Some Properties of Dirichlet *L*-Functions Associated with their Nontrivial Zeros I.

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SOME PROPERTIES OF DIRICHLET L-FUNCTIONS ASSOCIATED WITH THEIR NONTRIVIAL ZEROS I.

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ABSTRACT. New results associated with the Extended Riemann Hypothesis on the zeros of the Dirichlet L-functions are obtained. The presentation of our work consists of two parts. In the present first part the connection between values of L-functions and Gauss sums is studied. This leads to a sufficient condition for the value $s = \frac{1}{2}$ to be a zero of a given L-function. A necessary condition for the validity of the Extended Riemann Hypothesis is found. This involves the signs of the even derivatives of the analogue $\xi(s, \chi)$ of the well-known $\xi(s)$ function associated with the Riemann zeta-function. It is also proved that if the absolute value of a real even primitive character χ does not exceed the value 15 or the absolute value of a real odd primitive character χ does not exceed the value 7, then the value of the corresponding L-function and all its even derivatives at the point $s = \frac{1}{2}$ are positive. In the second part of the presentation asymptotic formulas will be given for the even derivatives of the function $\xi(s, \chi)$ at $s = \frac{1}{2}$ as the order of the derivative tends to infinity.

CONTENTS

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1. Introduction

In recent years, new results concerning the Riemann hypothesis on the zeros of the classical Riemann zeta function $\zeta(s)$ have been obtained (for background see, e.g., ([3],[9],[10],[23],[24],[25]). Some of these results concern new zero free regions [4] other results concern new relations of the Riemann hypothesis with other conjectures or results from other areas, see e.g. [2], [3], [5], [22] (and references therein). Other types of recent results are related to the work of the second author of this paper. These results can be divided into two groups. The results relating to the first group are associated with the construction of an operator acting in a Hilbert space such that the Riemann problem is equivalent to the problem of the existence of an eigenvector with eigenvalue $\lambda = -1$ for this operator ([13],[18]). The results relating to the second group ([14] – [17]) are associated with the behavior of the Riemann ξ-function $\xi(s) = \frac{1}{2}s(s-1)\,\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ and its derivatives at the point $s = \frac{1}{2}$. Because of the equality $\xi(s) = \xi(1-s)$ all the odd derivatives of $\xi(s)$ at $s = \frac{1}{2}$ are equal to zero. It turned out that the signs of the even derivatives of $\xi(s)$ at $s = \frac{1}{2}$ plays an important role in the Riemann hypothesis. Namely, if at least one even derivative of the function $\xi(s)$ at the point $s = \frac{1}{2}$ were not positive, the Riemann hypothesis on the zeros of $\zeta(s)$ would be false: in this case there would exist a complex zero of $\zeta(s)$ that does not lie on the line Re $s = \frac{1}{2}$ ([14], Theorem 2). Further, however, it was proved ([14], Theorem 1), that all the even derivatives of $\xi(s)$ at the point $s = \frac{1}{2}$ are strictly positive. Moreover, an asymptotics for values of the even derivatives at $s = \frac{1}{2}$ as the order of the derivative tends to infinity was found ([15], [16]). These results permitted to show that the analogue of the Riemann hypothesis does not hold for an arbitrarily sharp approximation of $\zeta(s)$ satisfying the same functional equation as the function $\zeta(s)$ ([17]). This approximation has the same unique pole at $s = 1$ and the real (trivial) zeros at negative even values of s, and assumes real values for real values of s [10].

The aim of the present work is to generalize the mentioned results relating to the values of $\zeta(s)$ and its derivatives at $s = \frac{1}{2}$ to the case of Dirichlet L-functions $L(s, \chi)$, where χ is the Dirichlet character. As for previous work on the zeros of L-functions see, e.g., [6],[7],[8],[11],[12],[19],[20],[21],[22].

