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Solution concepts in vector optimization. A fresh look at an old story

Frank Heyde Andreas Löhne

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Abstract

Over the last decades various solution concepts for vector optimization problems have been established and used, among them efficient, weakly efficient and properly efficient solutions. In contrast to the classical approach, we define a solution to be a set of efficient solutions on which the infimum of the objective function with respect to an appropriate complete lattice (the space of self-infimal sets) is attained. The set of weakly efficient solutions is not considered to be a solution, but, weak efficiency is essential in the construction of the complete lattice. In this way, two classic concepts are involved in a common approach.

Several different notions of semicontinuity are compared. Using the space of self-infimal sets, we can show that various originally different concepts coincide. A Weierstrass existence result is proved for our solution concept.

A slight relaxation of the solution concept yields a relationship to properly efficient solutions.

1 Introduction

The theory of vector optimization [19, 16] has a number of structural differences in contrast to scalar optimization. For instance, the objective space, which is a partially ordered linear space extended by $\pm\infty$, is in contrast to the extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ not necessarily totally ordered. Moreover, there are several "competing" solution concepts, such as efficient solutions, weakly efficient solutions and properly efficient solutions. In the literature one can observe a long discussion about the advantages and disadvantages of the different concepts. Efficient solutions are usually motivated by applications and weakly or properly efficient solutions are motivated to be beneficial for the theory and sometimes easier to calculate.

Another difference is the fact that in the vectorial case one is usually interested in the set of all solutions (of certain type), and not only in a single one. The idea of vector optimization is to present the decision maker all or at least a representative choice of efficient solutions. The task of the decision maker is to select one efficient solution. A single efficient solution can be calculated by scalarization, which is not more complicated than solving a comparable scalar optimization problem. However, it is a priori not clear how to choose the "right" or the "best" scalarization.

The concept of infimum (and supremum) is important in (scalar) optimization. It is used to describe a solution, is involved in important concepts like conjugate function, directional derivative and support function. It is also important for duality theory. But, even the infimum (in the sense of a greatest lower bound in the complete lattice) is not appropriate to define the most of the mentioned concepts and results if the vectorial objective space is a complete lattice (which is only true in special cases); see e.g. [22] for such an approach. As a consequence, the infimum (and supremum) with respect to a complete lattice is usually not taken into account in vector optimization. Therefore, the formulations of several important results in vector optimization, such as duality theorems, existence results and optimality conditions differ from their scalar counterparts.

One has started to overcome these structural shortcomings in some recent works [18, 13] by a set-valued approach. It turned out that it is possible to formulate and prove vectorial duality theorems very similar to the corresponding scalar results if the vectorial image space is replaced by a certain, more flexible subset of the power set of this vector space, the space of self-infimal subsets. This space provides a complete lattice and the infimum with respect to this complete lattice is closely related to the classic solution concepts in vector optimization. Nevertheless, neither of the classical notions can be characterized as the attainment of a certain infimum. A solution concept which meets this requirement is developed in this article.

Our solution concept is not a further modification of the existing ones, but rather it involves several classic concepts. It can be motivated by both a theoretic as well as an applicatory point of view. We demonstrate its advantages by a Weierstrass existence result, which is based on a variety of semicontinuity concepts. In contrast to the traditional approach, these semicontinuity concepts can be shown to be equivalent in our theory.

The article is organized as follows. The solution concept is developed in two steps, in Section 2 and 4. Section 3 is devoted to the construction of the complete lattice, the space of self-infimal sets. In Section 5 we discuss several general semicontinuity concepts as well as their relationships. These results are applied in Section 6 to the case of vector optimization. The equivalence of several concepts as well as a Weierstrass existence result are shown. A relaxed solution concept, its relationship to properly efficient solutions and its relevance for the duality theory is discussed in Section 7.

2 Solution concept. Part I

In this section we consider a general complete-lattice-valued optimization problem and discuss a solution concept which is based on the *attainment of the infimum*. At the end of this section we see that the direct application of this approach to vector optimization is not yet useful. The problems, however, will be solved in the subsequent sections.

Throughout this article let $f : X \to Z$, where X is an arbitrary set and $(Z, \leq)$ is a partially ordered set. For a nonempty subset $A \subseteq X$, called *feasible set*, we consider the following optimization problem:

$$\text{Minimize } f : X \to Z \text{ with respect to } \leq \text{ over } A. \tag{1}$$

A classical solution concept is the following.

Definition 2.1. An element $\bar{x} \in A$ is called an *efficient solution* of (1) if

$$[x \in A \;\wedge\; f(x) \leq f(\bar{x})] \quad \Longrightarrow \quad f(x) = f(\bar{x}).$$

The set of all efficient solutions of (1) is denoted by $\mathrm{Eff}\,(f, A)$.

For $B \subseteq Z$ we denote by

$$\operatorname{Min} B := \{z \in B |\ (y \in B \wedge y \leq z) \Rightarrow y = z\}$$

the set of *minimal elements* of B. Using the notation

$$f[A] := \{f(x) |\ x \in A\}$$

it holds

$$\operatorname{Min} f[A] = f[\operatorname{Eff}(f, A)].$$

As demonstrated by the following two examples, the set $\operatorname{Eff}(f, A)$ without any further requirement is unsatisfactory as a solution concept for vector optimization.

Example 2.2. Let $X = Z = \mathbb{R}^2$, Z partially ordered by the natural ordering cone $\mathbb{R}^2_+$. Let f be the identity map and

$$A = \left\{x \in \mathbb{R}^2 |\ x_1 > 0,\ x_2 > 0,\ x_1 + 100x_2 > 100\right\} \cup \left\{(100, 0)^T\right\}.$$

We have $\operatorname{Eff}(f, A) = \left\{(100, 0)^T\right\}$. But the nonempty set $\operatorname{Eff}(f, A)$ does not yield a sufficient amount of information about the problem. From a practical point of view, for instance, the feasible, non-efficient point $(1, 1)^T$ could be more interesting than the set of efficient solutions.

On the other hand, there are vector optimization problems where it is already sufficient for the decision maker to know a proper subset of $\operatorname{Eff}(f, A)$.

Example 2.3. Let $X = Z = \mathbb{R}^2$, Z partially ordered by $\mathbb{R}^2_+$, and

$$A = \left\{x \in \mathbb{R}^2 |\ x_1 \geq 0,\ x_2 \geq 0,\ 2x_1 + x_2 \geq 2,\ x_1 + 2x_2 \geq 2\right\}.$$

The objective $f : X \to Z$ is defined as $f(x) = (0, x_2)^T$. It holds

$$\operatorname{Eff}(f, A) = \left\{x \in \mathbb{R}^2 |\ x_1 \geq 2, x_2 = 0\right\}.$$

In typical applications the decision maker selects a point in the image $f[\operatorname{Eff}(f, A)]$ of $\operatorname{Eff}(f, A)$ with respect to f. We have $f[\operatorname{Eff}(f, A)] = \left\{(0, 0)^T\right\}$. But, the same image is already obtained by any nonempty subset of $\operatorname{Eff}(f, A)$.

Concerning Example 2.3, we can relax the solution concept to be a set $\bar{X} \subseteq A$ such that $f[\bar{X}] = \operatorname{Min} f[A]$. It is possible to avoid the situation in Example 2.2 by assuming the well-known domination property, which is recalled and discussed below. We do not proceed in this way. Instead, the *attainment of the infimum* is involved into the discussion. In order to ensure the existence of the infimum (and supremum), we need to assume $(Z, \leq)$ to be a *complete lattice*. Let us recall the definition.

