A Note on the Renormalized Square of Free Quantum Fields in Space-Time Dimension $d \geq 4$

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A note on the renormalized square of free quantum fields in space-time dimension $d \geq 4$

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Abstract

We consider the renormalized or Wick square of the free quantum field, which is well-defined as a Hida distribution. It is known that this is a random variable for space-time dimension $d \leq 3$. In this report, by considering the characteristic function, we show that this Wick square is not a random variable in dimension $d \geq 4$.

Keywords: quantum fields, Wick powers, free field model, Hida distribution, Lévy-Khinchine representation.

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1 Introduction

Let $m_0 > 0$ be a fixed number and $d \geq 4$. Let (ϕ, μ_0) be Nelson's Euclidean free field on $\mathbf{R}^d$ of mass m_0, *i.e.*, μ_0 is the centered Gaussian measure on

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$\mathcal{S}'(\mathbf{R}^d)$ with the covariance $(-\Delta + m_0^2)^{-1}$. Also, let $f \in \mathcal{S}(\mathbf{R}^d)$, $\mathcal{S}(\mathbf{R}^d)$ being Schwartz' function space of rapidly decreasing smooth test functions. Then $\phi(f) \equiv \int_{\mathbf{R}^d} \phi_x f(x) dx$ (in the distributional sense) is well-defined as a random variable.

We also consider the quantity $:\phi^2:(f)$, which is well-defined as a Hida distribution for all d (for this concept see, *e.g.*, [12]). Here $:\cdot:$ stands for the Wick power. In Section 2, by using analytic continuation of the relativistic free field, we give the precise definition of $:\phi^2:(f)$. $:\phi^2:(f)$ is by definition the Wick power of the Euclidean (quantum) free field (over the Euclidean space-time $\mathbf{R}^d$).

In Section 3, we recall the definition and the basic properties of Hida distributions, and use them to confirm that $:\phi^2:(f)$ is a Hida distribution.

This Wick square field $:\phi^2:(f)$ has been studied in the $d \leq 3$ dimensional case, for example, in [1], [3], [10], [11], [14]. In these references, it is showed in particular that $:\phi^2:(f)$ is a random variable for $d \leq 3$. It is a natural question to ask whether this quantity $:\phi^2:(f)$ is still a random variable for $d \geq 4$.

One can easily see that it can not be square integrable, moreover, in Section 4, we shall see that the natural candidate for its characteristic function turns out to be the function identically zero for $d \geq 4$, which implies that indeed $:\phi^2:(f)$ is not a random variable (see Proposition 4.1 and Theorem 4.2).

Remark 1 *We want to emphasize that, for $d \leq 3$, $\phi^2:(f)$ is a random variable with a representation of the characteristic function (at $\frac{1}{2}$)) as $e^{\frac{1}{2}M(f)}$, with $M(f)$ expressed as in (4.3) (this has been discussed in [11]). (4.3) can indeed be proven by a limit procedure, starting from a regularization of $:\phi^2:(f)$). From this, it follows that for $d \leq 3$, $:\phi^2:(f)$ is an infinitely divisible random variable, and in particular one can construct from it other Euclidean fields of the form $\sum_{i=1}^N :\phi_j^2:(f)$, with $:\phi_j^2:(f)$ independent fields distributed as $:\phi^2:(f)$, and $N \in \mathbf{N}$; these in turn can be extended to any $N \in \mathbf{R}$. This motivated our study, in the hope of proving (4.3) also for $d \geq 4$ and thus obtaining a rich class of (infinitely divisible) Euclidean random fields for $d \geq 4$. However, our rigorous results leave little hope for such a construction. We also want to remark that [3] showed that the relativistic Wick square is not infinitely divisible (in a sense explained in [3]) in any space-time dimension.*

2 Definition of the relativistic resp. Euclidean Wick square

In this section, we first recall the definition of the Wick square of the relativistic free field ϕ_R. We denote this Wick square by $:\phi_R^2:$. We then recall the definition $:\phi_E^2:$ of the corresponding Euclidean Wick square, by analytic continuation of the relativistic free field (see also [2], [10], [12], [7], and the references therein).

Let us first recall the relativistic free field. We consider the Fock space $F = \oplus_{n=0}^{\infty} F^{(n)}$, where $F^{(n)}$ is the tensor product of n one particle spaces $F^{(n)} = \mathrm{Sy}[F_1 \otimes \cdots \otimes F^{(1)}]$, $F^{(0)} \equiv \mathbf{C}$. Here "Sy" stands for the symmetrizer. $F^{(1)}$ is the space of complex valued functions on $\mathbf{R}^d$ square integrable with respect to the measure $d\Omega^+(p) = \delta(p^2 + m_0^2)\theta(p_0)d^dp$ (mod the class of functions of 0 norm with respect to this measure), where δ is the Delta-measure at 0, $p = (p_0, p_1, \cdots, p_{d-1})$, $p^2 \equiv -p_0^2 + (p_1^2 + \cdots + p_{d-1}^2)$, and $\theta(p_0) = 1_{\{p_0>0\}}$.