The present work consists of two parts. In the first part (which constitutes the present paper) the connection between the values of L-functions and the Gauss sums is studied, which leads to a sufficient condition for the validity of the equality $L(\frac{1}{2}, \chi) = 0$. A necessary condition for the validity of the Extended Riemann Hypothesis for L-function is also found. This involves an analysis of the signs of the even derivatives of the analogue $\xi(s, \chi)$ of the function Riemann ξ-function $\xi(s)$. It is also proved that if the modulo of a real even primitive character χ does not exceed the value 15 or the modulo of a real odd primitive character χ does not exceed the value 7, then the values of the corresponding L-functions $L(s, \chi)$ and all its even derivatives at $s = \frac{1}{2}$ are positive. In the second part of our study [1] asymptotic formulas for the even derivatives of the function $\xi(s, \chi)$ at $s = \frac{1}{2}$ as the order of the derivatives tends to infinity are found. As a consequence of these asymptotic formulas we obtain that all these even derivatives of sufficiently large order are positive. These results are then applied to the case of L-functions with characters which are given by a Legendre symbol.

2. The connection between Dirichlet L-Functions and Gauss sums

In this section we recall the main definition and the basic facts used later in the paper concerning L-functions and Gauss sums. For more details see, e.g., [3],[9],[10].

Definition 1. *A character $\chi(n)$, $n \in \mathbb{Z}$ modulo $k \in \mathbb{N}, k \geqslant 2$ is by definition a function defined on all integer numbers that satisfies the following properties:*

1) *$\chi(n)$ is not identically zero and is periodic with period k; moreover, $\chi(n) = 0$, if the greatest common divisor (n, k) of n and k is strictly larger than 1, and $\chi(n) \neq 0$, if $(n, k) = 1$.*
2) *$\chi(n)$ is completely multiplicative, that is $\chi(nm) = \chi(n)\chi(m)$ for all integers n and m.*

In the following we shall speak shortly of characters instead of characters modulo k.

Definition 2. *The principal character, denoted $\chi_0(n)$, is the character that is equal to 1 whenever $(n, k) = 1$.*

Definition 3. *A non-principal character $\chi(n)$ modulo k is called primitive if its smallest period is equal to k. All other non-principal characters modulo k are called imprimitive.*

Definition 4. *A character χ is called real if $\chi(n)$ is a real number, for all $n \in \mathbb{Z}$. A character that takes complex values is called complex, and the character whose value at n is the complex conjugate of $\chi(n)$ is called the complex conjugate of χ and is denoted $\bar{\chi}$.*

Because of the multiplicativity of characters, we have

$$\chi^2(-1) = 1, \qquad \text{i.e. } \chi(-1) = \pm 1 \ .$$

Definition 5. *A character $\chi(n)$ for which $\chi(-1) = +1$ is called even and a character for which $\chi(-1) = -1$ is called odd.*

Example. *Let $p > 2$ be a prime number. We consider the real primitive character $\chi(n) = \left(\frac{n}{p}\right)$, where $\left(\frac{n}{p}\right)$ is the Legendre symbol: this symbol is equal to 1 if n is a quadratic residue modulo p, it is equal to -1 if n is a non-quadratic residue modulo p, and it is equal to 0 if d is divisible by p.*

Definition 6. *If $\chi(n)$ is a character modulo k one calls Gauss sum $\tau(\chi)$ (associated with χ) the value*

$$\tau(\chi) = \sum_{a=1}^{k} \chi(a)\ e^{2\pi i \frac{a}{k}}$$

Definition 7. *Let k be a natural number, and let χ be a character modulo k. An L-function is a series of the form*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\ , \qquad \mathbb{R}e\ s > 1 \quad .$$

Lemma 1. *Let $\chi(n) = \chi_0(n)$ be a principal character modulo k. Then for $\mathbb{R}e\ s > 1$*

$$L(s, \chi_0) = \zeta(s) \prod_{p \backslash k} \left(1 - \frac{1}{p^s}\right).$$

Proof. See, e.g. [10].

Lemma 2. *Let $\chi(n)$ be a primitive character modulo k, set $\delta \equiv \begin{cases} 0, & if\ \chi(-1) = +1 \\ 1, & if\ \chi(-1) = -1 \end{cases}$, and*

$$\xi(s, \chi) = \left(\frac{\pi}{k}\right)^{-\frac{s+\delta}{2}} \Gamma\left(\frac{s+\delta}{2}\right) L(s, \chi).$$

Then

$$\xi(1 - s, \bar{\chi}) = \frac{i^\delta \sqrt{k}}{\tau(\chi)} \xi(s, \chi) \quad , \tag{2.1}$$

where $\tau(\chi)$ is the Gauss sum.

