Ito Formula for Stochastic Integrals w.r.t. Compensated Poisson Random Measures on Separable Banach Spaces

B. Rüdiger, G. Ziglio

no. 187

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereiches 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

Bonn, Oktober 2004
Ito formula for stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces

B. Rüdiger*, G. Ziglio**

* Mathematisches Institut, Universität Koblenz-Landau, Campus Koblenz, Universitätsstrasse 1, 56070 Koblenz, Germany;
* SFB 611, Institut für Angewandte Mathematik, Abteilung Stochastik, Universität Bonn, Wegelerstr. 6, D -53115 Bonn, Germany.
** Facoltà di Scienze, Università di Trento, Via Sommarive 14, I-38050 Povo (Trento) Italia

Abstract

We prove the Ito formula (3) for Banach valued stochastic processes with jumps, the martingale part given by a stochastic integral w.r.t. a compensated Poisson random measure. Such stochastic integrals have been discussed in [35]. The Ito formula is computed here for Banach valued functions.

AMS -classification (2000): 60H05, 60G51, 60G57, 46B09, 47G99

Keywords: Stochastic integrals on separable Banach spaces, random martingales measures, compensated Poisson random measures, additive processes, random Banach valued functions, type 2 Banach spaces, M-type 2 Banach spaces, Ito formula, Lévy -Ito decomposition theorem.

1 Introduction

In [35] the definition of stochastic integrals

\[ Z_t(\omega) := \int_0^t \int_A f(s, x, \omega) \left( N(dsdx)(\omega) - \nu(dsdx) \right) \]  

of Hilbert and Banach valued functions \( f(s, x, \omega) \), w.r.t. compensated Poisson random measures (cPrms) \( N(dsdx)(\omega) - \nu(dsdx) \), has been discussed. (See also [7], [8], [9], [10], [13], [19], [25], [24], [26], [31], [32], [33], [34], [37], [38] for the discussion of stochastic integrals w.r.t. cPrms on infinite dimensional spaces). The state space of \( f \) is denoted with \( F \), and is a separable Banach space. \( s, t \in \mathbb{R}_+ \), \( x \) is defined on some separable Banach space \( E \), \( A \) is a Borel set of \( E \) \( \setminus \) \{0\}, and \( \omega \) is defined on a probability space \((\Omega, \mathcal{F}, P)\). The cPrm \( q(dsdx)(\omega) := N(dsdx)(\omega) - \nu(dsdx) \) is associated to some \( E \) - valued additive process \((X_t)_{t \in \mathbb{R}_+} \) on \((\Omega, \mathcal{F}, P)\) (see Section 2 for the concept of cPrm, and Section 3 for the discussion of the integrals (1)). The problem of giving an
appropriate meaning to (1) comes from the property of the cPrms to be only $\sigma$-finite (in general not finite) random measures. The cPrms are in general not finite on the sets $(0, T] \times \Lambda$, if it does not hold that $0 \notin \overline{\Lambda}$.

For real valued functions this problem was already discussed e.g. in [14], where the integrals are defined by approximating the integral (1) in $L^p(\Omega, \mathcal{F}, P)$, $p = 1, 2$, with integrals on the set $A \cap B^c(0, \epsilon)$, where with $B^c(0, \epsilon)$ we denote the complementary of the ball of radius $\epsilon > 0$, centered in 0 (we refer to Definition 3.20 for a precise statement).

The definition of the integrals (1) for the real valued case was discussed in a different way than [14] in [36] and [6]. In [36] and [6] the "natural" definition of "Ito integrals" (consisting of an $L^p(\Omega, \mathcal{F}, P)$-approximation of integrals of appropriate "simple functions"), is extended to the case where integration is performed w.r.t. cPrms instead of the Brownian motion. (In [36] cPrm associated to $\alpha$-stable processes are considered, while [6] discusses general martingale measures on $\mathbb{R}^d$).

In [35] the definition of stochastic integrals (1) for real valued random functions, given in [14], and resp. [36], [6], has been generalized to the case of Hilbert and Banach valued random functions. We call them simple -$p$-, and resp. strong -$p$-integrals. It has been proven that the definition of "strong -$p$- integral" (which for the real valued case coincides with the one used in [36], [6]) is stronger than the definition of "simple -$p$- integral" (which in the real valued case coincides with the one of [14]). (In a similar way like e.g. the definition of a "Lebegues integral" is a stronger definition than the definition of "Riemann integral"). In fact, we need one more sufficient condition for the integrand $f$, if we require the existence of the "simple -$p$-integral". We need that $f$ has a left -continuous version. (In [14] it is required that for the existence of the simple -$p$-integral the integrand has to be predictable, in the sense that it is in the smallest $\sigma$-algebra generated by the left -continuous functions). In [35] it has been proven that if however this condition holds, then the simple -$2$- and strong -$2$-integrals coincide (see also Section 3, for precise statements). The results in [35], beside being a generalization to Hilbert and Banach spaces, unify therefore the definition of the integral (1) proposed in [14] with the one of [36], [6], for the case of integrands which have a left continuous version.

The definition of the integral (1) on infinite dimensional spaces, like Hilbert or Banach spaces, is necessary to discuss stochastic differential equations (SDEs) with non Gaussian noise on such spaces (see e.g. [4], [3], where this problem is discussed). In fact, from the Lévy -Ito decomposition theorem [15], [7], [1] it follows that "additive non Gaussian noise" is naturally defined through integrals of the form (1). In [22], [23], existence and uniqueness of SDEs with non Gaussian additive noise has been proven on separable Banach spaces, under local Lipschitz conditions for the drift- and noise - coefficients. (We refer to [22] for precise statements). In [23] the continuous dependence of the solutions on the coefficients of such SDEs has been shown and the Markov properties have been
discussed. Applications of such SDEs related to financial models, are given in [23], too.

In this article we prove the Ito formula for the Banach valued processes

\[ Y_t(\omega) := Z_t(\omega) + \int_0^t g(s, \omega) \, dh(s, \omega) + \int_0^t \int_A k(s, x, \omega) \, N(dsdx)(\omega), \quad (2) \]

\( Z_t(\omega) \) being the stochastic integral in (1). We assume that \( h(s, \omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R} \)

is a real valued, càdlàg process, whose almost all paths are of bounded variation, \( g \) is a Banach valued random (càdlàg or càg-làd) function depending on time (see Section 4 for precise statements). We consider \( H : \mathbb{R}_+ \times F \to G, \) with \( G \) a separable Banach space, \( H(s, y), \ s \in \mathbb{R}_+, \ y \in F \) twice Fréchet differentiable. We shall prove (see Section 4, Section 5, Section 6 for precise statements) that the following Ito formula holds:

\[ \mathcal{H}(t, Y_t(\omega)) - \mathcal{H}(\tau, Y_\tau(\omega)) = \]

\[ + \int_\tau^t \frac{\partial_s \mathcal{H}(s, Y_s(\omega))}{\partial s} ds + \int_\tau^t \int_A \{ \mathcal{H}(s, Y_s(\omega) + f(s, x, \omega)) - \mathcal{H}(s, Y_s(\omega)) \} q(dsdx)(\omega) \]

\[ + \int_\tau^t \int_A \{ \mathcal{H}(s, Y_s(\omega) + f(s, x, \omega)) - \mathcal{H}(s, Y_s(\omega)) - \partial_y \mathcal{H}(s, Y_s(\omega)) f(s, x, \omega) \} \nu(dsdx) \]

\[ + \int_\tau^t \int_A \{ \mathcal{H}(s, Y_s(\omega) + k(s, x, \omega)) - \mathcal{H}(s, Y_s(\omega)) \} N(dsdx)(\omega) + \int_\tau^t \partial_y \mathcal{H}(s, Y_s(\omega)) g(s, \omega) dh(s, \omega) \]

\[ + \sum_{\tau < s \leq t} \{ \mathcal{H}(s, Y_s(\omega) + g(s, \omega) \Delta h(s, \omega)) - \mathcal{H}(s, Y_s(\omega)) - \partial_y \mathcal{H}(s, Y_s(\omega)) g(s, \omega) \Delta h(s, \omega) \} \]

\[ \mathcal{P} \quad a.s., \quad (3) \]

where \( \partial_s \mathcal{H}(s, y) \) denotes the Fréchet derivative of \( \mathcal{H} \) w.r.t. \( s \in \mathbb{R}_+ \) for \( y \) fixed, and \( \partial_y \mathcal{H}(s, y) \) denotes the Fréchet derivative of \( \mathcal{H} \) w.r.t. \( y \in F, \) for \( s \in \mathbb{R}_+ \) fixed and

\[ q(dsdx)(\omega) := N(dsdx)(\omega) - \nu(dsdx) \]

Remark 1.1 The simplest example for a process (2), for which the Ito formula (3) can be applied, is when \( Y_t \) coincides with a pure jump additive process \( (\mathcal{A}_t)_{t \in \mathbb{R}_+} \) with values in a Banach space \( E. \) From the Lévy -Ito decomposition theorem [15], [7], [1] it follows that

\[ \mathcal{A}_t(\omega) = \int_0^t \int_{|x| \leq 1} x(N(dsdx)(\omega) - \nu(dsdx)) + \alpha_t + \int_0^t \int_{|x| > 1} xN(dsdx)(\omega) \mathcal{P} - a.s. \quad (4) \]

where \( \alpha_t \in E. \) The stochastic integral w.r.t. \( q(dsdx)(\omega) \) is a strong \(-p\) and simple \(-p\) -integral with \( p = 1, \) resp. \( p = 2, \) if

\[ \int_0^t \int_{|x| \leq 1} ||x||^p \nu(dsdx) < \infty, \quad (5) \]
and if, in case of \( p = 2 \), the Banach space is of type 2 \([1]\). (See Definition 3.19 or \([5]\), \([20]\) for the definition of type 2 Banach spaces.) In the general case it is only a simple \(-p\) -integral, as follows from \([7]\) (see also final Remark in \([1]\)). A series of other examples for which the Ito formula (3) can be applied on infinite dimensional Banach spaces are in \([22]\), \([23]\) \([2]\) (see e.g. also \([12]\), \([25]\)).

Suppose \( F = G = \mathbb{R} \), and \( \mathcal{H} \) depends not on the time \( s \in \mathbb{R} \). Suppose that \( f \) is left continuous in time, so that the integral in (1) is a simple \(-2\) -integral. Suppose also that the term \( \int_0^t g(s, \omega) dh(s, \omega) \) in (2) is skipped. The Ito formula in (41) coincides for this case with the Ito -formula found in \([14]\). In \([25]\), \([13]\) an Ito formula is also given for Hilbert -valued stochastic integrals of the form (1). The stochastic integrals (1) are there defined in a more abstract way, by assuming some conditions which control the second moment of the integrals of approximating functions . These conditions are satisfied by our strong \(-2\) -integrals (used also in \([36]\), \([6]\) for the real valued case), so that it follows that the Ito -formula in \([25]\), \([13]\) is satisfied by such integrals. The Ito formula in \([25]\), \([13]\) is however not given in terms of the cPrm \( q(dsdx)(\omega) \) and compensator \( \nu(dsdx) \), like done in this paper (3), where we also generalize to the case of Banach valued functions. An Ito formula in terms of the cPrm \( q(dsdx)(\omega) \) and compensator \( \nu(dsdx) \) is however necessary to establish the generator of some jump Markov process \( Y_t(\omega) \) of the form (2), like e.g. the problems discussed in \([23]\).

2 Compensated Poisson random measures on separable Banach spaces

We assume that a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq +\infty}, P)\), satisfying the ”usual hypothesis”, is given:

i) \( \mathcal{F}_t \) contain all null sets of \( \mathcal{F} \), for all \( t \) such that \( 0 \leq t < +\infty \)

ii) \( \mathcal{F}_t = \mathcal{F}_t^+ \), where \( \mathcal{F}_t^+ = \cap_{u \geq t} \mathcal{F}_u \), for all \( t \) such that \( 0 \leq t < +\infty \), i.e. the filtration is right continuous.

In this Section we recall the definition of compensated Poisson random measures associated to additive processes on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq +\infty}, P)\) with values in \((E, \mathcal{B}(E))\), where in the whole paper we assume that \( E \) is a separable Banach space with norm \( \| \cdot \| \) and \( \mathcal{B}(E) \) is the corresponding Borel \( \sigma \)-algebra (see also \([12]\), \([1]\), \([35]\)).

**Definition 2.1** A process \((X_t)_{t \geq 0}\) with state space \((E, \mathcal{B}(E))\) is an \( \mathcal{F}_t \) - additive process on \((\Omega, \mathcal{F}, P)\) if

i) \((X_t)_{t \geq 0}\) is adapted (to \((\mathcal{F}_t)_{t \geq 0}\))

ii) \(X_0 = 0\) a.s.
iii) \((X_t)_{t \geq 0}\) has increments independent of the past, i.e. \(X_t - X_s\) is independent of \(F_s\) if \(0 \leq s < t\).

iv) \((X_t)_{t \geq 0}\) is stochastically continuous.

v) \((X_t)_{t \geq 0}\) is càdlàg.

An additive process is a Lévy process if the following condition is satisfied:

vi) \((X_t)_{t \geq 0}\) has stationary increments, that is \(X_t - X_s\) has the same distribution as \(X_{t-s}\), \(0 \leq s < t\).

Remark 2.2 That any process satisfying i) - iv) has a càdlàg version follows from its being, after compensation, a martingale (see e.g. [11], [16], [29]).

Let \((X_t)_{t \geq 0}\) be an additive process on \((E, \mathcal{B}(E))\) (in the sense of Definition 2.1).

Set \(X_t - \lim_{s \uparrow t} X_s\) and \(\Delta X_s := X_s - X_{s-}\).