The fields ϕ_R and $:\phi_R^2:$ are defined according to the following: For any $n \in \mathbf{N}$, let $\mathcal{M}^{(n)}$ be the set of all functions in $F^{(n)}$ such that $|\Phi(p_1, \cdots, p_n)|(1+\sum_{i=1}^n |p_i^0|^\alpha)$ is bounded for $p_0 \geq 0$, for each $\alpha \in \mathbf{N}$. Let $\mathcal{D}$ be the linear span of the sets $\mathcal{M}^{(n)}$ $(n \geq 0)$. For any $\Phi \in \mathcal{D}$, the component of $\phi_R(f)\Phi \equiv \int \phi_R(x)f(x)dx\Phi$ (the integral being in the distributional sense, *i.e.*, ϕ_R is an operator-valued distribution acting on the vector Φ) in $F^{(n)}$ is given by

$$\begin{aligned}(\phi_R(f)\Phi)^n &= \sqrt{\pi}\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^n \tilde{f}(-p_k)\Psi(p_1, \cdots, \hat{p_k}, \cdots, p_n) \\ &\qquad +\sqrt{n+1}\int d\Omega^+(n)\tilde{f}(n)\Psi(n, p_1, \cdots, p_n)\Big),\end{aligned}$$

and the component of $:\phi_R^2:(f)\Phi$ in $F^{(n)}$ is given by

$$(:\phi_R^2:(f)\Phi)^{(n)} = \frac{1}{2}\left(T_{(2,n,0)}(\tilde{f})\Phi^{(n-2)} + T_{(2,n,1)}(\tilde{f})\Phi^{(n)} + T_{(2,n,2)}(\tilde{f})\Phi^{(n+2)}\right),$$

with $\tilde{f}$ being the Fourier transform of f (defined by $\tilde{f}(p) = \int e^{ixp}f(x)dx$, $f \in \mathcal{S}(\mathbf{R}^d)$), and the operators $T_{(2,n,j)}(g) : \mathcal{M}^{(n-2+2j)} \to \mathcal{M}^{(n)}$, $j = 0, 1, 2$, given by

$$(T_{(2,n,0)}(g)\Psi)(p_1, \cdots, p_n) = (n(n-1))^{-1/2}$$

$$\times \sum_{k_1 \neq k_2, k_1, k_2 = 1}^{n} g(-p_{k_1} - p_{k_2}) \Psi(p_1, \cdots, \hat{p_{k_1}}, \cdots, \hat{p_{k_2}}, \cdots, p_n), \quad \Psi \in \mathcal{M}^{(n-2)},$$

$$(T_{(2,n,1)}(g)\Psi)(p_1, \cdots, p_n) =$$

$$\sum_{k_1=1}^{n} \int d\Omega^+(n_1) g(n_1 - p_{k_1}) \Psi(n_1, p_1, \cdots, \hat{p_{k_1}}, \cdots, p_n), \quad \Psi \in \mathcal{M}^{(n)},$$

$$(T_{(2,n,2)}(g)\Psi)(p_1, \cdots, p_n) = ((n+2)(n+1))^{1/2}$$

$$\times \int \int d\Omega^+(n_1) d\Omega^+(n_2) g(n_1 + n_2) \Psi(n_1, n_2, p_1, \cdots, p_n), \quad \Psi \in \mathcal{M}^{(n)},$$

where $\hat{}$ means that the corresponding component is excluded. (See also [14] and [20]).

Under the above definition, it is known that the corresponding Wightman functions

$$W_n(f_1, \cdots, f_n) = (\Omega_0, :\phi_R^2: (f_1) \cdots :\phi_R^2: (f_n)\Omega_0),$$

with $\Omega_0 \equiv (1, 0, \cdots, 0, \cdots) \in F^{(0)}$ the (relativistic) Fock vacuum and $(\cdot, \cdot)$ the scalar product in F, satisfy the Wightman Axioms [21] (see also [19], [13]). Therefore, the corresponding Schwinger functions (of the Euclidean field) satisfy the Osterwalder-Schrader Axioms (see [16] and *e.g.*, [18]). More precisely, by axioms, there exist $W_n(x_1, \cdots, x_n) = (\Omega_0, :\phi_R^2: (x_1) \cdots :\phi_R^2: (x_n)\Omega_0)$ such that (in the distributional sensen)

$$W_n(f_1, \cdots, f_n) = \int_{\mathbf{R}^{dn}} f_1(x_1) \cdots f_n(x_n) W_n(x_1, \cdots, x_n) dx_1 \cdots dx_n.$$