Given a partially ordered set $(Z, \leq)$, we say $\bar{z} \in Z$ is a *lower bound* of $Y \subseteq Z$ if $\bar{z} \leq y$ for all $y \in Y$. The element $\bar{z} \in Z$ is called the infimum of $Y \subseteq Z$ (written $\bar{z} = \inf Y$) if $\bar{z}$ is a lower bound of Y and for every other lower bound $\hat{z}$ of Y it holds $\hat{z} \leq \bar{z}$. As the ordering $\leq$ is antisymmetric, the infimum, if it exists, is uniquely defined. The partially ordered set $(Z, \leq)$ is called a *complete lattice* if every subset of Z has an infimum and a supremum, where the supremum is defined analogously and is denoted by $\sup Y$. For more details about complete lattices; see, e.g., [2, 26].

From now on, let $(Z, \leq)$ be a complete lattice. The infimum of f over $A \subseteq X$ is defined as

$$\inf_{x \in A} f(x) := \inf \{f(x) |\ x \in A\} = \inf f[A].$$

Definition 2.4. Let $A \subseteq X$ and $\bar{x} \in X$. We say *the infimum of f over A is attained in $\bar{x}$* if

$$\bar{x} \in A \quad \wedge \quad f(\bar{x}) = \inf_{x \in A} f(x).$$

In case such a point $\bar{x}$ exists (does not exist), we say the infimum of f over A is (not) attained.

The attainment of the infimum is an important concept in optimization. In vector optimization it is, however, very hard to fulfill as the following example shows.

Example 2.5. Let $X = \mathbb{R}^2$, $Z = \mathbb{R}^2 \cup \{+\infty\} \cup \{-\infty\}$, $\mathbb{R}^2$ partially ordered by the cone $\mathbb{R}^2_+$. The ordering is denoted by $\leq$ and extended to Z by setting $-\infty \leq z \leq +\infty$ for all $z \in Z$. Then, $(Z, \leq)$ is a complete lattice. Let

$$A = \left\{x \in \mathbb{R}^2 |\; x_1 \geq 0,\; x_2 \geq 0,\; 2x_1 + x_2 \geq 2,\; x_1 + 2x_2 \geq 2\right\}$$

and let f be the identity map. Then the infimum of f over A is not attained. Indeed, we have $\inf_{x \in A} f(x) = 0$, but there is no $\bar{x} \in A$ with $f(\bar{x}) = 0$.

A further aspect can be observed in the previous example. We enforce that the infimum is attained in a single vector. As already discussed in the introduction, we expect a solution to be a set of vectors. To this end, we introduce the *canonical extension* of the function f, namely the function

$$F : 2^X \to Z, \quad F(U) := \inf_{x \in U} f(x).$$

Of course, it holds $f(x) = F(\{x\})$ for all $x \in X$. Working with the canonical extension F instead of f, we make the following two observations. The attainment of the infimum is now easier to realize. And, the infimum is now attained in a set, not in a single element of X.

We now give a characterization of the attainment of the infimum of the canonical extension F in terms of f.

Proposition 2.6. *Let $A \subseteq X$. The following statements are equivalent:*

(i) The infimum of F over 2^A is attained in $\bar{X}$, i.e.,

$$\bar{X} \in 2^A \quad \wedge \quad F(\bar{X}) = \inf_{U \in 2^A} F(U).$$

(ii) $\bar{X} \subseteq A \quad \wedge \quad \inf_{x \in \bar{X}} f(x) = \inf_{x \in A} f(x).$

Proof. It remains to prove the equality

$$\inf_{U \in 2^A} F(U) = \inf_{x \in A} f(x). \tag{2}$$

For all $x \in A$ we have

$$\inf_{U \in 2^A} F(U) \leq F(\{x\}) = f(x).$$

The infimum over $x \in A$ yields $\leq$ in (2). For all $U \subseteq A$ it holds

$$F(U) = \inf_{x \in U} f(x) \geq \inf_{x \in A} f(x).$$

Taking the infimum over all $U \in 2^A$ we get $\geq$ in (2). $\square$

Next we define a solution concept for the complete-lattice-valued problem (1).

Definition 2.7. A nonempty set $\bar{X}$ with $f[\bar{X}] = \operatorname{Min} f[A]$ is called a *solution* of (1) if the infimum of the canonical extension F over 2^A is attained in $\bar{X}$.

It is easily seen that, if a solution exists, then $\operatorname{Eff}(f, A)$ is a solution. Of course, if $\operatorname{Eff}(f, A)$ is a solution, every subset $\bar{X}$ of $\operatorname{Eff}(f, A)$ with $f[\bar{X}] = \operatorname{Min} f[A]$ is a solution, too. If $\bar{X} = \operatorname{Eff}(f, A)$ is the only solution of (1), we say $\bar{X}$ is a *unique* solution.

In Example 2.2 (where a complete lattice Z is obtained by extending $\mathbb{R}^2$ by two elements $\pm\infty$), $\operatorname{Eff}(f, A)$ is not a solution; whence a solution does not exist. A natural condition ensuring that $\operatorname{Eff}(f, A)$ is a solution is the well-known *domination property*, see, e.g., [4].

Proposition 2.8. *The set* $\operatorname{Eff}(f, A)$ *is a solution of* (1) *if the* domination property *holds, that is,*

$$\forall x \in A,\ \ \exists \bar{x} \in \operatorname{Eff}(f, A): \ \ f(\bar{x}) \leq f(x). \tag{3}$$

Proof. Set $\bar{X} := \operatorname{Eff}(f, A)$. Of course, we have $\bar{X} \in 2^A$. According to Proposition 2.6, the attainment of the infimum of the canonical extension F over 2^A in $\bar{X}$ is equivalent to

$$\inf_{x \in \bar{X}} f(x) = \inf_{x \in A} f(x). \tag{4}$$

From (3) we get $\inf_{x \in \bar{X}} f(x) \leq \inf_{x \in A} f(x)$ and the opposite inequality in (4) follows from $\bar{X} \subseteq A$. □

The domination property is not necessary for a solution to exist. An example is given below (Example 4.5).

In order to apply our solution concept to vector optimization, we need a suitable complete lattice $(Z, \leq)$. Originally, the image space of a vector optimization problem is a partially ordered vector space $(Y, \leq)$. In some cases, Y can be extended to a complete lattice by setting

$$Z := Y \cup \{-\infty\} \cup \{+\infty\},$$

where the ordering is extended in the usual way:

$$\forall z \in Z: \ \ -\infty \leq z \leq +\infty.$$

This procedure has two disadvantages. On the one hand, in many cases we do not obtain a complete lattice in this way. On the other hand, even if a complete lattice is obtained, our solution concept is unsatisfactory with this choice of Z. This is demonstrated by the following two examples.

Example 2.9. Let $Y = \mathbb{R}^3$ and let C be the polyhedral convex cone which is spanned by the vectors $(0,0,1)^T$, $(0,1,1)^T$, $(1,0,1)^T$, $(1,1,1)^T$. Then $(Z, \leq_C)$ is not a complete lattice. For instance there is no supremum of the set $\{(0,0,0)^T, (1,0,0)^T\}$.

Example 2.10. Let $X = \mathbb{R}^2$, $(Z, \leq)$ the complete lattice from Example 2.5, f the identity map and

$$A := \left\{x \in \mathbb{R}^2 |\ \ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 > 1\right\} \cup \{x \in \mathbb{R}^2 |\ \ x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq 2\}.$$

Then $\bar{X} := \left\{(0,2)^T, (2,0)^T\right\}$ is a solution. This is unsatisfactory from the viewpoint of vector optimization, because this set does not contain enough information.

In the next section we introduce a suitable complete lattice $(Z, \leq)$ which allows us to treat vector optimization problems.

