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Lévy Black-Scholes Setting

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no. 240
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Abstract

The price of an Asian Option driven by Lévy type noise is computed. A rather explicit formula is given in the case of a Variance Gamma driving process.

Keywords: Asian Option Pricing, Lévy Processes, Bessel Processes, Variance Gamma Processes, Laplace Transformation, Esscher Transform

AMS-Classification: 60G10, 60G51, 44A10, 60H15, 60H30, 90A09, 90A12

1 Introduction

In this paper we discuss the pricing problem of Asian options in a general Lévy Black-Scholes setting. That is we consider a stock prices process $S_t$ which is driven by a Lévy process $Y_t$ with drift:

$$dS_t = \sigma S_t dY_t + \mu S_t dt,$$

having a positive deterministic initial value $S_0$ at time $t_0$. From the Doléans-Dade exponential formula we derive a solution of such a stochastic differential equation which is of the form

$$S_t = S_0 \exp(Y_t^*),$$

where $Y_t^*$ is again a Lévy process. The generating triplet of this process $Y_t^*$ can be derived from the generating triplet of the process $Y_t$. We use an equivalent martingale measure, derived by means of the Esscher transform, such that the discounted stock price is a martingale and $Y$ is still a Lévy process under this new measure. This latter property is important since we only should change the distribution of the asset price but not the property of independent and stationary increments of the driving process. Then we consider an arithmetic Asian option, the value of which at the terminal time $T$ depends on the mean of the whole path of the price process up to time $T$, i.e.

$$\frac{1}{T-t_0} \int_{t_0}^{T} S_t dt, \quad t_0 < T.$$

The problem of pricing a path dependent option is to determine the option price at inception time $t_0$. This leads to the problem of computing a discounted condition expectation of the form

$$C_F(K, t) = e^{-r(T-t)} \cdot E_{P(e)} \left[ \left( \frac{1}{T-t_0} \int_{t_0}^{T} S_0 \exp(Y_u^*) du - K \right)_+ \mid \mathcal{F}_t \right],$$
$\mathcal{F}_t$ being the filtration underlying the process $(Y_t)_{t \geq 0}$ at time $t$. To calculate this, we use the method of Laplace transformation introduced by [GemYo] for the classical Black-Scholes setting. Then we can reduce the problem to the task of computing a conditional expectation of the form

$$E \left[ e^{-\lambda \tau_q} \exp \left( Y^*_\tau_q \right) \right] = s,$$

where $\tau_q$ is a stopping time defined by

$$\tau_q = \inf \left\{ s \geq 0 : A_s := \int_0^s \exp(Y^*_u) du > q \right\}, \quad q > 0.$$

Finally, we consider the case of a Variance Gamma process $Y_t$. In that situation we derive an explicit pricing formula in terms of an inverse Laplace transform for the price of an Asian option of the stock price process driven by a Variance Gamma process with drift. Here we use the fact that any Variance Gamma process can be expressed as a standard Brownian motion with independent gamma subordinator. Moreover we use a characterization in terms of time changed Bessel processes for the exponential of Brownian motion. This idea was first used in [GemYo]. The inverse Laplace transform describing the Asian option price can then be computed numerically.

2 Lévy Processes

In this section we recall basic concepts and results of Lévy processes. For more details and proofs see, e.g. [Ap], [Ber], [IW], [Ru], [Sa].

**Definition 2.1** A stochastic process $\{X_t\}_{t \geq 0}$ on $\mathbb{R}^d$ is called a Lévy process if the following conditions are satisfied:

(i) For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent (independent increments);

(ii) $X_0 = 0$ a.s.;

(iii) The distribution of $X_{s-t} - X_s$ does not depend on $s$, i.e. it equals the law of $Y_t - Y_0 = Y_t$ (stationary increments);

(iv) it is stochastically continuous, i.e. for every $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s \to t} P(|X_s - X_t| > \epsilon) = 0;$$

(v) there is an $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $X_t(\omega)$ is càdlàg, i.e. $X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

A stochastic process is called a Lévy process in law, if it satisfies (i)-(iv). For any Lévy process in law there exists a unique modification which is càdlàg and which is also a Lévy process. For these concepts and results see, e.g., [Ap], [Sa], [Be]. In this case (i)-(v) hold, and we have a Lévy process.
2.1 Lévy-Itô Decomposition

Any Lévy process on \( \mathbb{R}^d \) can be written as the linear combination of a \( d \)-dimensional Brownian motion with a deterministic constant drift and a \( d \)-dimensional jump process. This property is called the Lévy-Itô decomposition. We will denote the drift process by \( \gamma_t \) for some \( \gamma \in \mathbb{R}^d \) and the jump process by \( M_t \). Then the Lévy process \( Y_t \) can be written in its decomposed form as

\[
Y_t = B_t + \gamma t + M_t,
\]

where \( B_t \) is a \( d \)-dimensional Brownian motion with covariance matrix \( A \in \mathbb{R}^{d \times d} \).

The value of the jump process \( Y \) just before the jump at time \( t > 0 \) is denoted by \( Y_t^- := \lim_{s \uparrow t} Y_s \). The jump of \( Y \) at time \( t \) is \( \Delta Y_t = Y_t - Y_t^- \). Since any Lévy process is stochastically continuous, one has that for a fixed time \( t > 0 \), the jump \( \Delta Y_t = 0 \) a.s..

The Lévy process can, in general, jump infinitely often. However, as we shall see later, there exists only a finite number of jumps of size greater than 1. Thus a Lévy process can only possess, in general, an infinite number of "small jumps" (i.e., jumps of size less than 1).

To specify the jump process \( M_t \) we count the jumps of a certain size. Therefore we define for each \( \omega \in \Omega \) the counting measure

\[
J(t, A)(\omega) := \#\{0 \leq s \leq t : \Delta Y_s(\omega) \in A\} = \sum_{0 \leq s \leq t} 1_A(\Delta Y_s(\omega)).
\]

**Definition 2.2** \( A \in \mathcal{B}(\mathbb{R}^d\{0\}) \) is bounded from below if \( 0 \neq A \).

A set \( A \in \mathcal{B}(\mathbb{R}^d\{0\}) \) which is bounded from below satisfies \( \inf\{|x|, x \in A\} > 0 \) and thus \( J(1, A) < \infty \) a.s..

**Definition 2.3** Let \( f \) be a Borel measurable function from \( \mathbb{R}^d \) to \( \mathbb{R}^d \) and let \( A \) be a set which is bounded from below, then for each \( t > 0, \omega \in \Omega \), we may define the Poisson integral of \( f \) on \( A \) as a random finite sum by

\[
\int_A f(x)J(t, dx)(\omega) = \sum_{x \in A} f(x)J(t, \{x\})(\omega).
\]

As a consequence we obtain that

\[
\int_A xJ(t, dx)(\omega) = \sum_{0 \leq s \leq t} \Delta Y_s(\omega)
\]

defines the height of all jumps of \( Y_s(\omega) \) up to time \( t \).

For each \( \omega \in \Omega, t \geq 0 \), the set function \( A \mapsto J(t, A)(\omega) \) is a counting measure on \( \mathcal{B}(\mathbb{R}^d\{0\}) \) and hence

\[
E[J(t, A)] = \int_{\Omega} J(t, A)(\omega) dP(\omega)
\]
is a Borel measure on \( \mathcal{B}(\mathbb{R}^d\{0\}) \). We define the intensity measure associated with \( Y \) by

\[
\nu := E[J(1, \cdot)].
\]

As already mentioned before a Lévy process can have infinitely many jumps. Thus the intensity measure associated with \( Y \) is not necessarily finite, i.e. \( \int_{\mathbb{R}^d} \nu(dx) = \infty \). The
function \( s \mapsto Y_s(\omega) \) has only a finite number of jumps greater than one on \([0, t]\) for each fixed \( \omega \), since the process has càdlàg paths. Hence there must be an infinite number of small jumps which cause the series of the norm of jumps, \( \sum_{0 \leq s \leq t} |\Delta Y_s| \), to diverge for all \( t > 0 \). Moreover, \( \int_{|x| < 1} x \nu(dx) = \infty \).

One can show, see e.g. [Ap], that \( \sum_{0 \leq s \leq t} |\Delta Y_s|^2 \) converges, such that \( \int_{\mathbb{R}^d} x^2 \nu(dx) < \infty \) always holds. This implies that, for all \( t > 0 \), \( \sum_{0 \leq s \leq t} (|\Delta Y_s|^2 \wedge 1) \) is locally integrable, and therefore \( \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty \).

