Inverse Spectral Problems for Coupled Oscillating Systems: Reconstruction by Three Spectra

Sergio Albeverio, Rostyslav Hryniv, Yaroslav Mykytyuk

no. 236
INVERSE SPECTRAL PROBLEMS FOR COUPLED OSCILLATING SYSTEMS: RECONSTRUCTION BY THREE SPECTRA

SERGIO ALBEVERIO∗ ROSTYSŁAV HRYNIV†‡ YAROSŁAV MYKYTYUK†

ABSTRACT. We study an inverse spectral problem for a compound oscillating system consisting of a singular string and \( N \) masses joined by springs. The operator \( \mathcal{A} \) corresponding to this system acts in \( L_2(0,1) \times \mathbb{C}^N \) and is composed of a Sturm–Liouville operator in \( L_2(0,1) \) with a distributional potential and a Jacobi matrix in \( \mathbb{C}^N \), which are coupled in a special way. We solve the problem of reconstructing the system by three spectra—namely, by the spectrum of \( \mathcal{A} \) and the spectra of its decoupled parts. A complete description of possible spectra is given.

1. INTRODUCTION

The main aim of the present paper is to solve an inverse spectral problem for a class of oscillating systems composed of a singular string and \( N \) masses joined by springs. Mathematically such a system is described by a Sturm–Liouville operator \( S \) coupled in a special way to a Jacobi operator \( J \).

Namely, assume that \( q \) is a real-valued distribution from \( W_2^{-1}(0,1) \) and denote by \( S \) a Sturm–Liouville operator in \( L_2(0,1) \) that is formally given by the differential expression

\[
l := -\frac{d^2}{dx^2} + q
\]

and the Robin or the Dirichlet boundary condition at the point \( x = 0 \). The precise definition of \( S \) is based on regularisation of \( l \) by quasi-derivatives \([19, 20]\) and goes as follows. We fix a real-valued distributional primitive \( \sigma \in L_2(0,1) \) of \( q \) and rewrite \( ly \) as

\[
l_\sigma y := -(y' - \sigma y)' - \sigma y'
\]

on the natural domain

\[
\mathcal{D}(l_\sigma) = \{ y \in W_1^1(0,1) \mid y' - \sigma y \in W_1^1(0,1), l_\sigma y \in L_2(0,1) \}.
\]

In what follows, we shall abbreviate the quasi-derivative \( y' - \sigma y \) to \( y^{[1]}_\sigma \) or simply to \( y^{[1]} \) when \( \sigma \) is fixed by the context. We define now the operator \( S \) by \( Sy = l_\sigma y \) on the
domain
\[ \mathcal{D}(S) = \{ y \in \mathcal{D}(l_\sigma) \mid y^{[1]}(0) = hy(0) \} \]
for some \( h \in \mathbb{R} \cup \{\infty\} \), \( h = \infty \) corresponding to the Dirichlet boundary condition \( y(0) = 0 \).

Assume that \( J \) is a Jacobi matrix in \( \mathbb{C}^N \), \( N \in \mathbb{N} \), i.e., that \( J \) in the standard basis \( e_1, \ldots, e_N \) of \( \mathbb{C}^N \) is a symmetric matrix with with real entries \( b_1, \ldots, b_N \) on the main diagonal and positive entries \( a_1, \ldots, a_{N-1} \) on the main sub- and super-diagonals.

Denote also by \( B \) the intertwining operator between \( L_2(0,1) \) and \( \mathbb{C}^N \) given on \( \mathcal{D}(S) \) by \( By = a_0 y^{[1]}(1)e_1 \) for some \( a_0 > 0 \).

Finally, we consider the operator
\begin{equation}
\mathcal{A} := \begin{pmatrix} S & 0 \\ B & J \end{pmatrix}
\end{equation}
that acts in the product space \( \mathcal{H} := L_2(0,1) \times \mathbb{C}^N \) on the domain
\[ \mathcal{D}(\mathcal{A}) := \{ (y, d) \in \mathcal{H} \mid y \in \mathcal{D}(S), \ d = (d(1), \ldots, d(N)), \ y(1) = a_0 d(1) \}. \]

It is known [1] that \( \mathcal{A} \) is self-adjoint and bounded below in \( \mathcal{H} \) and has a simple discrete spectrum. Adding if necessary a sufficiently large constant to the potential \( q \) and to the numbers \( b_1, \ldots, b_m \), we can make the operator \( \mathcal{A} \) positive and shall assume this without loss of generality.

**Remark 1.1.** We observe that although the Sturm–Liouville differential expression \( l_\sigma \) is independent of the particular choice of the primitive \( \sigma \), the quasi-derivative \( y^{[1]} \) in the interface condition and in the boundary condition for \( S \) if \( h \) is finite—and thus the whole operator \( \mathcal{A} \)—depends on \( \sigma \). We notice, however, that \( \mathcal{A} \) is invariant under the simultaneous change \( \sigma \mapsto \sigma + C, \ h \mapsto h + C \), and \( b_i \mapsto b_i + a_0 C \) for any real \( C \). This invariance will be used in Section 4.

Along with \( \mathcal{A} \) we consider two operators, \( \mathcal{A}_0 := S_N \oplus J \) and \( \mathcal{A}_\infty := S_D \oplus J_{(1)} \), where \( S_N \) and \( S_D \) are the restrictions of \( S \) by the “Neumann” boundary condition \( y^{[1]}(1) = 0 \) and the Dirichlet boundary condition \( y(1) = 0 \) respectively, and \( J_{(1)} \) is the Jacobi matrix obtained by removing the top row and the most-left column of \( J \).

The operators \( \mathcal{A}_0 \) and \( \mathcal{A}_\infty \) formally correspond to two extreme cases of the coupling not allowed in \( \mathcal{A} \): first with no coupling at all, and the second with infinite, i.e., rigid coupling. It is easily seen that \( \mathcal{A} \) and \( \mathcal{A}_0 \) are self-adjoint extensions of the same symmetric operator with defect indices \( (1,1) \) specifying the interface condition at the point \( x = 1 \), and the same holds for \( \mathcal{A} \) and \( \mathcal{A}_\infty \). Therefore, as in the papers [8,13,16,17], it is natural to study the question, to which extent \( \mathcal{A} \) is determined by the spectra of \( \mathcal{A} \) and \( \mathcal{A}_0 \), or those of \( \mathcal{A} \) and \( \mathcal{A}_\infty \). As in the purely continuous case of a Sturm–Liouville operator [8,13,17] or of a purely discrete case of a Jacobi matrix [16], one has to know the spectra of \( S_N \) and \( J \) or those of \( S_D \) and \( J_{(1)} \) separately—and not just their union—in order to reconstruct \( \mathcal{A} \).

Thus the **inverse spectral problem** we are going to solve is that of the reconstruction of the operator \( \mathcal{A} \) from the spectra of \( \mathcal{A}, S_N, \) and \( J \) or from those of \( \mathcal{A}, S_D, \) and \( J_{(1)} \). It generalizes the inverse spectral problems by three spectra for the standard Sturm–Liouville operators or for Jacobi matrices treated in the above-cited papers and is related to the inverse spectral problem for Sturm–Liouville operators with rationally dependent boundary conditions, see [1,3–6].

We shall solve the above inverse problem by reducing it to that of reconstructing \( \mathcal{A} \) from its spectrum and the sequence of the corresponding norming constants. The latter
problem was studied in detail in [1] (see also [3–5] for the related inverse problem for a Sturm–Liouville operator with rationally dependent boundary conditions), and this allows a complete description of the spectra for the operators involved. We shall prove that the operator \( \mathcal{A} \) is recovered uniquely if and only if the three spectra do not intersect. This establishes in this special case the conjecture raised in [8] for Sturm–Liouville operators, which was later established in [13]; the case of finite Jacobi matrices was studied in [16].