Proof. See, e.g. [10].

Theorem 1. *Let χ be a real primitive character modulo $k > 2$ such that $L(\frac{1}{2}, \chi) \neq 0$. Then*

1) *if k is odd and $\chi(-1) = 1$, then*
$$\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \cos \frac{2\pi a}{k} = \frac{\sqrt{k}}{2} \ ;$$
2) *if k is even and $\chi(-1) = 1$, then*
$$\sum_{a=1}^{\frac{k}{2}-1} \chi(a) \cos \frac{2\pi a}{k} - \frac{1}{2}\chi(\frac{k}{2}) = \frac{\sqrt{k}}{2} \ ;$$
3) *if k is odd and $\chi(-1) = -1$, then*
$$\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \sin \frac{2\pi a}{k} = \frac{\sqrt{k}}{2} \ ;$$
4) *if k is even and $\chi(-1) = -1$, then*
$$\sum_{a=1}^{\frac{k}{2}-1} \chi(a) \sin \frac{2\pi a}{k} = \frac{\sqrt{k}}{2} \ .$$

Proof. The proof uses the following Lemma 3:

Lemma 3. *Let χ be as in Theorem 1. If $\chi(-1) = 1$, then $\tau(\chi) = \sqrt{k}$, and if $\chi(-1) = -1$, then $\tau(\chi) = i\sqrt{k}$.*

Proof. **of Lemma 3.** Substituting the value $s = \frac{1}{2}$ in (2.1) and using the relations $\chi = \bar{\chi}$, $L\left(\frac{1}{2}, \chi\right) \neq 0$, we obtain the equality $\tau(\chi) = i^{\delta}\sqrt{k}$, which proves Lemma 3.

Let $\chi(-1) = 1$. Then for any integer a the equality $\chi(a) = \chi(-a)$ holds. From definition 6 it follows that, if k is odd, then

$$\tau(\chi) = 2\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \cos \frac{2\pi a}{k} \ , \tag{2.2}$$

and, if k is even, then

$$\tau(\chi) = 2\sum_{a=1}^{\frac{k}{2}-1} \chi(a) \cos \frac{2\pi a}{k} - \chi\left(\frac{k}{2}\right) . \tag{2.3}$$

Hence, the statements 1) and 2) of Theorem 1 follow from (2.2), (2.3) and Lemma 3. Let $\chi(-1) = -1$. Then for any integer a the equality $\chi(a) = -\chi(-a)$ holds, and, if k is odd, then

$$\tau(\chi) = 2i \sum_{a=1}^{\frac{k-1}{2}} \chi(a) \sin \frac{2\pi a}{k} \,, \tag{2.4}$$

and, if k is even, then

$$\tau(\chi) = 2i \sum_{a=1}^{\frac{k}{2}-1} \chi(a) \sin \frac{2\pi a}{k}. \tag{2.5}$$

The statements 3) and 4) of Theorem 1 follow now from (2.4), (2.5) and Lemma 3. Theorem 1 is thus proved. ∎

Corollary 1. *Assume that χ is real primitive character modulo $k > 2$ and at least one following condition holds:*

1') *if k odd, $\chi(-1) = 1$, then*

$$\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \cos \frac{2\pi a}{k} \neq \frac{\sqrt{k}}{2} \;;$$

2') *if k is even, $\chi(-1) = 1$, then*

$$\sum_{a=1}^{\frac{k}{2}-1} \chi(a) \cos \frac{2\pi a}{k} - \frac{1}{2}\,\chi\left(\frac{k}{2}\right) \neq \frac{\sqrt{k}}{2} \;;$$

3') *if k is odd, $\chi(-1) = -1$, then*

$$\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \sin \frac{2\pi a}{k} \neq \frac{\sqrt{k}}{2} \;;$$

4') *if k is even, $\chi(-1) = -1$, then*

$$\sum_{a=1}^{\frac{k}{2}-1} \chi(a) \sin \frac{2\pi a}{k} \neq \frac{\sqrt{k}}{2} \;.$$

Then the value $s = \frac{1}{2}$ is a zero of the function $L(s, \chi)$.