We denote with \(N(dt \, dx) (\omega)\) the Poisson random measure associated to the additive process \((X_t)_{t \geq 0}\), and with \(\nu(dt \, dx)\) its compensator. (See e.g. [12] or [35].)

\[
q(dt \, dx)(\omega) := N(dt \, dx)(\omega) - \nu(dt \, dx) \tag{6}
\]

is the compensated Poisson random measure associated to \((X_t)_{t \geq 0}\). (We sometimes omit to write the dependence on \(\omega\).) We refer to e.g. [12] or [35] for the properties of \(q(dt \, dx)\). We recall however that \(N(dt \, dx)(\omega)\), for each \(\omega\) fixed, resp. \(\nu(dt \, dx)\), are \(\sigma\)-finite measures on the \(\sigma\)-algebra \(\mathcal{B}(\mathbb{R}_+ \times E \setminus \{0\})\) generated by the product semi-ring \(\mathcal{S}(\mathbb{R}_+ \times \mathcal{B}(E \setminus \{0\}))\) of product sets \((t_1, t_2] \times A\), with \(0 \leq t_1 < t_2\), and \(A \in \mathcal{B}(E \setminus \{0\})\), where \(\mathcal{B}(E \setminus \{0\})\) is the trace \(\sigma\)-algebra on \(E \setminus \{0\}\) of the Borel \(\sigma\)-algebra \(\mathcal{B}(E)\) on \(E\). If however \(\Lambda \in \mathcal{F}(E \setminus \{0\})\), with

\[
\mathcal{F}(E \setminus \{0\}) := \{\Lambda \in \mathcal{B}(E \setminus \{0\}) : 0 \in (\overline{\Lambda})^c\}, \tag{7}
\]

then \(\forall \omega \in \Omega\)

\[
N((t_1, t_2] \times \Lambda)(\omega) = \sum_{t_1 < s \leq t_2} 1_\Lambda(\Delta X_s) < \infty \tag{8}
\]

and

\[
P(N((t_1, t_2] \times \Lambda) = k) = \exp(-\nu((t_1, t_2] \times \Lambda)) \frac{(\nu((t_1, t_2] \times \Lambda))_k}{k!} \tag{9}
\]

with

\[
\nu((t_1, t_2] \times \Lambda) := E[N((t_1, t_2] \times \Lambda)] < \infty \quad \tag{10}
\]
3 Stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces

Let $F$ be a separable Banach space with norm $\| \cdot \|_F$. (When no misunderstanding is possible we write $\| \cdot \|$ instead of $\| \cdot \|_F$.) Let $F_t := \mathcal{B}(\mathbb{R}_+ \times (E \setminus \{0\})) \otimes \mathcal{F}_t$ be the product $\sigma$-algebra generated by the semi-ring $\mathcal{B}(\mathbb{R}_+ \times (E \setminus \{0\})) \times \mathcal{F}_t$ of the product sets $A \times F$, $A \in \mathcal{B}(\mathbb{R}_+ \times E \setminus \{0\})$, $F \in \mathcal{F}_t$. Let $T > 0$, and

$$M^T(E/F) := \{ f : \mathbb{R}_+ \times E \setminus \{0\} \times \Omega \rightarrow F, \text{ such that } f \text{ is } F_t/\mathcal{B}(F) \text{ measurable} \}
\text{f(t, x, \omega) is } \mathcal{F}_t \text{- adapted } \forall x \in E \setminus \{0\},\ t \in (0, T]\} \quad (11)$$

Here we recall the definition of stochastic integrals of random functions $f(t, x, \omega) \in M^T(E/F)$ with respect to the compensated Poisson random measures $q(dt \, dx)(\omega) := N(dt \, dx)(\omega) - \nu(dt \, dx)$ associated to an additive process $(X_t)_{t \geq 0}$ introduced in [35] (and [1] for the case of deterministic functions $f(x)$ not depending on time $t \in \mathbb{R}_+$ and on $\omega \in \Omega$).

There is a "natural definition" of stochastic integral w.r.t. $N(dt \, dx)(\omega)$, resp. $q(dt \, dx)(\omega)$, on those sets $(0, T) \times \Lambda$ where the measures $N(dt \, dx)(\omega)$ (with $\omega$ fixed) and resp. $\nu(dt \, dx)$ are finite, i.e. $0 \notin \overline{\Lambda}$. According to [35] we give the following definition

**Definition 3.1** Let $t \in (0, T], \Lambda \in \mathcal{F}(E \setminus \{0\})$ (defined in (7)), $f \in M^T(E/F)$. The natural integral of $f$ on $(0, t] \times \Lambda$ w.r.t. Poisson random measure $N(dt \, dx)(\omega)$ is

$$\int_0^t \mathbf{1}_\Lambda f(s, x, \omega) N(ds \, dx)(\omega) := \sum_{0 < s \leq t} f(s, \Delta X_s)(\omega) \mathbf{1}_\Lambda(\Delta X_s(\omega)) \quad \omega \in \Omega \quad (12)$$

**Definition 3.2** Let $t \in (0, T], \Lambda \in \mathcal{F}(E \setminus \{0\})$ (defined in (7)), $f \in M^T(E/F)$. Assume that $f(\cdot, \cdot, \omega)$ is Bochner integrable on $(0, T] \times \Lambda$ w.r.t. $\nu$, for all $\omega \in \Omega$ fixed. The natural integral of $f$ on $(0, t] \times \Lambda$ w.r.t. the compensated Poisson random measure $q(dt \, dx)$ := $N(dt \, dx)(\omega) - \nu(dt \, dx)$ is

$$\int_0^t \mathbf{1}_\Lambda f(s, x, \omega) (N(ds \, dx)(\omega) - \nu(ds \, dx)) := \sum_{0 < s \leq t} f(s, \Delta X_s)(\omega) \mathbf{1}_\Lambda(\Delta X_s(\omega)) - \int_0^t \mathbf{1}_\Lambda f(s, x, \omega) \nu(ds \, dx) \quad \omega \in \Omega \quad (13)$$

where the last term is understood as a Bochner integral, (for $\omega \in \Omega$ fixed) of $f(s, x, \omega)$ w.r.t. the measure $\nu$. 

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It is more difficult to define the stochastic integral on those sets \((0, T] \times A, A \in \mathcal{B}(E \setminus \{0\})\), s.th. \(\nu((0, T] \times A) = \infty\). For real valued functions this problem was already discussed e.g. in [36] and [6], [14], [39] (for general martingale measures), where two different definitions of stochastic integrals were introduced. In [35] the definitions of stochastic integrals of Hilbert and Banach valued random functions, which generalize (in a natural way) the definitions of such stochastic integrals (of real valued random functions) have been given, for the case where integration is performed w.r.t. compensated Poisson random measures on separable Banach spaces. Like in [35], we shall call them strong \(-p\) -, resp. simple \(-p\) -integrals. The strong \(-p\) -integral is the limit in \(L^p_F(\Omega, \mathcal{F}, P)\) (the space of \(\mathcal{F}\)-valued random variables \(Y\), with \(E[\|Y\|^p] < \infty\), defined in Definition 3.8) of the "natural integrals" (13) of the "simple functions" defined in (14). (We refer to Definition 3.9 for a precise statement.) The simple \(-p\) -integral is the limit in \(L^p_F(\Omega, \mathcal{F}, P)\) of the "natural integrals" over the sets \((0, T] \times \Lambda_{\epsilon} \), with \(T > 0\) and \(\Lambda_{\epsilon} := A \cap B^c(0, \epsilon)\), when \(\epsilon \to 0\), where with \(B^c(0, \epsilon)\) we denote the complementary of the ball of radius \(\epsilon > 0\) centered in 0 (we refer to Definition 3.20 for a precise statement). In [35] sufficient conditions for the strong \(-1\) - and simple \(-1\) -integral are found on any separable Banach space, and sufficient conditions of the strong \(-2\) - and simple \(-2\) -integral are found on separable Banach spaces of \(M\)-type 2 and type 2 (see Theorems 3.12 -3.16 and Theorems 3.22-3.26). (See Definition 3.19 or [5], [20] for the definition of type 2 Banach spaces, Definition 3.17 or [27], [28] for the definition of \(M\)-type 2 Banach spaces). Here we first introduce the notion of strong \(-p\), as it turns out from [35] that the strong \(-p\) integral is a stronger definition, than the simple \(-p\) -integral. In fact, if the sufficient conditions in Theorems 3.12 -3.16 for the existence of the strong \(-p\) -integrals for \(f(t, x, \omega)\) are satisfied, and, moreover, \(f(\cdot, x, \omega)\) is left continuous on \(t \in \mathbb{R}_+\) for each \(x \in E \setminus \{0\}\) and \(\omega \in \Omega\) fixed, then \(f\) is also simple \(-p\) integrable, and the simple \(-p\) and strong \(-p\) -integrals coincide (see Theorems 3.22 -3.26). We first introduce the simple functions and recall the definition of "strong \(-p\) -integral", \(p \geq 1\).

**Definition 3.3** A function \(f\) belongs to the set \(\Sigma(E/F)\) of simple functions , if \(f \in M_T(E/F)\), \(T > 0\) and there exist \(n \in \mathbb{N}\), \(m \in \mathbb{N}\), such that

\[
f(t, x, \omega) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} 1_{A_{k,l}}(x)1_{F_{k,l}}(\omega)1_{(t_k, t_{k+1})}(t)a_{k,l}
\]

(14)

where \(A_{k,l} \in \mathcal{F}(E \setminus \{0\})\) (i.e. \(0 \notin \overline{A_{k,l}}\)), \(t_k \in (0, T]\), \(t_k < t_{k+1}\), \(F_{k,l} \in \mathcal{F}_{t_k}\), \(a_{k,l} \in F\). For all \(k \in 1, \ldots, n-1\) fixed, \(A_{k,l_1} \times F_{k,l_1} \cap A_{k,l_2} \times F_{k,l_2} = \emptyset\) if \(l_1 \neq l_2\).

**Remark 3.4** Let \(f \in \Sigma(E/F)\) be of the form (14), then the natural integral defined in Definition 3.2 is given also in the following form:
\[
\int_0^T \int_A f(t,x,\omega) q(dt, dx)(\omega) = \sum_{k=1}^{n-1} \sum_{l=1}^m a_{k,l} 1_{F_{k,l}}(\omega) q(\{t_k, t_{k+1}\} \cap (0,T] \times A_{k,l} \cap A)(\omega).
\]

for all \( A \in \mathcal{B}(E \setminus \{0\}) \), \( T > 0 \). This is an easy consequence of the Definition of the random measure \( q(dt, dx)(\omega) \).

We recall here the definition of strong \(-\) integral (Definition 3.9 below) given in [35] through approximation of the natural integrals of simple functions. Before we establish some properties of the functions \( f \in M^T_{p,q}(E/F) \), \( p \geq 1 \), with

\[
M^T_{p,q}(E/F) := \{ f \in M^T(E/F) : \int_0^T \int E[\|f(t,x,\omega)\|^p] q(dt, dx) < \infty \},
\]

where with \( E[f] \) we denote the expectation with respect to the probability measure \( P \).

Let \( \nu \times P \) be the product measure on the semi-ring \( \mathcal{B}(\mathbb{R}_+ \times (E \setminus \{0\})) \times \mathcal{F}_\infty \) of the product sets \( A \times F \), \( A \in \mathcal{B}(\mathbb{R}_+ \times (E \setminus \{0\})) \), \( F \in \mathcal{F}_\infty \). Let us also denote by \( \nu \otimes P \) the unique extension of \( \nu \times P \) on the product \( \sigma \)-algebra \( F_\infty := \mathcal{B}(\mathbb{R}_+ \times (E \setminus \{0\})) \otimes \mathcal{F}_\infty \) generated by \( \mathcal{B}(\mathbb{R}_+ \times (E \setminus \{0\})) \otimes \mathcal{F}_\infty \).

Starting from here we assume in the whole article that the following hypothesis \( A \) is satisfied:

**Hypothesis A**: \( \nu \) is a product measure \( \nu = \alpha \otimes \beta \) on the \( \sigma \)-algebra generated by the semi-ring \( \mathcal{S}(\mathbb{R}_+) \times \mathcal{B}(E \setminus \{0\}) \), of a finite measure \( \alpha \) on \( \mathcal{S}(\mathbb{R}_+) \) and a \( \sigma \)-finite measure \( \beta \) on \( \mathcal{B}(E \setminus \{0\}) \). \( \alpha(dt) \) is absolutely continuous w.r.t. the Lebesgues measure \( dt \) on \( \mathcal{B}(\mathbb{R}_+) \), i.e. \( \alpha(dt) \propto dt \).

**Definition 3.5** Let \( f : \mathbb{R}_+ \times (E \setminus \{0\}) \times \Omega \rightarrow F \) be given. A sequence \( \{f_n\}_{n \in \mathbb{N}} \) of \( F_T/\mathcal{B}(F) \)-measurable functions is \( L^p \)-approximating \( f \) on \( (0,T] \times A \times \Omega \) w.r.t. \( \nu \otimes P \), if \( f_n \) is \( \nu \otimes P \)-a.s. converging to \( f \), when \( n \rightarrow \infty \), and

\[
\lim_{n \rightarrow \infty} \int_0^T \int_A E[\|f_n(t,x,\omega) - f(t,x,\omega)\|^p] \, d\nu = 0,
\]

i.e. \( \|f_n - f\| \) converges to zero in \( L^p((0,T] \times A \times \Omega, \nu \otimes P) \), when \( n \rightarrow \infty \).

**Theorem 3.6** [35] Let \( p \geq 1 \), \( T > 0 \), then for all \( f \in M^T_{p,q}(E/F) \) and all \( A \in \mathcal{B}(E \setminus \{0\}) \), there is a sequence of simple functions \( \{f_n\}_{n \in \mathbb{N}} \) which satisfies the following

**Property P**: \( f_n \in \Sigma(E/F) \forall n \in \mathbb{N} \), and \( f_n \) is \( L^p \)-approximating \( f \) on \( (0,T] \times A \times \Omega \) w.r.t. \( \nu \otimes P \).

**Remark 3.7** From the proofs of Theorem 3.6 in [35] it follows that if \( f \in M^T_{p,q}(E/F) \cap M^T_{p,q}(E/F) \) with \( p, q \geq 1 \), \( T > 0 \), then, for all \( \nu \in \mathcal{B}(E \setminus \{0\}) \) there is a sequence of simple functions \( \{f_n\}_{n \in \mathbb{N}} \), \( f_n \in \Sigma(E/F) \forall n \in \mathbb{N}, \) s.th. \( f_n \) is both \( L^p \) - and \( L^q \) - approximating \( f \) on \( (0,T] \times A \times \Omega \) w.r.t. \( \nu \otimes P \).
Definition 3.8 Let $p \geq 1$, $L^p_p(\Omega, \mathcal{F}, P)$ is the space of $F$-valued random variables, such that $E\|Y\|^p = \int \|Y\|^p dP < \infty$. We denote by $\| \cdot \|_p$ the quasi-norm ([20]) given by $\|Y\|_p = (E\|Y\|^p)^{1/p}$. Given $(Y_n)_{n \in \mathbb{N}}, Y \in L^p_p(\Omega, \mathcal{F}, P)$, we write $\lim_{n \to \infty} Y_n = Y$ if $\lim_{n \to \infty} \|Y_n - Y\|_p = 0$.