The treatments of the Dirichlet boundary condition (\( h = \infty \)) and the Robin boundary condition (\( h \in \mathbb{R} \)) at the point \( x = 0 \) are completely analogous, and we shall consider in detail only the Dirichlet case. In the next section we shall derive some useful formulae (e.g., for the resolvent of \( \mathcal{A} \) and the norming constants) that will be used in the subsequent analysis. In Sections 3 and 4 we reconstruct the operator \( \mathcal{A} \) from the spectra of \( \mathcal{A} \), \( S_D \), and \( J^{(1)} \) and from the spectra of \( \mathcal{A} \), \( S_N \), and \( J \) respectively.

Notations. Throughout the paper, the prime will denote the derivative in \( x \in [0, 1] \), and the overdot will stand for differentiation in the complex variable \( \lambda \) or \( z \). Given two strictly increasing (finite or infinite) sequences \((a_n)\) and \((b_n)\), we shall denote by \((c_n) := (a_n) \sqcup (b_n)\) the non-decreasing sequence obtained by amalgamating the sequences \((a_n)\) and \((b_n)\) and listing the common elements twice. We shall write \( \sigma(T) \) for the spectrum of a linear operator \( T \) acting in a Hilbert space.

2. Preliminaries

It is known [1] that the operator \( \mathcal{A} \) of (1.1) is self-adjoint, lower semi-bounded, and has discrete spectrum \( \lambda_1 < \lambda_2 < \ldots \); we recall our standing and nonrestrictive assumption that \( \lambda_1 > 0 \).

For every nonzero \( \lambda \in \mathbb{C} \), we define the “fundamental system of solutions” \( Y_{-}(:, \lambda) \) and \( Y_{+}(:, \lambda) \) corresponding to the eigenvalue problem \( \mathcal{A} Y = \lambda Y \). Namely, the element \( Y_{-}(0, \lambda) = 0, \ Y_{+}^{(1)}(0, \lambda) = \sqrt{\lambda} \), and satisfies the system \( \mathcal{A} Y = \lambda Y \) in the \( L_2(0, 1) \)-component and in the first \( N - 1 \) components of \( \mathbb{C}^N \). In other words, there is a unique \( c = c(\lambda) \in \mathbb{C} \) such that

\[
(\mathcal{A} - \lambda) Y_{-}(:, \lambda) = \begin{pmatrix} 0 \\ e e_N \end{pmatrix} ;
\]

in particular, \( c(\lambda) = 0 \) if and only if \( \lambda \) is in the spectrum of \( \mathcal{A} \), in which case \( Y_{-}( :, \lambda ) \) is a corresponding eigenelement. The element \( Y_{+}( :, \lambda ) := (y_{+}(:, \lambda), d_{+}(:, \lambda))^t \) is normalized by the terminal condition \( d_{+}(N, \lambda) = 1 \), satisfies the system

\[
l y_{+} - \lambda y_{+} = 0 ,
\]

\[
a_0 y_{+}^{(1)}(1)e_1 + (J - \lambda)d_{+} = 0 ,
\]

and the interface condition \( y_{+}(1, \lambda) = a_0 d_{+}(1, \lambda) \), but need not satisfy the initial condition \( y_{+}(0, \lambda) = 0 \). Moreover, \( y_{+}(0, \lambda) = 0 \) holds if and only if \( \lambda \) is in the spectrum of \( \mathcal{A} \), in which case \( Y_{+}( :, \lambda ) \) is a corresponding eigenelement.

Using the elements \( Y_{\pm}( :, \lambda ) \), it is possible to construct the Green function of the operator \( \mathcal{A} \) and to find the explicit form of its resolvent, similarly to such constructions for a Sturm–Liouville equation.
Lemma 2.1. Assume that $\lambda \in \mathbb{C}$ belongs to the resolvent set of the operators $J$ and $\mathcal{A}$ and that $(g, v)^{t}$ is an arbitrary element of $\mathcal{H}$. Then the element

$$(y, d) := (\mathcal{A} - \lambda)^{-1}(g, v)$$

is given by

$$y(x) = \frac{y_{-}(x, \lambda)}{W(\lambda)} \int_{x}^{1} y_{+} + (v, d_{+}(\cdot, \lambda))_{CN} + \frac{y_{+}(x, \lambda)}{W(\lambda)} \int_{0}^{x} y_{-} - g$$

$$d(k) = (J - \lambda)^{-1} v(k) + \frac{d_{+}(k, \lambda)}{W(\lambda)} \left[ \int_{0}^{1} y_{-} - g + \frac{y_{+}[1](1, \lambda)}{y_{+}[1](1, \lambda)} (v, d_{+}(\cdot, \lambda))_{CN} \right],$$

where $W(\lambda) := y_{+}(x, \lambda)y_{+}[1](x, \lambda) - y_{+}[1](x, \lambda)y_{-}(x, \lambda)$ is the Wronskian of the solutions $y_{+}$ and $y_{-}$.

Proof. The function $y$ solves the equation $Sy = \lambda y + g$ and thus is equal to $y_{0} + \alpha y_{-}$, with

$$y_{0}(x) := \frac{y_{-}(x, \lambda)}{W(\lambda)} \int_{x}^{1} y_{+} + \frac{y_{+}(x, \lambda)}{W(\lambda)} \int_{0}^{x} y_{-} - g$$

being a particular solution to the above non-homogeneous problem and $\alpha$ some complex number. Since $d_{+}(\cdot, \lambda) = -a_{0}y_{+}[1](1, \lambda)(J - \lambda)^{-1}e_{1}$, the relation

$$(2.1) \quad a_{0}y_{+}[1](1)e_{1} + (J - \lambda)d = v$$

implies that $d = d_{0} + \beta d_{+}(\cdot, \lambda)$ with $d_{0} := (J - \lambda)^{-1}v$ and some $\beta \in \mathbb{C}$.

The constants $\alpha$ and $\beta$ must be such that the interface condition $y(1) = a_{0}d(1)$ and relation (2.1) hold. By virtue of the relation

$$d_{0}(1) = (J - \lambda)^{-1} v, \quad e_{1} = -\frac{(v, d_{+}(\cdot, \lambda))_{CN}}{a_{0}y_{+}[1](1, \lambda)},$$

the interface condition transforms into

$$\alpha y_{-}(1, \lambda) - \beta y_{+}(1, \lambda) = -\frac{y_{+}(1, \lambda)}{W(\lambda)} \int_{0}^{1} y_{-} - g - \frac{(v, d_{+}(\cdot, \lambda))_{CN}}{y_{+}[1](1, \lambda)}.$$

Similarly, equation (2.1) can be recast as

$$\alpha y_{+}[1](1, \lambda) - \beta y_{-}[1](1, \lambda) = -\frac{y_{+}[1](1, \lambda)}{W(\lambda)} \int_{0}^{1} y_{-} - g.$$

The above two equations form a linear system for $\alpha$ and $\beta$, solving which we find that

$$\alpha = \frac{(v, d_{+}(\cdot, \lambda))_{CN}}{W(\lambda)}, \quad \beta = \frac{\int_{0}^{1} y_{-} - g}{W(\lambda)} + \frac{y_{+}[1](1, \lambda)}{y_{+}[1](1, \lambda)} (v, d_{+}(\cdot, \lambda))_{CN},$$

and the required formula for $(y, d)^{t}$ follows. \hfill \Box

The sequence $(Y_{\cdot}(\cdot, \lambda_{n}))_{n \in \mathbb{N}}$ forms an orthogonal basis of the space $\mathcal{H}$. We denote by $\alpha_{n} := \|Y(\cdot, \lambda_{n})\|^{-2}$ the norming constant corresponding to the eigenvalue $\lambda_{n}$. A useful formula for the norming constants is given by the following lemma.
Lemma 2.2. Assume that $\lambda_n \in \sigma(A)$ is not in the spectrum of $J$. Then the corresponding norming constant $\alpha_n := \|Y_-(\cdot, \lambda_n)\|^{-2}$ satisfies the equalities

\[ \alpha_n = -\frac{y_+^{[1]}(1, \lambda_n)}{\sqrt{\lambda_n y_-^{[1]}(1, \lambda_n) y_+(0, \lambda_n)}}. \]  