**Definition 2.4** Let \( \nu \) be a Borel measure defined on \( \mathbb{R}^d \). It is a Lévy measure if

\[
\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty.
\]

Since the Lévy process can have an infinite number of small jumps, in which case \( \int_{|x| < 1} x J(t, dx) \) does not necessarily exist, we cannot define the jump process \( M_t \) by \( M_t = \int_{\mathbb{R}^d} x J(t, dx) \).

We will first split the jump process into two processes, i.e. in one process which has only jumps larger than one,

\[
M_t^{(2)} := \int_{|x| \geq 1} x J(t, dx),
\]

and a process with jumps less than 1,

\[
M_t^{(1)} := \int_{|x| < 1} x (J(t, dx) - t \nu(dx)) =: \int_{|x| < 1} x \tilde{J}(t, dx),
\]

with \( \tilde{J}(t, dx) = J(t, dx) - t \nu(dx) \). The term \( -x t \nu(dx) \) is called the compensator and is necessary in order to ensure the existence of the integral in \( M_t^{(1)} \).

**Definition 2.5** For each \( f \in L^1(A, \nu), t \geq 0 \), define the compensated Poisson integral by

\[
\int_A f(x) \tilde{J}(t, dx) = \int_A f(x) J(t, dx) - t \int_A f(x) \nu(dx).
\]

We write the jump process of the Lévy process as

\[
M_t = M_t^{(1)} + M_t^{(2)} = \int_{|x| < 1} x \tilde{J}(t, dx) + \int_{|x| \geq 1} x J(t, dx).
\]

**Theorem 2.6 (Lévy-Itô Decomposition)** Let \( Y \) be a Lévy process, then there exists a vector \( \gamma \in \mathbb{R}^d \), a Brownian motion \( B \) with covariance \( A \in \mathbb{R}^{d \times d} \) and an independent Poisson random measure \( J \) on \( H = (0, \infty) \times (\mathbb{R}^d \setminus \{0\}) \) such that for each \( t \geq 0 \),

\[
Y_t = B_t + \gamma t + \int_{|x| < 1} x (J(t, dx) - t \nu(dx)) + \int_{|x| \geq 1} x J(t, dx)
\]

\[
= B_t + \gamma t + \int_{|x| < 1} x \tilde{J}(t, dx) + \int_{|x| \geq 1} x \tilde{J}(t, dx)
\]

\[
= B_t + \gamma t + M_t,
\]

where \( \gamma = E \left[ Y_1 - \int_{|x| \geq 1} x J(t, dx) \right] \).

**Proof.** See, for example [Ap], Theorem 2.4.6., or [Ru].\[\square\]
2.2 Lévy-Khintchine Representation

**Definition 2.7** The characteristic function \( \hat{\mu} : \mathbb{R}^d \to \mathbb{C} \) of a probability measure \( \mu \) on \( \mathbb{R}^d \) is defined as
\[
\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i \langle z, y \rangle} \mu(dy), \quad z \in \mathbb{R}^d,
\]
where \( \langle \cdot, \cdot \rangle \) denotes a scalar product of two vectors.

The characteristic function of the distribution \( P_{Y_1} \) of a random process \( Y_t \) on \( \mathbb{R}^d \) is denoted by
\[
\hat{P}_{Y_1}(z) = \int_{\mathbb{R}^d} e^{i \langle z, y \rangle} P_{Y_1}(dy) = E \left[ e^{i \langle z, Y_1 \rangle} \right] =: \Phi_{Y_1}(z), \quad z \in \mathbb{R}^d.
\]

In this section we want to derive an expression for the characteristic function of a \( d \)-dimensional Lévy process which can be written (as we saw in the previous subsection) as a linear combination of a Brownian motion, a linear process and a jump process. Since all these processes are independent we know that the characteristic function of a Lévy process will be the product of the separate characteristic functions of each component. The characteristic function of the Brownian motion can easily be computed and is given by
\[
\Phi_{B_t}(z) = E \left[ e^{i \langle z, B_t \rangle} \right] = e^{-\frac{1}{2} t \langle z, A z \rangle}, \quad z \in \mathbb{R}^d.
\]

The characteristic function of a linear process \( \gamma_t, \gamma \in \mathbb{R}^d \), is
\[
E \left[ e^{i \langle z, \gamma t \rangle} \right] = e^{1 \langle z, \gamma \rangle t}.
\]

It remains to compute the characteristic function of the jump process
\[
M_t = M^{(1)}_t + M^{(2)}_t
\]
with
\[
M^{(1)}_t := \int_{|x|<1} x (J(t, dx) - t \nu(dx)) \quad \text{and} \quad M^{(2)}_t := \int_{|x|\geq 1} x J(t, dx).
\]

**Theorem 2.8** Let \( A \in \mathcal{B}(\mathbb{R}^d) \) be bounded from below. Then \( (J(t, A), t \geq 0) \) is a Poisson process with intensity \( \nu(A) \).

**Proof.** See for example [Ap], Theorem 2.3.5, page 88. \( \square \)

**Lemma 2.9** The characteristic function of \( M^{(2)}_t \) is given by
\[
\Phi_{M^{(2)}_t}(z) = E \left[ e^{i \langle z, M^{(2)}_t \rangle} \right] = \exp \left( t \int_{|x|\geq 1} \left( e^{i \langle z, x \rangle} - 1 \right) \nu(dx) \right), \quad z \in \mathbb{R}^d.
\]

**Proof.** Follows from [Ap], Theorem 2.3.8, page 91. See also [Be], Lemma 2, page 11. \( \square \)

**Lemma 2.10** The characteristic function of \( M^{(1)}_t \) is given by
\[
\Phi_{M^{(1)}_t}(z) = E \left[ e^{i \langle z, M^{(1)}_t \rangle} \right] = \exp \left( t \int_{|x|<1} \left( e^{i \langle z, x \rangle} - 1 - i \langle z, x \rangle \right) \nu(dx) \right), \quad z \in \mathbb{R}^d.
\]
Proof. Follows immediately from Lemma 2.9 and Theorem 2.3.8 in [Ap], page 91. □

The above Lemmata show that the characteristic function of the jump process $M_t = M_t^{(1)} + M_t^{(2)}$ is given by

$$\Phi_{M_t}(z) = \exp \left( t \int_{\mathbb{R}^d} \left( e^{izx} - 1 - i(z, x)1_{\{|x|<1\}}(x) \right) \nu(dx) \right), \quad z \in \mathbb{R}^d.$$ 

Since the Lévy measure $\nu$ integrates $(|x|^2 \wedge 1)$ the integral $\int_{\mathbb{R}^d} \left( e^{izx} - 1 - i(z, x)1_{\{|x|<1\}}(x) \right) \nu(dx)$ exists and is finite.

The characteristic function of the Lévy process $Y_t$ is the multiplication of the characteristic functions of each independent component of the decomposition and is thus given by

$$E \left[ e^{iz_Y} \right] = \exp \left( t \left( -\frac{1}{2}(z, Az) + i(\gamma, z) + \int_{\mathbb{R}^d} \left( e^{izx} - 1 - i(z, x)1_{\{|x|<1\}}(x) \right) \nu(dx) \right) \right).$$

Because of the independent increments property the distribution of a Lévy process is uniquely determined by $P_{Y_t}$. Therefore the Lévy process $Y_t$ can be characterized by its characteristic function for $t = 1$. This is known as the Lévy-Khintchine representation (see, e.g. [Ap], [Sa],[Ber]).

Theorem 2.11 (Lévy-Khintchine representation) The characteristic function of the distribution of a $d$-dimensional Lévy process is given by

$$\Phi_{Y_1}(z) = E \left[ e^{iz_Y} \right]$$

where $A$ is a symmetric nonnegative-definite $d \times d$-matrix, $\nu$ is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty$, and $\gamma \in \mathbb{R}^d$. The representation (3) by $A, \nu$ and $\gamma$ is unique. The measure $\nu$ is called the Lévy measure of $P_{Y_1}$ and $(\gamma, A, \nu)$ is called the generating triplet of $P_{Y_1}$.