Definition 3.9 Let $p \geq 1$, $T > 0$. We say that $f$ is strong-$p$-integrable on $(0,T] \times A$ w.r.t. $q(dtdx)(\omega)$, $A \in \mathcal{B}(E \setminus \{0\})$, if there is a sequence $(f_n)_{n \in \mathbb{N}} \in \Sigma(E/F)$, s.t. $f_n$ is $L^p$-approximating $f$ on $(0,T] \times A \times \Omega$ w.r.t. $\nu \otimes P$, and for any such sequence the limit of the natural integrals of $f_n$ w.r.t. $q(dtdx)$ exists in $L^p_p(\Omega, \mathcal{F}, P)$ for $n \to \infty$, i.e.

$$\int_0^T \int_A f(t, x, \omega)q(dtdx)(\omega) := \lim_{n \to \infty} \int_0^T \int_A f_n(t, x, \omega)q(dtdx)(\omega) \quad (18)$$

exists. Moreover, the limit (18) does not depend on the sequence $(f_n)_{n \in \mathbb{N}} \in \Sigma(E/F)$, which is $L^p$-approximating $f$ on $(0,T] \times A \times \Omega$ w.r.t. $\nu \otimes P$. We call the limit in (18) the strong-$p$-integral of $f$ w.r.t. $q(dtdx)$ on $(0,T] \times A$.

The strong-$p$-integral of $f$ w.r.t. $q(dtdx)$ on $(\tau, T] \times A$, with $0 < \tau < T$, is defined as

$$\int_{\tau}^T \int_A f(t, x, \omega)q(dtdx)(\omega) := \int_0^T \int_A f(t, x, \omega)q(dtdx)(\omega) - \int_0^\tau \int_A f(t, x, \omega)q(dtdx)(\omega) \quad (19)$$

Remark 3.10 Let $f \in M^{r,q}_T(E/F) \cap M^{T,q}_r(E/F), r, q \geq 1$. If $f$ is both $r$- and $q$-strong-integrable on $(0,T] \times A, r, q \geq 1$, then from Remark 3.7 it follows that the strong-$r$-integral coincides with the strong-$q$-integral. In fact from any sequence $f_n$, for which the limit in (18) holds with $p = r$ and $p = q$, it is possible to extract a subsequence for which the convergence (18) holds also $P$-a.s..

Proposition 3.11 [35] Let $p \geq 1$. Let $f$ be strong-$p$-integrable on $(0,T] \times A, A \in \mathcal{B}(E \setminus \{0\})$. Then the strong-$p$-integral $\int_0^t \int_A f(s, x)q(dsdx), t \in [0,T]$, is an $\mathcal{F}_t$-martingale with mean zero.

Theorem 3.12 [35] Let $f \in M^{T,1}_r(E/F)$, then $f$ is strong-$1$-integrable w.r.t. $q(dtdx)$ on $(0,t] \times A$, for any $0 < t \leq T, A \in \mathcal{B}(E \setminus \{0\})$. Moreover

$$E[\| \int_0^t \int_A f(s, x, \omega)q(dsdx)(\omega) \|] \leq 2 \int_0^t \int_A E[\|f(s, x, \omega)\|] \nu(dsdx) \quad (20)$$
Theorem 3.13 [35] Suppose \((F, \mathcal{B}(F)):= (H, \mathcal{B}(H))\) is a separable Hilbert space. Let \(f \in M_{t, 2}^{\mathcal{F}}(E/H)\), then \(f\) is strong \(2\)-integrable w.r.t. \(q(dtdx)\) on \((0, t] \times A\), for any \(0 < t \leq T\), \(A \in \mathcal{B}(E \setminus \{0\})\). Moreover
\[
E[\| \int_0^t \int_A f(s, x, \omega)q(dtdx)(\omega) \|^2] = \int_0^t \int_A E[\| f(s, x, \omega) \|^2] \nu(dsdx) \quad (21)
\]

Theorem 3.14 [23] Suppose that \(F\) is a separable Banach space of \(M\)-type 2. Let \(f \in M_{t, 2}^{\mathcal{F}}(E/F)\), then \(f\) is strong \(2\)-integrable w.r.t. \(q(dtdx)\) on \((0, t] \times A\), for any \(0 < t \leq T\), \(A \in \mathcal{B}(E \setminus \{0\})\). Moreover
\[
E[\| \int_0^t \int_A f(s, x, \omega)q(dsdx)(\omega) \|^2] \leq K_2^2 \int_0^t \int_A E[\| f(s, x, \omega) \|^2] \nu(dsdx). \quad (22)
\]
where \(K_2\) is the constant in the Definition 3.17 of \(M\)-type 2 Banach spaces.

Remark 3.15 Theorem 3.14 was first proven in [35], however only for functions \(f(t, x, \omega) = f(t, \omega)\), i.e. which do not depend on the variable \(x \in E\).

Theorem 3.16 [35] Suppose that \(F\) is a separable Banach space of type 2. Let \(f \in M_{t, 2}^{\mathcal{F}}(E/F)\), and \(f\) be a deterministic function, i.e. \(f(t, x, \omega) = f(t, x)\), then \(f\) is strong \(2\)-integrable w.r.t. \(q(dtdx)\) on \((0, t] \times A\), for any \(0 < t \leq T\), \(A \in \mathcal{B}(E \setminus \{0\})\). Moreover inequality (22) holds, with \(K_2\) being the constant in the Definition 3.19 of type 2 Banach spaces.

We recall here the definitions of \(M\)-type 2 and type 2 separable Banach spaces (see e.g. [27], [5]).

Definition 3.17 A separable Banach space \(F\), with norm \(\| \cdot \|\), is of \(M\)-type 2, if there is a constant \(K_2\), such that for any \(F\)-valued martingale \((M_k)_{k \in 1,...,n}\) the following inequality holds:
\[
E[\| M_n \|^2] \leq K_2 \sum_{k=1}^n E[\| M_k - M_{k-1} \|^2] \quad (23)
\]
with the convention that \(M_{-1} = 0\).

We remark that a separable Hilbert space is in particular a separable Banach space of \(M\)-type 2. In fact, any \(2\)-uniformly smooth separable Banach space is of \(M\)-type 2 [28], [40]. We recall here the definition of \(2\)-uniformly smooth separable Banach space.

Definition 3.18 A separable Banach space \(F\), with norm \(\| \cdot \|\), is \(2\)-uniformly smooth if there is a constant \(K_2 > 0\), s.th. for all \(x, y \in F\)
\[
\| x + y \|^2 + \| x - y \|^2 \leq 2\| x \|^2 + K_2\| y \|^2
\]

Definition 3.19 A separable Banach space $F$ is of type 2, if there is a constant $K_2$, such that if $\{X_i\}_{i=1}^n$ is any finite set of centered independent $F$-valued random variables, such that $E[\|X_i\|^2] < \infty$, then

$$E[\| \sum_{i=1}^n X_i \|^2] \leq K_2 \sum_{i=1}^n E[\|X_i\|^2]$$  \hspace{1cm} (24)$$

We remark that any separable Banach space of $M$-type 2 is a separable Banach space of type 2. Moreover, a separable Banach space is of type 2 as well as of cotype 2 if and only if it is isomorphic to a separable Hilbert space [18], where a Banach space of cotype 2 is defined by putting $\geq$ instead of $\leq$ in (24) (see [5], or [20]).

We recall the definition of "simple -$p$-integrals" w.r.t. the compensated Poisson random measures $\nu(\text{d}t \text{d}x)$, introduced in [35].

Definition 3.20 Let $p \geq 1$, $T > 0$, $A \in \mathcal{B}(E \setminus \{0\})$. Let $f \in M_T^{1}(E/F)$ and $f$ bounded $\nu \otimes P$-a.s. on $(0, T] \times A \times \Omega$. $f$ is simple -$p$ integrable on $(0, T] \times A \times \Omega$, if for all sequences $\{\delta_n\}_{n \in \mathbb{N}}$, $\delta_n \in \mathbb{R}_+$, such that $\lim_{n \to \infty} \delta_n = 0$, the limit

$$\lim_{n \to \infty} \sum_{0<s\leq T} \mathbb{1}_{\Delta X_s(\omega) \in \Lambda_{\delta_n} \cap A} f(s, \Delta X_s(\omega), \omega) - \int_0^T \int_{\Lambda_{\delta_n} \cap A} f(t, x, \omega) \nu(\text{d}t \text{d}x)$$  \hspace{1cm} (25)$$

with

$$\Lambda_{\delta_n} := \{x \in E \setminus \{0\} : \delta_n < \|x\]\}$$ \hspace{1cm} (26)$$

exists and does not depend on the choice of the sequences $\{\delta_n\}_{n \in \mathbb{N}}$, satisfying the above properties.

We call the limit (25) the simple -$p$ integral of $f$ on $(0, T] \times A$ w.r.t. the compensated Poisson random measure $N(\text{d}t \text{d}x) - \nu(\text{d}t \text{d}x)$, and denote it with

$$\int_0^T \int_A f(t, x, \omega)(N(\text{d}t \text{d}x)(\omega) - \nu(\text{d}t \text{d}x))$$ \hspace{1cm} (27)$$

Remark 3.21 If $f$ is both $r$- and $q$- simple integrable on $(0, T] \times A$, $p, q \geq 1$, then the limit in (25) with $p = r$ coincides $P$-a.s., with the limit in (25) with $p = q$, as there exist in both cases a subsequence of a sequence $\{\delta_n\}_{n \in \mathbb{N}}$, such that $\lim_{n \to \infty} \delta_n = 0$, for which the convergence (25) holds also $P$- a.s..

The following Theorems 3.22 - 3.27 and Corollary 3.28 have been proven in [35].

Theorem 3.22 Let $A \in \mathcal{B}(E \setminus \{0\})$, $f \in M_T^{1,1}(E/F)$, and assume $f$ is left continuous in the time interval $(0, T]$ for every $x \in A$ and $P$- a.e. $\omega \in \Omega$ fixed. Then $f$ is simple 1 -integrable on $(0, T] \times A$ and the simple 1 -integral coincides with the strong 1 -integral.
Theorem 3.23 Let $F$ be a separable Hilbert space. Let $A \in \mathcal{B}(E \setminus \{0\})$, $f \in M^{T, 2}_0(E/F)$, and let $f$ be left continuous in the time interval $(0, T]$ for every $x \in A$ and $P$-a.e. $\omega \in \Omega$ fixed. Then $f$ is simple $2$-integrable on $(0, T] \times A$. The simple $2$-integral coincides with the strong $2$-integral.

Theorem 3.24 Let $F$ be a separable Banach space of $M$-type 2. Let $A \in \mathcal{B}(E \setminus \{0\})$, $f \in M^{T, 2}_0(E/F)$. Let $f$ be left continuous in the time interval $(0, T]$ for $P$-a.e. $\omega \in \Omega$ fixed. Then $f$ is simple $2$-integrable on $(0, T] \times A$. The simple $2$-integral coincides with the strong $2$-integral.

Remark 3.25 Theorem 3.24 has been stated in [35] for functions $f$, such that $f(s, x, \omega) = f(s, \omega)$, i.e. such that $f$ does not depend on the variable $x \in E$. The proof of the Theorem goes through however for general functions, like stated in 3.24, and is in this case a consequence of Theorem 3.14 (see Remark 3.15).

Theorem 3.26 Let $F$ be a separable Banach space of type 2. Let $A \in \mathcal{B}(E \setminus \{0\})$, $f \in M^{T, 2}_0(E/F)$, and let $f$ be left continuous in the time interval $(0, T]$ for every $x \in A$ fixed. Let $f$ be a deterministic function, i.e. $f(t, x, \omega) = f(t, x)$. Then $f$ is simple $2$-integrable on $(0, T] \times A$. The simple $2$-integral coincides with the strong $2$-integral.

The statements in Theorems 3.22-3.26 are proven in [35] by first proving the following Theorem 3.27 and its Corollary 3.28, that we shall also use in the next Section of this article.

Theorem 3.27 Let $\Lambda \in \mathcal{F}(E \setminus \{0\})$ (defined in (7)). Let $f$ be uniformly bounded on $(0, T] \times \Lambda \times \Omega$, and left continuous in the time interval $(0, T]$ for every $x \in \Lambda$ and for $P$-a.e. $\omega \in \Omega$. Let $p \geq 1$. For $t \in (0, T]$ the strong $p$-integral of $f$, coincides with the natural integral of $f$ on $(0, t] \times \Lambda$.

Corollary 3.28 Let $\Lambda \in \mathcal{F}(E \setminus \{0\})$, and let $f$ be left continuous in the time interval $(0, T]$, for every $x \in \Lambda$ and for $P$-a.e. $\omega \in \Omega$ fixed. Suppose that one of the following three hypothesis is satisfied:

i) $f \in M^{T, 1}_0(E/F)$,

ii) $f \in M^{T, 2}_0(E/F)$ and $F$ is a separable Banach space of $M$-type 2.

iii) $f \in M^{T, 2}_0(E/F)$, $f(s, x, \omega) = f(s, x)$, i.e. $f$ is a deterministic function, and $F$ is a Banach space of type 2.

Then the natural integral of $f$ on $(0, t] \times \Lambda$ is $\nu \otimes P$-a.s. bounded $\forall t \in (0, T]$ and coincides with the strong 1-integral of $f$, if condition i) is satisfied, with the strong 2-integral of $f$, if condition ii), or iii) is satisfied.
Remark 3.29 The condition ii) in Corollary 3.28 has been proven in [35] for the case where \( f \) does not depend on the variable \( x \in E \). In fact, ii) follows from Theorem 3.14, which in [35] was still proven only for this case (see Remark 3.15. The same proof holds however for the general case, like stated in ii).