Similarly, if $\lambda_n \in \sigma(A)$ is not in the spectrum of $J(1)$, then

\[ \alpha_n = -\frac{y_+(1, \lambda_n)}{\sqrt{\lambda_n y_-(1, \lambda_n) y_+(0, \lambda_n)}}. \]

Proof. We take an arbitrary function $g \in L_2(0, 1)$, put $G := (g, 0)^t$, and calculate the $L_2$-component $\hat{g}$ of the element $(A - \lambda)^{-1}G$ in two ways. On the one hand, the resolution of identity of the operator $A$ gives

\[ \hat{g}(x) = \sum_{k=1}^{\infty} \alpha_k (g, y_-(\cdot, \lambda_k))_{CN} y_-(x, \lambda_k). \]

On the other hand, using Lemma 2.1, we find that

\[ \hat{g}(x) = \frac{y_-(x, \lambda)}{W(\lambda)} \int_x^1 y_+g + \frac{y_+(x, \lambda)}{W(\lambda)} \int_0^x y_+y. \]

Equating the residues at the point $\lambda = \lambda_n$ and noting that the functions $y_-(\cdot, \lambda_n)$ and $y_+(\cdot, \lambda_n)$ are collinear and that $\lambda_n$ is a simple zero of $W$, we conclude that

\[ \alpha_n y_-(x, \lambda_n) (g, y_-(\cdot, \lambda_n))_{CN} = -\frac{y_+(x, \lambda_n)}{W(\lambda_n)} (g, y_-(\cdot, \lambda_n))_{CN}, \]

or, on account of the relation $W(\lambda) \equiv \sqrt{\lambda} y_+(0, \lambda)$,

\[ \alpha_n = -\frac{y_+(x, \lambda_n)}{\sqrt{\lambda_n y_-(x, \lambda_n) y_+(0, \lambda_n)}} \frac{1}{y_+(0, \lambda_n)}. \]

Finally, the ratio $y_+(x, \lambda_n)/y_-(x, \lambda_n)$ does not depend on $x$, and, moreover,

\[ \frac{y_+(x, \lambda_n)}{y_-(x, \lambda_n)} = \frac{y_+(1, \lambda_n)}{y_-(1, \lambda_n)} \]

if $y_-(1, \lambda_n) \neq 0$ and

\[ \frac{y_+(x, \lambda_n)}{y_-(x, \lambda_n)} = \frac{y_+^{[1]}(1, \lambda_n)}{y_-^{[1]}(1, \lambda_n)}, \]

if $y_+^{[1]}(1, \lambda_n) \neq 0$, and the required formulae follow. It remains to recall [1] that, for $\lambda_n$ in the spectrum of $A$, the equality $y_-(1, \lambda_n) = 0$ holds if and only if $\lambda_n$ is an eigenvalue of $J$ and that $y_+^{[1]}(1, \lambda_n) = 0$ if and only if $\lambda_n$ is an eigenvalue of $J(1)$.

It is known (see [6] for the case $q \in L_1(0, 1)$ and [1] for the case $q \in W^{-1}_2(0, 1)$) that the eigenvalues $(\lambda_n)$ and the corresponding norming constants $(\alpha_n)$ determine the operator $A$ uniquely. Moreover, the cited papers give the algorithm of reconstruction of $A$ from these spectral data. The next proposition gives also the complete description of the spectral data, cf. [1, 6].

Proposition 2.3. The eigenvalues $(\lambda_n)$ of $A$ and the corresponding norming constants $(\alpha_n)$ obey the asymptotics

\[ \lambda_n = [\pi(n - N) + \hat{\lambda}_n]^2, \quad \alpha_n = 2 + \hat{\alpha}_n, \]
where the sequences \((\tilde{\lambda}_n)\) and \((\tilde{\alpha}_n)\) belong to \(\ell_2\).

Conversely, any sequences \((\lambda_n)\) and \((\alpha_n)\) of real numbers such that

(a) the \(\lambda_n\) strictly increase and have the representation \(\lambda_n = [\pi(n - N) + \tilde{\lambda}_n]^2\) for some \(N \in \mathbb{N}\) and an \(\ell_2\)-sequence \((\tilde{\lambda}_n)\);

(b) the \(\alpha_n\) are positive and equal \(2 + \tilde{\alpha}_n\) for some \(\ell_2\)-sequence \((\tilde{\alpha}_n)\)

are the sequences of eigenvalues and the norming constants for a unique operator \(\mathcal{A}\) of the form (1.1).

In the following, we denote by \(\mu_{n,D}\) (resp. \(\mu_{n,N}\)) the eigenvalues of the operator \(S_D\) (resp. of the operator \(S_N\)), and by \(\nu_{1,j}, \ldots, \nu_{N,j}\) (resp. \(\nu_{1}^{(1)}, \ldots, \nu_{N-1}^{(1)}\)) the eigenvalues of \(J\) (resp. of \(J_{(1)}\)), all labelled in increasing order. It is well known that the operators \(S_D\) and \(S_N\) and the Jacobi matrices \(J\) and \(J_{(1)}\) have simple discrete spectra. We recall and derive next some properties of these spectra.

**Proposition 2.4** ([10, 19, 20]). There exist sequences \((\tilde{\mu}_{n,D})\) and \((\tilde{\mu}_{n,N})\) belonging to \(\ell_2(\mathbb{N})\) such that

(a) \(\mu_{n,D} = [\pi n + \tilde{\mu}_{n,D}]^2\);

(b) \(\mu_{n,N} = [\pi(n - \frac{1}{2}) + \tilde{\mu}_{n,N}]^2\).

We observe that the numbers \(\mu_{n,D}\) are zeros of the function \(y_{-}(1, \lambda)\) and \(\mu_{n,N}\)—those of \(y_{-}^{[1]}(1, \lambda)\). Since both functions are exponential in \(\lambda\) of order \(\frac{1}{2}\), they can be reconstructed from their zeros in the following way.

**Proposition 2.5** ([11]). The following equalities hold:

\[
y_{-}(1, \lambda) = \sqrt{\lambda} \prod_{k=1}^{\infty} \frac{\mu_{k,D} - \lambda}{\pi^2 k^2}, \quad y_{-}^{[1]}(1, \lambda) = \sqrt{\lambda} \prod_{k=1}^{\infty} \frac{\mu_{k,N} - \lambda}{\pi^2 (k - \frac{1}{2})^2}.
\]

Simple considerations show that the functions \(y_{+}(1, \lambda)\) and \(y_{+}^{[1]}(1, \lambda)\) are related to the eigenvalues of \(J\) and \(J_{(1)}\) as follows.

**Lemma 2.6.** The following equalities hold:

\[
y_{+}(1, \lambda) = \frac{a_0}{a_1 \cdots a_{N-1}} \prod_{k=1}^{N-1} (\lambda - \nu_k^{(1)}), \quad y_{+}^{[1]}(1, \lambda) = \frac{1}{a_0 a_1 \cdots a_{N-1}} \prod_{k=1}^{N} (\lambda - \nu_k^{(1)}).
\]

**Proof.** To find the representation for \(y_{+}^{[1]}(1, \lambda)\), it suffices to establish an analogous formula for \(d_{+}(1, \lambda)\). Using the relation \((J - \lambda) d_{+}(:, \lambda) = -a_0 y_{+}^{[1]}(1, \lambda) e_1\) and the normalization \(d_{+}(N, \lambda) = 1\), we find recursively that \(d_{+}(N - k, \lambda)\) is a polynomial in \(\lambda\) of degree \(k\) with leading coefficient \((a_{N-1} \cdots a_{N-k})^{-1}\). Therefore \(y_{+}^{[1]}(1, \lambda)\) is a polynomial in \(\lambda\) of degree \(N\) with leading coefficient \((a_0 a_1 \cdots a_{N-1})^{-1}\), and since it vanishes at the points \(\nu_{1,j}, \ldots, \nu_{N,j}\), the above formula follows.

