|\dot{u}|\,|v| \le \varepsilon_1|\dot{u}|^2 + \frac{C_1^2}{4\varepsilon_1}\,|v|^2,$$

for all $\varepsilon_1 > 0$. Further,

$$|\langle \nabla\mathcal{E}(u), \dot{f}\rangle| \le \varepsilon_2|\nabla\mathcal{E}(u)|^2 + \frac{1}{4\varepsilon_2}\,|\dot{f}|^2, \quad \varepsilon_2 > 0.$$

Letting

$$G_3(t) = G_2(t) + 2M|u_1|^2\int_t^\infty (k(\tau)^2 + \dot{k}(\tau)^2)\,d\tau + \frac{1}{4\varepsilon_2}\int_t^\infty |\dot{f}(\tau)|^2\,d\tau, \quad t \ge T_2,$$

we infer from (33) and the preceding estimates that

$$\dot{G}_3 \le -\frac{b_\infty}{8}\,|z|^2 - \frac{\gamma}{8}\,a * |z|^2 + \varepsilon_1|\dot{u}|^2 + \varepsilon_2|\nabla\mathcal{E}(u)|^2 + \frac{C_1^2}{4\varepsilon_1}\,|v|^2, \quad t > T_2. \tag{34}$$

3. Estimating $|\dot{u}|^2$. The $|\dot{u}|^2$ term can be controlled by what we have obtained so far. In fact, observe first that (31) gives

$$\begin{aligned}
\dot{u} &= b_0 z + a * z + \gamma(1 * a * k * \ddot{u}) + u_1\\
&= b_0 z + a * z + \gamma(a * k * [\dot{u} - u_1]) + u_1\\
&= b_0 z + a * z + \gamma(a * v) - \gamma(1 * a * k)u_1 + (b_0 k + b * k)u_1\\
&= b_0 z + a * z + \gamma(a * v) + (b_0 k + a * k)u_1.
\end{aligned}$$

Convolving the identity $a + \gamma(1 * a) = b$ with k we see that the function $\zeta = a * k$ is subject to

$$\zeta + \gamma(1 * \zeta) = 1 - b_0 k, \quad t \ge 0.$$

In case $b_0 = 0$ this immediately implies $(a * k)(t) = e^{-\gamma t}$, $t \ge 0$. If $b_0 > 0$, ζ solves the problem

$$\dot{\zeta} + \gamma\zeta = -b_0\dot{k}, \;\; t > 0, \quad \zeta(0) = 0,$$

hence

$$\zeta(t) = -b_0(e^{-\gamma\cdot} * \dot{k})(t) = -b_0 k(t) + e^{-\gamma t} + b_0\gamma(e^{-\gamma\cdot} * k)(t), \quad t \ge 0.$$

Consequently, we have in the general case ($b_0 \ge 0$)

$$\dot{u} = b_0 z + a * z + \gamma(a * v) + \Big(e^{-\gamma t} + b_0\gamma(e^{-\gamma\cdot} * k)(t)\Big)u_1, \tag{35}$$

which in turn implies

$$\begin{aligned}
|\dot{u}|^2 &\le 4b_0^2|z|^2 + 4|a|_1(a * |z|^2 + \gamma^2 a * |v|^2) + 4\Big(e^{-\gamma t} + b_0\gamma(e^{-\gamma\cdot} * k)(t)\Big)^2|u_1|^2\\
&\le C_2(b_0^2|z|^2 + a * |z|^2 + a * |v|^2 + \psi(t)|u_1|^2),
\end{aligned} \tag{36}$$

where $C_2 = C_2(\gamma, |a|_1, |k|_1, b_0)$ is a positive constant (recall Remark 2.1(iv)), and

$$\psi(t) = e^{-2\gamma t} + (e^{-2\gamma\cdot} * k)(t), \quad t \geq 0.$$

We next choose

$$\varepsilon_1 = \min\Big\{\frac{b_\infty}{16C_2 b_0^2}, \frac{\gamma}{16C_2}\Big\}$$