Remark. *For a given real primitive character conditions 1') – 4') can be checked numerically.*

Theorem 2. *Let χ be a complex primitive character modulo $k>2$ such that*

$$L\left(\frac{1}{2},\chi\right) \overset{def}{=} |L\left(\frac{1}{2},\chi\right)|\ e^{i\psi(\chi)} \neq 0\ .$$

Then ,

1) *if $\chi(-1)=1$, then*

$$\tau(\chi)=e^{2i\psi(\chi)}\sqrt{k}\ ;$$

2) *if $\chi(-1)=-1$, then*

$$\tau(\chi)=e^{(2\psi(\chi)+\frac{\pi}{2})i}\sqrt{k}\ .$$

Proof. From the definitions of the functions $L(s,\chi)$ and $\xi(s,\chi)$ (see Lemma 2) it follows that $L(\frac{1}{2},\chi) = \overline{L(\frac{1}{2},\overline{\chi})}$ and $\frac{\xi(\frac{1}{2},\chi)}{\xi(\frac{1}{2},\overline{\chi})} = e^{2i\psi(\chi)}$. Hence, substituting the value $s=\frac{1}{2}$ in (2.1), we obtain the statements of Theorem 2. ∎

3. Necessary conditions for the validity of the Extended Riemann Hypothesis for L-functions

Theorem 3. *Set*

$$\xi_0(s,\chi) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi)\zeta_k(s)\ ,\ \text{where } \zeta_k(s) = \prod_{p\backslash k}\left(1-\frac{1}{p^s}\right)^{-1}.$$

Let $\chi_0(n)$ be a principal character modulo k. Then the equality

$$\xi_0(s,\chi_0) = \xi_0(1-s,\chi_0) \tag{3.1}$$

holds, moreover, if at least one even derivative of the function $\xi_0(s,\chi_0)$ at the point $s=\frac{1}{2}$ were not positive, then the Extended Riemann Hypothesis would be false: in this case there would exist a complex zero of $L(s,\chi_0)$ that does not lie on the line $\mathbb{R}e\ s = \frac{1}{2}$.

Proof. By Lemma 1 the equality $\xi_0(s,\chi_0) = \xi(s)$ holds, where $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, $\zeta(s)$ is the classical Riemann zeta-function. Hence, the equality 3.1 follows from the equality $\xi(s)=\xi(1-s)$, and the statement of Theorem 3 on even derivatives is a corollary of Theorem 2 from [14] on even derivatives of the function $\xi(s)$. Theorem 3 is proved. ∎

Theorem 4. *Let χ be a real primitive character modulo $k>2$ such that $L\big(\frac{1}{2},\chi\big)\neq 0$,* $\xi(s,\chi) = \left(\frac{\pi}{k}\right)^{-\frac{s+\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi)$, *where* $\delta = \begin{cases} 0, & if\ \chi(-1)=+1\ ; \\ 1, & if\ \chi(-1)=-1\ . \end{cases}$
Then the equality

$$\xi(s,\chi) = \xi(1-s,\chi) \tag{3.2}$$

holds, and if the sign of at least one even derivative of the function $\xi(s,\chi)$ at the point $s=\frac{1}{2}$ differs from the sign of the number $L\big(\frac{1}{2},\chi\big)$, then the Extended Riemann Hypothesis is false: in this case there exists a nontrivial zero of $L(s,\chi)$ that does not lie on the line $\mathbb{R}\mathrm{e}\ s=\frac{1}{2}$.

Proof. From the fact that the character χ is real and primitive, by Lemma 2 we have:

$$\xi(1-s,\chi)=\frac{i^{\delta}\sqrt{k}}{\tau(\chi)}\xi(s,\chi) \quad , \tag{3.3}$$

and since $L\left(\frac{1}{2},\chi\right)\neq 0$, then substituting $s=\frac{1}{2}$ in (3.3), we obtain the equality $\frac{i^{\delta}\sqrt{k}}{\tau(\chi)}=1$. Hence, (3.2) follows from (3.3). We consider the function $\Xi(t,\chi)=\xi\left(\frac{1}{2}+it,\chi\right)$ of a complex variable t. By (3.2) $\Xi(t,\chi)$ is an even entire function of first order taking real values for real values of t, and any nontrivial zero z of $L(s,\chi)$ has the form $z=\frac{1}{2}+i\rho$, where ρ is any zero of the function $\Xi(t,\chi)$. Therefore it follows from Weierstrass formula for entire functions that $\Xi(t,\chi)$ can be represented in the form