Let

$$S_n((x_1^0, \vec{x_1}), \cdots, (x_n^0, \vec{x_n})) \equiv W_n((ix_1^0, \vec{x_1}), \cdots, (ix_n^0, \vec{x_n})),$$

with $(x_j^0, \vec{x_j}) \in \mathbf{R} \times \mathbf{R}^{d-1}$, $j = 1, \cdots, n$, and define the Schwinger functions corresponding to the Wightman functions by

$$S_n(f_1, \cdots, f_n) \equiv \int_{\mathbf{R}^{dn}} f_1(x_1) \cdots f_n(x_n) S_n(x_1, \cdots, x_n) dx_1 \cdots dx_n,$$

for any $(f_1, \cdots, f_n) \in \mathcal{S}_{\neq}(\mathbf{R}^d)$ with $\mathcal{S}_{\neq}(\mathbf{R}^d)$ denoting the set of all functions which vanish (together with their derivatives) on each hyperplane $y_i - y_j = 0$, $i, j = 1, \cdots, n$. Then S_n satisfies the Osterwalder-Schrader Axioms. Therefore, there exists a Hilbert space $\mathcal{H}_E$, a (Euclidean) vacuum $\Omega_{0,E}$ and a field $:\phi_E^2: (f)$ such that $S_n(f_1, \cdots, f_n) = (\Omega_{0,E}, :\phi_E^2: (f_1) \cdots :\phi_E^2: (f_n)\Omega_{0,E})_{\mathcal{H}_E}$.

One can raise the question whether these Schwinger functions also satisfy Nelson's Axioms [15], [18] (in particular whether they satisfy Nelson's positivity condition), so that the corresponding (Euclidean Wick square) field $(:\phi_E^2:(f), f \in \mathcal{S}(\mathbf{R}^d))$ is not only an "abstract operator field family", but also a family of random variables (defined on a common probability space). We show however in Section 4 that this is not the case, since the only canditate for the corresponding characteristic function is trivial (see Proposition 4.1 and Theorem 4.2).

3 Hida distributions

For some purposes like describing "singular interactions", the space $L^2(d\mu_0)$ is too small, and we need to define a bigger space (this corresponds to extending $L^2(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$, the space of Schwartz tempered distributions, which is given as the dual of the subspace $\mathcal{S}(\mathbf{R}^d)$, the Schwartz space of rapidly decreasing smooth test functions, in $L^2(\mathbf{R}^d)$. The latter extension is needed to describe "singular functions"). We will use the Kondratiev triple

$$S \subset L^2(d\mu_0) \subset S_{-1}.$$

The definition is given below (see also, *e.g.*, [4], [10], [12], and the references therein).

For any $p, q \in \mathbf{Z}$, we define the Hilbertian norm of a smooth Wick polynomial $\varphi(\phi) = \sum_{n=1}^{N} \langle :\phi^{\otimes n}:, \varphi^{(n)}\rangle$, $\phi \in \mathcal{S}'(\mathbf{R}^d)$, by

$$\|\varphi\|_{p,q,1}^2 \equiv \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi^{(n)}|_p^2.$$

For any $p, q, \in \mathbf{N}_0$, let $(\mathcal{H}_p)_q^1$ be the Hilbert space given by

$$(\mathcal{H}_p)_q^1 = \Big\{ f \in L^2(d\mu_0) \Big| f(\phi) = \sum_{n=0}^{\infty} \langle :\phi^{\otimes n}:, f^{(n)}\rangle, \|\varphi\|_{p,q,1} < \infty \Big\}.$$

Finally, the space S of test functions is defined as

$$S = \cap_{p,q\geq 0} (\mathcal{H}_p)_q^1,$$

and S_{-1} is defined as the dual of S with respect to $L^2(d\mu_0)$. The elements in S_{-1} are called Hida distributions. It is clear that $S_{-1} = \cup_{p,q\geq 0} (\mathcal{H}_{-p})_{-q}^{-1}$, where

$(\mathcal{H}_{-p})^{-1}_{-q}$ denotes the dual of $(\mathcal{H}_p)^1_q$ with respect to $L^2(d\mu_0)$. The bilinear dual pairing $\langle\langle\cdot,\cdot\rangle\rangle$ between S and S_{-1} is given by

$$\langle\langle f, \varphi\rangle\rangle = (\overline{f}, \varphi)_{L^2(\mu_0)}, \qquad f \in L^2(\mu_0), \varphi \in S.$$