Analogously, \(y_{+}(1, \lambda) = a_0 d_{+}(1, \lambda)\) is a polynomial in \(\lambda\) of degree \(N - 1\) and leading term \(a_0 / (a_1 \cdots a_{N-1})\) that vanishes at the points \(\nu_1^{(1)}, \ldots, \nu_{N-1}^{(1)}\), and the result follows.

Finally, we find below an explicit expression for \(y_{+}(0, \lambda)\) in terms of the eigenvalues \(\lambda_n\) of the operator \(\mathcal{A}\).

**Lemma 2.7.** The following holds:

\[
y_{+}(0, \lambda) = -(a_0 a_1 \cdots a_{N-1})^{-1} \prod_{k=1}^{N} (\lambda - \lambda_k) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2}.
\]
Proof. In what follows, \( \lambda \) is an arbitrary nonzero complex number. We recall that \( y_+(x, \lambda) \) is a solution of the equation \( ly = \lambda y \) satisfying the terminal conditions \( y(1) = y_+(1, \lambda) \) and \( y^{[1]}(1) = y_+^{[1]}(1, \lambda) \), whence

\[
y_+(x, \lambda) = y_+(1, \lambda)u(x, \lambda) + y_+^{[1]}(1, \lambda)v(x, \lambda),
\]
where \( u(\cdot, \lambda) \) and \( v(\cdot, \lambda) \) are solutions of the problems

\[
\begin{align*}
L_\sigma u &= \lambda u, \\
u(1) &= 1, \\
u^{[1]}(1) &= 0,
\end{align*}
\]

Recalling [12] that \( u(x, \lambda) \) and \( v(x, \lambda) \) have the integral representations

\[
\begin{align*}
u(x, \lambda) &= \cos \sqrt{\lambda}(x - 1) + \int_x^1 k_1(x, t) \cos \sqrt{\lambda}(t - 1) \, dt, \\
v(x, \lambda) &= \sin \sqrt{\lambda}(x - 1) + \int_x^1 k_2(x, t) \sin \sqrt{\lambda}(t - 1) \, dt 
\end{align*}
\]
for some upper-diagonal kernels \( k_j \) such that \( k(x, \cdot) \) belongs to \( L_2(0, 1) \) for every \( x \in [0, 1] \), and that by Lemma 2.6 \( y_+^{[1]}(1, \lambda) \) and \( y_+(1, \lambda) \) are polynomials in \( \lambda \) of degrees \( N \) and \( N - 1 \) respectively, we find that

\[
y_+(0, \lambda) = -\frac{\lambda^{N-\frac{1}{2}} \sin \sqrt{\lambda}}{a_0 a_1 \cdots a_{N-1}} [1 + o(1)]
\]
as \( \lambda \to -\infty \).

Since \( y_+(0, \lambda) \) is an entire function of \( \lambda \) of exponential type \( \frac{1}{2} \) and since its zeros coincide with the eigenvalues of \( \mathcal{A} \), we conclude that

\[
y_+(0, \lambda) = C_1 \prod_{k \in \mathbb{N}} \left( 1 - \frac{\lambda}{\lambda_n} \right)
\]
for some constant \( C_1 \in \mathbb{C} \). Now we find that

\[
\begin{align*}
-1 &= \lim_{\lambda \to -\infty} \frac{a_0 a_1 \cdots a_{N-1} y_+(0, \lambda)}{\lambda^{N-\frac{1}{2}} \sin \sqrt{\lambda}} \\
&= \lim_{\lambda \to -\infty} \frac{C_1 a_0 a_1 \cdots a_{N-1}}{\lambda^N} \prod_{k \in \mathbb{N}} \left( 1 - \frac{\lambda}{\lambda_k} \right) \prod_{k \in \mathbb{N}} \left( 1 - \frac{\lambda}{\pi^2 k^2} \right) \\
&= \lim_{\lambda \to -\infty} \frac{C_1 a_0 a_1 \cdots a_{N-1}}{\lambda^N} \prod_{k=1}^{N} \left( 1 - \frac{\lambda}{\lambda_k} \right) \frac{\prod_{k \in \mathbb{N}} \lambda_{k+N} - \lambda \pi^2 k^2}{\lambda_{k+N} \pi^2 k^2} \\
&= (-1)^N \frac{C_1 a_0 a_1 \cdots a_{N-1}}{\lambda_1 \cdots \lambda_N} \prod_{k \in \mathbb{N}} \frac{\pi^2 k^2}{\lambda_{k+N}},
\end{align*}
\]
since the above products converge uniformly on \( \mathbb{C} \), whence

\[
y_+(0, \lambda) = \frac{(-1)^N \lambda_1 \cdots \lambda_N}{a_0 a_1 \cdots a_{N-1}} \prod_{k=1}^{N} \left( 1 - \frac{\lambda}{\lambda_k} \right) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2} \\
= \frac{a_0 a_1 \cdots a_{N-1}}{(-1)^N} \prod_{k=1}^{N} \left( \lambda - \lambda_k \right) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2}.
\]
The lemma is proved. \hfill \Box

We shall use several times the following statement about integral representations of some entire functions, cf. [15, Lemma 3.4.2].

**Proposition 2.8** ([11]). Assume that the numbers $a_n$ and $b_n$ are such that $a_n = \pi n + \bar{a}_n$ and $b_n = \pi (n - \frac{1}{2}) + \bar{b}_n$ with some $\ell_2$-sequences $(\bar{a}_n)$ and $(\bar{b}_n)$. Put

$$\phi(z) := \sqrt{z} \prod_{n \in \mathbb{N}} \frac{a_n^2 - z}{\pi^2 n^2}, \quad \psi(z) := \prod_{n \in \mathbb{N}} \frac{b_n^2 - z}{\pi^2 (n - \frac{1}{2})^2},$$

then there exist functions $\hat{\phi}$ and $\hat{\psi}$ in $L_2(0, 1)$ such that

$$\phi(z) = \sin \sqrt{z} + \int_0^1 \hat{\phi}(t) \sin \sqrt{zt} \, dt, \quad \psi(z) = \cos \sqrt{z} + \int_0^1 \hat{\psi}(t) \cos \sqrt{zt} \, dt.$$

3. **Reconstruction from $\mathcal{A}$, $S_D$, and $J_{(1)}$**

Given an arbitrary operator matrix $\mathcal{A}$ of the form (1.1), we denote by $(\lambda_n)_{n \in \mathbb{N}}$, $(\mu_n, D)_{n \in \mathbb{N}}$, and $(\nu_n)_{n=1}^N$ the eigenvalue sequences of $\mathcal{A}$, the operator $S_D$, and the Jacobi matrix $J_{(1)}$ respectively. Put also $(\lambda'_n)_{n \in \mathbb{N}} := (\mu_n, D) \Pi (\nu_n)_{n=1}^N$, where the amalgamation operation $\Pi$ was defined in the Introduction. An interesting property of the spectra involved is that every multiple element of $(\lambda'_n)$ is an eigenvalue of $\mathcal{A}$ and any eigenvalue of $\mathcal{A}$ that belongs also to $(\lambda'_n)$ occurs therein twice. In other words, the following statement holds true.

**Proposition 3.1** ([1]). $\sigma(\mathcal{A}) \cap \sigma(S_D) = \sigma(\mathcal{A}) \cap \sigma(J_{(1)}) = \sigma(S_D) \cap \sigma(J_{(1)})$.

This allows us to establish the weak interlacing property of the sequences $(\lambda_n)$ and $(\lambda'_n)$ in the following sense.