(with the obvious interpretation in case $b_0 = 0$) and define the higher energy function G by

$$G(t) := G_3(t) + \varepsilon_1 C_2 |u_1|^2 \int_t^\infty \psi(\tau)\, d\tau, \quad t \geq T_2.$$

Then (34) and (36) yield

$$\dot{G} \leq -\frac{b_\infty}{16}\,|z|^2 - \frac{\gamma}{16}\, a * |z|^2 + \frac{C_1^2}{4\varepsilon_1}\,|v|^2 + \varepsilon_1 C_2\, a * |v|^2 + \varepsilon_2 |\nabla\mathcal{E}(u)|^2, \quad t > T_2. \tag{37}$$

4. Modifying the higher energy to control $|\nabla\mathcal{E}(u)|^2$. By (1), we have

$$\begin{aligned}\frac{d}{dt}\,\langle\nabla\mathcal{E}(u), \dot{u}\rangle &= \langle\nabla^2\mathcal{E}(u)\dot{u}, \dot{u}\rangle + \langle\nabla\mathcal{E}(u), \ddot{u}\rangle\\ &= \langle\nabla^2\mathcal{E}(u)\dot{u}, \dot{u}\rangle + \langle\nabla\mathcal{E}(u), -\nabla\mathcal{E}(u) - v + f\rangle\\ &\leq C_1|\dot{u}|^2 - \frac{1}{2}\,|\nabla\mathcal{E}(u)|^2 + |v|^2 + |f|^2.\end{aligned}$$

Setting

$$H(t) = \langle\nabla\mathcal{E}(u(t)), \dot{u}(t)\rangle + \int_t^\infty |f(\tau)|^2\, d\tau + C_1C_2|u_1|^2 \int_t^\infty \psi(\tau)\, d\tau, \quad t \geq T_2,$$

and using (36) it follows that

$$\dot{H} \leq -\frac{1}{2}\,|\nabla\mathcal{E}(u)|^2 + C_1C_2(a * |v|^2 + a * |z|^2 + b_0^2|z|^2) + |v|^2, \quad t > T_2. \tag{38}$$

Next, we set (with the obvious interpretation in case $b_0 = 0$)

$$\delta = \frac{1}{32C_1C_2}\,\min\Big\{\gamma, \frac{b_\infty}{b_0^2}\Big\}, \qquad \varepsilon_2 = \frac{\delta}{4}\,.$$

Combining (37) and (38) then yields

$$\frac{d}{dt}\Big(G + \delta H\Big) \leq -C_3\Big(|z|^2 + a * |z|^2 + |\nabla\mathcal{E}(u)|^2\Big) + C_4\Big(|v|^2 + a * |v|^2\Big), \quad t > T_2, \tag{39}$$

where C_3, C_4 are positive constants that depend only on $\gamma, |a|_1$, and C_1.

5. Constructing a suitable new Lyapunov function. It is easy to see that the C_4 term on the right of (39) can be absorbed when adding a term ωF to $G + \delta H$ with $\omega > 0$ sufficiently large. In fact, putting

$$\omega = \frac{4(C_3 + C_4)}{\min\{b_\infty, \gamma\}},$$

we conclude from (27) and (39) that

$$\frac{d}{dt}\Big(\omega F + G + \delta H\Big) \leq -C_3\Big(|v|^2 + a * |v|^2 + |z|^2 + a * |z|^2 + |\nabla\mathcal{E}(u)|^2\Big), \quad t > T_2. \tag{40}$$