$$\Xi(t,\chi)=\pm e^{\alpha+\beta t}\prod_{n=1}^{\infty}\left(1-\frac{t}{\rho_n}\right)e^{\frac{t}{\rho_n}} \ , \tag{3.4}$$

where α and β are constants and the numbers $\rho_n (n\in\mathbb{N})$ run over all zeros of $\Xi(t,\chi)$. Since if ρ_n is a zero also $-\rho_n$ is a zero of $\Xi(t,\chi)$, the assumption that the Extended Riemann Hypothesis is satisfied implies by (3.4) and the evenness of $\Xi(t,\chi)$ that

$$\Xi(t,\chi)=\pm e^{\alpha}\prod_{n=1}\left(1-\frac{t^2}{\tilde{\rho}_n^2}\right) \ , \tag{3.5}$$

where the product extends over all positive zeros $\tilde{\rho}_n$ of $\Xi(t,\chi)$. If we now expand (3.5) in a Taylor series $\Xi(t)=\sum_{k=0}^{\infty}c_k t^{2k}$ at the point $t=0$, then it will follow from (3.5) that for al integers $\mathrm{v}\geqslant 0$ the adjacent coefficients c_{v} and $c_{\mathrm{v}+1}$ have opposite signs, and according to the definition of $\Xi(t,\chi)$ this means that all the even derivatives of $\xi(s,\chi)$ at the point $s=\frac{1}{2}$ have the same sign as $\xi\left(\frac{1}{2},\chi\right)$, which according to the condition of Theorem 4 is not equal to zero. Theorem 4 is proved. ∎

4. Estimates of the value of the function $\xi(s,\chi)$ and its derivatives at the point $s=\frac{1}{2}$

In this section we consider only real primitive characters modulo k such that $L(\frac{1}{2},\chi)\neq 0$. In this case for $\delta=\begin{cases}0, & if\ \chi(-1)=+1\\ 1, & if\ \chi(-1)=-1\end{cases}$ the function $\xi(s,\chi)=\left(\frac{\pi}{k}\right)^{-\frac{s+\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi)$ satisfies the equality $\xi(s,\chi)=\xi(1-s,\chi)$, and therefore all odd derivatives of $\xi(s,\chi)$ at the point $s=\frac{1}{2}$ are equal to zero.

Theorem 5. *If χ is an even character modulo $k\leqslant 15$, then the value of $\xi(s,\chi)$ and all its even derivatives at the point $s=\frac{1}{2}$ are positive.*

Proof. We first state and prove Lemma 4.

Lemma 4. *If $k\leqslant 15$, then for all $x\geqslant 1$ the inequality*

$$\sum_{n=1}^{\infty}\chi(n)e^{-\frac{n^2\pi x}{k}}>0$$

holds.

Proof. **of Lemma 4** Let $\kappa=\frac{\pi x}{k}$, $\delta=5\kappa$, $\omega=e^{-\delta}$. Then for $n\geqslant 2$ the inequality

$$\frac{e^{-\kappa(n+1)^2}}{e^{-\kappa n^2}}=\exp(-\kappa(2n+1))\leqslant e^{-\delta}$$

holds, and, consequently,

$$\sum_{n=2}^{\infty}e^{-\frac{n^2\pi x}{k}}<\frac{e^{-\frac{4\pi x}{k}}}{1-\exp\left(-\frac{5\pi x}{k}\right)}=\frac{e^{-\frac{4\pi x}{k}}}{1-\omega}\ . \tag{4.1}$$

We have:

$$-\ln(1-\omega)=\sum_{\nu=1}^{\infty}\frac{\omega^{\nu}}{\nu!}<\frac{\omega}{1-\omega}=\frac{1}{e^{\delta}-1}\ .$$