Proof. See for example [Sa], Chapter 2.8, Theorem 8.1 (ii), page 37. □

Example 2.12 (Lévy measure of a Variance Gamma process) The Variance Gamma process is a Lévy process $X_t$ that has a Variance Gamma law $VG(\sigma, \nu, \theta)$. Its characteristic function is

$$E \left[ \exp(auX_t) \right] = \left( 1 - iu\theta\nu + \frac{1}{2}\sigma^2 \nu u^2 \right)^{-\frac{t}{\nu}}, \quad u \in \mathbb{R}, \quad \theta \in \mathbb{R}, \quad \sigma \in \mathbb{R}.$$ 

It can be characterized as a time changed Brownian motion with drift

$$X_t = \theta\gamma(t) + \sigma W_{\gamma(t)},$$

where $W$ is a Brownian motion and $\gamma$ is a Gamma $G(1/\nu, 1/\nu)$ process with parameters $1/\nu$ and $1/\nu$ (see, e.g., [Ap]). The Variance Gamma process is a finite variation process, hence it is the difference of two increasing processes, and in particular (see [MaCaCh]) of two Gamma processes

$$X_t = G(t; \mu_1, \gamma_1) - G(t; \mu_2, \gamma_2),$$
where \( G(t; \mu, \gamma) \) denotes a Gamma process with parameters \( \mu \) and \( \gamma \) at time \( t \). The characteristic function can be factorized

\[
E[\exp(iuX_t)] = \left( 1 - \frac{iu}{\nu_1} \right)^{-\frac{\mu}{\gamma}} \left( 1 + \frac{iu}{\nu_2} \right)^{-\frac{\mu}{\gamma}}
\]

with

\[
\nu_1^{-1} = \frac{1}{2} \left( \nu + \sqrt{\nu^2 + 2\nu^2} \right) \quad \text{and} \quad \nu_2^{-1} = \frac{1}{2} \left( \nu - \sqrt{\nu^2 + 2\nu^2} \right).
\]

The Lévy density of \( X \) is

\[
\frac{1}{\gamma} \frac{1}{\lvert x \rvert} \exp(-\nu_1 \lvert x \rvert) \quad \text{for} \quad x < 0
\]

and

\[
\frac{1}{\gamma} \frac{1}{x} \exp(-\nu_2 x) \quad \text{for} \quad x > 0.
\]

The density of \( X_1 \) is

\[
\frac{2e^{\sigma^2 x^2}}{\gamma^{1/2} \sqrt{2\pi\sigma^2 \Gamma(1/2)}} \left( \frac{x^2}{\theta^2 + 2\sigma^2} \right)^{1/2} \frac{K_{1/2} \left( \frac{1}{\sigma^2} \sqrt{x^2 \left( \theta^2 + 2\sigma^2 \right)} \right)}
\]

where \( K_\alpha \) is the modified Bessel function.

Stock prices given by (1) respectively (2) when driven by a Variance Gamma process get the special form

\[
S_t = S_0 \exp \left( rt + X(t; \sigma, \nu, \theta) + \frac{t}{\nu} \ln \left( 1 - \theta\nu - \frac{\sigma^2}{2} \right) \right).
\]

From \( E[e^{X_t}] = \exp \left( -\frac{t}{\nu} \ln \left( 1 - \theta\nu - \frac{\sigma^2}{2} \right) \right) \), we get that \( S_te^{-rt} \) is a martingale.

The Lévy measure of a Variance Gamma process is

\[
\nu(x) = \frac{C}{|x|^{1/2}} e^{x(\lambda - \lambda_+)/2} \cdot K_{1/2}(|x|(|\lambda_- - \lambda_+|)/2),
\]

where \( \lambda_-, \lambda_+, C \) are some positive constants and \( K_{1/2} \) is a Bessel function of the third kind, with index \( \frac{1}{2} \) (see, e.g. [Ap], page 290).

**Definition 2.13** The moment generating function of the \( d \)-dimensional random variable \( Y_t \) is defined as

\[
\operatorname{mgf}_{Y}(\theta) = E \left[ e^{i\langle \theta, Y_t \rangle} \right]
\]

for all \( \theta \in \mathbb{R}^d \), provided \( E \left[ e^{i\langle \theta, Y_t \rangle} \right] < \infty \). Or equivalently,

\[
\operatorname{mgf}_{Y}(\theta)^t = E \left[ e^{i\langle \theta, Y_t \rangle} \right].
\]

Thus \( \operatorname{mgf}_{Y}(\theta)^t \) is the Laplace transform of the probability distribution of \( Y_t \) evaluated at \( \theta \). The following relations between the moment generating function (whenever it exists) and the characteristic function hold

\[
E \left[ e^{i(z,Y_t)} \right] = \operatorname{mgf}_{Y}(iz) = \Phi_{Y_t}(z), \quad z \in \mathbb{R}^d,
\]

\[
E \left[ e^{i(z,Y_t)} \right] = \operatorname{mgf}_{Y}(iz)^t = \Phi_{Y_t}(z), \quad z \in \mathbb{R}^d,
\]

The moment generating function does not necessarily exist. However, the above connection holds for all \( z \in \mathbb{R}^d \) since the moment generating function always exists on \( i\mathbb{R}^d \), i.e., \( \operatorname{mgf}_{Y}(iz) < \infty \), for all \( z \in \mathbb{R}^d \).
Theorem 2.14 (Exponential Moment) Let $Y_t$ be a Lévy process in $\mathbb{R}^d$ with generating triplet $(A, \gamma, \nu)$. Let

$$C := \left\{ c \in \mathbb{R}^d : \int_{|x|>1} e^{c\langle x \rangle} \nu(dx) < \infty \right\}.$$ 

(i) The set $C$ is convex and contains the origin.

(ii) $c \in C$ if and only if $E \left[ e^{c\langle Y_t \rangle} \right] < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$.

(iii) If $z \in \mathbb{C}^d$ is such that $\Re z \in C$, then

$$\Psi(z) = \frac{1}{2} \langle z, A z \rangle + \langle \gamma, z \rangle + \int_{\mathbb{R}^d} \left( e^{\langle z, x \rangle} - 1 - \langle z, x \rangle 1_{\{|x|\leq 1\}}(x) \right) \nu(dx)$$

is definable, $E \left[ e^{\langle z, Y_t \rangle} \right] < \infty$, and

$$E \left[ e^{\langle z, Y_t \rangle} \right] = e^{t\Psi(z)}, \quad t > 0.$$

Proof. See for example [Sa], Theorem 25.17, page 165. 

3 The Black-Scholes Model in a Lévy Setting

We consider a stock price $S_t$ driven by a general geometric Lévy process

$$dS_t = \sigma S_t dY_t + \mu S_t dt,$$

having a positive initial value $S_0$ at time $t_0$, where $Y_t$ is a Lévy process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. Here $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the minimal filtration generated by $Y_t$. The drift $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$ are constant. By Theorem 2.6 the Lévy process $Y_t$ can be decomposed into

$$Y_t = B_t + \gamma t + M_t = \sqrt{c}W_t + \gamma t + M_t$$

where $c > 0$ is the variance of the one-dimensional Brownian motion $B_t$ and $W_t$ is a standard Brownian motion. $M_t$ is a jump process. Then the stock price process satisfies the following stochastic differential equation

$$dS_t = S_t d\sigma B_t + S_t dM_t + (\sigma \gamma + \mu) dt.$$ 

Applying the Doléans-Dade exponential formula we know that the solution of this stochastic differential equation is given by

$$S_t = S_0 \cdot \exp \left\{ \sigma B_t + \sigma M_t + (\sigma \gamma + \mu) t - \frac{1}{2} \left[ \sigma B, \sigma B \right]_t \right\} \times \prod_{t_0 < s \leq t} (1 + \sigma \Delta M_s) \exp \{-\sigma \Delta M_s\}$$

$$= S_0 \cdot \exp \left\{ \sigma B_t + \sigma M_t + \left( \sigma \gamma + \mu - \frac{\sigma^2}{2} \right) t \right\} \times \prod_{t_0 < s \leq t} (1 + \sigma \Delta M_s) \exp \{-\sigma \Delta M_s\}.$$ 