Recalling the definitions of F, G, and H, it follows from (40) that there exist constants $M_1, C > 0$ such that the function V defined on $[T_2, \infty)$ by

$$\begin{aligned} V = &\frac{1}{2}\,|\dot u|^2 + \mathcal{E}(u) + \frac{b_0}{2}\,|v|^2 + \frac{1}{2}\,a*|v|^2 - \langle f, b_0 v + a*v\rangle + \frac{\delta}{\omega}\,\langle \nabla\mathcal{E}(u), \dot u\rangle \\ &+ \frac{1}{\omega}\Big(\frac{1}{2}\,|v-f|^2 + \frac{b_0}{2}\,|z|^2 + \frac{1}{2}\,a*|z|^2 - \langle \dot f, b_0 z + a*z\rangle + \langle \nabla\mathcal{E}(u), v\rangle \\ &+ \langle k(t)u_1, b_0 z + a*z\rangle\Big) + M_1|u_1|^2\int_t^\infty (\psi(\tau) + k(\tau)^2 + \dot k(\tau)^2)\,d\tau \\ &+ M_1\int_t^\infty (|f(\tau)|^2 + |\dot f(\tau)|^2 + |\ddot f(\tau)|^2)\,d\tau, \end{aligned} \tag{41}$$

satisfies

$$\dot V \le -C\Big(|v|^2 + a*|v|^2 + |z|^2 + a*|z|^2 + |\nabla\mathcal{E}(u)|^2\Big), \quad t > T_2. \tag{42}$$

6. Strictness of the Lyapunov functions. The functions F and V are *strict* Lyapunov functions in the sense that if F resp. V is constant on some interval $[t_0, t_1]$ ($t_0 < t_1$) in the domain of the function considered, then this implies that u is constant on $[0, t_1]$. In fact, if F is constant on $[t_0, t_1] \subset [0, \infty)$, then by (27), this means that $(a*|v|^2)(t) = 0$ for all $t \in (t_0, t_1)$, which in turn entails $v = 0$ in $[0, t_1]$, by (K2). By definition of v, we have

$$b*v = b*k*\dot u = (1 - b_0 k)*\dot u = u - u_0 - b_0 v,$$

and hence $u = u_0$ in $[0, t_1]$. The argument for V is the same.

Summarizing we have proved

Proposition 4.1 *Let (K1), (K2), (K3), (H$\mathcal{E}$), and (Hf) be satisfied. Let u_0, $u_1 \in \mathbb{R}^n$. Assume that u is a global bounded solution of (1). Then the functions F and V defined by (26) and (41), respectively, are locally absolutely continuous and nonincreasing on $\mathbb{R}_+$ resp. $[T_2, \infty)$ with $T_2 = \max\{T_0, T_1\}$, and the estimates (27) and (42) hold in the a.e. sense. Both F and V are strict Lyapunov functions in the sense described in Step 6 above.*

5 Properties of the ω-limit set

We recall that the ω-limit set of a global solution u of (1) is defined by

$$\omega(u) = \{u_* \in \mathbb{R}^n : \text{ there exist } t_n \uparrow \infty \text{ s.t. } \lim_{n\to\infty} u(t_n) = u_*\}.$$

For every global bounded solution u of (1), $\omega(u)$ is nonempty, compact, and connected.

Proposition 5.1 *Suppose (K1), (K2), (K3), (H$\mathcal{E}$), and (Hf) are fulfilled. Let u_0, $u_1 \in \mathbb{R}^n$ and assume that u is a global bounded solution of (1). Then*

(i) $|v|^2$, $a*|v|^2$, $|z|^2$, $a*|z|^2 \in L_1(\mathbb{R}_+)$.

(ii) $\dot u \in L_2(\mathbb{R}_+;\mathbb{R}^n)$ *and* $\lim_{t\to\infty} |\dot u(t)| = 0$.

(iii) The potential $\mathcal{E}$ is constant on $\omega(u)$ and $\lim_{t\to\infty} \mathcal{E}(u(t))$ exists.

(iv) $\nabla\mathcal{E}(u_*) = 0$ *for every* $u_* \in \omega(u)$.

(v) Each of the terms $|v(t)|$, $(a*|v|^2)(t)$, $b_0|z(t)|$, and $(a*|z|^2)(t)$ tends to 0 as $t \to \infty$.

(vi) $\lim_{t\to\infty} \nabla\mathcal{E}(u(t)) = 0$.