Hence, by (4.1)

$$\ln\left(e^{\frac{-\pi x}{k}}\right)-\ln\sum_{n=2}^{\infty}e^{-\frac{n^2\pi x}{k}}>\frac{3\delta}{5}-\ln(1-\omega)=\frac{3\delta}{5}-\frac{1}{e^{\delta}-1}>0\ , \tag{4.2}$$

if $\delta\geqslant 1$. Since, according to the condition $k\leqslant 15$ for $x\geqslant 1$ the inequality $\delta=\frac{5\pi x}{k}>1$ holds, then by 4.2 and by relations $\chi(1)=1$, $|\chi(n)|\leqslant 1$

we have:

$$\sum_{n=1}^{\infty} \chi(n) e^{-\frac{n^2\pi x}{k}} > e^{-\frac{\pi x}{k}} - \sum_{n=2}^{\infty} e^{-\frac{n^2\pi x}{k}} > 0 \ .$$

Lemma 4 is proved. ∎

According to the well-known equality ([10])

$$\xi(s,\chi) = \frac{1}{2}\int_1^{\infty} x^{\frac{s}{2}-1}\theta(x,\chi)dx + \frac{1}{2}\frac{\sqrt{k}}{\tau(\chi)}\int_1^{\infty} x^{\frac{s}{2}-\frac{1}{2}}\theta(x,\chi)dx \ ,$$

where χ is even primitive character modulo k and

$$\theta(x,\chi) = 2\sum_{n=1}^{\infty} \chi(n) e^{-\frac{n^2\pi x}{k}} \ , \tag{4.3}$$

and, consequently, by Lemma 3 we have:

$$\xi(s,\chi) = \frac{1}{2}\int_1^{\infty} \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} \right) \theta(x,\chi)dx \ . \tag{4.4}$$

By dominated convergence we have then, for any even number $m \geqslant 2$:

$$\frac{d^m\xi}{ds^m}\left(\frac{1}{2},\chi\right) = \left(\frac{1}{2}\right)^m \int_1^{\infty} (\ln^m x)\, x^{-\frac{3}{4}}\theta(x,\chi)dx \ . \tag{4.5}$$

Now the statements of theorem 5 follow from (4.3)-(4.5) and Lemma 4. Theorem 5 is proved. ∎

Theorem 6. *If χ is odd character modulo $k \leqslant 7$, then the value of the function $\xi(s,\chi)$ and all its even derivatives at the point $s = \frac{1}{2}$ are positive.*

Proof. We first state and prove the following :

Lemma 5. *If $k \leqslant 7$, then for all $x \geqslant 1$ the inequality $\sum_{n=1}^{\infty} n\chi(n)e^{-\frac{n^2\pi x}{k}} > 0$ holds.*

Proof. **of Lemma 5.** Let $\kappa = \frac{\pi x}{k}$, $\delta = 5\kappa$, $\varepsilon = \ln\frac{3}{2}$. For $n \geqslant 2$ we obtain:

$$\frac{(n+1)e^{-\kappa(n+1)^2}}{ne^{-\kappa n^2}} < \frac{3}{2}\exp(-\delta) = \exp\left(-\delta+\varepsilon\right) \ ,$$

and, consequently,

$$\sum_{n=2}^{\infty} ne^{-\frac{n^2\pi x}{k}} < \frac{2e^{-\frac{4\pi x}{k}}}{1-\exp(-\delta+\varepsilon)} \ . \tag{4.6}$$

We have:

$$-\ln(1-\exp(-\delta+\varepsilon)) < \frac{1}{\exp(\delta-\varepsilon)-1} \ .$$

Hence, by (4.6),

$$\ln\left(e^{-\frac{\pi x}{k}}\right) - \ln\sum_{n=2}^{\infty} ne^{-\frac{n^2\pi x}{k}} \ > \ \frac{3\delta}{5} - \ln 2 - \frac{1}{\exp(\delta-\varepsilon)-1} \ . \tag{4.7}$$

We prove, that, if $\delta \geqslant 2$, then the right-hand side in (4.7) is greater then zero. To this end, we use the estimates

$$\ln 2 < 0,7 \ , \qquad \varepsilon = \ln\frac{3}{2} < 0,41 \ . \tag{4.8}$$

Since $\delta \geqslant 2$, then, by (4.8), we have:

$$\frac{5}{3\delta - 5\ln 2} < 2 \ , \ \text{and} \qquad \exp(\delta-\varepsilon) > e^{1,59} - 1 > 3,8 \ .$$