3 The complete lattice $\mathcal{I}$ of self-infimal sets

The *infimal set* (resp. *supremal set*) of a given set $A \subseteq \mathbb{R}^q$ was introduced by Nieuwenhuis [21]. The concept was extended by Tanino [24], and slightly modified with respect to the elements $\pm\infty$ by Löhne and Tammer [18]. The space $\mathcal{I}$ of self-infimal sets (A is self-infimal, if A equals its infimal set) was shown in [18] to be a complete lattice. In [18, 13] it was demonstrated that the complete lattice $\mathcal{I}$ is useful for the duality theory in vector optimization.

Throughout let $C \subsetneq \mathbb{R}^q$ be a closed convex cone with nonempty interior. The set of *weakly minimal* points of a subset $A \subseteq \mathbb{R}^q$ (with respect to C) is defined by

$$\operatorname{wMin} A := \{y \in A |\ (\{y\} - \operatorname{int} C) \cap A = \emptyset\}.$$

The *upper closure* (with respect to C) of $A \subseteq \mathbb{R}^q$ is defined (see e.g. [4]) to be the set

$$\operatorname{Cl}_+ A := \{y \in \mathbb{R}^q |\ \{y\} + \operatorname{int} C \subseteq A + \operatorname{int} C\}.$$

It is an easy task to show that for every (not necessarily convex) set $A \subseteq \mathbb{R}^q$ it holds

$$\operatorname{Cl}_+ A = \operatorname{cl}(A + \operatorname{int} C) = \operatorname{cl}(A + C).$$

If $A \neq \emptyset$ we have [21, Th. I-18]

$$\operatorname{wMin} \operatorname{Cl}_+ A = \emptyset \iff A + \operatorname{int} C = \mathbb{R}^q \iff \operatorname{Cl}_+ A = \mathbb{R}^q.$$

The definition of the upper closure can be extended for subsets of the space $\overline{\mathbb{R}^q} := \mathbb{R}^q \cup \{-\infty\} \cup \{+\infty\}$. For a subset $A \subseteq \overline{\mathbb{R}^q}$ we set

$$\operatorname{Cl}_+ A := \begin{cases} \mathbb{R}^q & \text{if} \quad -\infty \in A \\ \{y \in \mathbb{R}^q |\ \{y\} + \operatorname{int} C \subseteq A \setminus \{+\infty\} + \operatorname{int} C\} & \text{else.} \end{cases}$$

Clearly, the upper closure of a subset of $\overline{\mathbb{R}^q}$ is always a subset of $\mathbb{R}^q$. The *infimal set* of $A \subseteq \overline{\mathbb{R}^q}$ (with respect to C) is defined by

$$\operatorname{Inf} A := \begin{cases} \operatorname{wMin} \operatorname{Cl}_+ A & \text{if} \quad \emptyset \neq \operatorname{Cl}_+ A \neq \mathbb{R}^q \\ \{-\infty\} & \text{if} \quad \operatorname{Cl}_+ A = \mathbb{R}^q \\ \{+\infty\} & \text{if} \quad \operatorname{Cl}_+ A = \emptyset. \end{cases}$$

We see that the infimal set of $A \subseteq \mathbb{R}^q$ with respect to C coincides essentially with the set of weakly minimal elements of the set $\operatorname{cl}(A+C)$. By our conventions, $\operatorname{Inf} A$ is a nonempty set for every $A \subseteq \overline{\mathbb{R}^q}$. Clearly, if $-\infty$ belongs to A, we have $\operatorname{Inf} A = \{-\infty\}$, in particular, $\operatorname{Inf}\{-\infty\} = \{-\infty\}$. Furthermore, it holds $\operatorname{Inf} \emptyset = \operatorname{Inf}\{+\infty\} = \{+\infty\}$ and $\operatorname{Cl}_+ A = \operatorname{Cl}_+(A \cup \{+\infty\})$ and hence $\operatorname{Inf} A = \operatorname{Inf}(A \cup \{+\infty\})$ for all $A \subseteq \overline{\mathbb{R}^q}$. For all $A, B \subseteq \overline{\mathbb{R}^q}$ it holds

$$(A \subseteq B \implies \operatorname{Cl}_+ A \subseteq \operatorname{Cl}_+ B) \qquad \text{and} \qquad \operatorname{Cl}_+ \operatorname{Inf} A = \operatorname{Cl}_+ A \tag{5}$$

Moreover, for $A \subseteq \mathbb{R}^q$ with $\emptyset \neq \operatorname{Cl}_+ A \neq \mathbb{R}^q$ it holds

$$\operatorname{Inf} A = \{y \in \mathbb{R}^q |\ y \notin A + \operatorname{int} C,\ \{y\} + \operatorname{int} C \subseteq A + \operatorname{int} C\} \tag{6}$$

$$A + \operatorname{int} C = \operatorname{Inf} A + \operatorname{int} C \tag{7}$$

More details about infimal and upper closed sets can be found in [21, 24, 18]. One can define analogously, using the set $\operatorname{wMax} A := -\operatorname{wMin}(-A)$ of weakly maximal elements of $A \subseteq \mathbb{R}^q$, the lower closure $\operatorname{Cl}_- A$ and the set $\operatorname{Sup} A$ of supremal elements of $A \subseteq \overline{\mathbb{R}^q}$ and there are analogous statements.

Let $\mathcal{I} := \mathcal{I}_C\left(\overline{\mathbb{R}^q}\right)$ be the family of all self-infimal subsets of $\overline{\mathbb{R}^q}$, i.e., all sets $A \subseteq \overline{\mathbb{R}^q}$ satisfying $\operatorname{Inf} A = A$. In $\mathcal{I}$ we introduce an order relation $\preccurlyeq$ as follows:

$$A \preccurlyeq B :\iff \operatorname{Cl}_+ A \supseteq \operatorname{Cl}_+ B.$$

We denote by $\mathcal{F} := \mathcal{F}_C(\mathbb{R}^q)$ the space of all upper closed subsets of $\mathbb{R}^q$ (with respect to C). Of course, $(\mathcal{F}, \supseteq)$ is complete lattice and for any nonempty subset $\mathcal{A} \subseteq \mathcal{F}$ it holds

$$\inf \mathcal{A} = \operatorname{cl} \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

It is sometimes easier to work with the complete lattice $(\mathcal{F}, \supseteq)$ instead of $(\mathcal{I}, \preccurlyeq)$. A corresponding statement for $(\mathcal{I}, \preccurlyeq)$ is then easily obtained using the following result.

Theorem 3.1 ([18]). *The map $A \overset{j}{\longmapsto} \operatorname{Cl}_+ A$ is an isotone (i.e. order-preserving) one-to-one map from $(\mathcal{I}, \preccurlyeq)$ onto $(\mathcal{F}, \supseteq)$. The inverse map is $B \overset{j^{-1}}{\longmapsto} \operatorname{Inf} B$.*

One of the central preliminary result is the following.

Theorem 3.2 ([18]). *The partially ordered set $(\mathcal{I}, \preccurlyeq)$ is a complete lattice. For nonempty sets $\mathcal{A} \subseteq \mathcal{I}$ it holds*

$$\inf \mathcal{A} = \operatorname{Inf} \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \operatorname{Sup} \bigcup_{A \in \mathcal{A}} A.$$

In every complete lattice the infimum (supremum) over the empty set equals to the largest (smallest) element. In our case this means $\inf \emptyset = \{+\infty\}$ and $\sup \emptyset = \{-\infty\}$. Note that an addition and a multiplication by nonnegative real numbers can be introduced in $\mathcal{I}$ so that it is possible to deal with convexity; see [18, 13] for more details.

4 Solution concept. Part II

The ideas of Section 2 and Section 3 are now combined in order to get an adequate solution concept for vector optimization problems.