Proof: We show first that V is bounded below on $[T_2,\infty)$. Clearly, $\mathcal{E}(u)$ enjoys this property. The modulus of each of the mixed terms $\langle f, b_0 v + a*v\rangle$, $\langle \nabla\mathcal{E}(u), \dot u\rangle$, $\langle \dot f, b_0 z + a*z\rangle$, and $\langle k(t)u_1, b_0 z + a*z\rangle$ can be estimated by a small multiple of some nonnegative term appearing in V and some bounded quantity. Likewise,

$$\begin{aligned}|\langle \nabla\mathcal{E}(u), v\rangle| &\le \varepsilon|v|^2 + \frac{1}{4\varepsilon}\,|\nabla\mathcal{E}(u)|^2\\ &\le 2\varepsilon|v-f|^2 + 2\varepsilon|f|^2 + \frac{1}{4\varepsilon}\,|\nabla\mathcal{E}(u)|^2,\end{aligned}$$

where the last two terms are bounded. The remaining terms in V are obviously nonnegative, and so V is bounded below. Since V is nonincreasing, by Proposition 4.1, the limit $\lim_{t\to\infty} V(t) = \inf_{t\ge T_2} V(t) =: V_\infty$ exists. Assertion (i) is then an immediate consequence of estimate (42). The first part of (ii) follows from (i) and (36).

A similar (even simpler) argument shows that F is bounded below and nonincreasing, too, and so the limit $\lim_{t\to\infty} F(t) = \inf_{t\ge 0} F(t) =: F_\infty$ exists as well.

Next, let $u_* \in \omega(u)$ and $t_n \uparrow \infty$ such that $\lim_{n\to\infty} u(t_n) = u_*$. Since $\dot u \in L_2(\mathbb{R}_+;\mathbb{R}^n)$, we have for every $n\in\mathbb{N}$ and any $s\in[0,1]$,

$$\begin{aligned}|u(t_n+s)-u_*| &\le |u(t_n)-u_*| + \int_{t_n}^{t_n+s} |\dot u(\tau)|\,d\tau\\ &\le |u(t_n)-u_*| + (\int_{t_n}^{t_n+s} |\dot u(\tau)|^2\,d\tau)^{1/2},\end{aligned} \tag{43}$$

where both terms on the right-hand side of (43) tend to zero as $n\to\infty$. Thus $u(t_n+s)\to u_*$ as $n\to\infty$ for all $s\in[0,1]$. By continuity of $\mathcal{E}$, this in turn yields $\mathcal{E}(u(t_n+s))\to\mathcal{E}(u_*)$ as $n\to\infty$ for all $s\in[0,1]$, and therefore

$$\mathcal{E}(u_*) = \lim_{n\to\infty}\int_0^1 \mathcal{E}(u(t_n+s))\,ds, \tag{44}$$

by the dominated convergence theorem. Integrating $F(t_n+\cdot)$ (where $t_n\ge T_2$), we obtain

$$\begin{aligned}\int_0^1 F(t_n+s)\,ds &= \int_0^1 \mathcal{E}(u(t_n+s))\,ds + \frac12\int_{t_n}^{t_n+1}\Big(|\dot u(s)|^2 + b_0|v(s)|^2 + (a*|v|^2)(s)\Big)\,ds\\ &\quad + \int_{t_n}^{t_n+1}\Big(-\langle f(s), b_0 v(s) + (a*v)(s)\rangle + M\int_s^\infty (|f(\tau)|^2 + |\dot f(\tau)|^2)\,d\tau\Big)\,ds,\end{aligned}$$

which shows that

$$F_\infty = \lim_{n\to\infty}\int_0^1 F(t_n+s)\,ds = \mathcal{E}(u_*), \tag{45}$$

where we use (44), (Hf), (i), and $\dot u\in L_2(\mathbb{R}_+;\mathbb{R}^n)$ as well as the simple estimate

$$|\int_{t_n}^{t_n+1}\langle f(s), (a*v)(s)\rangle\,ds|^2 \le |a|_1 \int_{t_n}^{t_n+1} |f(s)|^2\,ds \int_{t_n}^{t_n+1} (a*|v|^2)(s)\,ds.$$