**Lemma 3.2.** The sequences $(\lambda_n)$ and $(\lambda'_n)$ weakly interlace, i.e., $\lambda_1 < \lambda'_1$ and for every $n \in \mathbb{N}$ either $\lambda'_n < \lambda_{n+1} < \lambda'_n + 1$ or $\lambda'_n = \lambda_{n+1} = \lambda'_n + 1$.

**Proof.** Denote by $\delta_x$ the Dirac delta-function at the point $x = 1$ and put $D_1 := (\delta_1, 0)^1$. It is known that the domain of the Sturm–Liouville operator $S$ is contained in $W_2^1(0, 1)$; in particular, the functions $y_{\pm}(:, \lambda)$ belong to $W_2^1(0, 1)$. The explicit formula for the resolvent of the operator $\mathcal{A}$ derived in Lemma 2.1 shows that the expression

$$f(\lambda) := ((\mathcal{A} - \lambda)^{-1} D_1, D_1)$$

makes sense for any $\lambda$ not in the spectrum of $\mathcal{A}$ and, moreover, that

$$f(\lambda) = \frac{y_+(1, \lambda) y_-(1, \lambda)}{W(\lambda)}.$$

Since $f(\lambda)$ is a Nevanlinna function, its zeros and poles interlace. On the other hand, the zeros of $f$ coincide with $\lambda'_k$ and the poles with those $\lambda_k$ which do not appear in $(\lambda'_n)$. In view of Proposition 3.1 and the known asymptotics of $\lambda_n$ and $\mu_n, D$ this justifies the claim. \hfill \Box

The asymptotics of $\lambda_n$ and $\mu_n, D$ shows that $\lambda_{n+N} - \mu_n, D = o(n)$ as $n \to \infty$. In fact, this result can be improved, cf. [5] for the case $q \in L_1(0, 1)$.

**Lemma 3.3.** There exists an $\ell_2$-sequence $(b_n)$ such that

$$\lambda_{n+N} - \mu_n, D = 2a_0^2 (1 + b_n).$$
By Propositions 2.5 and 2.8, there is also \( g \) due to the asymptotics of (3.1) the norming constant \( \alpha_{n+N} \) holds. Using the representation of the functions \( y_+(0, \lambda) \) and \( y_+(1, \lambda) \), we find that

\[
\alpha_{n+N} = \frac{a_0^2 \prod_{k=1}^{N-1} (\lambda_{n+N} - \nu_k)}{y_-(1, \lambda_{n+N}) \prod_{k=1}^{N} (\lambda_{n+N} - \lambda_k)} \int \frac{d}{d\lambda} \left( \sqrt{\lambda} \prod_{k=1}^{N} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2} \right) \bigg|_{\lambda = \lambda_{n+N}}.
\]

Due to the asymptotics of \( \lambda_n \) and Proposition 2.8 the function

\[
\phi(\lambda) := \sqrt{\lambda} \prod_{k \in \mathbb{N}} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2}
\]

can be represented in the form

\[
\phi(\lambda) = \sin \sqrt{\lambda} + \int_0^1 f(t) \sin \sqrt{\lambda} \, dt
\]

for some \( f \in L_2(0, 1) \), whence

\[
\dot{\phi}(\lambda_{n+N}) = \frac{1}{2\sqrt{\lambda_{n+N}}} \left( \cos \sqrt{\lambda_{n+N}} + \int_0^1 tf(t) \cos \sqrt{\lambda_{n+N}} \, dt \right).
\]

By Propositions 2.5 and 2.8, there is also \( g \in L_2(0, 1) \) such that

\[
\psi(\lambda) := y_-(1, \lambda) = \sin \sqrt{\lambda} + \int_0^1 g(t) \sin \sqrt{\lambda} \, dt.
\]

In view of the mean value theorem there are numbers \( \xi_n \) between \( \mu_{n,D} \) and \( \lambda_{n+N} \) such that

\[
\psi(\lambda_{n+N}) = (\lambda_{n+N} - \mu_{n,D}) \dot{\psi}(\xi_n)
\]

\[
= \frac{\lambda_{n+N} - \mu_{n,D}}{2\sqrt{\xi_n}} \left( \cos \sqrt{\xi_n} + \int_0^1 t g(t) \cos \sqrt{\xi_n} \, dt \right).
\]

Due to the asymptotics of \( \lambda_n \) and \( \xi_n \) the sequences \( \cos \sqrt{\lambda_{n+N}} t \) and \( \cos \sqrt{\xi_n} t \) form Riesz bases of \( L_2(0, 1) \) [9] and hence

\[
\cos \sqrt{\lambda_{n+N}} + \int_0^1 tf(t) \cos \sqrt{\lambda_{n+N}} \, dt = (-1)^{n+N}(1 + c_n),
\]

\[
\cos \sqrt{\xi_n} + \int_0^1 t g(t) \cos \sqrt{\xi_n} \, dt = (-1)^{n+N}(1 + d_n)
\]

with square summable sequences \( (c_n)_{n \in \mathbb{N}} \) and \( (d_n)_{n \in \mathbb{N}} \). Therefore (3.1) can be recast as

\[
\alpha_{n+N}(1 + c_n)(1 + d_n) = \frac{4a_0^2}{\lambda_{n+N} - \mu_{n,D} \lambda_{n+N} - \lambda_N} \prod_{k=1}^{N-1} \frac{\lambda_{n+N} - \nu_k}{\lambda_{n+N} - \lambda_k},
\]

which, on account of the asymptotics of \( \alpha_n \) of Proposition 2.3, implies that

\[
\frac{2a_0^2}{\lambda_{n+N} - \mu_{n,D}} = 1 + \hat{\alpha}_n, \quad (\hat{\alpha}_n) \in \ell_2,
\]

and the result follows.

\( \square \)

**Definition 3.4.** We denote by \( \mathcal{L}_N \) the set of all triples \( \Lambda := (\lambda_n)_{n=1}^\infty, (\mu_n)_{n=1}^\infty, (\nu_n)_{n=1}^{N-1} \) of strictly monotone sequences such that the following holds:

1. there is an \( \ell_2 \)-sequence \( (\hat{\lambda}_n) \) such that \( \lambda_n = [\pi (n - N) + \hat{\lambda}_n]^2; \)
Lemma 3.3 shows that $\beta$.

Due to the weak interlacing property of $\Lambda$ the numbers $\beta$.

(3.3) $\Phi(z)$ satisfies the relation holds for all $n$.

In the reverse direction, we shall prove that any element of $L_N$ the corresponding spectral triple $((\lambda_n), (\mu_n, D), (\nu^1_n))$ forms an element of $L_N$. In the reverse direction, we shall prove that any element of $L_N$ is the spectral triple of the above form.

**Theorem 3.5.** For any $\Lambda := ((\lambda_n), (\mu_n), (\nu_n)) \in L_N$ there exists an operator $A$ of the form $1.1$ such that $\lambda_n$, $\mu_n$, and $\nu_n$ are the eigenvalues of the operators $A$, $S_D$, and $J_{(1)}$ respectively. Such an operator $A$ is unique if and only if the set $A$ is empty.

**Proof.** We start with constructing the functions

$$
\phi(z) := \sqrt{\frac{z}{2}} \prod_{n \in \mathbb{N}} \frac{\lambda_n + z - x}{\pi^2 n^2}, \quad \psi(z) := \sqrt{\frac{z}{2}} \prod_{n \in \mathbb{N}} \frac{\mu_n - z}{\pi^2 n^2},
$$

and for $n \in B_A$ put (cf. (3.1))

$$
\beta_n := \frac{\gamma_0 \prod_{k=1}^{N-1} (\lambda_n - \mu_k)}{2 \psi(\lambda_n)} \left\{ \frac{d}{d\lambda} \left( \phi(\lambda) \prod_{k=1}^{N} (\lambda - \lambda_k) \right) \right\}_{\lambda = \lambda_n}.
$$

Due to the weak interlacing property of $\Lambda$ the numbers $\beta_n$ are positive and the proof of Lemma 3.3 shows that $\beta_n = 2 + \beta_n$ for a sequence $(\beta_n)$ belonging to $l_2(B_A)$.