Here $\left[ , \right]_t$ denotes the quadratic variation process (see, e.g., [Sa]). The term

$$\prod_{t_0 < s \leq t} (1 + \sigma \Delta M_s) \exp \{-\sigma \Delta M_s\}$$ 

is definable, $E \left[ e^{\langle z, Y_t \rangle} \right] < \infty$, and

$$E \left[ e^{\langle z, Y_t \rangle} \right] = e^{t\Psi(z), \quad t > 0}.$$
is to be understood in the sense that we multiply over all times \( s \) for which \( \Delta M_s \) is positive, that is we multiply over all jump times. We assume that the company never goes bankrupt, otherwise the analysis gets more complicated. Thus we assume that the stock price \( S_t \) is strictly positive for all times \( t > t_0 \). This restriction is equivalent to

\[
1 + \sigma \Delta M_s > 0 \quad \text{for all} \quad s > t_0
\]

\[
\Leftrightarrow \quad \Delta M_s > -\frac{1}{\sigma} =: \alpha \quad \text{for all} \quad s > t_0
\]

where we assumed that \( \sigma \neq 0 \). If \( 0 < \sigma < 1 \), then \( \alpha < -1 \). The restriction on the jump part of the Lévy process yields

\[
Y_t = B_t + \gamma t + \int_0^\infty x (J(t, dx) - t\nu(dx)) + \int_{[1\infty)} x\nu(dx)t
\]

\[
Y_t = B_t + mt + \int_0^\infty x (J(t, dx) - t\nu(dx)),
\]

where \( m = \gamma + \int_{[1\infty]} x\nu(dx) \). The Fourier transform of the Lévy process with restricted jumps is given by

\[
\hat{P}_{Y_t}(z) = \exp\left(-\frac{1}{2}cz^2 + izz + \int_0^\infty (e^{izx} - 1 - izx) \nu(dx)\right),
\]

which means that the generating triplet of \( Y_t \) is \((m, c, \nu)\).

As proved in [Be], Lemma 4, page 31, the solution of the stochastic differential equation (4) takes the form

\[(5) \quad S_t = S_0 \exp(Y_t^*),\]

where \( Y_t^* \) is a Lévy process with generating triplet

\[(6) \quad (\sigma m + \mu - c\sigma^2/2 + \int_0^\infty (\ln(1 + \sigma x) - \sigma x)\nu(dx), \sigma^2 c, \nu \circ f^{-1}) \equiv (\gamma^*, c^*, \nu^*),\]

where \( f(x) = \ln(1 + \sigma x) \) and \( \sigma < x < \infty \). This follows mainly by an application of the Doléans-Dade exponential formula. Consequently we can describe the stock price process by means of an exponential Lévy process \( Y_t^* \). We denote the discounted stock price by \( \bar{S}_t \), i.e.

\[(7) \quad \bar{S}_t := e^{-rt}S_t = S_0 \exp(Y_t^* - rt) = S_0 \exp(\bar{Y}_t), \quad \text{where} \quad \bar{Y}_t = Y_t^* - rt.\]

As a straightforward computation shows, \( \bar{Y}_t \) is a Lévy process with generating triplet

\[(8) \quad (\sigma m + \mu - c\sigma^2/2 - r + \int_0^\infty (\ln(1 + \sigma x) - \sigma x)\nu(dx), \sigma^2 c, \nu \circ f^{-1}) \equiv (\gamma^*, c^*, \nu^*),\]

with \( f(x) = \ln(1 - \sigma x) \).

**Definition 3.1** Let \( L \) be a Lévy process on some filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) with an existing moment generating function \( \text{mgf}_L(\theta) \neq 0 \) for some \( \theta \in \mathbb{R} \). An Esscher transform is any change of \( P \) to a locally equivalent measure \( P^{(\theta)} \) with a density process \( Z_t^{(\theta)} = \frac{dP^{(\theta)}}{dP} \mid_{\mathcal{F}_t} \) of the form

\[
Z_t^{(\theta)} = \frac{\exp(\theta L_t)}{\text{mgf}_L(\theta)}, \quad \theta \in \mathbb{R}.
\]
Proposition 3.2  

$Z_t$ defined as above describes a density process for all $\theta \in \mathbb{R}$ such that $E[\exp(\theta L_t)] < \infty$. $L$ is again a Lévy process under the new measure $P^{(\theta)}$.

Proof. For the proof see [Ra], Proposition 1.8, page 7.

In [Be] an equivalent martingale measure is derived by means of the Esscher transform such that the discounted stock price $\tilde{S}_t$ is a martingale and $Y$ is still a Lévy process under this new measure. This latter property is important since we only should change the distribution of the asset price but not the property of independent and stationary increments of the driving process. We recall that [GerSh] were the first who proposed to use the Esscher transforms. Bernoth in [Be] derived an explicit formula for the characteristic function of the Lévy process under the new probability measure and for the generating triplet of the Lévy process. This result is summarized in the following Proposition.

Proposition 3.3  

Let $Y^*$ be a Lévy process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with corresponding characteristic function

$\Phi_{Y^*}(z) = E\left[e^{izY_t^*}\right] = \exp\left\{-\frac{1}{2}c^*z^2 + i\gamma^*z + \int_{\alpha}^{\infty} (e^{izx} - 1 - izx) \nu^ *(dx)\right\}$, $z \in \mathbb{R}$,

where $(c^*, \gamma^*, \nu^*)$ denotes the generating triplet of the process $Y^*$. Let $P^{(\theta)}$ be an equivalent probability measure corresponding to the Esscher transform $Z^{(\theta)}$ with $\theta \in (a, b-1)$. Then under the measure $P^{(\theta)}$ the characteristic function of the Lévy process, denoted by $\Phi_{Y^*}^{(\theta)}$, takes the form

$\Phi_{Y^*}^{(\theta)}(z) = \frac{\Phi_{Y^*}^{(\theta)}(z - i\theta)}{\Phi_{Y^*}^{(\theta)}(-i\theta)}$.  

Moreover, the generating triplet of the Lévy process $Y^*$ under the probability measure $P^{(\theta)}$ is given by

$c^{(\theta)}_\theta = c^*$,

$\nu^{(\theta)}_\theta(dx) = e^{x\theta}\nu^*(dx)$,

$\gamma^{(\theta)}_\theta = \gamma^* + c^*\theta - \int_{\alpha}^{\infty} \left(1 - e^{x\theta}\right) x\nu^*(dx)$.

Proof. The proof of (9) is by a single computation. The rest uses (8) and simple change of variables. Details are given in [Be] (Lemma 7 and Proposition 7, page 40/41).

4 The Pricing Problem of Asian Options

We consider a stock price process $S_t$ driven by the stochastic differential equation (4) where $Y_t$ is a Lévy process as before. As we already mentioned the solution of such a stochastic differential equation is given in form of an exponential Lévy process (5) where the generating triplet of $Y^*$ can be constructed from the generating triplet of $Y$.

The problem we will consider in the following is to calculate the price of an Asian Option which is a path dependent option, that means that the value of such an option at maturity time $T$ can depend on the whole path of the price process from inception time up to time $T$. We will consider only arithmetic Asian options of European type which are defined by the arithmetic mean of the stock price over the time period $T$. For geometric Asian options there exists a generalized Black Scholes formula, hence a closed form solution for
the pricing problem (see for example [Hu], page 467). However, the pricing problem for arithmetic Asian options is still under investigation since no closed form solution is known up to now. The value of an arithmetic Asian Option at the terminal time $T$ depends on the mean of the whole path of the price process up to time $T$,

$$\frac{1}{T - t_0} \int_{t_0}^T S_t dt, \quad t_0 < T.$$ 

Consider the process $\{A(\tau), \tau > t_0\}$, where

$$A(\tau) := \frac{1}{\tau - t_0} \int_{t_0}^{\tau} S_t dt, \quad \tau > t_0.$$ 

which is a time integral over the stock price process (5). We will sometimes write $A_t$ instead of $A(t)$. Now the problem of pricing a path dependent option is to determine the option price at inception time $t_0$. We will denote the payoff function of a path dependent option by $F(A_T)$. In the risk-neutral framework of the classical Black-Scholes model (i.e. of an asset following a geometric Brownian motion) the option price at any time $t$ equals the value of a self-financing and replicating portfolio $V_t$ at time $t$. Under the equivalent martingale measure $P^{(\theta)}$ (defined via the Escher transform of the original probability measure $P$ for some $\theta \in \mathbb{R}$) the discounted value of the portfolio, $e^{-rt}V_t$ is a $P^{(\theta)}$-martingale (see [Be], Theorem 9, Chapter 3). Hence we obtain

$$e^{-rt}V_t(\phi) = E_{P(\theta)}[e^{-rT}V_t(\phi)|\mathcal{F}_t] = E_{P(\theta)}[e^{-rT}F(A_T)|\mathcal{F}_t]$$

$$\Leftrightarrow V_t(\phi) = e^{-r(T-t)}E_{P(\theta)}\left[F\left(\frac{1}{T - t_0} \int_{t_0}^{T} S_t dt\right) | \mathcal{F}_t\right].$$