Since u_* was an arbitrary element of $\omega(u)$, (45) implies that $\mathcal{E}$ is constant on $\omega(u)$. Moreover, by the relative compactness of the orbit of u, we see that $\lim_{t\to\infty}\mathcal{E}(u(t)) = F_\infty$, that is $\lim_{t\to\infty} F(t) - \mathcal{E}(u_*) = 0$. Since $f(t)$ and the last term in the definition of F tend to 0 as $t\to\infty$, it follows that $\frac{1}{2}|\dot{u}|^2 + \frac{1}{4}(b_0|v|^2 + a * |v|^2)$ is bounded above by a continuous function that goes to 0 as $t \to \infty$. Hence $\lim_{t\to\infty} |\dot{u}(t)| = \lim_{t\to\infty}(a * |v|^2)(t) = 0$.

To establish (iv), let $u_* \in \omega(u)$ and choose $t_n \uparrow \infty$ such that $\lim_{n\to\infty} u(t_n) = u_*$. We know already that this entails $u(t_n+s) \to u_*$ as $n \to \infty$ for all $s \in [0,1]$. Thus $\nabla\mathcal{E}(u(t_n+s)) \to \nabla\mathcal{E}(u_*)$ as $n \to \infty$ for all $s \in [0,1]$. Using the dominated convergence theorem and the equation for u we have

$$\begin{aligned}\nabla\mathcal{E}(u_*) &= \lim_{n\to\infty} \int_0^1 \nabla\mathcal{E}(u(t_n+s))\,ds \\ &= \lim_{n\to\infty} \int_0^1 \Big(- \ddot{u}(t_n+s) - v(t_n+s) + f(t_n+s)\Big)\,ds \\ &= \lim_{n\to\infty} \Big(\dot{u}(t_n) - \dot{u}(t_n+1)\Big) + \lim_{n\to\infty} \int_{t_n}^{t_n+1} \Big(- v(s) + f(s)\Big)\,ds = 0,\end{aligned}$$

by (i), (ii), (Hf), and Hölder's inequality. Thus (iv) is proved. By the relative compactness of the orbit of u, we have $\lim_{t\to\infty} \nabla\mathcal{E}(u(t)) = \nabla\mathcal{E}(u^*)$, which together with (iv) implies (vi).

It remains to prove (v). Letting $u_* \in \omega(u)$, we integrate $V(t_n + \cdot)$ (with $t_n \geq T_2$) from 0 to 1 and argue similarly as before for F, making use of (i), (Hf), and (K3), to the result that

$$V_\infty = \lim_{n\to\infty} \int_0^1 V(t_n+s)\,ds = \mathcal{E}(u_*). \tag{46}$$

Therefore $\lim_{t\to\infty} V(t) - \mathcal{E}(u_*) = 0$. We can then estimate

$$\begin{aligned}V - \mathcal{E}(u_*) \geq\ & \frac{1}{2}\,|\dot{u}|^2 + \Big(\mathcal{E}(u) - \mathcal{E}(u_*)\Big) + \frac{b_0}{4}\,|v|^2 + \frac{\delta}{\omega}\,\langle \nabla\mathcal{E}(u), \dot{u}\rangle \\ & + \frac{1}{4}\Big(a * |v|^2 + \frac{1}{\omega}\,|v|^2 + \frac{1}{\omega}\,a * |z|^2 + \frac{b_0}{\omega}\,|z|^2\Big) \\ & - C_0\Big(|f|^2 + |\dot{f}|^2 + |\nabla\mathcal{E}(u)|^2 + k(t)^2|u_1|^2\Big) \\ & + M_1|u_1|^2 \int_t^\infty (\psi(\tau) + k(\tau)^2 + \dot{k}(\tau)^2)\,d\tau \\ & + M_1 \int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2 + |\ddot{f}(\tau)|^2)\,d\tau,\end{aligned}$$