Let X be a set and $A \subseteq X$. Let $C \subsetneq \mathbb{R}^q$ be a closed convex pointed (i.e. $C \cap (-C) = \{0\}$) cone with nonempty interior and let $f : X \to \overline{\mathbb{R}^q}$. The cone C induces a partial ordering $\leq_C$ on $\mathbb{R}^q$, which can be extended to $\overline{\mathbb{R}^q}$. The extended ordering is again denoted by $\leq_C$. We consider the following vector optimization problem:

$$\text{Minimize } f : X \to \overline{\mathbb{R}^q} \text{ with respect to } \leq_C \text{ over } A. \tag{8}$$

Note that $(\overline{\mathbb{R}^q}, \leq_C)$ is not necessarily a complete lattice, compare Example 2.9. As already discussed in Section 2, a solution in the sense of Definition 2.7 is not suitable for problem (8), despite whether $(\overline{\mathbb{R}^q}, \leq_C)$ is a complete lattice or not.

Thus we assign to (8) a corresponding $\mathcal{I}$-valued-problem, i.e., a problem of type (1), where the complete lattice $(Z, \leq) = (\mathcal{I}, \preccurlyeq)$ is used. Note that $(\mathcal{I}, \preccurlyeq)$ is defined with respect to the ordering cone C of the vector optimization problem. The space $(\mathcal{I}, \preccurlyeq)$ (with respect to C) is always a complete lattice, even if $(\overline{\mathbb{R}^q}, \leq_C)$ is not.

Given a function $f : X \to \overline{\mathbb{R}^q}$, we set $\bar{f} : X \to \mathcal{I}$, $\bar{f}(x) := \operatorname{Inf}\{f(x)\}$ and we assign to (8) the following problem.

$$\text{Minimize } \bar{f} : X \to \mathcal{I} \text{ with respect to } \preccurlyeq \text{ over } A. \tag{9}$$

Problem (9) is said to be the *lattice extension* of the vector optimization problem (8). This terminology can be motivated by the fact that the lattice extension of the vector optimization problem allows us to handle the problem in the framework of complete lattices. Next, we see that both problems (8) and (9) are related by having the same efficient solutions.

Proposition 4.1. *A feasible element $\bar{x} \in A$ is an efficient solution of the vector optimization problem* (8) *if and only if it is an efficient solution of its lattice extension* (9)*.*

Proof. Follows from the fact that $f(x) \leq_C f(v)$ if and only if $\operatorname{Inf}\{f(x)\} \preccurlyeq \operatorname{Inf}\{f(v)\}$. □

We now define our solution concept for the vector optimization problem (8).

Definition 4.2. A set $\bar{X} \subseteq X$ is called a *solution* of the vector optimization problem (8), if $\bar{X}$ is a solution of the corresponding lattice extension (9).

We next investigate some properties of a solution of (8).

Theorem 4.3. *A set $\bar{X} \subseteq X$ is a solution of the vector optimization problem* (8) *if and only if the following three conditions are satisfied:*

(i) $\bar{X} \subseteq A$,

(ii) $f[\bar{X}] = \operatorname{Min} f[A]$,

(iii) $\operatorname{Inf} f[\bar{X}] = \operatorname{Inf} f[A]$.

Proof. This is a direct consequence of Proposition 2.6 and Theorem 3.2. □

Example 4.4. Consider the vector optimization problem (8) with a linear objective function f and a polyhedral convex feasible set A. Then, the set $\operatorname{Eff}(f, A)$ is a solution, whenever it is nonempty. As shown in [12, Lemma 2.1] (note that the cone has to be pointed there) the domination property is fulfilled in this case. Thus Proposition 2.8 yields that $\operatorname{Eff}(f, A)$ is a solution.

Example 4.5. Consider the vector optimization problem (8) with $f : \mathbb{R}^2 \to \mathbb{R}^2$ the identity map, $C = \mathbb{R}^2_+$ and let

$$A := \{x \in \mathbb{R}^2 |\ x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \geq 1\} \setminus \{(0,1)^T\}.$$

Then $\bar{X} := \operatorname{Eff}(f, A) = \{\lambda\,(0,1)^T + (1-\lambda)\,(1,0)^T |\ 0 \leq \lambda < 1\}$ is a solution, but the domination property is not satisfied.

The following example shows that a our solution concept is also relevant for problems which are not a lattice extension of a given vector optimization problem.

Example 4.6. (Duality theory for linear vector optimization problems [13]) Consider the vector optimization problem (8) with the following specification to a linear problem; let $C = \mathbb{R}^q_+$, $f(x) = Mx$ for some matrix $M \in \mathbb{R}^{q \times n}$ and $A = \{x \in \mathbb{R}^n |\ Bx \geq b\}$ for some matrix $B \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following $\mathcal{I}$-valued dual problem was introduced in [13]:

$$\text{Maximize } H : \mathbb{R}^m \times \mathbb{R}^q \to \mathcal{I} \text{ with respect to } \preccurlyeq \text{ over } U. \tag{10}$$

where we set $k := (1, \ldots, 1)^T$ and

$$H : \mathbb{R}^m \times \mathbb{R}^q \to \mathcal{I}, \quad H(u, c) := \operatorname{Inf} \left\{ y \in \mathbb{R}^q |\ c^T y = b^T u \right\},$$

$$U := \left\{ (u, c) \in \mathbb{R}^m \times \mathbb{R}^q |\ (u, c) \geq 0,\ k^T c = 1,\ B^T u = M^T c \right\}.$$

In [13], strong duality between the lattice extension of the given problem and the dual problem (10) is shown. Furthermore, it is proved that the supremum of the dual problem is attained in the set of efficient solutions, whenever $A \neq \emptyset$ and $U \neq \emptyset$. Using the terminology of the present paper, this means that a solution of the dual problem exists.

5 Semicontinuity

In this section we analyse several notions of lower semicontinuity for functions with values in Z, where Z is a partially ordered set and sometimes even a complete lattice. We are mainly interested in the general case without any a priori topology on Z but also in the special case that $Z = \bar{Y} := Y \cup \{\pm\infty\}$ is the extension of a partially ordered topological vector space Y. In particular we examine $\overline{\mathbb{R}^q}$- and $\mathcal{F}$-valued functions (note that $\mathcal{F}$ is isomorphic to $\mathcal{I}$, see Theorem 3.1). Moreover we state a Weierstrass type result ensuring the validity of the domination property for the general optimization problem (1).

If $f : X \to Z = \bar{Y}$ is a function from a topological space X into the topological lattice of extended real numbers (i.e., $Y = \mathbb{R}$) then the following five properties are equivalent characterizations of lower semicontinuity of the function f.

(a) For all $z \in Z$ the level sets $\mathcal{L}_f(z) := \{x \in X |\ f(x) \leq z\}$ are closed.

(b) For all $y \in Y$ the level sets $\mathcal{L}_f(y)$ are closed.

(c) For all $x_0 \in X$,

$$f(x_0) \leq \sup_{U \in \mathcal{N}(x_0)} \inf_{x \in U} f(x) =: \liminf_{x \to x_0} \bar{f}(x)$$

holds true, where $\mathcal{N}(x_0)$ is a fundamental system of neighborhoods of x_0.

(d) For every $x_0 \in X$, every $y_0 \in Y$ with $y_0 \leq f(x_0)$ and every neighborhood V of y_0 there is some neighborhood U of x_0 such that

$$\forall x \in U\ \exists y \in V : f(x) \geq y.$$

(e) The epigraph of f, $\operatorname{epi} f := \{(x, y) \in X \times Y |\ f(x) \leq y\}$, is closed.

In more general cases of Z these five properties do not coincide any longer. If Z is merely a complete lattice without any additional structure then only the properties (a) and (c) are applicable. Following [8] and [20] we introduce the following notions.