For an Asian Call Option the payoff function is

$$F(S_T, t_0) = \max\{A_T - K, 0\} = \max\left\{\frac{1}{T - t_0} \int_{t_0}^{T} S_t dt - K, 0\right\},$$

where $K$ is the strike price. Hence by the previous calculations the price of an Asian Call Option at any time $t$ with strike price $K$ is given by the discounted conditional expectation

$$C_F(K, t) = e^{-r(T-t)} \cdot E_{P(\theta)}\left[\left(\frac{1}{T - t_0} \int_{t_0}^{T} S_t \exp(Y_u^*) du - K\right) \right] | \mathcal{F}_t].$$

To calculate the conditional expectation in equation (10) we want to derive closed-form expressions for all moments of the arithmetic average

$$A_T = \frac{1}{T - t_0} \int_{t_0}^{T} S_t dt.$$ 

For this we use the Laplace transformation method introduced by Geman and Yor in [GemYo] in the classical Black-Scholes setting.

We will first simplify expression (10) above. Set $\bar{K} := K \cdot (T - t_0)$ and

$$\bar{A}_T := \int_{t_0}^{T} S_t dt.$$ 

Then

$$C_F(K, t) = \frac{e^{-r(T-t)}}{T - t_0} \cdot \bar{C}_F(K, t),$$

where $\bar{C}_F(K, t)$ is the discounted conditional expectation of a path dependent option with strike price $\bar{K}$ and terminal time $T - t_0$. 

The pricing problem (see for example [Hu], page 467). However, the pricing problem for arithmetic Asian options is still under investigation since no closed form solution is known up to now. The value of an arithmetic Asian Option at the terminal time $T$ depends on the mean of the whole path of the price process up to time $T$,
where

\[ (12) \quad \tilde{C}_F(K, t) := E_{P(t)} \left[ \left( \tilde{\Delta}_T - \tilde{K} \right) + \left| \mathcal{F}_t \right| \right] = E_{P(t)} \left[ \left( \int_{t_0}^{T} S_0 \exp(Y_u^* du - \tilde{K}) \right) + \left| \mathcal{F}_t \right| \right]. \]

This can be reduced to a deterministic function. Indeed, we have for \( t_0 \leq t < T, \)

\[ \tilde{C}_F(K, t) = E_{P(t)} \left[ \left( \int_{t_0}^{T} S_0 \exp(Y_u^* du - \tilde{K}) \right) + \left| \mathcal{F}_t \right| \right] = E_{P(t)} \left[ \left( S_t \int_{0}^{T-t} \exp(Y_u^* du - \left( \tilde{K} - \tilde{\Delta}_t \right)) + \left| \mathcal{F}_t \right| \right], \]

where we used that, for the definition of \( \tilde{\Delta}_T \) and the formula (7) for \( S_t, \) for any \( t \geq t_0 : \)

\[ \tilde{\Delta}_T = \int_{t_0}^{T} S_0 \exp(Y_u^* du) \]
\[ = \int_{t_0}^{T} S_0 \cdot \exp(Y_t^*) \cdot \exp(Y_u^* - Y_t^*) du. \]

Using that the law of \( \exp(Y_u^* - Y_t^*) \) is the same as the one of \( Y_{u-t}^* \) and \( \exp(Y_t^*) \) is independent of \( \exp(Y_u^* - Y_t^*) \) we obtain the following equalities (here \( = \mathcal{L} \) means equality in law):

\[ \tilde{\Delta}_T = \mathcal{L} \int_{t_0}^{T} S_t \exp(Y_{u-t}^*) du \]
\[ = \mathcal{L} S_t \int_{t_0-t}^{T-t} \exp(Y_u^*) du \]
\[ = \mathcal{L} S_t \int_{0}^{T-t} \exp(Y_u^*) du + S_t \int_{t_0-t}^{0} \exp(Y_u^*) du \]
\[ = \mathcal{L} S_t \int_{0}^{T-t} \exp(Y_u^*) du + \int_{t_0-t}^{0} S_0 \exp(Y_u^* - Y_{u-t}^*) du \]
\[ = \mathcal{L} \tilde{\Delta}_t + S_t \int_{0}^{T-t} \exp(Y_u^*) du. \]

Since \( Y_t^* \), which is equal in law to \( Y_{u-t}^* - Y_t^* \), is a Lévy process independent of \( \mathcal{F}_t, \) we obtain for equation (12) the following equalities in law:

\[ \tilde{C}_F(K, t) = \mathcal{L} E_{P(t)} \left[ \left( \int_{t_0}^{T} S_0 \exp(Y_u^* du - \tilde{K}) \right) + \left| \mathcal{F}_t \right| \right] \]
\[ = \mathcal{L} S_t \cdot E_{P(t)} \left[ \left( \int_{0}^{T-t} \exp(Y_u^* du - \left( \tilde{K} - \tilde{\Delta}_t \right) \frac{S_t}{S_t} \right) \right] + \left| \mathcal{F}_t \right| \]

(here we also used \( \tilde{\Delta}_T = \mathcal{L} \tilde{\Delta}_t + S_t \int_{0}^{T-t} \exp(Y_u^*) du. \) Hence, from (11) and the equation above, we get that the price of an Asian Call Option at time \( t \) with strike price \( K \) is given by

\[ (13) \quad C_F(K, t) = \frac{e^{-r(T-t)}}{T-t_0} \cdot S_t \cdot E_{P(t)} \left[ \left( \int_{0}^{T-t} \exp(Y_u^* du - \frac{K(T-t_0) - \int_{t_0}^{t} S_u du}{S_t} \right) \right], \]

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where $Y^*$ is a Lévy process with generating triplet $(\gamma^*, c^*, \nu^*)$ given by equation (6). To calculate this we need to know the joint distribution of the average $A^*_t$ and the underlying asset $S_t$, where

$$A^*_t := \int_0^t \exp(Y^*_u)du.$$ 

The moments of $A^*_t$ give full information about its law by the moment problem. To compute the moments we use the method of the Laplace transformation in time, following the method used by Geman and Yor in [GemYo] for the case of the continuous Black-Scholes model.

**Definition 4.1** Let $f$ be a measurable function from $\mathbb{R}^d$ into $\mathbb{R}$ respectively from $\mathbb{R}^d$ into $\mathbb{C}$. Then the Laplace transform of $f$, whenever it exists, is defined as

$$L[f](z) := \int_{\mathbb{R}^d} e^{-\langle z, x \rangle} f(x) dx, \; z \in \mathbb{C}^d.$$ 

For the coupling $\langle z, x \rangle$ to make sense, $x \in \mathbb{R}^d$ is understood as an element of $\mathbb{C}^d$. For a random variable $X$ on $\mathbb{R}^d$ with density function $\varphi$, the following connection between the Fourier transform and the Laplace transform (whenever $L$ exists, and $\Phi_X$ has an analytic continuation from $z \in \mathbb{R}^d$ to $iz \in i\mathbb{R}^d$) of $X$ holds

$$L[\varphi](z) = \int_{\mathbb{R}^d} e^{-\langle z, x \rangle} \varphi(x) dx = \int_{\mathbb{R}^d} e^{-\langle z, x \rangle} P_X(dx) = E \left[ e^{-\langle z, X \rangle} \right] = \Phi_X(iz).$$

Relating to Definition 2.13 we see that for a Lévy process $X_t$ with density function $\varphi_t$ the Laplace transform of $X_t$ at a point $\theta \in \mathbb{R}^d$ equals the moment generating function of $X_t$ evaluated at $-\theta$ (provided that both functions exists for $X_t$), i.e.

$$L[\varphi_t](\theta) = E \left[ e^{-\langle z, X_t \rangle} \right] = (\text{mgf}_X(-\theta))^t, \; \theta \in \mathbb{R}^d, \; t_0 \leq t \leq T.$$ 