provided $C_0 > 0$ is selected sufficiently large. Thus, in view of (ii), (vi), the assumptions on k and f, and since $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \mathcal{E}(u(t)) = \mathcal{E}(u_*)$, the nonnegative term

$$\frac{1}{4}\Big(a * |v|^2 + \frac{1}{\omega}\,|v|^2 + \frac{1}{\omega}\,a * |z|^2 + \frac{b_0}{\omega}\,|z|^2\Big)$$

is dominated by a continuous function that goes to 0 as $t \to \infty$. Hence (v) is satisfied. The proof is complete. $\square$

6 Convergence to equilibrium

We will now show that any global bounded solution of (1) converges to a solution $u_* \in \mathbb{R}^n$ of $\nabla\mathcal{E}(u_*) = 0$ as $t \to \infty$, thereby proving our main result, Theorem 2.1. The argument relies on Propositions 4.1, 5.1, and the Łojasiewicz inequality.

Proof of Theorem 2.1: Suppose u is a global bounded solution of (1), and let $u_* \in \omega(u)$ be as in the statement of Theorem 2.1. Set

$$W(t) = V(t) - \mathcal{E}(u_*), \quad t \geq T_2.$$

Then, by Proposition 4.1 and relation (46), W is nonincreasing on $[T_2, \infty)$, and $\lim_{t\to\infty} W(t) = 0$. Moreover, there exists a constant $C > 0$ such that

$$\dot{W} \leq -C\Big(|v|^2 + a * |v|^2 + |z|^2 + a * |z|^2 + |\nabla\mathcal{E}(u)|^2\Big), \quad t > T_2. \tag{47}$$

If $W(t_1) = 0$ for some $t_1 \geq T_2$, then $W(t) = 0$ for all $t \geq t_1$, and hence $u(t) = u_*$, since V is a strict Lyapunov function. So we may assume that $W(t) > 0$ for all $t \geq T_2$.

By the definitions of V and W, the property $a \in L_1(\mathbb{R}_+)$, Lemma 3.1, and Young's inequality, we may estimate on $[T_2, \infty)$,

$$\begin{aligned} W^{1-\theta} \leq & C_1\Big\{|\mathcal{E}(u) - \mathcal{E}(u_*)|^{1-\theta} + |\dot{u}|^{2(1-\theta)} + (a * |v|^2)^{\frac{2(1-\theta)}{2}} + |\nabla\mathcal{E}(u)|^{2(1-\theta)} \\ & + |v|^{2(1-\theta)} + (a * |z|^2)^{\frac{2(1-\theta)}{2}} + (k(t)|u_1|)^{2(1-\theta)} + |f|^{2(1-\theta)} + |\dot{f}|^{2(1-\theta)} \\ & + b_0|z|^{2(1-\theta)} + |u_1|^{2(1-\theta)}(\int_t^\infty (\psi(\tau) + k(\tau)^2 + \dot{k}(\tau)^2)\, d\tau)^{1-\theta} \\ & + (\int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2 + |\ddot{f}(\tau)|^2)\, d\tau)^{1-\theta}\Big\}, \end{aligned} \tag{48}$$

with some constant $C_1 > 0$. Note that $\theta \in (0, 1/2]$ implies $2(1-\theta) \geq 1$. Using the assumptions on k and f as well as Proposition 5.1 (ii), (v), (vi), it follows that for t sufficiently large, say $t \geq T \geq T_2$, we have

$$\begin{aligned} W^{1-\theta} \leq & C_2\Big\{|\mathcal{E}(u) - \mathcal{E}(u_*)|^{1-\theta} + |\dot{u}| + (a * |v|^2)^{\frac{1}{2}} + |\nabla\mathcal{E}(u)| \\ & + |z| + |v| + (a * |z|^2)^{\frac{1}{2}} + \Lambda(t)\Big\}, \end{aligned} \tag{49}$$

where $C_2 > 0$ is some constant, and

$$\begin{aligned} \Lambda(t) = & k(t)|u_1| + |f| + |\dot{f}| + |u_1|(\int_t^\infty (\psi(\tau) + k(\tau)^2 + \dot{k}(\tau)^2)\, d\tau)^{\frac{1}{2}} \\ & + (\int_t^\infty (|f(\tau)|^2 + |\dot{f}(\tau)|^2 + |\ddot{f}(\tau)|^2)\, d\tau)^{\frac{1}{2}}, \quad t \geq T_2. \end{aligned}$$