Definition 5.1. Let X be a topological space, $(Y, \leq)$ and $(Z, \leq)$ partially ordered sets. A function $f : X \to Z$ is called *level closed* if property (a) holds. $f : X \to \bar{Y}$ is called *weakly level closed* if property (b) holds. In case Z is a complete lattice a function $f : X \to Z$ is called *lattice-l.s.c.* if property (c) holds. In case Y is a partially ordered topological space $f : X \to \bar{Y}$ is called *topologically l.s.c.* if property (d) holds and *epi-closed* if (e) holds.

Remark 5.2. In [20] the concept of level closedness is used for property (b). We call a function level closed if all level sets are closed and denote the weaker property (b) by weak level closedness.

The notions of lattice- and topological semicontinuity are introduced in [8]. Our definition of lattice semicontinuity coincides with that of [8] but the definition of topological semicontinuity differs slightly from that in [8] since we do not require a topological structure on the whole set $\bar{Y}$ but it coincides with the concept denoted simply by lower semicontinuity in [20]. Note also that [8] deals with upper not with lower semicontinuity.

In the following we investigate the relationships between these properties. First we clarify the connection between the two notions that do not require further structural assumption for the image space Z in addition to the lattice property.

Proposition 5.3. *Let X be a topological space and $(Z, \leq)$ a complete lattice. Then a function $f : X \to Z$ is level closed if it is lattice-l.s.c..*

Proof. Assume that f is lattice-l.s.c. but not level closed, i.e., there is some $z_0 \in Z$ such that $\mathcal{L}_f(z_0)$ is not closed. Then there is some $x_0 \in X$ with $x_0 \notin \mathcal{L}_f(z_0)$ such that for all $U \in \mathcal{N}(x_0)$ there exists some $x \in U$ with $x \in \mathcal{L}_f(z_0)$. This implies

$$\sup_{U \in \mathcal{N}(x_0)} \inf_{x \in U} f(x) \leq z_0,$$

hence $f(x_0) \leq z_0$ by (c), contradicting $x_0 \notin \mathcal{L}_f(z_0)$. □

The converse is not true in general as the following example shows. But it was shown in [17, Theorem 3.6.] that every level closed function $f : X \to Z$ is lattice-l.s.c. if and only if Z is a completely distributive lattice.

Example 5.4. Let $X := \mathbb{R}$, $Z := \overline{\mathbb{R}^2}$ with $\mathbb{R}^2$ partially ordered by $\mathbb{R}^2_+$. The function $f : X \to Z$, defined by

$$f(x) = \begin{cases} (1,0)^T & \text{if } x \geq 0, \\ (0, -1/x)^T & \text{if } x < 0 \end{cases}$$

is level closed since

$$\mathcal{L}_f(y) = \{x \in X |\ f(x) \leq y\} = \begin{cases} [0, +\infty) & \text{if } y_2 = 0, y_1 \geq 1 \\ (-\infty, -1/y_2] \cup [0, +\infty) & \text{if } y_2 > 0, y_1 \geq 1 \\ (-\infty, -1/y_2] & \text{if } y_2 > 0, 0 \leq y_1 < 1 \\ \mathbb{R} & \text{if } y = +\infty \\ \emptyset & \text{otherwise.} \end{cases}$$

But f is not lattice-l.s.c. at $x_0 = 0$. If we take the set of open ε-intervals as fundamental system of neighborhoods of $x_0 = 0$, i.e., $\mathcal{N}(0) = \{(-\varepsilon, +\varepsilon) |\ \varepsilon > 0\}$, we obtain, for every $U = (-\varepsilon, +\varepsilon) \in \mathcal{N}(0)$,

$$\inf_{x \in U} f(x) = \inf \left\{ (0, 1/\varepsilon)^T, (1,0)^T \right\} = (0,0)^T$$

hence

$$\sup_{U\in\mathcal{N}(0)}\inf_{x\in U} f(x) = (0,0)^T \not\geq (1,0)^T = f(0).$$

The following relations between epi-closedness, weak level closedness and level closedness follow immediately from the definitions.

Proposition 5.5. *Let X be a topological space, $(Y, \leq)$ a partially ordered topological space and $f : X \to \bar{Y}$. The following statements hold.*

(i) If f is epi-closed, then it is weakly level closed.

(ii) If f is level closed, then it is weakly level closed.

The converse implications are only true under some additional assumptions.

Proposition 5.6. *Let X be a topological space.*

(i) Assume that $(Y, \leq_C)$ is a topological vector space ordered by a pointed closed convex cone C with nonempty interior. If a function $f : X \to \bar{Y}$ is weakly level closed, then it is epi-closed.

(ii) Assume that $(Y, \leq)$ is a partially ordered set having no least element. If a function $f : X \to \bar{Y}$ is weakly level closed, then it is level closed.

Proof. (i) Assume that f is weakly level closed, i.e., for every $x_0 \in X$ and $y \in Y$ it holds

$$(\forall U \in \mathcal{N}(x_0)\ \exists x \in U : f(x) \leq_C y) \quad \Longrightarrow \quad f(x_0) \leq_C y. \tag{11}$$

In order to prove that f is epi-closed we assume that $(x_0, y_0) \in \operatorname{cl}(\operatorname{epi} f)$, i.e.,

$$\forall U \in \mathcal{N}(x_0), V \in \mathcal{V}\ \exists x \in U, y \in V : f(x) \leq_C y_0 + y \tag{12}$$

where $\mathcal{V}$ denotes a fundamental system of neighborhoods of 0 in Y. We must show that $f(x_0) \leq_C y_0$. Since for every $e \in \operatorname{int} C$ there is some $V \in \mathcal{V}$ with $V \subseteq e - C$ we obtain from (12),

$$\forall e \in \operatorname{int} C, U \in \mathcal{N}(x_0)\ \exists x \in U : f(x) \leq_C y_0 + e.$$

Now, (11) implies that $f(x_0) \leq_C y_0 + e$ for all $e \in \operatorname{int} C$ hence $f(x_0) \leq_C y_0$ since C is closed.

(ii) It remains to show that $\mathcal{L}_f(+\infty)$ and $\mathcal{L}_f(-\infty)$ are closed. $\mathcal{L}_f(+\infty) = X$ is closed by definition. Since Y has no least element, for $z \in \bar{Y}$ we have $z = -\infty$ if and only if $z \leq y$ for all $y \in Y$. Hence

$$\mathcal{L}_f(-\infty) = \bigcap_{y\in Y} \mathcal{L}_f(y)$$

is a closed set as well. □

In the appendix of [8] several examples are given showing that in general no inclusion holds between the sets of lattice-l.s.c., topologically l.s.c. and epi-closed functions. Sufficient conditions under which some of the inclusions hold can be found in [23], [8] and [20]. In this context we only mention the following two results.

Proposition 5.7. *Let X be a topological space, $(Y, \leq)$ be a partially ordered topological space that has no greatest element. If the ordering of Y is closed (i.e., the set $G := \{(z, y) \in Y \times Y |\ z \leq y\}$ is closed) then every topologically l.s.c. function $f : X \to \bar{Y}$ is epi-closed.*

Proof. In order to prove that $\operatorname{epi} f$ is closed we take a pair $(x_0, y_0) \in (X \times Y) \setminus (\operatorname{epi} f)$ and show that there are neighborhoods $U \in \mathcal{N}(x_0)$ and $W \in \mathcal{V}(y_0)$ such that $(U \times W) \cap (\operatorname{epi} f) = \emptyset$.