We set

$$q := \frac{K(T - t_0) - \int_{t_0}^t S_u du}{S_t}.$$ 

Recall that we want to compute the value at time $t_0 \leq t < T$ of an Asian Option on an underlying asset $S_t$ with initial value $S_0$ at time $t_0$. Hence we can observe the values of $S_u$ for $t_0 \leq u \leq t$, and thus we know whether $q$ (defined above) is positive (i.e. $q > 0$) or $q \leq 0$. In the case $q \leq 0$, equation (13) reduces to

$$C_F(K, t) = \frac{e^{-r(T-t)}}{T - t_0} \cdot S_t \cdot \left( E_{P(\theta)} \left[ \int_0^{T-t} \exp(Y^*_u)du \right] - q \right).$$

We have to compute the first moment

$$E_{P(\theta)} [A_t^*] = E_{P(\theta)} \left[ \int_0^t \exp(Y^*_u)du \right]$$

$$= \int_{\Omega} \int_0^t \exp(Y^*_u)du dP(\theta)$$

$$= \int_0^t \int_{\mathbb{R}} \exp(y) \varphi_u(y) dy du$$

$$= \int_0^t L[\varphi_u](-1) du,$$
where \( \varphi_t(y) \) denotes the distribution of \( Y_t^* \) and \( L[\varphi_u](-1) \) the evaluation of \( L[\varphi_u] \) at \( z = -1 \). Thus we can compute the Asian Option price for \( q \leq 0 \) explicitly and we obtain

\[
C_F(K, t) = \frac{e^{-r(T-t)}}{T-t_0} \cdot S_t \cdot (E_{P(\theta)}[A^*_T] - q)
\]

(14)

Thus the valuation problem of an Asian Option reduces to the problem of computing the moment generating function of the Lévy process \( Y \). Let

\[
\tau_q := \inf\{t : A^*_t < q\}
\]

we have

\[
E_{P(\theta)}[(A^*_t - q)_+] = \int_0^t \exp\left\{ \frac{1}{2} \varphi_s + \gamma^*_s u + u \int_{t_0}^{\infty} (e^x - 1 - x) \nu^*(dx) \right\} du - q
\]

we obtain \( \nu^*(1) \) explicitly and we obtain

\[
\int_{|x| > 1} e^{Y^*_t \nu^*(dx)} < \infty
\]

we can apply Theorem 2.14 (iii) to obtain that

\[
\text{mgf}_{Y^*}(1) = E_{P(\theta)}[e^{Y^*_t}] = e^{u\left(\frac{1}{2} c^* + \gamma^* + \int_{t_0}^{\infty} (e^x - 1 - x) \nu^*(dx)\right)}
\]

Inserting this into equation (14) yields

\[
C_F(K, t) = \frac{e^{-r(T-t)}}{T-t_0} \cdot S_t \cdot \left( \int_0^t \exp\left\{ \frac{1}{2} \varphi_s + \gamma^*_s u + u \int_{t_0}^{\infty} (e^x - 1 - x) \nu^*(dx) \right\} du - q \right)
\]

Since \( (\gamma^*, c^*, \nu^*) \) can be constructed from the generating triplet \( (\gamma, c, \nu) \) of the original Lévy process \( Y \), we obtained a closed Form solution for the price of an Asian Option when \( q \leq 0 \) and \( \int_{|x| > 1} e^{Y^*_t \nu^*(dx)} < \infty \).

Now consider the case \( q > 0 \). We want to derive a formula for the Laplace transform with respect to the variable \( t \) of

\[
E_{P(\theta)}[(A^*_t - q)_+].
\]

By inversion of the Laplace transform and inserting the result into equation (13) one would then obtain the desired value of the Asian Option.

We recall that we have

\[
E_{P(\theta)}[A^*_t] = \int_0^t \exp\left\{ \frac{1}{2} uc^* + \gamma^*_u u + u \int_{t_0}^{\infty} (e^x - 1 - x) \nu^*(dx) \right\} du,
\]

where \( \gamma^*, c^*, \nu^* \) is the generating triplet of the Lévy process \( Y^* \).

Let \( \tau_q := \inf\{s : A^*_s > q\} \). Then

\[
A^*_t = A^*_{\tau_q} + \int_{\tau_q}^t \exp(Y_s^*)ds
\]

on the set \( \{A^*_t \geq q\} \). Since \( A^*_{\tau_q} = q \) we obtain

\[
E\left[(A^*_t - q)_+ \mid \mathcal{F}_{\tau_q}\right] = E\left[(A^*_t - q)_+ 1_{\{A^*_t \geq q\}} + (A^*_t - q)_+ 1_{\{A^*_t < q\}} \mid \mathcal{F}_{\tau_q}\right]
\]

\[
= E\left[(A^*_t - q)_+ 1_{\{A^*_t \geq q\}} \mid \mathcal{F}_{\tau_q}\right]
\]

\[
= E\left[A^*_t - q \mid \mathcal{F}_{\tau_q}\right]
\]

\[
= E\left[A^*_t - q + \int_{\tau_q}^t \exp(Y_s^*)ds - q \mid \mathcal{F}_{\tau_q}\right]
\]

\[
= E\left[1_{\{A^*_t \geq q\}} \cdot \int_{\tau_q}^t \exp(Y_s^*)ds \mid \mathcal{F}_{\tau_q}\right]
\]
By the strong Markov property and the independence of the increments of the Lévy process $Y_t^*$ the above expression can be simplified as follows

$$E \left[ \left( A_t^* - q \right)_+ \bigg| \mathcal{F}_{\tau_q} \right] = E \left[ 1_{\{A_t^* \geq q\}} \cdot \int_{\tau_q}^t \exp(Y_s^*) \, ds \right]$$

$$= E \left[ 1_{\{A_t^* \geq q\}} \cdot \int_{\tau_q}^t \exp(Y_{\tau_q}^* + Y_{s-\tau_q}^*) \, ds \right]$$

$$= E \left[ 1_{\{A_t^* \geq q\}} \cdot \exp(Y_{\tau_q}^*) \cdot \int_{\tau_q}^t \exp(Y_{s-\tau_q}^*) \, ds \right]$$

$$= \exp(Y_{\tau_q}^*) \cdot E \left[ 1_{\{A_t^* \geq q\}} \cdot \int_{\tau_q}^t \exp(Y_{s-\tau_q}^*) \, ds \right]$$

$$= \exp(Y_{\tau_q}^*) \cdot E \left[ \int_{\tau_q}^t \exp(Y_{s-\tau_q}^*) \, ds \right]$$

$$= \exp(Y_{\tau_q}^*) \cdot E \left[ A_t^* - \tau_q \right] ,$$

where we used that $\exp(Y_{\tau_q}^*)$ is $\mathcal{F}_{\tau_q}$-measurable. The first moment of $A_t^* - \tau_q$ can be computed, as seen before, and we finally obtain

$$E \left[ \left( A_t^* - q \right)_+ \bigg| \mathcal{F}_{\tau_q} \right] = \exp(Y_{\tau_q}^*) \cdot \int_0^{t-\tau_q} \exp \left\{ \frac{1}{2} \nu^* u + \gamma^* u + u \int_\alpha^\infty (e^x - 1 - x) \nu^*(dx) \right\} \, du .$$

It follows that

$$E \left[ \left( A_t^* - q \right)_+ \right] = E \left[ E \left[ \left( A_t^* - q \right)_+ \bigg| \mathcal{F}_{\tau_q} \right] \right]$$

$$= E \left[ \exp(Y_{\tau_q}^*) \cdot \int_0^{t-\tau_q} \exp \left\{ \frac{1}{2} \nu^* u + \gamma^* u + u \int_\alpha^\infty (e^x - 1 - x) \nu^*(dx) \right\} \, du \right] .$$