Observe that the assumptions on k and f ensure that $\Lambda \in L_1((T_2, \infty))$. Here we use Young's inequality for convolutions to estimate the convolution term appearing in ψ:

$$\int_t^\infty (e^{-2\gamma\cdot} * k)(\tau)\, d\tau \leq |k|_{L_1((t,\infty))} \int_t^\infty e^{-2\gamma\tau}\, d\tau \leq \frac{|k|_1}{2\gamma}\, e^{-2\gamma t}, \quad t \geq T_2.$$

In view of (36), there holds

$$|\dot{u}| \leq C_3\Big((a * |v|^2)^{\frac{1}{2}} + (a * |z|^2)^{\frac{1}{2}} + |z| + \Lambda(t)\Big), \quad t \geq T_2, \tag{50}$$

with some constant $C_3 > 0$.

Define next the set $\Omega_\sigma \subset (T, \infty)$ by

$$\Omega_\sigma = \{t \in (T, \infty) : \ |u(t) - u_*| < \sigma\}.$$

By continuity of u, Ω_σ is an open set in $\mathbb{R}$. Restricting t in (49) to Ω_σ, we may employ the Łojasiewicz inequality for $\mathcal{E}$ near u_* to obtain

$$W^{1-\theta} \leq C_4\Big\{|\nabla\mathcal{E}(u)| + |\dot u| + (a*|v|^2)^{\frac12} + |v| + (a*|z|^2)^{\frac12} + |z| + \Lambda(t)\Big\}, \quad t \in \Omega_\sigma, \tag{51}$$

for some constant $C_4 > 0$. From (47), (50), and (51) we then deduce that

$$\begin{aligned}
-\frac{d}{dt}W^\theta &= -\theta W^{\theta-1}\dot W \\
&\geq \frac{\theta C\{|v|^2 + a*|v|^2 + |z|^2 + a*|z|^2 + |\nabla\mathcal{E}(u)|^2\}}{C_5\{|\nabla\mathcal{E}(u)| + (a*|v|^2)^{\frac12} + |v| + (a*|z|^2)^{\frac12} + |z| + \Lambda(t)\}} \\
&\geq C_6\Big(|v|^2 + a*|v|^2 + a*|z|^2 + |z|^2 + |\nabla\mathcal{E}(u)|^2\Big)^{\frac12} - C_7\Lambda(t) \\
&\geq C_8\Big(|v| + (a*|v|^2)^{\frac12} + (a*|z|^2)^{\frac12} + |z| + |\nabla\mathcal{E}(u)|\Big) - C_7\Lambda(t), \quad t \in \Omega_\sigma,
\end{aligned} \tag{52}$$

where $C_i > 0$, $i = 5, \ldots, 8$ are constants. Integrating (52) over Ω_σ and using $\Lambda \in L_1((T_2, \infty))$ shows that each of the functions $(a*|v|^2)^{\frac12}$, $(a*|z|^2)^{\frac12}$, and $|z|$ belongs to $L_1(\Omega_\sigma)$. This together with (50) yields $\dot u \in L_1(\Omega_\sigma; \mathbb{R}^n)$. By a standard argument, see e.g. [19] or [27, p. 301, 302], we then conclude that $\dot u \in L_1(\mathbb{R}_+; \mathbb{R}^n)$, which implies $\lim_{t\to\infty} u(t) = u_*$. Finally, Proposition 5.1 yields $\nabla\mathcal{E}(u_*) = 0$. □

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