Now we define the sequence $(\alpha_n)$ with $\alpha_n = \beta_n$ if $n \in B_A$ and take $\alpha_n$ to be an arbitrary positive number if $n \in A_A$. The sequences $(\lambda_n)$ and $(\alpha_n)$ satisfy all the requirements of Proposition 2.3 and thus there exists an operator $A$ of the form (1.1) whose eigenvalues and norming constants coincide respectively with $(\lambda_n)$ and $(\alpha_n)$.

It remains to prove that the sequences $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu^n_{n=1})_{n=1}$ we have started with coincide with the eigenvalues $\mu_n, D_{n=1}$ and $(\nu^1_n)_{n=1}^{N-1}$ of the related operator $S_D$ and Jacobi matrix $J_{(1)}$ respectively. Since for the norming constants $\alpha_n$ with $n \in B_A$ formula (2.3) holds, we conclude that, for such $n$,

$$
\frac{a^2_0 \prod_{k=1}^{N-1} (\lambda_n - \mu_k^1)}{\psi(\lambda_n)} = \frac{\gamma_0 \prod_{k=1}^{N-1} (\lambda_n - \mu_k)}{2 \psi(\lambda_n)},
$$

i.e., that

$$
\frac{2a^2_0 \psi(\lambda_n)}{\sqrt{\lambda_n}} \prod_{k=1}^{N-1} (\lambda_n - \mu_k^1) - \frac{\gamma_0 y_{(1, \lambda_n)}}{\sqrt{\lambda_n}} \prod_{k=1}^{N-1} (\lambda_n - \mu_k) = 0.
$$

Recalling that $\psi(\lambda_n) = 0 = \prod_{k=1}^{N-1} (\lambda_n - \mu_k)$ for $n \in A_A$, we conclude that equality (3.2) holds for all $n \in \mathbb{N}$. We observe that (3.2) takes the form $\Phi(\lambda_n) = 0$, where the function $\Phi$ satisfies the relation

$$
\Phi(z) = O(|z|^{N-3/2}e^{\text{Im} \sqrt{|z|}})
$$

as $|z| \to \infty$. We shall prove that $\Phi \equiv 0$.
Assume not, and observe that $\Phi$ has then no zeros other than $\lambda_n$, $n \in \mathbb{N}$. Indeed, in view of (3.3) Jensen’s formula gives

\begin{equation}
\int_1^r \frac{n(t)}{t} \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} \log |\Phi(re^{i\theta})| \, d\theta + C_1
\end{equation}

\begin{equation}
\leq (N - \frac{3}{2}) \log r + \frac{\sqrt{r}}{2\pi} \int_0^{2\pi} |\sin \theta/2| \, d\theta + C_2
\end{equation}

\begin{equation}
= (N - \frac{3}{2}) \log r + \frac{2\sqrt{r}}{\pi} + C_2,
\end{equation}

where $n(t)$ denotes the number of zeros of $\Phi$ in the closed circle of radius $t$ centered at the origin and $C_1$ and $C_2$ are some positive constants. On the other hand, if $\Phi$ had at least one additional zero, then for any $\varepsilon > 0$ and all sufficiently large $t$ we would have

$$n(t) \geq \left[ \frac{\sqrt{t}}{\pi} - \varepsilon \right] + N + 1 \geq \frac{\sqrt{t}}{\pi} + N - \varepsilon,$$

which contradicts (3.4). Now $\Phi$, being of exponential type $\frac{1}{2}$, equals

$$\Phi(z) = C_3 \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)$$

for some constant $C_3$. Using the canonical product for $\sin \sqrt{z}$ and the asymptotics of $\lambda_k$, we conclude that

$$\lim_{z \to -\infty} \frac{\Phi(z)}{z^{N-\frac{1}{2}} \sin \sqrt{z}} =: C_4 \neq 0,$$

which contradicts (3.3).

Thus we have proved that $\Phi \equiv 0$, i.e., that

$$2a_0^2 \prod_{k \in \mathbb{N}} \frac{\mu_k - z}{\pi^2 k^2} \prod_{k=1}^{N-1} (z - \nu^1_k) \equiv 0 \prod_{k=1}^{\infty} \frac{\mu_k, D - z}{\pi^2 k^2} \prod_{k=1}^{N-1} (z - \nu_k).$$

It follows that every $\nu_n$ that does not occur in $(\mu_k)$ is an eigenvalue of $J_{(1)}$ and, similarly, every $\mu_n$ that does not occur in $(\nu_k)$ is an eigenvalue of $S_D$. Since the sequences $(\lambda_n)$ and $(\mu_{n,D})$ $(\nu^1_n)$ weakly interlace in the sense of Lemma 3.2, and since the same is true of $(\lambda_n)$ and $(\lambda'_n)$, simple considerations show that every multiple element of $(\lambda_n)$ belongs to the spectra of both $S_D$ and $J_{(1)}$, cf. [13, Sect. 6]. Thus all $\mu_n$ are eigenvalues of $S_D$ and all $\nu_n$—those of $J_{(1)}$. Since neither $J_{(1)}$ nor $S_D$ can have other eigenvalues due to the size and asymptotics limitations respectively, $\Lambda$ is the spectral triple for the operator $\mathcal{A}$ found.

If the set $A_{\Lambda}$ is empty, then the norming constants $\alpha_n$ are uniquely determined by $\Lambda$, so that $\mathcal{A}$ is unique in view of Proposition 2.3. If $A_{\Lambda}$ is non-empty, then different choices of $\alpha_n$ for $n \in A_{\Lambda}$ lead to different operators $\mathcal{A}$. The proof is complete. \qed

**Remark 3.6.** It follows from the proof of Theorem 3.5 that the set of $\Lambda$-isospectral operators $\mathcal{A}$ of the form (1.1) (i.e., the set of operators $\mathcal{A}$ such that the spectra of $\mathcal{A}$, $S_D$, and $J_{(1)}$ form the prescribed triple $\Lambda \in \mathcal{L}_N$) is a manifold of dimension equal to the cardinality of the set $A_{\Lambda}$. 


4. Reconstruction from the Spectra of $\mathcal{A}$, $S_N$, and $J$

Treatment of the inverse problem of reconstructing the operator $\mathcal{A}$ from the spectra of the operators $\mathcal{A}$, $S_N$, and the Jacobi matrix $J$ parallels in general that of the inverse problem of Section 3. One essential difference is that the invariance of $\mathcal{A}$ with respect to changing the primitive $\sigma$ to $\sigma + C$ and $b_1$ to $b_1 + a_0 C$ (mentioned in Remark 1.1) is important here as it changes the spectra of both the operator $S_N$ and the Jacobi matrix $J$. Thus the more correct inverse problem should be not only to reconstruct the operator $\mathcal{A}$ per se, but also to fix the appropriate quasi-derivative $\sigma$ of the potential $q$ and the corresponding Jacobi matrix $J$.

Given an arbitrary operator matrix $\mathcal{A}$ of the form (1.1) (with fixed $\sigma$), we denote by $(\lambda_n)_{n \in \mathbb{N}}, (\mu_{n,N})_{n \in \mathbb{N}}$, and $(\nu_{n,3})_{n=1}^N$ the eigenvalue sequences of $\mathcal{A}$, $S_N$, and $J$ respectively. Put also $(\lambda'_n)_{n \in \mathbb{N}} := (\mu_{n,D}) \Pi (\nu_{n,3})_{n=1}^N$. The above three spectra have the same intersection property as those of Section 3, namely

**Proposition 4.1 (\[1\]).** $\sigma(\mathcal{A}) \cap \sigma(S_N) = \sigma(\mathcal{A}) \cap \sigma(J) = \sigma(S_N) \cap \sigma(J)$.