Hence, $\frac{3\delta-5\ln 2}{5} > \frac{1}{\exp(\delta-\varepsilon)-1}$, and the right-hand side in the inequality (4.7) is positive for $\delta \geqslant 2$. Since, according to condition of Lemma 5 the inequality $k \leqslant 7$ is valid, then for $x \geqslant 1$ the inequality $\delta = \frac{5\pi x}{k} \geqslant 2$ holds, and by (4.7) and by relations $\chi(1) = 1$, $|\chi(n)| \leqslant 1$ we have: $\sum_{n=1}^{\infty} n\chi(n)e^{-\frac{n^2\pi x}{k}} \ > \ e^{-\frac{\pi x}{k}} - \sum_{n=2}^{\infty} ne^{-\frac{n^2\pi x}{k}} \ > 0$.
Lemma 5 is proved. ∎

The following well-known equality holds (see, e.g., [10]):

$$\xi(s,\chi) = \frac{1}{2}\int_1^{\infty} \theta_1(x,\chi)x^{\frac{s-1}{2}} + \frac{i\sqrt{k}}{2\tau(\chi)}\int_1^{\infty} \theta_1(x,\chi)x^{-\frac{s}{2}}dx \ ,$$

where χ is an odd primitive character modulo k and

$$\theta_1(x,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n)e^{-\frac{n^2\pi x}{k}} \quad , \tag{4.9}$$

and, consequently,

$$\xi(s,\chi) = \frac{1}{2}\int_1^{\infty}\left(x^{\frac{s-1}{2}} + x^{-\frac{s}{2}}\right)\theta_1(x,\chi)dx \quad . \tag{4.10}$$

Using dominated convergence we derive, for any even number $m \geqslant 2$:

$$\frac{d^m\xi}{ds^m}\left(\frac{1}{2},\chi\right) = \left(\frac{1}{2}\right)^m \int_1^{\infty} (\ln^m x)\, x^{-\frac{1}{4}}\theta_1(x,\chi)dx \quad . \tag{4.11}$$

Since the character χ is odd, then $\chi(-1) = -1$, and, by (4.9), we have:

$$\theta_1(x,\chi) = 2\sum_{n=1}^{\infty} n\chi(n)e^{-\frac{n2\pi x}{k}} \quad . \tag{4.12}$$

Now the statements of Theorem 6 follow from (4.10)-(4.12) and Lemma 5. Theorem 6 is proved. ∎

5. THE CASE WHERE THE CHARACTER IS A LEGENDRE SYMBOL

In this section we apply the result of section 4 for functions $L(s,\chi)$ with the character $\chi(n)$ equal to Legendre's symbol $\left(\frac{n}{k}\right)$ modulo the prime number $k \geqslant 3$. In this connection we will not assume the validity of the inequality $L\left(\frac{1}{2},\chi\right) \neq 0$, in contrast to section 4.

Theorem 7. *Assume that k is a prime number satisfying inequalities $3 \leqslant k \leqslant 13$, $\left(\frac{k-1}{k}\right) = 1$, and $\chi = \chi(n) = \left(\frac{n}{k}\right)$. Then the value of the function $\xi(s,\chi)$ and all its derivatives at the point $s = \frac{1}{2}$ are positive.*

Proof. Only numbers $k = 5$ and $k = 13$ are the only prime numbers with $3 \leqslant k \leqslant 13$ and $\left(\frac{k-1}{k}\right) = \left(\frac{-1}{k}\right) = 1$. Hence, it is enough to prove Theorem 7 for $k = 5$ and $k = 13$. In this cases $\chi(n) = \left(\frac{n}{k}\right)$ is an even character (see definition 5), and therefore, the Gauss sum $\tau(\chi)$ (see definition 6) satisfies the equality (2.2). It is well-known (see [25]) that $\tau(\chi)$ satisfies the equality