Similar to what is discussed for the Ito -integral w.r.t. the Brownian motion, the concept of strong -p -integral (1) can be generalized by approximating (1) with the natural integral of simple functions which converge to the integrand \( f \) in probability, instead of converging in \( L^p((0,T] \times A \times \Omega, \nu \otimes P) \). This is discussed in [36] for the case of stochastic integrals (1) of real valued functions w.r.t. cPrm associated to \( \alpha \)-stable processes, and in [35] for Banach valued functions and general cPrms. Such stochastic integrals are called in [35] ”strong integrals of type p”, to distinguish these from the strong -p- integrals. We report here the definition of strong integrals of type p, and some related results [35] needed in Section 6, where we shall prove that the Ito formula (3) holds also for such integrals.

Definition 3.30 Let \( p \geq 1 \), \( T > 0 \), \( A \in \mathcal{B}(E\setminus\{0\}) \). We say that \( f \in M^T(E/F) \) is strong integrable on \( (0,T] \times A \), with strong integral of type \( p \), if for all \( N \in \mathbb{N} \)

\[
f^N(t, x, \omega) := f(t, x, \omega) \mathbf{1}_{t_0} \int_A \|f\|_p d\nu \leq N \]

is strong \( p \)-integrable and the following limit (28) exists

\[
\lim_{N \to \infty} \int_0^T \int_A f(t, x, \omega) q(dt dx)(\omega) = \int_0^T \int_A f(t, x, \omega) q(dt dx)(\omega)
\]

in probability ,

(28)

where \( f^N(t, x, \omega) \mathbf{1}_{t_0} \int_A \|f\|_p d\nu \leq N \)

is the strong -p integral of the function \( f^N(t, x, \omega) \).

Remark 3.31 Let \( p \geq 1 \). Let \( f, g \) be strong integrable on \( (0,T] \times A \), \( A \in \mathcal{B}(E\setminus\{0\}) \), \( T > 0 \), with strong integral of type \( p \). For any \( a, b \in \mathbb{R} \), \( af + bg \) is strong integrable on \( (0,T] \times A \) with strong integral of type \( p \), and

\[
a \int_0^T \int_A f(s, x, \omega) q(ds dx) + b \int_0^T \int_A g(s, x, \omega) q(ds dx) = \int_0^T \int_A (af(s, x, \omega) + bg(s, x, \omega)) q(ds dx)
\]

(29)

Let

\[
N^T_p(E/F) := \{ f \in M^T(E/F) : \int_0^T \int_A \|f(t, x, \omega)\|_p^p \nu(dt dx) < \infty \quad P - a.s. \}
\]

(30)
Proposition 3.32 [35] Suppose
\( i' \) \( f \in N^{T,1}_{\nu}(E/F), \)
then \( f \) is strongly integrable on \((0,t] \times A\) with strongly integrable of type 1, for all \( t \in (0,T), A \in \mathcal{B}(E \setminus \{0\}) \).

Proposition 3.33 [35] Suppose that one of the following two hypotheses is satisfied:
\( ii' \) \( f \in N^{T,2}_{\nu}(E/F) \) and \( F \) is a separable Banach space of \( M \)-type 2,
\( iii' \) \( f \in N^{T,2}_{\nu}(E/F), f(s,x,\omega) = f(s,x) \), i.e. \( f \) is a deterministic function, and \( F \) is a Banach space of type 2.
then \( f \) is strongly integrable on \((0,t] \times A\) with strongly integrable of type 2, for all \( t \in (0,T), A \in \mathcal{B}(E \setminus \{0\}) \).

Theorem 3.34 [35] Let \( T > 0, p \geq 1 \), then for all \( f \in N^{T,p}_{\nu}(E/F) \), for all \( A \in \mathcal{B}(E \setminus \{0\}) \), there exists a sequence of simple functions \( \{ f_n \}_{n \in \mathbb{N}} \) satisfying the following property:
Property Q: \( f_n \in \Sigma(E/F) \forall n \in \mathbb{N} \), \( f_n \) converges \( \nu \otimes \mathcal{P} \)-a.s. to \( f \) on \((0,T] \times A \times \Omega\), when \( n \to \infty \), and
\[
\lim_{n \to \infty} \int_A \|f_n(t,x) - f(t,x)\|^p d\nu = 0 \quad \mathcal{P} \text{-a.s.} \quad (31)
\]

Theorem 3.35 [35] Let \( T > 0, A \in \mathcal{B}(E \setminus \{0\}) \). Define \( p = 1 \) if the hypothesis \( i' \) in Proposition 3.32 are satisfied, or \( p = 2 \), if the hypothesis \( ii' \) or \( iii' \) in Proposition 3.33 are satisfied. Suppose that \( \{ f_n \}_{n \in \mathbb{N}} \subseteq N^{T,p}_{\nu}(E/F) \) satisfies property Q in Theorem 3.34, and there exists \( g \in N^{T,p}_{\nu}(E/F) \), s.th.
\[
\int_0^T \int_A \|f_n\|^p d\nu \leq \int_0^T \int_A \|g\|^p d\nu \quad \mathcal{P} \text{-a.s.} \quad (32)
\]
then
\[
\int_0^T \int_A f(t,x)q dt dx = \lim_{n \to \infty} \int_0^T \int_A f_n(t,x)q dt dx \quad \text{in probability} \quad (33)
\]

Remark 3.36 Taking \( \{ f_n \}_{n \in \mathbb{N}} \subseteq \Sigma(E/F) \) instead of \( \{ f_n \}_{n \in \mathbb{N}} \subseteq N^{T,p}_{\nu}(E/F) \) in Theorem 3.35, we can see that the strongly integrals of type \( p \) have similar properties compared with the strongly \( p \)-integrals.

Remark 3.37 Under the hypothesis \( ii' \) Proposition 3.33 and Theorem 3.35 have been proven in [35] for the case where \( f \) does not depend on the variable \( x \in E \). In fact, under the condition \( ii' \) the statements follow from Theorem 3.14, which in [35] was still proven only for this case (see Remark 3.15). The same proofs hold however for the general case.
4 The Ito formula for Banach valued functions

Like in the previous Sections, we assume that $E$ and $F$ are separable Banach spaces, $q(dt dx) = N(dt dx)(\omega) - \nu(dt dx)$ is the compensated Poisson random measure associated to the additive process $(X_t)_{t \geq 0}$ with values on $E$.

Let $0 < t \leq T$, $A \in \mathcal{B}(E \setminus \{0\})$ and $Z_t(\omega) := \int_0^t \int_A f(s, x, \omega)q(ds dx)(\omega)$. (34)

We assume that one of the following hypothesis is satisfied:

i) $f \in M_T(E/F)$,

ii) $f \in M_T(E/F)$ and $F$ is a separable Banach space of $M$-type 2,

iii) $f \in M_T(E/F)$, $f(s, x, \omega) = f(s, x)$, i.e. $f$ is deterministic, and $F$ is a Banach space of type 2.

$Z_t(\omega)$ is then the strong $-p$-integral of $f$ w.r.t. $q(ds dx)(\omega)$, with $p = 1$ if i) holds, and $p = 2$ if ii) or iii) holds.

In this and the coming Sections we analyze the Ito-formula for the $F$-valued random process $(Y_t)_{t \geq 0}$, with

$$Y_t(\omega) := Z_t(\omega) + \int_0^t g(s, \omega)dh(s, \omega) + \int_0^t \int_{\Lambda} k(s, x, \omega)N(ds dx)(\omega).$$ (35)

We assume that $h(s, \omega) : [0, T] \times \Omega \to \mathbb{R}$ is a real valued, càdlàg process, whose almost all paths are of bounded variation, moreover $h(s, \omega)_{s \in [0, T]}$ is $\mathcal{F}_s$-adapted for all $s \in [0, T]$, and jointly measurable in all its variables. The random process $g(s, \omega) : [0, T] \times \Omega \to F$ is $\mathcal{F}_s$-adapted for all $s \in [0, T]$, and jointly measurable in all its variables, and is either càdlàg or càgàl (i.e. left continuous with right limit, for the definition see e.g. [30]). We assume that, for $P$-a.e. $\omega \in \Omega$, $g(s, \omega)$ is Bochner integrable w.r.t. the signed Lesbegues measure induced by the bounded variation process $(h(s, \omega))_{s \geq 0}$, i.e.

$$\int_0^t \|g(s, \omega)\| dh(s, \omega) < \infty \quad P - a.e.$$ (36)

where $(\|h(s, \omega)\|)_{s \geq 0}$ is the variation process associated to $(h(s, \omega))_{s \geq 0}$ (see e.g. [30] for the definition of variation process).

Moreover, we assume that

$$\Lambda \in \mathcal{F}(E \setminus \{0\}) \quad \text{so that} \quad 0 \notin \overline{\Lambda},$$ (37)

$k(s, x, \omega) \in M^T(E/F)$ and $k(s, x, \omega)$ is finite $P$-a.s. for every $s \in [0, T]$, $x \in \Lambda$.

Moreover $k(s, x, \omega)$ is càdlàg or càgàl.

We also assume that $\forall s \in [0, T]$

$$1_{\Lambda}(\Delta X_s(\omega))\Delta k(s, \Delta X_s(\omega), \omega) = 0 \quad 1_{\Lambda}(\Delta X_s(\omega))\Delta f(s, \Delta X_s(\omega), \omega) = 0 \quad P - a.s.,$$ (38)
\[ \forall s \in [0, T] \quad \Delta X_s(\omega) \Delta h(s, \omega) = 0 \quad P - a.s., \quad (39) \]

where

\[
\begin{align*}
\Delta h(s, \omega) &:= \lim_{t \downarrow s} h(t, \omega) - \lim_{t \uparrow s} h(t, \omega) \quad P - a.s. \\
\Delta k(s, x, \omega) &:= \lim_{t \downarrow s} k(t, x, \omega) - \lim_{t \uparrow s} k(t, x, \omega) \quad P - a.s. \\
\Delta f(s, x, \omega) &:= \lim_{t \downarrow s} f(t, x, \omega) - \lim_{t \uparrow s} f(t, x, \omega) \quad P - a.s.
\end{align*}
\]

Moreover

\[ \forall s \in [0, T] \quad 1_A(\Delta X_s(\omega)) 1_A(\Delta X_s(\omega)) = 0 \quad P - a.s., \quad (40) \]

Let \( \mathcal{H} : F \to G \), \( G \) be a separable Banach space, \( 0 \leq t < T \). (In the next Section we shall also consider \( \mathcal{H} \) depending on time.) In this Section we shall prove that the following Ito formula holds (see Theorem 4.4 and Theorem 4.6 for a precise statement):

\[
\mathcal{H}(Y_\tau(\omega)) - \mathcal{H}(Y_\tau(\omega)) = \\
\int_t^\tau \int_X \{ \mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \} \, g(dsdx)(\omega) \\
+ \sum_{\tau < s \leq \tau} \{ \mathcal{H}(Y_{s-}(\omega) + g(s-, D h(s, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \} \, N(dsdx)(\omega) + \int_t^\tau \mathcal{H}''(Y_{s-}(\omega)) g(s-, \omega) dh(s, \omega)
\]

\[ P - a.s., \quad (41) \]

where \( \mathcal{H}(y) \) is twice Fréchet differentiable \( \forall y \in F \). \( \mathcal{H}'(y) \) (resp. \( \mathcal{H}''(y) \)) denotes the first (resp. second) Fréchet derivative in \( y \).

Obviously \( \mathcal{H}' : F \to \mathcal{L}(F/G) \), and \( \mathcal{H}'' : F \to \mathcal{L}(\mathcal{L}(F/G)) \), where, as usual, we denote with \( \mathcal{L}(K/L) \) the Banach space of linear bounded operators from a Banach space \( K \) to a Banach space \( L \), with the usual supremum norm. With \( \| \cdot \|_K \) we denote the norm of a Banach space \( K \), so that in particular \( \| F \|_{\mathcal{L}(K/L)} = \sup_{y \in K} \| F(y) \|_L \), if \( F \in \mathcal{L}(K/L) \). When no misunderstanding is possible, we shall write simply \( \| \cdot \| \), leaving the subscription which denotes the Banach space. In the above example we would e.g. write \( \| F \| \) instead of \( \| F \|_{\mathcal{L}(K/L)} \), and \( \| y \| \), resp. \( \| F(y) \| \), instead of \( \| y \|_K \), resp. \( \| F(y) \|_L \).

We shall use in this Section the following inequalities, which can be found e.g. in [17], Chap. X.

\[
\| \mathcal{H}(y) - \mathcal{H}(y_0) - \mathcal{H}'(y_0)(y - y_0) \| \leq \| y - y_0 \| \sup_{0 < \theta \leq 1} \| \mathcal{H}'(y_0 + \theta(y - y_0)) - \mathcal{H}'(y_0) \| \\
\]

\[ (42) \]

\[
\| \mathcal{H}(y) - \mathcal{H}(y_0) \| \leq \| y - y_0 \| \sup_{0 < \theta \leq 1} \| \mathcal{H}'(y_0 + \theta(y - y_0)) \| \\
\]

\[ (43) \]
\[\|H'(y) - H'(y_0)\| \leq \|y - y_0\| \sup_{0 < \theta \leq 1} \|H''(y_0 + \theta(y - y_0))\| \]  \quad (44)

We now prove the properties 1)-6) in the following Theorem 4.1 and Lemma 4.3, which imply that all the integrals in equation (41) are well defined.

**Theorem 4.1** Let the following hypothesis hold

hypothesis B: (At least) one of the hypothesis i) - iii) is satisfied. \( G \) is a separable Banach space. If ii) or iii) is satisfied, we make the additional hypothesis that \( G \) is a separable Banach space of \( M \)-type 2

Let \( H : F \rightarrow G \), with \( H(y) \) twice Fréchet differentiable for all \( y \in F \). Let the second Fréchet derivative \( H'' \) be uniformly bounded on each centered ball of radius \( R \), i.e. \( B(0, R) \) with \( R \geq 0 \), and the Fréchet derivative \( H' \) be uniformly bounded.

Then

1) \( H(Y_s - (\omega)) + f(s, x, \omega) - H(Y_s - (\omega)) - H'(Y_s - (\omega)) f(s, x, \omega) \) is \( P \)-a.s. Bochner integrable w.r.t \( \nu \) on \( (0, T] \times A \), \( \forall A \in \mathcal{B}(\{0\}) \).

2) \( H(Y_s - (\omega)) + f(s, x, \omega) - H(Y_s - (\omega)) \in M_T^{T,1}(E/G) \), resp. \( \in M_T^{T,2}(E/G) \), if i), resp. ii) or iii), is satisfied.