**Lemma 4.2.** The sequences $(\lambda'_n)$ and $(\lambda_n)$ weakly interlace, i.e., for every $n \in \mathbb{N}$ either $\lambda'_n < \lambda_n < \lambda'_{n+1}$ or $\lambda'_n = \lambda_n = \lambda'_{n+1}$.

**Proof.** We observe that $(\lambda'_n)$ is the sequence of eigenvalues of the operator $\mathcal{A}_0 = S_N \oplus J$ counting multiplicities and that $\mathcal{A}$ and $\mathcal{A}_0$ are self-adjoint extensions of the symmetric operator $\mathcal{A}'$, which is the restriction of $\mathcal{A}'$ onto the domain

$$\mathcal{D}(\mathcal{A}') := \{(y, d)^t \in \mathcal{D}(\mathcal{A}) \mid y^{[1]}(1) = 0\}$$

and has deficiency indices (1,1). We denote by $\mathcal{H}'$ a maximal subspace of $\mathcal{D}(\mathcal{A}')$ that is invariant with respect to $\mathcal{A}'$ and put $\mathcal{H}'' := \mathcal{H} \ominus \mathcal{H}'$. The restrictions of the operators $\mathcal{A}$ and $\mathcal{A}_0$ onto $\mathcal{H}'$ coincide (with $\mathcal{A}'$) and $\dim \mathcal{H}' \leq N$ since if $Y = (y, d)^t$ is an eigenvector of $\mathcal{A}$ that belongs to $\mathcal{H}'$, then $d$ is an eigenvector of $J$. It follows from [7] (see also [2, Ch. 1.2]) that the spectra of the restrictions of $\mathcal{A}$ and $\mathcal{A}_0$ onto the subspace $\mathcal{H}''$ strictly interlace. Combining the two parts together, we see that either $\lambda'_n \leq \lambda_n$ for all $n \in \mathbb{N}$ or $\lambda'_n \geq \lambda'_n$ for all $n \in \mathbb{N}$; however, the inequality $\lambda_n \leq \lambda'_n$ is ruled out for all $n$ sufficiently large by the asymptotics of $\lambda_n$ and $\mu_{n,N}$, see Propositions 2.3 and 2.4. Taking into account the intersection property of Proposition 4.1, we conclude that the spectra weakly interlace in the specified sense. \hfill \Box

**Definition 4.3.** We denote by $\mathcal{L}'_N$ the set of all triples of strictly monotone sequences $\Lambda := ((\lambda_n)_{n \in \mathbb{N}}, (\mu_{n,N})_{n \in \mathbb{N}}, (\nu_{n,3})_{n=1}^N)$ satisfying the following properties:

1. there is an $\ell_2$-sequence $\tilde{\lambda}_n$ such that $\lambda_n = \lceil \pi(n - N) + \tilde{\lambda}_n \rceil^2$;
2. there is an $\ell_2$-sequence $\tilde{\mu}_n$ such that $\mu_n = \lceil \pi(n - \frac{1}{2}) + \tilde{\mu}_n \rceil^2$;
3. the sequences $(\lambda_n)$ and $(\lambda'_n) := (\mu_{n,D}) \Pi (\nu_{n,3})$ weakly interlace in the sense of Lemma 4.2.

We denote by $A_\Lambda$ the set of $n \in \mathbb{N}$ such that $\lambda_n = \lambda'_n$ and put $B_\Lambda := \mathbb{N} \setminus A_\Lambda$.

The results obtained so far show that, for any operator $\mathcal{A}$ of the form (1.1), the corresponding spectral triple $((\lambda_n), (\mu_{n,N}), (\nu_{n,3}))$ form an element of $\mathcal{L}'_N$. In the reverse direction, we shall prove that any element of $\mathcal{L}'_N$ is the spectral triple of the above form. The approach lies in reducing the problem to that of reconstruction of $\mathcal{A}$ from the eigenvalues and the norming constants. Lemmata 2.2, 2.6, and 2.7 imply that the three spectra determine uniquely the norming constants $\alpha_n$ for $n \in B_\Lambda$. Hence, if a given triple $\Lambda \in \mathcal{L}'_N$ is composed of the spectra of some $\mathcal{A}$ and its two parts, then the corresponding norming constants must be related to $\Lambda$ by the corresponding formula.
As a preliminary, we show that any triple in $L_N'$ produces in this way the numbers with correct asymptotics.

**Lemma 4.4.** Assume that $\Lambda = ((\lambda_n), (\mu_n), (\nu_n)) \in L_N'$ and define the functions $\phi, \psi,$ and $\chi$ by the formulae

\[
\phi(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k) \prod_{k=1}^{\infty} \frac{\lambda_{k+N} - \lambda}{\pi^2 k^2},
\]

\[
\psi(\lambda) = \sqrt{\lambda} \prod_{k=1}^{\infty} \frac{\mu_k - \lambda}{\pi^2 (k - \frac{1}{2})^2},
\]

\[
\chi(\lambda) = \prod_{k=1}^{N} (\lambda - \nu_k).
\]

Then the numbers

\[
\beta_n := \frac{\chi(\lambda_n)}{\sqrt{\lambda_n} \phi(\lambda_n) \psi(\lambda_n)}, \quad n \in B_\Lambda,
\]

have the asymptotics

\[
\beta_n = 2 + \tilde{\beta}_n
\]

where the sequence $\tilde{\beta}_n$ belongs to $\ell_2(B_\Lambda)$.

**Proof.** It clearly suffices to prove that

\[
\frac{1}{\beta_n} = \frac{\sqrt{\lambda_n} \phi(\lambda_n) \psi(\lambda_n)}{\chi(\lambda_n)} = \frac{1}{2} + \tilde{\beta}_n
\]

for some sequence $\tilde{\beta}_n \in \ell_2$. In view of the asymptotics of $(\lambda_k)$ and Proposition 2.8, there exists a function $f \in L_2(0, 1)$ such that

\[
\phi(\lambda) = \left( \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} + \int_0^1 f(t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt \right) \prod_{k=1}^{N} (\lambda - \lambda_k)
\]

and thus, for $n > N$,

\[
\dot{\phi}(\lambda_n) = \frac{\cos \sqrt{\lambda_n} + \int_0^1 t f(t) \cos \sqrt{\lambda_n} t dt}{2\lambda_n} \prod_{k=1}^{N} (\lambda - \lambda_k).
\]

Similarly, for some $g \in L_2(0, 1)$ it holds

\[
\psi(\lambda) = \sqrt{\lambda} \cos \sqrt{\lambda} + \sqrt{\lambda} \int_0^1 g(t) \cos \sqrt{\lambda} t dt.
\]

Since the system $\{\sin \sqrt{\lambda_n} t\}_{n>N}$ forms a Riesz basis of $L_2(0, 1)$ [9], for any $h \in L_2(0, 1)$ the sequence

\[
\int_0^1 h(t) \cos \sqrt{\lambda_n} t dt, \quad n > N,
\]

is square summable. The asymptotics of $\lambda_n$ implies that $\cos \sqrt{\lambda_n} = (-1)^{n+N}(1 + b_n)$, where the sequence $(b_n)$ is in $\ell_2$. Combining these relations, we arrive at the representation

\[
\sqrt{\lambda_n} \dot{\phi}(\lambda_n) \psi(\lambda_n) \chi(\lambda_n) = \frac{1}{2} \left( 1 + d_n \right) \prod_{k=1}^{N} \frac{\lambda_n - \lambda_k}{\lambda_n - \nu_k}
\]
with \((d_n) \in \ell_2\), which yields the result. \(\square\)

**Theorem 4.5.** For any \(\Lambda := (\lambda_n, (\mu_n), (\nu_n)) \in \mathcal{S}_N^\prime\) there exist \(a_0 > 0\), a function \(\sigma \in L_2(0, 1)\) and a Jacobi matrix \(J\) of size \(N\) such that \((\lambda_n)\) is the spectrum of the corresponding operator \(A\) in \(L_2(0, 1) \times \mathbb{C}^N\) of the form (1.1), \((\mu_n)\) is the spectrum of the operator \(S_N\), and \((\nu_n)\) is the spectrum of the Jacobi matrix \(J\). The operator \(A\) is unique if and only if the set \(A\) is empty.