$$|\tau(\chi)| = \sqrt{k} \ . \tag{5.1}$$

In view of (2.2), $\tau(\chi)$ is a real number, and it follows from (5.1) that $\tau(\chi)$ can take only two values : $\tau(\chi) = +\sqrt{k}$ and $\tau(\chi) = -\sqrt{k}$. In addition, according to equality (2.2) the equality $\tau(\chi) = +\sqrt{k}$ holds, if $\sum_{a=1}^{\frac{k-1}{2}} \chi(a)\cos\frac{2\pi a}{k} = \frac{\sqrt{k}}{2}$, and the equality $\tau(\chi) = -\sqrt{k}$ holds, if $\sum_{a=1}^{\frac{k-1}{2}} \chi(a)\cos\frac{2\pi a}{k} = -\frac{\sqrt{k}}{2}$. By numerical calculations we see that for $k = 5$ and $k = 13$ the equality $\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \cos \frac{2\pi a}{k} = \frac{\sqrt{k}}{2}$ holds. Therefore, in these cases the equality $\tau(\chi) = \sqrt{k}$ holds. Substituting this expression for $\tau(\chi)$ to the equality

$$\xi(s,\chi) = \frac{1}{2}\int_1^\infty x^{\frac{s}{2}-1}\theta(x,\chi)dx + \frac{1}{2}\frac{\sqrt{k}}{\tau(\chi}\int_1^\infty x^{-\frac{s}{2}-1}\theta(x,\chi)dx$$

with even character χ, in which $\theta(x,\chi)$ has the form of (4.3), we obtain the equality (4.4), and the equality (4.5) for any even number $m \geqslant 2$. Now the statement of Theorem 7 follows from (4.3)-(4.5) and Lemma 4. Theorem 7 is proved. ∎

Theorem 8. *Assume that k is a prime number satisfying the inequality $3 \leqslant k \leqslant 7$, $\left(\frac{k-1}{k}\right) = -1$, and $\chi = \chi(n) = \left(\frac{n}{k}\right)$. Then the value of the function $\xi(s,\chi)$ and all its derivatives at the point $s = \frac{1}{2}$ are positive.*

Proof. $k = 3$ and $k = 7$ are the only prime numbers for which $3 \leqslant k \leqslant 7$ and $\left(\frac{k-1}{k}\right) = \left(\frac{-1}{k}\right) = -1$. Hence, it is enough to prove Theorem 8 for $k = 5$ and $k = 13$. In these cases $\chi(n) = \left(\frac{n}{k}\right)$ is an odd character (definition 5), and therefore, the Gauss sum $\tau(\chi)$ (definition 6) satisfies the equality (2.4). According to (2.4) and (5.1) $\tau(\chi)$ is imaginary number, it can take only two values : $\tau(\chi) = i\sqrt{k}$, $\tau(\chi) = -i\sqrt{k}$. In addition, according to (2.4) the equality $\tau(\chi) = i\sqrt{k}$ holds, if the equality $\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \sin \frac{2\pi a}{k} = \frac{\sqrt{k}}{2}$ does holds, and the equality $\tau(\chi) = -i\sqrt{k}$ hold, if the equality $\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \sin \frac{2\pi a}{k} = -\frac{\sqrt{k}}{2}$ does hold. By numerical calculations, we see that for $k = 3$ and $k = 7$ the equality $\sum_{a=1}^{\frac{k-1}{2}} \chi(a) \sin \frac{2\pi a}{k} = \frac{\sqrt{k}}{2}$ holds. Therefore, in these cases the equality $\tau(\chi) = i\sqrt{k}$ holds. Substituting this expression for $\tau(\chi)$ in to the equality

$$\xi(s,\chi) = \frac{1}{2}\int_1^\infty x^{\frac{s}{2}-\frac{1}{2}}\theta_1(x,\chi)dx + \frac{i\sqrt{k}}{2\tau(\chi)}\int_1^\infty x^{-\frac{s}{2}}\theta_1(x,\chi)dx$$

with an odd character χ, in which $\theta_1(x,\chi)$ has the form (4.9), we obtain the equality (4.10) and the equality (4.11) for any even number $m \geqslant 2$. Now the statement of Theorem 8 follows from (4.10)-(4.12) and Lemma 5. Theorem 8 is proved. ∎

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