**Remark 4.2** Let \( C := \sup_{y \in F} \|H'(y)\| \), then from (43) it follows that

\[ \|H(y) - H(y_0)\| \leq C \|y - y_0\| \]  \quad (45)

From (42) and (44) it follows that for each \( R > 0 \), there exists a constant \( C_R \), depending on \( \omega \) \( \in \Omega \), s.th.

\[ \|H(y) - H(y_0) - H'(y_0)(y - y_0)\| \leq C_R \|y - y_0\|^2 \]  \quad (46)

**Proof of Theorem 4.1:**

Let us first prove property 1). Suppose that condition i) is satisfied, then property 1) follows from (45). In fact

\[
\int_0^T \int_A \|H(Y_s - (\omega)) + f(s, x, \omega) - H(Y_s - (\omega)) - H'(Y_s - (\omega)) f(s, x, \omega)\|\nu(dsdx) \\
\leq 2 \int_0^T \int_A C\|f(s, x, \omega)\|\nu(dsdx)
\]

where the r.h.s. in (47) is \( P \)-a.s. finite, as \( f \in M_T^{T,1}(E/F) \).

Suppose that condition ii) or iii) is satisfied, then property 1) follows from (46). In fact, from Remark 4.2, it follows that there is a finite constant \( C_R(\omega) > 0 \) depending on \( \omega \in \Omega \), s.th.

\[
\int_0^T \int_A \|H(Y_s - (\omega)) + f(s, x, \omega) - H(Y_s - (\omega)) - H'(Y_s - (\omega)) f(s, x, \omega)\|\nu(dsdx) \\
\leq \int_0^T \int_A C_R(\omega)\|f(s, x, \omega)\|^2\nu(dsdx),
\]

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where the r.h.s. in (41) is P -a.s. finite, as \( f \in M^{T,2}_{\nu}(E/F) \).

Property 2) follows from theorems 3.12 - 3.16, and

\[
\begin{align*}
\int_{0}^{T} \int_{A} E[|\mathcal{H}(Y_{s_0} + f(s, x, \omega)) - \mathcal{H}(Y_{s_0} + f(s, x, \omega))|]^{p} \nu(d\omega) & \leq C \int_{0}^{T} \int_{A} E[|f(s, x, \omega)|]^{p} \nu(d\omega) \\
\int_{0}^{T} \int_{A} E[\sup_{0<\theta \leq 1} |\mathcal{H}(Y_{s_0} + \theta f(s, x, \omega))|^{p}] \nu(d\omega) & \leq C \int_{0}^{T} \int_{A} E[|f(s, x, \omega)|]^{p} \nu(d\omega). \\
\end{align*}
\]

\[\text{(49)}\]

**Lemma 4.3** Suppose that all the hypothesis in Theorem 4.1 are verified. The following properties hold

3) \( \int_{0}^{T} |\mathcal{H}(Y_{s_0})g(s_0, \omega)|d|h(s, \omega)| < \infty \) \ P -a.s.

4) \( \sum_{0<s\leq t} |\mathcal{H}(Y_{s_0} + g(s_0, \omega)\Delta h(s, \omega)) - \mathcal{H}(Y_{s_0} + \Delta h(s, \omega))| < \infty \) \ P -a.s.

5) \( \sum_{0<s\leq t} |\mathcal{H}(Y_{s_0} + g(s_0, \omega)\Delta h(s, \omega))| < \infty \) \ P -a.s.

6) \( \int_{0}^{T} \int_{A} |\{\mathcal{H}(Y_{s_0} + k(s, x, \omega)) - \mathcal{H}(Y_{s_0} + \omega))\}|N(d\omega) < \infty \) \ P -a.s.

**Proof of Lemma:** Property 3) follows from the assumption that \( \mathcal{H} \) is uniformly bounded on \( F \), which implies

\[
\int_{0}^{T} |\mathcal{H}(Y_{s_0})g(s_0, \omega)|d|h(s, \omega)| \leq C \int_{0}^{T} |g(s_0, \omega)|d|h(s, \omega)| < \infty
\]

(50)

Property 4) follows from inequality (45), as this implies

\[
\sum_{0<s\leq t} |\mathcal{H}(Y_{s_0} + g(s_0, \omega)\Delta h(s, \omega)) - \mathcal{H}(Y_{s_0} + \Delta h(s, \omega))| \leq C \sum_{0<s\leq t} |g(s_0, \omega)\Delta h(s, \omega)|
\]

\[
\leq C \int_{0}^{T} |g(s_0, \omega)|d|h(s, \omega)|
\]

(51)

Property 5) is proven by similar arguments, and property 6) follows from the assumption that \( \mathcal{H} \) is uniformly bounded on \( F \), the assumption that \( k(s, x, \omega) \) is finite \( P -a.s. \) for every \( s \in [0, T] \), \( x \in \Lambda \) and the definition of natural integral w.r.t. \( N(dt\omega) \).

**Theorem 4.4** Suppose that the hypothesis in Theorem 4.1 are all satisfied. Let \( f(\cdot, x, \omega) \) be left continuous on \([0, T]\), for all \( x \in \Lambda \) and \( P -a.e. \) \( \omega \in \Omega \) fixed. Then on \((0, T) \times \Lambda \) \( \mathcal{H}(Y_{s_0}) + f(s, x, \omega) - \mathcal{H}(Y_{s_0}) \) is strong -p and simple -p- integrable with \( p = 1 \), resp. \( p = 2 \), if i), resp. ii) or iii), is satisfied.

Moreover (41) holds \( P -a.s. \).
Moreover, it is easily checked, that if i) (resp. ii), or iii)) is satisfied, follows from Theorem 3.22- 3.26 and Theorem 4.1. From hypothesis i), or ii), or iii), the above additional hypothesis A and Corollary 3.28, it follows that (Definition 3.2) of \( \omega \) then, for all \( \omega \in \Omega \),

\[
\mathcal{H}(Y_{\cdot}(\omega)) - \mathcal{H}(Y_{\tau}(\omega)) = \sum_{k=0}^{2^n-1} \mathcal{H}(Y_{\tau_{k+1}^n}(\omega)) - \mathcal{H}(Y_{\tau_{k}^n}(\omega)) = \sum_{k \in \Gamma_{A}(A) \cup \Gamma_{A}(\Lambda) \cup \Gamma_{n}^n} \mathcal{H}(Y_{\tau_{k+1}^n}(\omega)) - \mathcal{H}(Y_{\tau_{k}^n}(\omega)) + \sum_{k \notin \Gamma_{A}(A) \cup \Gamma_{A}(\Lambda) \cup \Gamma_{n}^n} \mathcal{H}(Y_{\tau_{k+1}^n}(\omega)) - \mathcal{H}(Y_{\tau_{k}^n}(\omega)) \tag{58}
\]

Remark 4.5 The integral defined in (34) is, according to Theorems 3.12 - 3.14 and Theorems 3.22 - 3.26, a strong -p and simple -p -integral, with \( p = 1 \), if condition i) is satisfied, resp. \( p = 2 \), if condition ii), or iii) is satisfied. From the inequalities (20), (21), (22) it follows that \( Z(\omega) \) is \( P \)-a.e. uniformly bounded on \([0,T] \).

Proof of Theorem 4.4: That on \((0,T] \times A, \mathcal{H}(Y_{\cdot}(\omega) + f(s,x,\omega)) - \mathcal{H}(Y_{\cdot}(\omega)) \) is strong -p- and simple -p- integrable with \( p = 1 \) (resp. \( p = 2 \)) if i) (resp. ii), or iii)) is satisfied, follows from Theorem 3.22- 3.26 and Theorem 4.1.

We now prove that (41) holds \( P \)-a.s.. First we assume the additional hypothesis that \( 0 \notin \Xi \), i.e. that \( A \in \mathcal{F}(E \setminus \{0\}) \).

From hypothesis i), or ii), or iii), the above additional hypothesis \( A \in \mathcal{F}(E \setminus \{0\}) \), and Corollary 3.28, it follows that

\[
Z_t(\omega) = \sum_{0<s \leq t} f(s, (\Delta X_s)(\omega), \omega) 1_A(\Delta X_s(\omega)) - \int_0^t \int_A f(s, x, \omega) \nu(dsdx). \tag{52}
\]

From 2) in Theorem 4.1 and Corollary 3.28 it follows that the natural integral (Definition 3.2) of \( \mathcal{H}(Y_{\cdot}(\omega) + f(s,x,\omega)) - \mathcal{H}(Y_{\cdot}(\omega)) \) exists, coincides with the strong -p -integral, with \( p = 1 \), resp. \( p = 2 \), and has the following form

\[
\int_0^t \int_A \{ \mathcal{H}(Y_{\cdot}(\omega) + f(s,x,\omega)) - \mathcal{H}(Y_{\cdot}(\omega)) \} q(dsdx)(\omega)
= \sum_{\tau<s \leq t} \mathcal{H}(Y_{\cdot}(\omega) + f(s,\Delta X_s(\omega),\omega)) - \mathcal{H}(Y_{\cdot}(\omega)) 1_A(\Delta X_s(\omega))
- \int_0^t \int_A \{ \mathcal{H}(Y_{\cdot}(\omega) + f(s,x,\omega)) - \mathcal{H}(Y_{\cdot}(\omega)) \} \nu(dsdx) \tag{53}
\]

Moreover, it is easily checked, that \( \mathcal{H}'(Y_{\cdot}(\omega))f(s,x,\omega) \) and \( \mathcal{H}(Y_{\cdot}(\omega) + f(s,x,\omega)) - \mathcal{H}(Y_{\cdot}(\omega)) \) are Bochner integrable for \( P \)-q.e. \( \omega \in \Omega \).

Let

\[
\Gamma_{\cdot}^n(A) := \{ k \in [0,\ldots,2^n - 1] : \exists s \in (\tau_{k}^n,\tau_{k+1}^n] : \Delta X_s(\omega) \in A \} \tag{54}
\]

then, for all \( \omega \in \Omega \),

\[
\mathcal{H}(Y_{\tau}(\omega)) - \mathcal{H}(Y_{\tau}(\omega)) = \sum_{k=0}^{2^n-1} \mathcal{H}(Y_{\tau_{k+1}^n}(\omega)) - \mathcal{H}(Y_{\tau_{k}^n}(\omega)) = \sum_{k \in \Gamma_{\cdot}^n(A) \cup \Gamma_{\cdot}(\Lambda) \cup \Gamma_{\cdot}^n} \mathcal{H}(Y_{\tau_{k+1}^n}(\omega)) - \mathcal{H}(Y_{\tau_{k}^n}(\omega)) + \sum_{k \notin \Gamma_{\cdot}^n(A) \cup \Gamma_{\cdot}(\Lambda) \cup \Gamma_{\cdot}^n} \mathcal{H}(Y_{\tau_{k+1}^n}(\omega)) - \mathcal{H}(Y_{\tau_{k}^n}(\omega)) \tag{58}
\]
As \( H \) is continuous on \( F \), it follows

\[
\lim_{n \to \infty} \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \mathcal{H}(Y_{\tau_{t+1}}(\omega)) - \mathcal{H}(Y_{\tau_{t}}(\omega)) = \sum_{\tau < s \leq t} \mathcal{H}(Y_{s-}(\omega) + f(s, \Delta X_s(\omega), \omega)) - \mathcal{H}(Y_{s-}(\omega)) + 1_A(\Delta X_s(\omega))

+ \sum_{\tau < s \leq t} \mathcal{H}(Y_{s-}(\omega) + k(s, \Delta X_s(\omega), \omega)) - \mathcal{H}(Y_{s-}(\omega)) + 1_A(\Delta X_s(\omega))

+ \sum_{\tau < s \leq t} \mathcal{H}(Y_{s-}(\omega) + g(s, \omega) \Delta h(s, \omega)) - \mathcal{H}(Y_{s-}(\omega))

= \int_T^t \int_A \{ \mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \} q(dsdx)(\omega)

+ \int_T^t \int_A \{ \mathcal{H}(Y_{s-}(\omega) + k(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \} N(dsdx)(\omega)

+ \sum_{\tau < s \leq t} \mathcal{H}(Y_{s-}(\omega) + g(s, \omega) \Delta h(s, \omega)) - \mathcal{H}(Y_{s-}(\omega))

+ \int_T^t \int_A \{ \mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \} \nu(dsdx) \quad (59)

P - a.s.