Now we want to compute the Laplace transform in $t$ of this. We obtain for $\lambda \in \mathbb{R}$, using Fubini-Tonelli’s Theorem:

$$\int_0^\infty e^{-\lambda t} E \left[ \left( A_t^* - q \right)_+ \right] \, dt$$

$$= \int_0^\infty e^{-\lambda t} E \left[ \exp(Y_{\tau_q}^*) \cdot \int_0^{t-\tau_q} \exp \left\{ \frac{1}{2} \nu^* u + \gamma^* u + u \int_\alpha^\infty (e^x - 1 - x) \nu^*(dx) \right\} \, du \right] \, dt$$

$$= E \left[ \exp(Y_{\tau_q}^*) \cdot \int_0^\infty e^{-\lambda t} \int_0^{t-\tau_q} \exp \left\{ \frac{1}{2} \nu^* u + \gamma^* u + u \int_\alpha^\infty (e^x - 1 - x) \nu^*(dx) \right\} \, du \, dt \right]$$

$$= E \left[ \exp(Y_{\tau_q}^*) \cdot e^{-\lambda \tau_q} \int_0^\infty e^{-\lambda t} \int_0^t \exp \left\{ \frac{1}{2} \nu^* u + \gamma^* u + u \int_\alpha^\infty (e^x - 1 - x) \nu^*(dx) \right\} \, du \, dt \right] .$$

Setting

$$f(\lambda) := \int_0^\infty e^{-\lambda t} \int_0^t \exp \left\{ \frac{1}{2} \nu^* u + \gamma^* u + u \int_\alpha^\infty (e^x - 1 - x) \nu^*(dx) \right\} \, du \, dt , \quad \lambda \in \mathbb{R},$$

we have obtained

$$\int_0^\infty e^{-\lambda t} E \left[ \left( A_t^* - q \right)_+ \right] \, dt = E \left[ \exp(Y_{\tau_q}^*) \cdot e^{-\lambda \tau_q} \cdot f(\lambda) \right] .$$

**Remark 4.2** $f(\lambda)$ could be computed numerically for given values of $\nu^*, \gamma^*$ and given $\nu^*$. 

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We denote the density at time $q$ of the transition semi group of the process $A^*_q = \exp(Y^*_q)$ from 1 at time 0 to $s$ at time $q$ by $\rho_q(1,s)$. We have:

\begin{equation}
\int_0^\infty e^{-\lambda t} E \left[ (A^*_s - q)_+ \right] dt = f(\lambda) \cdot \int_0^{\infty} s \cdot \rho_q(1,s) \cdot E \left[ e^{-\lambda \tau_q} \right] \exp(Y^*_s) = s \right] ds.
\end{equation}

The density $\rho_q(1,s)$ can be computed from the generating triplet of the process $Y^*_s$. Thus it remains to compute the conditional expectation

\begin{equation}
E \left[ e^{-\lambda \tau_q} \exp(Y^*_q) = s \right], \quad t_0 \leq s \leq T, \quad \lambda \in \mathbb{R}.
\end{equation}

To compute this, we first rewrite the stopping time $\tau_q$ as follows

\begin{align*}
\tau_q := \inf \{ s : A^*_s > q \} &= \int_0^q \mathbb{1}_{\{A^*_s > u\}} ds \\
&= \int_0^q \exp(-Y^*_s) \mathbb{1}_{\{A^*_s > u\}} \exp(Y^*_s) ds \\
&= \int_0^q \exp(-Y^*_s) \mathbb{1}_{\{A^*_s > u\}} dA^*_s \\
&= \int_0^q \exp(-Y^*_s) d\tau \\
&= \int_0^q \exp(-Y^*_s) du,
\end{align*}

where we substituted $s = \tau_u$. Using this identity the conditional expectation in (17) can be computed by expanding around $\lambda = 0$ as follows

\begin{align*}
E \left[ e^{-\lambda \tau_q} \exp(Y^*_q) = s \right] &= \sum_{n=1}^\infty \frac{1}{n!} E \left[ (\tau_q)^n \right] \right] \exp(Y^*_q) = s \\
&= \sum_{n=1}^\infty \frac{1}{n!} E \left[ \left( \int_0^q \exp(-Y^*_s) ds \right)^n \right] \exp(Y^*_q) = s \\
&= \sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty \left( \int_0^q \exp(-Y^*_s) ds \right)^n \mathbb{1}_{\{\exp(Y^*_q) = s\}} dP(\int_0^q \exp(-Y^*_s) ds) \otimes \exp(Y^*_q).
\end{align*}

Thus we have to compute the joint probability distribution of $\int_0^q \exp(-Y^*_s) ds$ and $\exp(Y^*_q)$. This is often a hard task. However, in some special situations we can explicitly compute the condition expectation (17) and thus derive a pricing formula for Asian options. In the classical Black Scholes model (driven by a Brownian motion) this task has been solved by [GemYo] via a technique using Bessel processes. In the following we will prove a pricing formula for Asian options in a Black-Scholes model driven by a Variance Gamma process. Here we will use the same methods as in [GemYo].

### 4.1 Explicit Computation in the Variance Gamma Case

We now consider the case where the stock price process is driven by a Variance Gamma $VG(\sigma, \nu, \theta)$ process $Y_t$. As mentioned in Example 2.12 the stock prices dynamics are given by

\begin{equation}
S_t = S_0 \exp \left( rt + Y(t; \sigma, \nu, \theta) + \frac{t}{\nu} \ln \left( 1 - \theta \nu - \frac{\sigma^2 \nu}{2} \right) \right).
\end{equation}
Thus our process $Y^*_t$ is defined as

$$Y^*_t = rt + Y(t; \sigma, \nu, \theta) + \frac{t}{\nu} \ln \left( 1 - \theta \nu - \frac{\sigma^2 \nu}{2} \right).$$

Now, going back to equation (13), we can rewrite this by using the identity

$$Y_t = \theta \gamma(t) + \sigma W_{\gamma(t)},$$

where $W$ is a Brownian motion and $\gamma$ is a Gamma $G(1/\nu, 1/\nu)$ process.

$$C_F(K, t) = \frac{e^{-r(T-t)}}{T-t_0} \cdot S_t \cdot E_{P(0)} \left[ \left( \int_0^{T-t} \exp (bu + \theta \gamma(u) + \sigma W_{\gamma(u)}) \, du - \kappa \right)_+ \right],$$

where

$$b := r + \frac{1}{\nu} \ln \left( 1 - \theta \nu - \frac{\sigma^2 \nu}{2} \right) \quad \text{and} \quad \kappa := \frac{K(T-t_0) - \int_{t_0}^t S_u \, du}{S_t}.$$

Using the scaling properties of Brownian motion and Gamma processes we obtain by substitution $u = cv$ with $c := \left( \frac{2}{\sigma} \right)^{2/\nu}$ the following

$$E_{P(0)} \left[ \left( \int_0^{cs} \exp \left( bu + \theta \gamma(u) + \sigma W_{\gamma(u)} \right) \, du - \kappa \right)_+ \right]$$

$$= E_{P(0)} \left[ \left( c \cdot \int_0^{cs} \exp \left( bcv + \theta \gamma(cv) + \sigma W_{\gamma(cv)} \right) \, dv - \kappa \right)_+ \right]$$

$$= E_{P(0)} \left[ \left( c \cdot \int_0^{cs} \exp \left( bcv + \theta c^{1/\nu} \gamma(v) + \sigma W_{\gamma(v)} \right) \, dv - \kappa \right)_+ \right]$$

$$= c \cdot E_{P(0)} \left[ \left( \int_0^{cs} \exp \left( 2 \left( \frac{bc}{2} + \frac{\theta c^{1/\nu}}{2} \gamma(v) + W_{\gamma(v)} \right) \right) \, dv - \kappa \right)_+ \right].$$

Thus we have

$$C_F(K, t) = c \cdot e^{-r(T-t)} \cdot \frac{1}{T-t_0} \cdot S_t \cdot E_{P(0)} \left[ \left( \int_0^{h} \exp \left( 2 \left( \frac{bc}{2} + \frac{\theta c^{1/\nu}}{2} \gamma(v) + W_{\gamma(v)} \right) \right) \, dv - \kappa \right)_+ \right],$$

with

$$b := r + \frac{1}{\nu} \ln \left( 1 - \theta \nu - \frac{\sigma^2 \nu}{2} \right) \quad \text{and} \quad \kappa := \frac{K(T-t_0) - \int_{t_0}^t S_u \, du}{S_t}$$

and

$$c := \left( \frac{2}{\sigma} \right)^{2/\nu} \quad \text{and} \quad h := c(T-t).$$