If $(x_0, y_0) \in (X \times Y) \setminus (\operatorname{epi} f)$ then $f(x_0) \neq -\infty$ and $(f(x_0), y_0) \notin G$. Let $\bar{y} \in Y$ be chosen such that $\bar{y} \leq f(x_0)$ and $(\bar{y}, y_0) \notin G$. Such an element $\bar{y}$ always exist. In fact, in case $f(x_0) \in Y$, $\bar{y} = f(x_0)$ would work and in case $f(x_0) = +\infty$, if no such $\bar{y}$ would exist then y_0 would be the greatest element of Y, a contradiction. Since $(\bar{y}, y_0) \notin G$ and G is closed there exist neighborhoods V of $\bar{y}$ and W of y_0 such that $(V \times W) \cap G = \emptyset$. Since f is topologically l.s.c. there exists a neighborhood $U \in \mathcal{N}(x_0)$ such that

$$\forall x \in U\ \exists y \in V : y \leq f(x). \tag{13}$$

This implies $(U \times W) \cap (\operatorname{epi} f) = \emptyset$ since otherwise there would exist $\widetilde{x} \in U, \widetilde{y} \in W$ with $f(\widetilde{x}) \leq \widetilde{y}$ and, by (13) there would exist $y \in V$ with $y \leq f(\widetilde{x})$ hence $y \leq \widetilde{y}$ in contradiction to $(V \times W) \cap G = \emptyset$. □

Remark 5.8. Proposition 5.7 is slightly different from [23], Proposition 1.3.a) but the proof goes essentially along the lines of the one in [23].

Proposition 5.9 ([20], Propositions 3.1 and 3.5)**.** *Let X be a topological space and Y be a topological vector space ordered by a pointed closed convex cone C with nonempty interior such that $\bar{Y}$ is a complete lattice. The following statements hold true.*

(i) Every topologically l.s.c. function $f : X \to \bar{Y}$ is lattice-l.s.c.

(ii) If C is Daniell then every lattice-l.s.c. function is topologically l.s.c.

Remark 5.10. In [20] X and Y are supposed to be metrizable spaces but the proofs of Propositions 3.1 and 3.5 do not require these assumptions.

In summary we can say that, if Y is a partially ordered topological space with a closed ordering that has no greatest element such that $\bar{Y}$ is a complete lattice, then for functions $f : X \to \bar{Y}$ the concept of weak level closedness is the weakest one. It is equivalent with level closedness if Y has no least element. In case $Y = \mathbb{R}^q$ is ordered by a closed convex pointed cone C with nonempty interior such that $(\overline{\mathbb{R}^q}, \leq_C)$ is a complete lattice, lattice- and topological lower semicontinuity coincide. Moreover, epi-closedness, level closedness and weak level closedness coincide and the first two concepts are stronger than the last three.

Next, we study the relationship of the concepts for functions with values in $Z = \mathcal{F}$. Since $\mathcal{F}$ is a priori not equipped with a topology we only consider the concepts of level closedness and lattice-semicontinuity. We are interested in characterizations of these concepts with the help of the topology of the underlying space $\mathbb{R}^q$. A topology for $\mathcal{F}$ is not considered in this article. We have the following characterizations.

Proposition 5.11. *A function $f : X \to \mathcal{F}$ is lattice-l.s.c. if and only if*

$$\operatorname{gr} f := \{(x, y) \in X \times \mathbb{R}^q |\ y \in f(x)\}$$

is closed.

Proof. We have
$$\sup_{U\in\mathcal{N}(x_0)}\inf_{x\in U} f(x) = \bigcap_{U\in\mathcal{N}(x_0)} \mathrm{cl} \bigcup_{x\in U} f(x).$$
Hence
$$y_0 \in \sup_{U\in\mathcal{N}(x_0)}\inf_{x\in U} f(x) \quad\Longleftrightarrow\quad \forall U \in \mathcal{N}(x_0), V \in \mathcal{V}(y_0)\ \exists x \in U, y \in V : y \in f(x) \tag{14}$$
where $\mathcal{V}(y_0)$ denotes a fundamental system of neighborhoods of y_0 in $\mathbb{R}^q$. Consequently f is lattice-l.s.c. if and only if for all (x_0, y_0) satisfying one of the equivalent conditions in (14) it holds $y_0 \in f(x_0)$. But, this is equivalent to $\mathrm{gr}\, f$ being closed. □

Proposition 5.12. *A function $f : X \to \mathcal{F}$ is level closed if and only if for all $y \in \mathbb{R}^q$ the sets $\{x \in X|\ y \in f(x)\}$ are closed.*

Proof. If f is level closed then the sets $\mathcal{L}_f(\mathrm{Cl}_+\{y\})$ are closed for all $y \in \mathbb{R}^q$. Since $y \in f(x)$ if and only if $\mathrm{Cl}_+\{y\} \subseteq \mathrm{Cl}_+ f(x) = f(x)$ the "only if"-part follows. The "if"-part follows from the fact that
$$\mathcal{L}_f(A) = \bigcap_{y\in A} \{x \in X|\ y \in f(x)\}$$
and that the intersection of closed sets is closed again. □

Corollary 5.13. *Let $f : X \to \overline{\mathbb{R}^q}$ be a vector function and $\tilde{f} : X \to \mathcal{F}$ its $\mathcal{F}$-valued extension, defined by $\tilde{f}(x) := \mathrm{Cl}_+\{f(x)\}$. Then $\tilde{f}$ is level closed if and only if f is weakly level closed.*

Proof. By Proposition 5.12 $\tilde{f}$ is level closed if and only if for all $y \in \mathbb{R}^q$ the sets $\{x \in X|\ y \in \tilde{f}(x)\}$ are closed. Since $y \in \tilde{f}(x)$ if and only if $y \geq f(x)$ the statement follows. □

We are able to show that for functions with values in $\mathcal{F}$, lattice-semicontinuity and level closedness are equivalent.

Proposition 5.14. *A function $f : X \to \mathcal{F}$ is lattice-l.s.c. if and only if it is level closed.*

Proof. By Proposition 5.3 each lattice-l.s.c. function is also level closed. We next prove the converse implication. Assume that f is level closed. We show that $\mathrm{gr}\, f$ is closed, hence f is lattice-l.s.c. by Proposition 5.11. Assume that $(x_0, y_0) \in X \times Y$ is given such that for all $U \in \mathcal{N}(x_0), V \in \mathcal{V}(y_0)$ there exist $x \in U, y \in V$ with $y \in f(x)$. We must show that $y_0 \in f(x_0)$.

Take $a \in \{y_0\} + \mathrm{int}\, C$ arbitrarily. Then there exists some neighborhood $\tilde{V} \in \mathcal{V}(y_0)$ such that $y \leq_C a$, i.e., $a \in \mathrm{Cl}_+\{y\}$ holds for all $y \in \tilde{V}$. Hence, for all $U \in \mathcal{N}(x_0)$ there exist some $x \in U$ and some $y \in \tilde{V}$ with $y \in f(x)$, hence $a \in \mathrm{Cl}_+\{y\} \subseteq f(x)$. This implies by Proposition 5.12
$$x_0 \in \mathrm{cl}\, \{x \in X|\ a \in f(x)\} = \{x \in X|\ a \in f(x)\}.$$
Thus we have $y_0 + \mathrm{int}\, C \subseteq f(x_0)$ and consequently $y_0 \in \mathrm{cl}\,(y_0 + \mathrm{int}\, C) \subseteq f(x_0)$. □

Next we formulate a sufficient condition for the domination property of the general optimization problem (1). As in the classical Weierstrass theorem, the assumptions are lower semicontinuity of f and compactness of the feasible set. The appropriate semicontinuity condition for the function f in the general case is level closedness.

Proposition 5.15. *Let X be a compact topological space, $(Z, \leq)$ be a partially ordered set and $f : X \to Z$ a level closed function. Then the domination property holds, i.e., for every $x \in X$ there exists a minimal point $y \in f[X]$ with $y \leq f(x)$.*

Proof. We must show that for every $x \in X$ the set $\{y \in f[X] |\, y \leq f(x)\} = f[\mathcal{L}_f(f(x))]$ has minimal points. Because of Zorn's lemma it suffices to show that every chain in $f[\mathcal{L}_f(f(x))]$ has a lower bound in $f[\mathcal{L}_f(f(x))]$. Since every lower bound (in $f[X]$) of any subset W of $f[\mathcal{L}_f(f(x))]$ is obviously in $f[\mathcal{L}_f(f(x))]$ itself, it is enough to prove that every chain in $f[X]$ has a lower bound.