(41) follows once shown that for some subsequence of \( \{n\}_{n \in \mathbb{N}} \), that for simplicity we still denote with \( \{n\} \), the following convergence holds for \( n \to \infty \)

\[
\lim_{n \to \infty} \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \{ \mathcal{H}(Y_{\tau_{t+1}}(\omega)) - \mathcal{H}(Y_{\tau_{t}}(\omega)) \}

= -\int_T^t \int_A \mathcal{H}^\prime(Y_{s-}(\omega)) f(s, x, \omega) \nu(dsdx)

+ \int_T^t \mathcal{H}^\prime(Y_{s-}(\omega)) g(s, \omega) dh(s, \omega)

- \sum_{\tau < s \leq t} \{ \mathcal{H}^\prime(Y_{s-}(\omega)) g(s, \omega) \Delta h(s, \omega) \} \quad P - a.s. \quad (60)

Proof of (60):

We make a Taylor expansion of the function \( \mathcal{H}(y) \):

\[
\sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \mathcal{H}(Y_{\tau_{t+1}}(\omega)) - \mathcal{H}(Y_{\tau_{t}}(\omega))

= \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \mathcal{H}^\prime(Y_{\tau_{t}}(\omega)) (Y_{\tau_{t+1}}(\omega) - Y_{\tau_{t}}(\omega))

+ \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \mathcal{H}^\prime(Y_{\tau_{t}}(\omega)) (Z_{\tau_{t+1}}(\omega) - Z_{\tau_{t}}(\omega))

+ \sum_{k=1}^{n-1} \mathcal{H}^\prime(Y_{\tau_{t}}(\omega)) (\int_{\tau_{t}}^{\tau_{t+1}} g(s, \omega) dh(s, \omega))

- \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \mathcal{H}^\prime(Y_{\tau_{t}}(\omega)) (\int_{\tau_{t}}^{\tau_{t+1}} g(s, \omega) dh(s, \omega))

+ \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \mathcal{H}^\prime(Y_{\tau_{t}}(\omega)) (\int_{\tau_{t}}^{\tau_{t+1}} g(s, \omega) dh(s, \omega))

= \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \mathcal{H}^\prime(Y_{\tau_{t}}(\omega)) (\int_{\tau_{t}}^{\tau_{t+1}} g(s, \omega) dh(s, \omega)) \quad (61)

We first prove that

\[
\lim_{n \to \infty} \sum_{k \in \Gamma^+_\mu(A) \cup \Gamma^-_\nu(A) \cup \Gamma^+_{\mu\nu}} \epsilon_r^\mu(\omega) = 0 \quad P - a.e. \quad (62)

From Remark 4.5 it follows that for \( P - a.e. \) \( \omega \in \Omega \) there is a finite constant \( R(\omega) \) such that \( Y_s(\omega) \leq R(\omega), \forall t \in (\tau, T] \). From (46) it follows that for \( P - a.e. \) \( \omega \in \Omega \) there is a finite constant \( C(\omega) > 0 \), s.th. 20
\[ \|e_k^n(\omega)\| \leq C(\omega) \|Y_{T_{k+1}}(\omega) - Y_{T_k}(\omega)\|^2 \] (63)

and

\[ \sum_{k \notin \Gamma_k(A)} C(\omega) \|Y_{T_{k+1}}(\omega) - Y_{T_k}(\omega)\|^2 \]
\[ \leq 2C(\omega) \sum_{k \notin \Gamma_k(A)} \|Z_{T_{k+1}}(\omega) - Z_{T_k}(\omega)\|^2 \]
\[ + 2C(\omega) \sum_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} g(s, \omega)dh(s, \omega)\|^2 \]
\[ \leq 2C(\omega) \sup_{k \notin \Gamma_k(A)} \int_{T_k}^{T_{k+1}} \int_A f(s, x, \omega)\nu(dsd\sigma) \]
\[ \times \sup_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} g(s, \omega)dh(s, \omega)\| \]
\[ \times \sum_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} g(s, \omega)dh(s, \omega)\| \] (64)

From i), or ii), or iii), and the assumption that \( A \in \mathcal{F}(E \setminus \{0\}) \), it follows that \( f(s, x, \omega) \) is Bochner integrable \( P \)-a.e. w.r.t. \( \nu \), it follows

\[ \limsup_{n \to \infty} \sup_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} \int_A f(s, x, \omega)\nu(dsd\sigma)\| \]
\[ \leq \limsup_{n \to \infty} \sup_{k \notin \Gamma_k(A)} \int_{T_k}^{T_{k+1}} \int_A \|f(s, x, \omega)\|\nu(dsd\sigma) = 0 \] \( P \)-a.s.

(65)

\[ \sum_{k=0}^{2^n-1} \int_{T_k}^{T_{k+1}} \int_A \|f(s, x, \omega)\|\nu(dsd\sigma) \] \[ \leq \int_{T}^{T_f} \int_A \|f(s, x, \omega)\|\nu(dsd\sigma) \] (66)

Moreover we have that

\[ \sup_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} g(s, \omega)dh(s, \omega)\| \]
\[ \times \sum_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} g(s, \omega)dh(s, \omega)\| \]
\[ \leq \sup_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} g(s, \omega)dh(s, \omega)\| \int_{T}^{T_f} \|g(s, \omega)\|dh(s, \omega) \] \( P \)-a.s.

(67)

and

\[ \limsup_{n \to \infty} \sup_{k \notin \Gamma_k(A)} \|\int_{T_k}^{T_{k+1}} g(s, \omega)dh(s, \omega)\| \]
\[ \leq \limsup_{n \to \infty} \sup_{k \notin \Gamma_k(A)} \int_{T_k}^{T_{k+1}} \|g(s, \omega)\|dh(s, \omega) = 0 \] \( P \)-a.s.

(68)
It follows that
\[
\lim_{n \to \infty} \sum_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \|Y_{t_{n+1}}^n(\omega) - Y_{t_n}^n(\omega)\|^2 = 0 \quad P - a.s. \quad (69)
\]
and hence from (63) it follows that (62) holds.

We have
\[
\lim_{n \to \infty} \sum_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \mathcal{H}'(Y_{t_n}^n(\omega))(Z_{t_{n+1}}^n(\omega)) - Z_{t_n}^n(\omega)) = - \lim_{n \to \infty} \sum_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \mathcal{H}'(Y_{t_n}^n(\omega)) \int_{t_n}^{t_{n+1}} f(s, x, \omega) \nu(dsdx)
\]
\[
= - \int_{t}^{T} \int_{A} \mathcal{H}'(Y_{s-}^n(\omega))f(s, x, \omega) \nu(dsdx) \quad P - a.e., \quad (70)
\]
where the last equality in (70) follows from the following estimates and the hypothesis $A$ that $\nu(dsdx) = \alpha(ds)\beta(dx)$, with $\alpha(ds) \prec ds$:
\[
\lim sup_{n \to \infty} \sum_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \|\int_{t_n}^{t_{n+1}} \int_{A} \mathcal{H}'(Y_{t_n}^n(\omega))f(s, x, \omega) \nu(dsdx)
\]
\[
\leq C(\omega) \sum_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \int_{t_n}^{t_{n+1}} \int_{A} \|Y_{t_n}^n(\omega) - Y_{s-}^n(\omega)\|f(s, x, \omega)\|\nu(dsdx)
\]
\[
\leq C(\omega) \{\sup_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \int_{t_n}^{t_{n+1}} \int_{A} \|f(s, x, \omega)\|\|dsdx\}
\]
\[
\times \sum_{k=0}^{2^n-1} \int_{t_n}^{t_{n+1}} \int_{A} \|f(s, x, \omega)\|\nu(dsdx) = 0 \quad P - a.s. \quad (71)
\]
where the first inequality and convergence to zero in (71) holds by the same arguments as in the proof of (62).

(60) is proven, once we have shown that
\[
\sum_{k=1}^{n-1} \mathcal{H}'(Y_{t_k}^n(\omega))(\int_{t_n}^{t_{k+1}} g(s, \omega) dh(s, \omega)) - \sum_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \mathcal{H}'(Y_{t_n}^n(\omega))(\int_{t_n}^{t_{k+1}} g(s, \omega) dh(s, \omega))
\]
\[
= \int_{t}^{T} \mathcal{H}'(Y_{s-}^n(\omega))g(s, \omega) \Delta h(s, \omega) \quad P - a.s. \quad (72)
\]
From the continuity of $\mathcal{H}'$ it follows that
\[
\lim_{n \to \infty} \sum_{k \notin \Gamma_n(A) \cup \Gamma_n'(A) \cup \Gamma_n''} \mathcal{H}'(Y_{t_n}^n(\omega))(\int_{t_n}^{t_{k+1}} g(s, \omega) dh(s, \omega))
\]
\[
= \sum_{t < s \leq T} \{\mathcal{H}'(Y_{s-}^n(\omega))g(s, \omega) \Delta h(s, \omega)\} \quad P - a.s. \quad (73)
\]
while from Theorem 21 Chapt. II, Paragraph 5 of [30] it follows

$$\lim_{n \to \infty} \sum_{k=1}^{2^{n-1}} \mathcal{H}'(Y_{\tau_k}^n(\omega)) \left( \int_{\tau_k}^{\tau_{k+1}} g(s, \omega) dh(s, \omega) \right) = \int_{\tau}^{t} \mathcal{H}'(Y_{s-}) g(s, \omega) dh(s, \omega),$$

(74)

where convergence holds in probability (see [30] for a stronger statement). It follows that for some subsequence of \( \{n\}_{n \in \mathbb{N}} \), that for simplicity we still denote with \( \{n\} \), convergence in (74) and hence in (60) holds \( P \)-a.s.. It follows that (41) holds for every \( A \in \mathcal{F}(E \setminus \{0\}) \), i.e. such that \( 0 \notin A \).

In order to prove that (41) holds for any \( A \in \mathcal{B}(E \setminus \{0\}) \), we first consider (41) with a set \( A \cap B^c(0, \epsilon) \) instead of \( A \), and then take the limit \( \epsilon \to 0 \). In fact, from Theorem 3.22 - 3.26, it follows that for some sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \), such that \( \epsilon_n \to 0 \) when \( n \to \infty \), the following limit holds.

$$\lim_{n \to \infty} \int_{\tau}^{t} \int_{A \cap B^c(0, \epsilon_n)} (\mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega))) q(dsdx)(\omega)$$

$$\to \int_{\tau}^{t} \int_{A} (\mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega))) q(dsdx)(\omega)$$

when \( n \to \infty \) \( P \)-a.e.

and from the definition of Bochner integral it follows that

$$\lim_{n \to 0} \int_{\tau}^{t} \int_{A \cap B^c(0, \epsilon)} (\mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) - \mathcal{H}'(Y_{s-}(\omega)) f(s, x, \omega)) \nu(dsdx)$$

$$= \int_{\tau}^{t} \int_{A} (\mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) - \mathcal{H}'(Y_{s-}(\omega)) f(s, x, \omega)) \nu(dsdx)$$

\( P \)-a.e.

so that (41) holds \( P \)-a.e.  

Theorem 4.6 Suppose that the same hypothesis as in Theorem 4.1 hold, then \( \mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \) is strong -1, resp. strong -2, - integrable on \( (0, T] \times A \) if condition i), resp. ii) or iii), is satisfied. (41) holds \( P \)-a.s..

Remark 4.7 \( Z_t(\omega) \), defined in (34), is in Theorem 4.6, according to Theorems 3.12 -3.16, a strong -p -integral, with \( p = 1 \), if condition i) is satisfied, resp. \( p = 2 \), if condition ii) or iii), is satisfied. Moreover \( E[\|Z_t\|] \) is uniformly bounded on \( [0, T] \).

Proof of Theorem 4.6:
That $\mathcal{H}(Y_{n}(\omega)+f(s,x,\omega))-\mathcal{H}(Y_{n}(\omega))$ is strong -1, resp. strong -2, - integrable on $(0,T] \times A$ if condition i), resp. ii) or iii), is satisfied, follows from Theorem 1.3-1.6 and Theorem 4.1.

Let us prove that (41) holds $P$-a.s.. Let $\{f_n\}_{n \in \mathbb{N}} \in \Sigma(E/F).$ Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is $L^p$-approximating $f$ on $(0,T] \times A \times \Omega$ w.r.t. $\nu \otimes P$ with $p = 1$ (resp. $p = 2$) if condition i) (resp. ii) or iii)) is satisfied. Let

$$Z^n_t(\omega) := \int_0^t \int_A f_n(s,x,\omega)q(dsdx)(\omega)$$

(75)

$$Y^n_t(\omega) := \int_0^t \int_A f_n(s,x,\omega)q(dsdx)(\omega)+\int_0^t g(s,\omega)dh(s,\omega)+\int_0^t \int_A k(s,x,\omega)N(dsdx)(\omega).$$

(76)

then from Theorem 4.4 it follows that $Y^n_t(\omega)$ satisfies (41) $P$- a.s. for all $0 \leq \tau \leq t \leq T$.

We know from Theorem 3.12 - Theorem 3.16 that $\lim_{n \to \infty} Z^n_t(\omega) = Z_t(\omega)$. It follows that there is a subsequence, that for simplicity we denote again with $\{Z^n_t(\omega)\}_{n \in \mathbb{N}},$ such that

$$\lim_{n \to \infty} Z^n_t(\omega) = Z_t(\omega) \quad P-a.s. \quad for \quad all \quad 0 \leq \tau \leq t \leq T,$$

(77)

so that

$$\lim_{n \to \infty} Y^n_t(\omega) = Y_t(\omega) \quad P-a.s. \quad for \quad all \quad 0 \leq \tau \leq t \leq T,$$

(78)

and hence

$$\lim_{n \to \infty} \mathcal{H}(Y^n_t(\omega)) - \mathcal{H}(Y_t(\omega)) = \mathcal{H}(Y_t(\omega)) - \mathcal{H}(Y^n_t(\omega)) \quad P-a.s. \quad for \quad all \quad 0 \leq \tau \leq t \leq T,$$

(79)

$$\lim_{n \to \infty} \int_\tau^t \int_A \{\mathcal{H}(Y^n_{s-}(\omega)+k(s,x,\omega)) - \mathcal{H}(Y^n_{s-}(\omega))\} N(dsdx)(\omega)$$

$$= \int_\tau^t \int_A \{\mathcal{H}(Y_{s-}(\omega)+k(s,x,\omega)) - \mathcal{H}(Y_{s-}(\omega))\} N(dsdx)(\omega)$$

$$P-a.s. \quad for \quad all \quad 0 \leq \tau \leq t \leq T,$$

(80)

$$\lim_{n \to \infty} \sum_{\tau \leq s \leq t} \{\mathcal{H}(Y^n_{s-}(\omega)+g(s,\omega)\Delta h(s,\omega)) - \mathcal{H}(Y^n_{s-}(\omega)) - \mathcal{H}(Y^n_{s-}(\omega)g(s,\omega)\Delta h(s,\omega))\}$$

$$= \{\mathcal{H}(Y_{s-}(\omega)+g(s,\omega)\Delta h(s,\omega)) - \mathcal{H}(Y_{s-}(\omega)) - \mathcal{H}(Y_{s-}(\omega)g(s,\omega)\Delta h(s,\omega))\}$$

$$P-a.s. \quad for \quad all \quad 0 \leq \tau \leq t \leq T.$$

(81)

We shall prove that there is a subsequence such that the following properties (82), (83) hold.

$$\lim_{n \to \infty} \int_\tau^t \mathcal{H}'(Y^n_{s-})(g(s,\omega)dh(s,\omega) = \int_\tau^t \mathcal{H}'(Y_{s-})(g(s,\omega)dh(s,\omega) \quad P-a.s..$$

(82)
\[ \lim_{n \to \infty} \int_0^T \int_A \{ \mathcal{H}(Y^n_{s-} + f_n(s, x, \omega)) - \mathcal{H}(Y^n_{s-} + f_n(s, x, \omega)) \} \nu(dsdx) \]
\[ = \int_0^T \int_A \{ \mathcal{H}(Y_{s-} + f(s, x, \omega)) - \mathcal{H}(Y_{s-} + f(s, x, \omega)) \} \nu(dsdx) \quad P - \text{a.s.} \]

Moreover, we shall prove that for some subsequence

\[ \lim_{n \to \infty} \int_0^T \int_A \{ \mathcal{H}(Y^n_{s-} + f_n(s, x, \omega)) - \mathcal{H}(Y^n_{s-} + f_n(s, x, \omega)) \} q(dsdx)(\omega) \]
\[ = \int_0^T \int_A \{ \mathcal{H}(Y_{s-} + f(s, x, \omega)) - \mathcal{H}(Y_{s-} + f(s, x, \omega)) \} q(dsdx)(\omega). \]

It then follows that there is a subsequence, so that the limit in (84) holds also \( P - \text{a.s.} \).