What remains to be computed is

$$E_{P(0)} \left[ (A_h - q)_+ \right],$$

where

$$A_h := \int_0^h \exp \left( 2 \left( \frac{bc}{2} + \frac{\theta c^{1/\nu}}{2} \gamma(s) + W_{\gamma(s)} \right) \right) \, ds \quad \text{and} \quad q := \frac{\kappa}{c}.$$
From equation (14) we obtain in case $q < 0$:

$$C_F(K, t) = \frac{e^{-r(T-t)}}{T-t_0} \cdot S_t \cdot \left( \int_0^t (\text{mgf}_{Y_t^*}(1)) \cdot du - q \right)$$

$$= \frac{e^{-r(T-t)}}{T-t_0} \cdot S_t \cdot \left( \int_0^t \left( \frac{1}{1 - \theta \nu + (\sigma^2 \nu/2)} \right) \cdot du - q \right)$$

if $(\sigma^2 \nu/2) \neq 1 - \theta \nu$. Here we used the identity $\text{mgf}_{Y_t^*}(iz) = \Phi_{Y_t^*}(z)$ for $z \in \mathbb{R}$, where $\Phi_{Y_t^*}$ is the characteristic function of the process $Y_t^*$. Since this characteristic function is given by

$$\Phi_{Y_t^*}(u) = \left( \frac{1}{1 - i\theta \nu u + (\sigma^2 \nu/2)u^2} \right)^{1/\nu},$$

which is an analytic function for $(\sigma^2 \nu/2)u^2 \neq 1 - i\theta \nu u$, the identity $\text{mgf}_{Y_t^*}(iz) = \Phi_{Y_t^*}(z)$ also holds for $u = -i$ whenever $(\sigma^2 \nu/2) \neq 1 - \theta \nu$. Thus we have

$$\text{mgf}_{Y_t^*}(1) = \Phi_{Y_t^*}(-i) = \left( \frac{1}{1 - \theta \nu + (\sigma^2 \nu/2)} \right)^{1/\nu}.$$

We now consider the more complicated case $q > 0$. For this, we again use the method of Laplace transformation with respect to the variable $h$. We will first write the process $X_s := \exp(\alpha s + \beta \gamma(s) + W_\gamma(s))$, $s \geq 0$, as

$$X_s = R(A_s),$$

where $(R(u), u \geq 0)$ is a Bessel process, starting at $R^{(\alpha, \beta)}(0) = 1$. Here we will use the fact that any Variance Gamma process $Y^*(t)$ can be expressed as a standard Brownian motion $W(T(t))$ with independent gamma subordinator $T(t)$. Hence we can write

$$\exp(Y^*_s) = \exp(\alpha s + \beta \gamma(s) + W_\gamma(s)) = \exp(W(T(s)))$$

Moreover we use a result by [Wi] stating that the exponential of Brownian motion with drift is a time-changed Bessel process, i.e.

$$\exp(W(t) + \delta t) = R^{(\delta)} \left( \int_0^t \exp(2(W(s) + \alpha s))ds \right), \quad t \geq 0,$$

where $(R^{(\delta)}(u), u \geq 0)$ is a Bessel process with index $\delta$. Hence we obtain

$$\exp(Y^*_t) = \exp(W(T(t))) = R \left( \int_0^{T(t)} \exp(2W(s))ds \right), \quad t \geq 0,$$

where $(R(u), u \geq 0)$ is a Bessel process with index $0$. Then, returning to equation (17), and considering the fact that

$$\tau_q = \int_0^q \frac{du}{(R(u))^2},$$

we obtain

$$E \left[ e^{-\lambda \tau_q} \exp(Y^*_{\tau_q}) \right] = s$$

(20)

$$= E \left[ e^{-\lambda \int_0^{\tau_q} \frac{du}{(R(u))^2}} R \left( \int_0^{T(\tau_q)} \exp(2W(u))du \right) \right] = s.$$
Proposition 4.3 For every $\delta \in \mathbb{R}, a > 0, x > 0, t > 0$,

$$E_a^{(0)} \left[ \exp \left( -\frac{\delta^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \right] R_t = x = \frac{I_{|\delta|}}{I_0} \left( \frac{ax}{t} \right),$$

where, on the right-hand side, $(f/g)(y)$ stands for $f(y)/g(y)$ and $I_4$ resp. $I_0$ denote Bessel functions. $E_a^{(0)}$ denotes the expectation relative to $P_a^{(0)}$, the low of the Bessel process $R^{(0)}_t$ starting at $a$.

Applying this to equation (20) yields

$$E \left[ e^{-\lambda \tau_*} \exp(Y_{\tau_*}^*) = s \right] = E \left[ e^{-\lambda \int_0^q \frac{du}{(R(u))^2}} \right] R \left( \int_0^{T(\tau_*)} \exp(2W(u))du \right) = s$$

$$= \frac{I_{|\sqrt{2\lambda}|}}{I_0} \left( \frac{s}{q} \right).$$

Thus the Laplace transform (16) of the Asian option price is given by

$$(21) \int_0^\infty e^{-\lambda t} E \left[ (A_t^* - q)_+ \right] dt = f(\lambda) \cdot \int_0^\infty s \cdot \rho_q(1, s) \cdot \frac{I_{|\sqrt{2\lambda}|}}{I_0} \left( \frac{s}{q} \right) ds.$$ 

where $\rho_q(1, s)$ denotes the density at time $q$ of the Bessel semigroup, with index 0 and starting position 1 at time 0. This density can be computed explicitly and is given by (see [GemYo], Proposition 2.2)

$$\rho_q(1, s) = \frac{s}{q} \exp \left( -\frac{1}{2q} (1 + s^2) \right) I_0 \left( \frac{s}{q} \right).$$

Hence the Laplace transform of $E \left[ (A_t^* - q)_+ \right]$ with respect to $t$ can be represented by

$$\int_0^\infty e^{-\lambda t} E \left[ (A_t^* - q)_+ \right] dt = f(\lambda) \cdot \int_0^\infty \frac{s^2}{q} \exp \left( -\frac{1}{2q} (1 + s^2) \right) I_{|\sqrt{2\lambda}|} \left( \frac{s}{q} \right) ds,$$

where

$$f(\lambda) := \int_0^\infty e^{-\lambda t} \int_0^t (\text{mgf}_{Y_*(1)})^n dudt = \int_0^\infty e^{-\lambda t} \int_0^t \left( \frac{1}{1 - \theta \nu + (\sigma^2 \nu/2)} \right)^{tu/\nu} dudt, \lambda > 0.$$ 

By inversion of this Laplace transform for a fixed $t$ we obtain $E \left[ (A_t^* - q)_+ \right]$ and thus by equation (19) the Asian option price. The inversion of the Laplace transform is given by

$$F(t) := \frac{1}{2\pi} \int_0^\infty e^{\lambda t} \left( f(\lambda) \cdot \int_0^\infty \frac{s^2}{q} \exp \left( -\frac{1}{2q} (1 + s^2) \right) I_{|\sqrt{2\lambda}|} \left( \frac{s}{q} \right) ds \right) d\lambda$$

$$= L^{-1} [f](t) \ast L^{-1} [\Psi](t),$$

where we denote the inverse Laplace transform of a function by $L^{-1}$, thus

$$L^{-1} [f](t) := \frac{1}{2\pi} \int_0^\infty e^{\lambda t} f(\lambda) d\lambda.$$ 

$\Psi$ is the function defined by

$$\Psi = \int_0^\infty \frac{s^2}{q} \exp \left( -\frac{1}{2q} (1 + s^2) \right) I_{|\sqrt{2\lambda}|} \left( \frac{s}{q} \right) ds.$$
Here we used that the inverse Laplace transform of a product $f \cdot g$ is the convolution (denoted by $\ast$) of the inverse Laplace transforms of $f$ and $g$. Hence we have to compute two inverse Laplace transformations. This can be expressed analytically in terms of power series involving quite complicated expressions. From a practical point of view a direct numerical computation of the inverse Laplace transform might be more efficient. However, since the parameter $\lambda$ of the inverse Laplace transformation appears in the index of the Bessel function, also a numerical computation is not that easy and there is no standard software for this task.

**Remark 4.4** The procedure we used to derive the pricing formula for Asian options on a Variance Gamma driven stock price, can be applied to any Black-Scholes type model with Lévy noise (in the sense of equation (1)), whenever the driving process $Y_t$ has a scaling property and the property that it can be written as a time changed Brownian motion with respect to some subordinator.

### 5 Acknowledgements

We are very grateful to David Applebaum for stimulating discussion, and in particular for the suggestion to look at the Variance Gamma process as an example.

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