Let W be a chain in $f[X]$. A subset W of $f[X]$ has a lower bound in $f[X]$ if and only if the set

$$\{x \in X |\, \forall w \in W : f(x) \leq w\} = \bigcap_{w \in W} \mathcal{L}_f(w)$$

is nonempty. If B is a finite subset of W then $\bigcap_{b \in B} \mathcal{L}_f(b)$ is nonempty since every finite chain in $f[X]$ has a smallest element hence a lower bound. Since all the sets $\mathcal{L}_f(w)$ are closed, X being compact implies that $\bigcap_{w \in W} \mathcal{L}_f(w)$ is nonempty, too. Hence W has a lower bound. $\square$

6 A vectorial Weierstrass theorem

In this section we apply the results of the previous section to vector optimization problems. In particular, we formulate an existence theorem of Weierstrass type for a solution of the vector optimization problem (8).

Consider the vector optimization problem (8) and its lattice extension (9). The objective function $\bar{f}$ of (9) is $\mathcal{I}$-valued. The semicontinuity concept that we need for the existence result can be characterized in the following ways:

Theorem 6.1. *For the objective function $f : X \to \overline{\mathbb{R}^q}$ of (8) and the objective function $\bar{f} : X \to \mathcal{I}$, $\bar{f}(x) := \operatorname{Inf}\{f(x)\}$ of the lattice extension (9) of (8) the following statements are equivalent:*

(i) f is epi-closed, i.e., the epigraph of f is closed.

(ii) f is level closed, i.e., f has closed level sets for all levels in $\overline{\mathbb{R}^q}$.

(iii) f is weakly level closed, i.e., f has closed level sets for all levels in $\mathbb{R}^q$.

(iv) $\bar{f}$ is level closed, i.e., $\bar{f}$ has closed level sets for all levels in $\mathcal{I}$.

(v) $\bar{f}$ is lattice-l.s.c., i.e., for all $x_0 \in X$ it holds $\bar{f}(x_0) \preccurlyeq \liminf_{x \to x_0} \bar{f}(x)$.

Proof. The equivalence of (i) - (iii) follows directly from Proposition 5.5 and Proposition 5.6. The equivalence of (iii) - (v) follows from Corollary 5.13, Proposition 5.14, the fact (see Theorem 3.1) that a function $g : X \to \mathcal{I}$ is level closed (lattice l.s.c.) if and only if $j \circ g : X \to \mathcal{F}$ is level closed (lattice l.s.c.) and the fact that for the $\mathcal{I}$-valued extension $\bar{f}$ and the $\mathcal{F}$-valued extension $\tilde{f}$ of a function $f : X \to \overline{\mathbb{R}^q}$, $\tilde{f} = j \circ \bar{f}$ holds true. $\square$

Applying Proposition 5.15 we can formulate the following existence result for a solution of a vector optimization problem. The result is a complete analogon of the classical existence theorem of Weierstrass.

Theorem 6.2. *If one of the equivalent characterizations of lower semicontinuity in the preceding theorem is satisfied for the objective function $f : X \to \overline{\mathbb{R}^q}$ of* (8) *and if A is a compact subset of X, then* (8) *has a solution.*

Proof. This is a direct consequence of Proposition 2.8, Proposition 5.15 and Theorem 6.1. $\square$

The preceding two results show a further advantage of our solution concept. For the $\mathcal{I}$-valued extension of a vector function f lattice-semicontinuity and level closedness always coincide whereas for f this is not the case in general. So the analogy between vectorial and scalar Weierstrass existence results was improved.

7 Mild solutions

For a solution $\bar{X}$ of (1) we assume that $f[\bar{X}] = \operatorname{Min} f[A]$. This condition is relaxed in this section.

Definition 7.1. A nonempty set $\hat{X}$ with $f[\hat{X}] \subseteq \operatorname{Min} f[A]$ is called a *mild solution* of (1) if the infimum of the canonical extension F over 2^A is attained in $\hat{X}$.

The principle of a mild solution can be explained as follows. A mild solution $\hat{X}$ is allowed to be a smaller set than a solution. But, as the attainment of the infimum is required, the set $\hat{X}$ cannot become arbitrarily small, which ensures that it contains a sufficient amount of information. Of course, every solution of (1) is also a mild solution of (1). In general, a mild solution can be a proper subset of a solution.

Theorem 7.2. *If a mild solution of* (1) *exists, then there exists a solution of* (1)*.*

Proof. Let $\hat{X}$ be a mild solution of (1). Set $\bar{X} := \operatorname{Eff}(f, A)$, then $A \supseteq \bar{X} \supseteq \hat{X} \neq \emptyset$. Since $\inf_{x\in A} f(x) = \inf_{x\in \hat{X}} f(x)$, we get $\inf_{x\in A} f(x) = \inf_{x\in \bar{X}} f(x)$. Thus $\bar{X}$ is a solution of (1). $\square$

Example 7.3. Consider again the $\mathcal{I}$-valued dual problem (10) of the lattice extension of a given linear vector optimization problem, which we already discussed in Example 4.6.

It was also shown in [13, Theorem 20] that the supremum of this problem is attained in a finite set, whenever $A \neq \emptyset$ and $U \neq \emptyset$. In fact, the supremum is attained in the set of those vertices of U that are efficient solutions. In the terminology of the present paper this means, that a mild solution of the dual problem exists, and, this mild solution is a finite set. The solution is not a finite set in general. In this example, a mild solution already contains the information which is necessary to give a dual description of the optimal value of both problems.

Definition 7.4. A set $\hat{X}$ is called *mild solution* of the vector optimization problem (8) if it is a mild solution of its lattice extension (9).

For the special case of a vector optimization problem, we have the following characterization of a mild solution.

Theorem 7.5. *Assume that a solution for* (8) *exists. A set $\hat{X} \subseteq A$ is a mild solution of* (8) *if and only if*

$$f[\hat{X}] \subseteq \operatorname{Min} f[A] \subseteq \operatorname{Inf} f[\hat{X}]. \tag{15}$$

Proof. If $f(x) = -\infty$ for some $x \in A$, then for every nonempty set $\hat{X} \subseteq \{x \in X |\; f(x) = -\infty\}$ it holds

$$\{-\infty\} = f[\hat{X}] = \operatorname{Min} f[A] = \operatorname{Inf} f[A] = \operatorname{Inf} f[\hat{X}].$$

In the case that $f(x) = +\infty$ for all $x \in A$ we have

$$\{+\infty\} = f[\hat{X}] = \operatorname{Min} f[A] = \operatorname{Inf} f[A] = \operatorname{Inf} f[\hat{X}]$$

for every nonempty subset $\hat{X} \subseteq A$. In both cases the statement is obvious. Since $\operatorname{Min} f[A] = \operatorname{Min}(f[A] \setminus \{+\infty\})$ and $\operatorname{Inf} f[A] = \operatorname{Inf}(f[A] \setminus \{+\infty\})$, we can assume that $f[A] \subseteq \mathbb{R}^q$.