From (79)- (84) it then follows that (41) holds \( P - \text{a.s.} \).

Proof of (82):

Let us first prove the following

*Statement:* \( Y^n_{\cdot}(\omega) \) converges for some subsequence, that for simplicity we denote still with \( Y^n_{\cdot}(\omega) \), \( ds \otimes P - \text{a.s.} \) to \( Y_{\cdot}(\omega) \) on \([0, T] \times \Omega \) (\( ds \) denoting the Lesbegues measure on \( \mathbb{R}_+ \)).

This is a consequence of Doob’s inequality applied to the sub -martingales \( |Z^n_t(\omega) - Z_t(\omega)|| \). (That \( |Z^n_t(\omega) - Z_t(\omega)|| \) is a sub -martingale follows from Proposition 3.11 and Lemma 8.11, Chapt. II, Part I, [21].) In fact,

\[ \sup_{t \in [0, T]} |Y^n_t(\omega) - Y_t(\omega)| = \sup_{t \in [0, T]} |Z^n_t(\omega) - Z_t(\omega)| \]

and if i) holds, then

\[ P(\sup_{t \in [0, T]} |Z^n_t(\omega) - Z_t(\omega)| \geq \epsilon) \leq \frac{1}{\epsilon} \int_0^T \int_A E|f_n(s, x, \omega) - f(s, x, \omega)|^2 \nu(dsdx) \]

while if ii) or iii) holds, then

\[ E[\sup_{t \in [0, T]} |Z^n_t(\omega) - Z_t(\omega)||^2] \leq \sup_{t \in [0, T]} E[|Z^n_t(\omega) - Z_t(\omega)||^2] \leq \int_0^T \int_A E|f_n(s, x) - f(s, x)|^2 \nu(dsdx) \to 0 \quad \text{when} \quad n \to \infty, \]

The *statement* is then a consequence of Lesbegue’s dominated convergence theorem. (82) follows then from inequality (44), which implies the existence of a constant \( R(\omega) \) depending on \( \omega \), such that

\[ \| \int_0^T \mathcal{H}(Y^n_{s-}) g(s-, \omega) dh(s, \omega) - \int_0^T \mathcal{H}(Y_{s-}) g(s-, \omega) dh(s, \omega) \| \]
\[ \leq R(\omega) \int_0^T \| Y^n_{s-} - Y_{s-} \| \| \sup_{s \in [0, T]} |g(s, \omega)| dh(s, \omega) \| \]

(87)
Suppose condition i) is satisfied. Let $C$ we obtain:

$$\int_t^T \int_A \{ \mathcal{H}(Y^n_s(\omega) + f_n(s, x, \omega)) - \mathcal{H}(Y^n_s(\omega)) - \mathcal{H}'(Y^n_s(\omega))f_n(s, x, \omega) \} \nu(dsdx)
- \{ \mathcal{H}(Y^n_s(\omega) + f(s, x, \omega)) - \mathcal{H}(Y^n_s(\omega)) - \mathcal{H}'(Y^n_s(\omega))f(s, x, \omega) \} \nu(dsdx)
\leq 4C \int_t^T \int_A \| f_n(s, x, \omega) - f(s, x, \omega) \| \nu(dsdx)$$

(88)

Since

$$\lim_{n \to \infty} E\left[ \int_0^T \int_A \| f_n(s, x, \omega) - f(s, x, \omega) \| \nu(dsdx) \right] = 0$$

by hypothesis, it follows that there is some subsequence, that we denote for simplicity again with $\{ f_n \}$, such that

$$\int_t^T \int_A \| f_n(s, x, \omega) - f(s, x, \omega) \| \nu(dsdx) \to 0 \text{ P-a.s. when } n \to \infty$$

(89)

Suppose condition ii), or iii) is satisfied (i.e. $p = 2$). By hypothesis $\{ f_n \}_{n \in \mathbb{N}}$ converges $\nu \otimes P$ - a.s. to $f$ on $[0, T] \times A \times \Omega$. Moreover $Y^n_s(\omega)$ converges for some subsequence $\nu \otimes P$ - a.s. to $Y_s(\omega)$. This is a consequence of the hypothesis $A$ that $\nu(dsdx) = \alpha(ds)\beta(dx)$ with $\alpha(ds) \prec ds$ and Doob’s inequality (86). It follows that there is a subsequence s.th. the integrand in (88) converges for some subsequence $\nu \otimes P$ - a.s. to zero when $n \to \infty$. Moreover using (46) and Remark 4.5 it follows that there is a constant $R(\omega)$ depending on $\omega \in \Omega$, s.th. for some subsequence

$$\int_t^T \int_A \| \mathcal{H}(Y^n_s(\omega) + f_n(s, x, \omega)) - \mathcal{H}(Y^n_s(\omega)) - \mathcal{H}'(Y^n_s(\omega))f_n(s, x, \omega) \| \nu(dsdx)
- \| \mathcal{H}(Y^n_s(\omega) + f(s, x, \omega)) - \mathcal{H}(Y^n_s(\omega)) - \mathcal{H}'(Y^n_s(\omega))f(s, x, \omega) \| \nu(dsdx)
\leq 4R(\omega) \int_t^T \int_A \| f_n(s, x, \omega) \|^2 + \| f(s, x, \omega) \|^2 \nu(dsdx)
\leq 16R(\omega) \int_t^T \int_A \| f_n(s, x, \omega) - f(s, x, \omega) \|^2 \nu(dsdx) + 16R(\omega) \int_t^T \int_A \| f(s, x, \omega) \|^2 \nu(dsdx)
\to 16R(\omega) \int_t^T \int_A \| f(s, x, \omega) \|^2 \nu(dsdx) \text{ P-a.s. when } n \to \infty$$

(90)

where convergence in (90) holds for some subsequence, by a similar argument as in (89). (83) is then a consequence of the Lebesgue dominated convergence theorem.

Proof of (84):

Let $p = 1$ if condition i) is satisfied, or $p = 2$ if condition ii) or iii) is satisfied.

From the Theorems 3.12-3.16 and inequality (45) it follows
\[ E\| \int_0^T \int_A \{ \mathcal{H}(Y^n_s \cdot (\omega) + f_n(s, x, \omega)) - \mathcal{H}(Y^n_{s-}(\omega)) \} q(dsdx)(\omega) - \int_0^T \int_A \{ \mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \} q(dsdx)(\omega) \| ^p \| \leq 2pK_p \int_0^T \int_A \| \{ \mathcal{H}(Y^n_s \cdot (\omega) + f_n(s, x, \omega)) - \mathcal{H}(Y^n_{s}(\omega)) \} \\
- \{ \mathcal{H}(Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(Y_{s-}(\omega)) \} \| ^p \| \nu(dsdx) \leq 4p^2K_p \int_0^T \int_A E[\| f_n(s, x, \omega) \| ^p + \| f(s, x, \omega) \| ^p \| \nu(dsdx) \\
\leq 8p^2K_p \int_0^T \int_A E[\| f_n(s, x) - f(s, x) \| ^p \| \nu(dsdx) + 8p^2K_p^2 \int_0^T \int_A E[\| f(s, x) \| ^p \| \nu(dsdx) \\
\to 8p^2K_p \int_0^T \int_A E[\| f(s, x) \| ^p \| \nu(dsdx) \text{ when } n \to \infty \tag{91} \]

where for \( p = 1, K_p = 1, \) while in case \( p = 2, K_p \) is the constant in the Definition 3.17 of \( M \)-type 2, if condition ii) holds, resp. \( K_p \) is the twice the constant in the Definition 3.19 of type 2 Banach space, if condition iii) holds. That (84) holds for some subsequence follows from the Lebesgue dominated convergence theorem and the fact that the integrand in the l.h.s. of (91) converges for some subsequence \( \nu \otimes P \) - a.s. to zero, on \([0, T] \times A \times \Omega. \tag{91} \]

5 The Ito formula for time dependent Banach valued functions

We shall consider in this article also the case where \( \mathcal{H} \) depends (continuously) on the time variable, too, i.e.

\[ \mathcal{H} : \mathbb{R}_+ \times F \to G \tag{92} \]

\( (s, y) \to G \)

We shall prove the Ito formula (3).

**Theorem 5.1** Suppose that hypothesis B (given in Theorem 4.1) is satisfied. Suppose that the Fréchet derivatives \( \partial_s \mathcal{H}(s, y) \) and \( \partial_y \mathcal{H}(s, y) \) exists and are uniformly bounded on \([\tau, t] \times F, \) and all the second Fréchet derivatives \( \partial_s \partial_y \mathcal{H}(s, y), \partial_y \partial_s \mathcal{H}(s, y), \partial_y \partial_y \mathcal{H}(s, y) \) and \( \partial_y \partial_y \mathcal{H}(s, y) \) exists and are uniformly bounded on \([\tau, t] \times B(0, R), \) for all \( R \geq 0. \) Then

1) \( \mathcal{H}(s, Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(s, Y_{s-}(\omega)) - \partial_y \mathcal{H}(s, \omega) f(s, x, \omega) \text{ is } P \text{-a.s. Bochner integrable w.r.t } \nu \text{ on } [0, T] \times A, \forall A \in \mathcal{B}(E \setminus \{0\}). \)

2) \( \mathcal{H}(s, Y_{s-}(\omega) + f(s, x, \omega)) - \mathcal{H}(s, Y_{s-}(\omega)) \in M^{T,1}_\nu(E/G), \text{ resp. } \in M^{T,2}_\nu(E/G), \text{ if i), resp. ii) or iii), is satisfied.} \)

3) \( \int_0^T \| \partial_y \mathcal{H}(s, Y_{s-})(\omega) g(s, \omega) \| d|h(s, \omega)| < \infty \text{ P -a.s.} \)
4) \[ \sum_{0 < s \leq t} \| \mathcal{H}(s, Y_s(\omega)) + g(s, \omega) \Delta h(s, \omega) - \mathcal{H}(s, Y_s(\omega)) \| < \infty \quad P \text{-a.s.} \]

5) \[ \sum_{0 < s \leq t} \| \partial_y \mathcal{H}(s, Y_s(\omega)) g(s, \omega) \Delta h(s, \omega) \| < \infty \quad P \text{-a.s.} \]

6) \[ \int_0^T \int_A \| \{ \mathcal{H}(s, Y_s(\omega)) + k(s, x, \omega) - \mathcal{H}(s, Y_s(\omega)) \} \| N(dsdx)(\omega) < \infty \quad P \text{-a.s.} \]

7) \[ \partial_y \mathcal{H}(s, Y_s(\omega)) \text{ is } P \text{-a.s. Bochner integrable w.r.t. the Lebesque measure as in } [\tau, t] \]

Moreover (3) holds \( P \text{-a.s.} \)

**Remark 5.2** The stochastic integral w.r.t. \( q(dsdx)(\omega) \) in (3) is a strong \(-p\) integral, with \( p = 1 \) if i) holds, resp. \( p = 2 \) if ii) or iii) holds. If \( f(\cdot, x, \omega) \) is left continuous on \((0, T] \), for all \( x \in A \) and \( P \text{-a.e. } \omega \in \Omega \) fixed, then it is a simple \(-p\) integral, too.

**Proof of Theorem 5.1:** The properties 1) -6) are proven like in Theorem 4.1. Property 7) is an obvious consequence of the fact that \( \partial_y \mathcal{H}(s, y) \) is uniformly bounded.

To prove that (3) holds \( P \text{-a.s.} \) we proceed in a similar way to the proof of the Theorems 4.4 and 4.6. We first make the assumption that \( f(\cdot, x, \omega) \) is left continuous on \((0, T] \), for all \( x \in A \) and \( P \text{-a.e. } \omega \in \Omega \) fixed, and that \( 0 \notin \overline{A} \), i.e. \( A \in \mathcal{F}(E \setminus \{0\}) \). Similar to the proof of Theorem 4.4 we obtain that for all \( \omega \in \Omega \),

\[
\mathcal{H}(t, Y_t(\omega)) - \mathcal{H}(\tau, Y_\tau(\omega)) = \sum_{k=0}^{n-1} \mathcal{H}(\tau_{k+1}^n, Y_{\tau_{k+1}^n}^n(\omega)) - \mathcal{H}(\tau_k^n, Y_{\tau_k^n}^n(\omega)) \\
= \sum_{k \in \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda)} \mathcal{H}(\tau_{k+1}^n, Y_{\tau_{k+1}^n}^n(\omega)) - \mathcal{H}(\tau_k^n, Y_{\tau_k^n}^n(\omega)) \\
+ \sum_{k \notin \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda)} \mathcal{H}(\tau_{k+1}^n, Y_{\tau_{k+1}^n}^n(\omega)) - \mathcal{H}(\tau_k^n, Y_{\tau_k^n}^n(\omega))
\]

(93)

where \( \Gamma_0^n(\Lambda) \), \( \Gamma_0^n(\Lambda) \) and \( \gamma_0^n \) are defined like in (54) -(57).