**Proof.** We start with constructing the functions \(\phi, \psi, \) and \(\chi\) of Lemma 4.4 and defining the numbers \(\beta_n\) as in (4.1). Next, we put \(\alpha_n = \beta_n\) for \(n \in B_A\), and take \(\alpha_n\) arbitrary positive for \(n \in A_A\). According to Lemma 4.4, \(\alpha_n\) obey the asymptotics \(\alpha_n = 2 + \tilde{\alpha}_n\) with some \((\tilde{\alpha}_n) \in \ell_2\).

By Proposition 2.3, there exists an operator \(A\) of the form (1.1), whose eigenvalues are \(\lambda_n\) and the corresponding norming constants are \(\alpha_n\). We claim that one can fix a primitive of the potential \(q\) of the operator \(S\) and a Jacobi matrix \(J\) in the representation of \(A\) in such a way that \(\mu_n\) are the eigenvalues of the operator \(S_N\) and \(\nu_n\) are the eigenvalues of \(J\).

We take \(k^*\) such that \(\mu_{k^*}\) is not an eigenvalue of \(A\) just found, fix the unique primitive \(\sigma\) of the potential \(q\) of the Sturm–Liouville operator \(S\) such that the relation \((y', - \sigma y')(1, \mu_{k^*}) = 0\) holds, and determine the corresponding Jacobi matrix \(J\) giving the representation (1.1) of \(A\). We denote by \(\mu_{n,N}\) and \(\nu_{n,j}\) the eigenvalues of \(S_N\) and \(J\) and observe that the above choice of \(\sigma\) makes \(\mu_{k^*}\) an eigenvalue of \(S_N\). Due to the construction of \(\beta_n\) and formula (2.2) for \(\alpha_n\), we have the equality

\[
\frac{\psi(\lambda_n)}{\chi(\lambda_n)} = \sqrt{\lambda_n} \prod_{k=1}^N \frac{\mu_{k,N} - \lambda_n}{\pi^2(k - \frac{1}{2})^2} \prod_{k=1}^N (\lambda_n - \nu_{k,j})
\]

for all \(n \in B_A\). Recalling that \(\psi(\lambda_n) = \chi(\lambda_n) = 0\) for \(n \in A_A\), we see that

\[
\psi(\lambda_n) \prod_{k=1}^N (\lambda_n - \nu_{k,j}) = \sqrt{\lambda_n} \chi(\lambda_n) \prod_{k=1}^N \frac{\mu_{k,N} - \lambda_n}{\pi^2(k - \frac{1}{2})^2}
\]

for all \(n \in \mathbb{N}\).

Put

\[
\Phi_1(z) := \frac{\psi(z)}{\sqrt{z}} \prod_{k=1}^N (z - \nu_{k,j}), \quad \Phi_2(z) := \chi(z) \prod_{k=1}^N \frac{\mu_{k,N} - z}{\pi^2(k - \frac{1}{2})^2};
\]

then \(\Phi_1(\lambda_n) = \Phi_2(\lambda_n)\) for all \(n \in \mathbb{N}\), and also \(\Phi_1(\mu_{k^*}) = \Phi_2(\mu_{k^*}) = 0\) (the latter relation follows from the fact that \(\mu_{k^*}\) is among \(\mu_{n,N}\) by the construction of \(S_N\)). In view of Proposition 2.8 the functions \(\Phi_j\) have the form

\[
\Phi_j(z) = p_j(z) \left( \cos \sqrt{z} + \int_0^1 g_j(t) \cos \sqrt{z} t dt \right)
\]

for some monic polynomials \(p_j\) of degree \(N\) and some functions \(g_j \in L_2(0, 1), j = 1, 2\). It follows that \(\Phi := \Phi_1 - \Phi_2\) is an entire function of exponential type \(\frac{1}{2}\) with zeros \(\{\lambda_n\}_{n \in \mathbb{N}} \cup \{\mu_{k^*}\}\) such that

\[
(4.2) \quad \Phi(z) = o\left( z^N e^{\text{Im} \sqrt{z}} \right)
\]

as \(|z| \to \infty\). Next we show as in the proof of Theorem 3.5 that \(\Phi \equiv 0\) by noticing that otherwise \(\Phi\) would have no zeros other than \(\lambda_n, n \in \mathbb{N}\), and \(\mu_{k^*}\), and that the canonical product for \(\Phi\) then would contradict the estimate (4.2).
Thus $\Phi_1 \equiv \Phi_2$, which together with the weak interlacing property of $(\lambda_n)$ and $(\lambda_n')$ as well as of $(\lambda_n)$ and $(\mu_{n,N})$ $(\nu_{n,1})$ shows that $\mu_n = \mu_{n,N}$ for all $n \in \mathbb{N}$ and that $\nu_k = \nu_{k,1}$ for $k = 1, \ldots, N$. Uniqueness statement follows from Proposition 2.3, and the proof is complete.

We remark that the set of $\Lambda$-isospectral operators $\mathcal{A}$ is again a manifold of dimension equal to the cardinality of the set $A_\Lambda$.

Acknowledgements. The research of R. H. was partially supported by the Alexander von Humboldt Foundation.

References


Verzeichnis der erschienenen Preprints ab No. 220

220. Otto, Felix; Rump, Tobias; Slepčev, Dejan: Coarsening Rates for a Droplet Model: Rigorous Upper Bounds

221. Gozzi, Fausto; Marinelli, Carlo: Stochastic Optimal Control of Delay Equations Arising in Advertising Models

222. Griebel, Michael; Oeltz, Daniel; Vassilevsky, Panayot: Space-Time Approximation with Sparse Grids

223. Arndt, Marcel; Griebel, Michael; Novák, Václav; Šitný, Petr; Roubíček, Tomáš: Martensitic Transformation in NiMnGa Single Crystals: Numerical Simulation and Experiments

224. DeSimone, Antonio; Knüpfer, Hans; Otto, Felix: 2-d Stability of the Néel Wall

225. Griebel, Michael; Metsch, Bram; Oeltz, Daniel; Schweitzer, Marc Alexander: Coarse Grid Classification: A Parallel Coarsening Scheme for Algebraic Multigrid Methods

226. De Santis, Emilio; Marinelli, Carlo: Stochastic Games with Infinitely many Interacting Agents


228. Verleye, Bart; Kiltz, Margrit; Croce, Roberto; Roose, Dirk; Lomov, Stepan; Verpoest, Ignasi: Computation of Permeability of Textile Reinforcements; erscheint in: Proceedings, Scientific Computation IMACS 2005, Paris (France), July 11-15, 2005

229. Albeverio, Sergio; Pustylnikov, Lev; Lokt, Tatiana; Pustylnikov, Roman: New Theoretical and Numerical Results Associated with Dirichlet L-Functions


231. Albeverio, Sergio; Pratsiovytyi, Mykola; Torbin, Grygoriy: Singular Probability Distributions and Fractal Properties of Sets of Real Numbers Defined by the Asymptotic Frequencies of their S-Adic Digits

232. Philipowski, Robert: Interacting Diffusions Approximating the Porous Medium Equation and Propagation of Chaos

233. Hahn, Atle: An Analytic Approach to Turaev's Shadow Invariant
234. Hildebrandt, Stefan; von der Mosel, Heiko: Conformal Representation of Surfaces, and Plateau’s Problem for Cartan Functionals

235. Grunewald, Natalie; Otto, Felix; Reznikoff, Maria G.; Villani, Cédric: A Two-Scale Proof of a Logarithmic Sobolev Inequality

236. Albeverio, Sergio; Hryniv, Rostyslav; Mykytyuk, Yaroslav: Inverse Spectral Problems for Coupled Oscillating Systems: Reconstruction by Three Spectra