(i) If $\hat{X}$ is a mild solution of (8), we have

$$\emptyset \neq \hat{X} \subseteq A, \qquad f[\hat{X}] \subseteq \operatorname{Min} f[A] \qquad \operatorname{Inf} f[\hat{X}] = \operatorname{Inf} f[A].$$

It remains to show $\operatorname{Min} f[A] \subseteq \operatorname{Inf} f[\hat{X}]$. Let $y \in \operatorname{Min} f[A]$, i.e.,

$$y \in f[A] \qquad \wedge \qquad y \notin f[A] + C \setminus \{0\}.$$

It follows

$$\{y\} + \operatorname{int} C \subseteq f[A] + \operatorname{int} C \qquad \wedge \qquad y \notin f[A] + \operatorname{int} C.$$

It follows that $\emptyset \neq \operatorname{Cl}_+ f[A] \neq \mathbb{R}^q$. By (6) we get $y \in \operatorname{Inf} f[A] = \operatorname{Inf} f[\hat{X}]$.

(ii) Let $\bar{X}$ be a solution of (8) and let (15) be satisfied. It follows $f[\hat{X}] \subseteq f[\bar{X}] \subseteq \operatorname{Inf} f[\hat{X}]$. From (5) we get $\operatorname{Cl}_+ f[\bar{X}] = \operatorname{Cl}_+ f[\hat{X}]$. Theorem 3.1 yields $\operatorname{Inf} f[\bar{X}] = \operatorname{Inf} f[\hat{X}]$. Hence $\hat{X}$ is a mild solution of (8). □

We next focus on a relationship to properly efficient solutions. The famous theorem by Arrow Barankin Blackwell [1] and related results state that, under certain assumptions, the set of properly minimal vectors is a dense subset of the set of minimal vectors. In the literature, there are a many density results for different types of proper efficiency; see e.g. [3, 6, 7, 10, 15]. The following theorem shows that the set of proper efficient solutions is an example of a mild solution, whenever (under certain assumptions) a corresponding density result holds.

Theorem 7.6. *Assume that a solution for* (8) *exists. Let $\hat{X} \subseteq A$ be a set such that $f[\hat{X}] \subseteq \mathbb{R}^q$ and*

$$f[\hat{X}] \subseteq \operatorname{Min} f[A] \subseteq \operatorname{cl} f[\hat{X}]. \tag{16}$$

Then is $\hat{X}$ is a mild solution of (8)*.*

Proof. Let $\bar{X}$ be a solution of (8). Then $\operatorname{Min} f[A]$ is nonempty hence $\operatorname{cl} f[\hat{X}]$ is nonempty and thus $\hat{X}$ is nonempty, too. It holds

$$f[\hat{X}] \subseteq \operatorname{Min} f[A] = f[\bar{X}] \subseteq \operatorname{cl} f[\hat{X}].$$

Using (5) and the fact $\operatorname{Cl}_+ \operatorname{cl} f[\hat{X}] = \operatorname{Cl}_+ f[\hat{X}]$, we get $\operatorname{Cl}_+ f[\bar{X}] = \operatorname{Cl}_+ f[\hat{X}]$. Theorem 3.1 yields $\operatorname{Inf} f[\bar{X}] = \operatorname{Inf} f[\hat{X}]$. Hence $\hat{X}$ is a mild solution of (8). □

In general, (16) does not hold for a mild solution $\hat{X}$ of (8).

Example 7.7. Let $X = \mathbb{R}^2$, $q = 2$, $C = \mathbb{R}^2_+$, f the identity map and

$$A := \left\{x \in \mathbb{R}^2 |\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 \geq 1\right\} \cup \{x \in \mathbb{R}^2 |\ x_1 = 0, x_2 \geq 2\}.$$

Then $\hat{X} := \{\lambda\,(0,1)^T + (1-\lambda)\,(1,0)^T |\ \lambda \in (0,1)\}$ is a mild solution. But $(0,2)^T \in \operatorname{Min} f[A] \setminus \operatorname{cl} f[\hat{X}]$, hence (16) is violated.

If we additionally assume that $f[A] + C$ is closed, we obtain that a mild solution satisfies (16). For instance, if A is a polyhedral convex set, C polyhedral and f linear, then $f[A] + C$ is closed. Also, the assumptions of the Weierstrass existence result, Theorem 6.2, imply that $f[A] + C$ is closed (this follows from $\operatorname{epi} f$ being closed and A compact).

Theorem 7.8. *Let $\hat{X}$ be a mild solution of* (8)*, $f[A] \subseteq \mathbb{R}^q$ and let $f[A] + C$ be closed. Then it holds*

$$f[\hat{X}] \subseteq \operatorname{Min} f[A] \subseteq \operatorname{cl} f[\hat{X}].$$

Proof. It remains to show the second inclusion. Let $y \in \operatorname{Min} f[A]$, i.e.,

$$y \in f[A] \subseteq \operatorname{cl}\,(f[A] + C) = \operatorname{Cl}_+ f[A].$$

and (taking into account that C is pointed and $f[A] + C$ is closed)

$$y \notin f[A] + C \setminus \{0\} = (f[A] + C) + C \setminus \{0\} = \operatorname{cl}\,(f[A] + C) + C \setminus \{0\} = \operatorname{Cl}_+ f[A] + C \setminus \{0\}.$$

This yields $y \in \operatorname{Min} \operatorname{Cl}_+ f[A]$. As $\hat{X}$ is a mild solution, we have $\operatorname{Inf} f[\hat{X}] = \operatorname{Inf} f[A]$; Theorem 3.1 implies $\operatorname{Cl}_+ f[\hat{X}] = \operatorname{Cl}_+ f[A]$. Thus, we have $y \in \operatorname{Min} \operatorname{Cl}_+ f[\hat{X}]$.

It remains to show that $\operatorname{Min} \operatorname{Cl}_+ f[\hat{X}] \subseteq \operatorname{cl} f[\hat{X}]$. Assuming the contrary, there exists some $y \in \operatorname{Cl}_+ f[\hat{X}] = \operatorname{cl}\,(f[\hat{X}] + C)$ such that $y \notin \operatorname{cl} f[\hat{X}]$ and

$$\big(y - C \setminus \{0\}\big) \cap \operatorname{cl}\,(f[\hat{X}] + C) = \emptyset. \tag{17}$$

Let (b_n) and (c_n) be sequences, respectively, in $f[\hat{X}]$ and C such that $b_n + c_n \to y$. There is no subsequence of c_n that converges to 0, because otherwise we get the contradiction $y \in \operatorname{cl} f[\hat{X}]$. Hence there exists $n_0 \in \mathbb{N}$ and $\alpha > 0$ such that $\|c_n\| \geq \alpha$ for all $n \geq n_0$. There is a subsequence $(c_n)_{n \in M}$ (M an infinite subset of $\{n \in \mathbb{N} |\ n \geq n_0\}$) such that

$$\tilde{c}_n := \frac{\alpha c_n}{\|c_n\|} \xrightarrow{M} \tilde{c} \in C \setminus \{0\}.$$

It follows

$$b_n + \left(1 - \frac{\alpha}{\|c_n\|}\right) c_n = b_n + c_n - \tilde{c}_n \xrightarrow{M} y - \tilde{c}.$$

We obtain $y - \tilde{c} \in \operatorname{cl}\,(f[\hat{X}] + C)$ which contradicts (17). $\square$

Note that, in contrast to all other results, in the proof of the preceding result arguments of finite dimensional vector spaces are used.

8 Final Remarks

This paper is a continuation of the articles [18] and [13], where it is shown that vector optimization can be treated in the framework of complete lattices. This approach has several advantages. On the one hand the analogy to the scalar optimization theory can be improved which makes vector optimization easier to understand. On the other hand this approach is promising for applications. This can be seen by the connections between the papers [13], [14] and [5].

The solution concept is based on two classical concepts, efficient solutions and weakly efficient solutions, where the first concept is part of the definition and the latter is involved in the construction of the complete lattice. This approach might shed a new light to the "endless competition" between the practically relevant efficient solutions and the theoretically more convenient weakly efficient solutions.

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