Similar to (59) it can be checked that

\[
\lim_{n \to -\infty} \sum_{k \in \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda) \cup \Gamma_0^n(\Lambda)} \mathcal{H}(\tau_{k+1}^n, Y_{\tau_{k+1}^n}^n(\omega)) - \mathcal{H}(\tau_k^n, Y_{\tau_k^n}^n(\omega)) \\
= I_1 \int_A \{ \mathcal{H}(s, Y_s(\omega)) + f(s, x, \omega) - \mathcal{H}(s, Y_s(\omega)) \} q(dsdx)(\omega) \\
+ I_2 \int_A \{ \mathcal{H}(s, Y_s(\omega)) + k(s, x, \omega) - \mathcal{H}(s, Y_s(\omega)) \} N(dsdx)(\omega) \\
+ \sum_{\tau < s \leq t} \mathcal{H}(s, Y_s(\omega)) + g(s, \omega) \Delta h(s, \omega) - \mathcal{H}(s, Y_s(\omega)) \\
+ \int_\tau^t \int_A \{ \mathcal{H}(s, Y_s(\omega)) + f(s, x, \omega) - \mathcal{H}(s, Y_s(\omega)) \} \nu(dsdx)
\]

(94)

\( P \text{-a.s.} \)

(3) follows once shown that for some subsequence of \( \{n\}_{n \in \mathbb{N}} \), that for simplicity we still denote with \( \{n\} \), the following convergence holds for \( n \to \infty \)
\[
\lim_{n \to \infty} \sum_{k \in \mathbb{Z}^2} \{ \mathcal{H}(\tau_{k+1}^n, Y_{\tau_{k+1}^n}(\omega)) - \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) \}
= \int_0^t \partial_s \mathcal{H}(s, Y_s(\omega)) ds - \int_0^t \int_A \partial_y \mathcal{H}(s, Y_s(\omega)) f(s, x, \omega) \nu(dsdx)
+ \int_0^t \partial_y \mathcal{H}(s, Y_s(\omega)) g(s, \omega) dh(s, \omega)
- \sum_{\tau < s \leq t} \{ \partial_y \mathcal{H}(s, Y_s(\omega)) g(s, \omega) \Delta h(s, \omega) \} \quad P - \text{a.s.} \quad (95)
\]

Proof of (95):
We make a Taylor expansion of the function \( \mathcal{H}(s, y) \):

\[
\sum_{k \in \mathbb{Z}^2} \partial_s \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega))(\tau_{k+1}^n - \tau_k^n) = \int_0^t \partial_s \mathcal{H}(s, Y_s(\omega)) ds \quad P - \text{a.s.} \quad (97)
\]

while

\[
\sum_{k \in \mathbb{Z}^2} \partial_y \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) (Y_{\tau_{k+1}^n}(\omega) - Y_{\tau_k^n}(\omega))
= \sum_{k \in \mathbb{Z}^2} \partial_y \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) (Z_{\tau_{k+1}^n}(\omega) - Z_{\tau_k^n}(\omega))
+ \sum_{k=1}^{n+1} \partial_y \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) (\int_{\tau_k^n}^{\tau_{k+1}^n} g(s, \omega) dh(s, \omega))
- \sum_{k \in \mathbb{Z}^2} \partial_y \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) (\int_{\tau_k^n}^{\tau_{k+1}^n} g(s, \omega) dh(s, \omega)) \quad (98)
\]

That

\[
\lim_{n \to \infty} \sum_{k \in \mathbb{Z}^2} \partial_y \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) (Z_{\tau_{k+1}^n}(\omega) - Z_{\tau_k^n}(\omega))
= - \int_0^t \int_A \partial_y \mathcal{H}(s, Y_s(\omega)) f(s, x, \omega) \nu(dsdx) \quad P - \text{a.e.} , \quad (99)
\]

is proven in a similar way as in (70). Moreover in a similar way as in the proof of (72) it can be shown that

\[
\sum_{k=1}^{n+1} \partial_y \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) (\int_{\tau_k^n}^{\tau_{k+1}^n} g(s, \omega) dh(s, \omega))
- \sum_{k \in \mathbb{Z}^2} \partial_y \mathcal{H}(\tau_k^n, Y_{\tau_k^n}(\omega)) (\int_{\tau_k^n}^{\tau_{k+1}^n} g(s, \omega) dh(s, \omega))
= \int_0^t \partial_y \mathcal{H}(s, Y_s(\omega)) g(s, \omega) dh(s, \omega)
- \sum_{\tau < s \leq t} \{ \partial_y \mathcal{H}(s, Y_s(\omega)) g(s, \omega) \Delta h(s, \omega) \} \quad P - \text{a.s.} \quad (100)
\]
(95) is proven, once shown that
\[
\lim_{n \to \infty} \sum_{k \in \mathbb{Z}_+} e r^n_k(\omega) = 0 \quad P - a.e. \tag{101}
\]
The proof of (101) is similar to the proof of (62), however, instead of getting inequality (63), we get
\[
\|e r^n_k(\omega)\| \leq C(\omega)(\|Y_{\tau_{k+1}}^n(\omega) - Y_{\tau_k}^n(\omega)\|^2 + |\tau_{k+1}^n - \tau_k^n|^2), \tag{102}
\]
which is obtained in a similar way as (62), however, by the following estimates
\[
\|\mathcal{H}(s, y) - \mathcal{H}(s_0, y_0) - \partial_s \mathcal{H}(s_0, y_0)(s - s_0) - \partial_y \mathcal{H}(s_0, y_0)(y - y_0)\| \leq
\]
\[
\|\mathcal{H}(s, y) - \mathcal{H}(s, y_0) - \partial_y \mathcal{H}(s, y_0)(y - y_0)\| +
\]
\[
\|\partial_y \mathcal{H}(s, y_0)(y - y_0) - \partial_y \mathcal{H}(s_0, y_0)(y - y_0)\| \leq
\]
\[
\sup_{0 < \theta \leq 1} \|\partial_y \mathcal{H}(s, y_0 + \theta(y - y_0)) - \partial_y \mathcal{H}(s, y_0)\| \|y - y_0\| +
\]
\[
\|\partial_y \mathcal{H}(s, y_0)(y - y_0) - \partial_y \mathcal{H}(s_0, y_0)(y - y_0)\| \leq
\]
\[
\sup_{0 < \theta \leq 1} \|\partial_y \partial_s \mathcal{H}(s_0 + \theta(s - s_0), y_0)\| |s - s_0|^2 +
\]
\[
\sup_{0 < \theta \leq 1} \|\partial_y \partial_y \mathcal{H}(s_0 + \theta(y - y_0), y_0)\| |y - y_0|^2 +
\]
\[
\sup_{s \in [\tau, t]} \|\partial_y \partial_y \mathcal{H}(s, y_0)\| |s - s_0| |y - y_0|
\]
\[
(103)
\]
(101) follows then from (69),(102) and
\[
\sum_{k=0}^{2^n-1} |\tau_{k+1}^n - \tau_k^n|^2 \leq \frac{(t - \tau)^2}{2^n} \tag{104}
\]
The rest of the proof of Theorem 5.1 is done by leaving out the hypothesis \(A \in \mathcal{F}(E \setminus \{0\})\) and the hypothesis that \(f\) is left continuous in the time variable.
This is done similar to the proof of the Theorems 4.1 and Theorem 4.4. ■

6 The Ito formula for the strong integrals of type p

We now prove the Ito formula for the strong integrals of type p on separable Banach spaces. Instead of assuming that i) or ii) or iii) is satisfied, we assume that either i') in Proposition 3.32, or ii') or iii') in Proposition 3.33 is satisfied: \(Z_t(\omega)\) in (1) is then the strong integral of type p, with \(p = 1\), if i') holds, and \(p = 2\), if ii') or iii') holds. We analyze the Ito -formula for the \(F\) -valued random process \((Y_t)_{t \geq 0}\) in (35). We assume again that the assumptions (36) - (40) hold, and prove that the Ito formula (3) holds. We prove in fact the following two theorems:
Theorem 6.1 We make the following hypothesis C: (At least) one of the hypothesis i’) - iii’) is satisfied. \( G \) is a separable Banach space. If ii’) or iii’) is satisfied, we make the additional hypothesis that \( G \) is a separable Banach space of \( M \)-type 2.

Let

\[
\mathcal{H} : \mathbb{R}_+ \times F \to G
\]

\[
(s, y) \to G
\] (105)

Suppose that the Fréchet derivatives \( \partial_s \mathcal{H}(s, y) \) and \( \partial_y \mathcal{H}(s, y) \) exists and are uniformly bounded on \([\tau, t] \times F\), and all the second Fréchet derivatives \( \partial_s \partial_y \mathcal{H}(s, y), \partial_y \partial_s \mathcal{H}(s, y) \) and \( \partial_y \partial_y \mathcal{H}(s, y) \) exists and are uniformly bounded on \([\tau, t] \times B(0, R)\), for all \( R \geq 0 \).

Then the properties 1),and 3)-7) in Theorem 5.1 hold. Moreover

2’) \( \mathcal{H}(s, Y_s(\omega)+ f(s, x, \omega)) - \mathcal{H}(s, Y_s(\omega)) \in N^{T,1}_\nu(E/G), \) resp. \( N^{T,2}_\nu(E/G), \) if i’), resp. ii’) or iii’), is satisfied.

Proof of Theorem 6.1: Suppose i’), resp. ii’) or iii’), is satisfied. 1) and 3) -7) are proven like in Theorem 5.1. 2’) is proven in a similar way as in Theorem 5.1, by using (43).

Theorem 6.2 Suppose that the same hypothesis as in Theorem 6.1 hold, then

\( \mathcal{H}(s, Y_s(\omega)+ f(s, x, \omega)) - \mathcal{H}(s, Y_s(\omega)) \) is strong integrable on \([0, T] \times A\), with strong integral of type 1, resp. strong integral of type 2, if condition i’), resp. ii’) or iii’), is satisfied. (3) holds \( P \)-a.s..

Proof of Theorem 6.2: The statement about the strong integrability follows from Proposition 3.32, Proposition 3.33 and Theorem 6.1. Let \( p = 1 \), if i’) holds, and \( p = 2 \), if ii’) or iii’) holds. Let \( f_n(s, x, \omega) := f(s, x, \omega) \mathbb{1}_{\|f\|_p \leq n}. \) From Theorem 4.6 it follows that the Ito formula (3) holds for \( Y^n_t(\omega) \) defined like in (76), where for this case \( Z^n_t \) denotes the strong -p- integral of \( f_n \), with \( p = 1 \), resp. \( p = 2 \). As

\[
\int_0^T \int_A \|f_n\|^p \, dv \leq \int_0^T \int_A \|f\|^p \, dv \quad P-\text{a.s.}
\]

and \( f_n \) satisfies property Q in Theorem 3.34, it follows from Theorem 3.35 that (77)- (78) hold for some subsequence. Moreover, it follows

\[
\lim_{n \to \infty} \mathcal{H}(t, Y^n_t(\omega)) - \mathcal{H}(\tau, Y^n_\tau(\omega)) = \mathcal{H}(t, Y_t(\omega)) - \mathcal{H}(\tau, Y_\tau(\omega)) \quad P-\text{a.s. for all } 0 \leq \tau \leq t \leq T,
\]

(106)

\[
\lim_{n \to \infty} \int_\tau^t \int_A \{\mathcal{H}(s, Y^n_s(\omega) + k(s, x, \omega)) - \mathcal{H}(s, Y^n_s(\omega))\} N(dsdx)(\omega)
\]

\[
= \int_\tau^t \int_A \{\mathcal{H}(s, Y_s(\omega) + k(s, x, \omega)) - \mathcal{H}(s, Y_s(\omega))\} N(dsdx)(\omega)
\]

\( P-\text{a.s. for all } 0 \leq \tau \leq t \leq T \),

(107)
Moreover the following property holds

\[
\lim_{n \to \infty} \sum_{\tau<s \leq t} \{ \mathcal{H}(s,Y^n_{s-}(\omega)+g(s_-,\omega)\Delta h(s,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega)) \} - \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))g(s_-,\omega)\Delta h(s,\omega) \}
\]

\[
= \sum_{\tau<s \leq t} \{ \mathcal{H}(s,Y^n_{s-}(\omega)+g(s_-,\omega)\Delta h(s,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega)) \} - \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))g(s_-,\omega)\Delta h(s,\omega) \}
\]

\[
P \text{ a.s. for all } 0 \leq \tau \leq t \leq T.
\]  

(108)

In a similar way, as in the proof of Theorem 4.6, where it has been shown that (82)-(83) hold for some subsequence, it can be proven that the properties (109)-(110) below hold. We leave the proof as an exercise.

\[
\lim_{n \to \infty} \int_\tau^t \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))g(s_-,\omega)dh(s,\omega) = \int_\tau^t \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))g(s_-,\omega)dh(s,\omega)
\]

\[
P \text{ a.s.}
\]  

(109)

\[
\lim_{n \to \infty} \int_\tau^t \int_A \{ \mathcal{H}(s,Y^n_{s-}(\omega)+f_n(s,x,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega)) \} - \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))f_n(s,x,\omega) \} \nu(dsdx)
\]

\[
= \int_\tau^t \int_A \{ \mathcal{H}(s,Y^n_{s-}(\omega)+f(s,x,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega)) \} - \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))f(s,x,\omega) \} \nu(dsdx)
\]

\[
P \text{ a.s.}
\]  

(110)

Moreover the following property holds

\[
\lim_{n \to \infty} \int_\tau^t \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))ds = \int_\tau^t \partial_y \mathcal{H}(s,Y^n_{s-}(\omega))ds
\]

\[
P \text{ a.s.}
\]  

(111)

The whole statement in Theorem 6.2 is then proven, once shown that

\[
\lim_{n \to \infty} \int_\tau^t \int_A \{ \mathcal{H}(s,Y^n_{s-}(\omega)+f_n(s,x,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega)) \} q(dsdx)(\omega)
\]

\[
= \int_\tau^t \int_A \{ \mathcal{H}(s,Y^n_{s-}(\omega)+f(s,x,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega)) \} q(dsdx)(\omega)
\]

\[
P \text{ a.s.}
\]  

(112)

\textbf{Proof of (112):}

As

\[
\int_0^T \int_A \| \mathcal{H}(s,Y^n_{s-}(\omega)+f_n(s,x,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega)) \|^p \nu(dsdx) \leq C \int_0^T \int_A \| f(s,x,\omega) \|^p \nu(dsdx)
\]

with \(C := \sup_{y \in F} ||\partial_y \mathcal{H}(s,y)||\), and as \(\mathcal{H}(s,Y^n_{s-}(\omega)+f_n(s,x,\omega)) - \mathcal{H}(s,Y^n_{s-}(\omega))\) satisfies property Q in Theorem 3.34, it follows that (112) holds.

\textbf{Acknowledgements} We thank Luciano Tubaro for many useful discussions related to this work and Sergio Albeverio for useful comments. The support and hospitality of the "Sonderforschungsbereich" SFB 611 in Bonn, as well as the Mathematics Department of the University of Trento, is gratefully acknowledged.
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Bestellungen nimmt entgegen:

Institut für Angewandte Mathematik
der Universität Bonn
Sonderforschungsbereich 611
Wegelerstr. 6
D - 53115 Bonn

Telefon: 0228/73 3411
Telefax: 0228/73 7864
E-mail: anke@iam.uni-bonn.de
Homepage: http://www.iam.uni-bonn.de/sfb611/

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