The set S_{-1} can be characterizated by S transform (see, *e.g.*, [12]). For $\Phi \in S_{-1}$, the S transform of Φ is given by the following:

$$S\Phi(g) \equiv \langle\langle \Phi, : \exp(\langle\cdot, g\rangle) :\rangle\rangle, \qquad g \in \mathcal{U} \subset \mathcal{S}(\mathbf{R}^d; \mathbf{C}), \tag{3.1}$$

where $\mathcal{U}$ is an open neighborhood of zero in $\mathcal{S}(\mathbf{R}^d; \mathbf{C})$ depending on $\Phi \in S_{-1}$, and $: \exp(\langle\cdot, g\rangle) :$ is given by

$$: \exp(\langle\phi, g\rangle) :\equiv \frac{\exp(\langle\phi, g\rangle)}{E^{\mu_0}[\exp(\langle\phi, g\rangle)]} = \sum_{n=1}^{\infty} \frac{1}{n!}\langle : \phi^{\otimes n} :, g^{\otimes n}\rangle$$

for $\phi \in \mathcal{S}'(\mathbf{R}^d), g \in \mathcal{S}(\mathbf{R}^d)$ (with E^{μ_0} the expectation with respect to μ_0).

Let $Hol_0(\mathcal{S}(\mathbf{R}^d; \mathbf{C}))$ denote the space of germs of holomorphic functions. Then we have the following (see, *e.g.*, [10]).

THEOREM 3.1 *1. If $\Phi \in S_{-1}$, then $S\Phi \in Hol_0(\mathcal{S}(\mathbf{R}^d; \mathbf{C}))$,*

2. For any $F \in Hol_0(\mathcal{S}(\mathbf{R}^d; \mathbf{C}))$, there exists a unique $\Phi \in S_{-1}$ such that $S\Phi = F$.

By using this result and the fact that $Hol_0(\mathcal{S}(\mathbf{R}^d; \mathbf{C}))$ is a algebra, for any $\Phi, \Psi \in S_{-1}$, it is now possible to define the Wick product of them

$$\Phi \diamond \Psi \equiv S^{-1}(S\Phi \cdot S\Psi).$$

By induction, this defines the Wick power

$$\Phi^{\diamond n} = S^{-1}((S\Phi)^n) \in S_{-1}.$$

It is known that for a Gaussian process φ, the Wick product $\varphi^{\diamond n}$ defined above coincides with the usual Wick product $: \varphi^n :$. Also, for any $x \in \mathbf{R}^d$, it is known that $\phi_x \in S_{-1}$, therefore, by the closedness property of S_{-1} under Wick products, we get that $: \phi_x^2 :\in S_{-1}$, too. Moreover, we have the following (see, *e.g.*, [10]):

PROPOSITION 3.2 *For every $x \in \mathbf{R}^d$, there exists a Hida distribution $:\phi_x^2: \in S_{-1}$ such that*

$$\langle\langle 1, :\phi_{x_1}^2: \cdots :\phi_{x_n}^2: \rangle\rangle = S_n(x_1, \cdots, x_n), \qquad x_i \neq x_j (i \neq j).$$

∎

Remark 2 *One has to be careful about the notations. Actually, from the fact that $:\phi_x^2: \in S_{-1}$, if one is not careful enough, one might think that, f being smooth, it would follow that $:\phi^2:(f) \equiv \int :\phi_x^2: f(x)dx$ is well-defined (in the sense of distribution over $\mathbf{R}^d$) and takes a finite value. However, the latter is NOT the case, since the fact $:\phi_x^2: \in S_{-1}$ only gives us that for any fixed $x \in \mathbf{R}^d$, $\langle\langle :\phi_x^2:, g\rangle\rangle$ is finite for any $g \in S$.*

We shall now see that $:\phi^2:(f)$ is a Hida distribution for any $f \in \mathcal{S}(\mathbf{R}^d)$, by using the definition (3.1) of S transform and Theorem 3.1. Actually, for any $g \in \mathcal{S}(\mathbf{R}^d; \mathbf{C})$ and $\lambda \in \mathbf{R}$, we have by definition and a simple calculation that

$$\begin{aligned}
& \langle\langle :\phi^2:(f), :\exp(\langle\phi, \lambda g\rangle):\rangle\rangle \\
= & \int :\phi^2:(f)\Big(\sum_{n=1}^{\infty} \frac{1}{n!} {}_{\mathcal{S}'}\langle :\phi^{\otimes n}:, (\lambda g)^{\otimes n}\rangle_{\mathcal{S}}\Big)\mu_0(d\phi) \\
= & \frac{\lambda^2}{2} \int :\phi^2:(f) :\phi(g)^2: \mu_0(d\phi) \\
= & \frac{\lambda^2}{2} \int_{\mathbf{R}^d} f(x)dx E^{\mu_0}[:\phi_x^2::\phi(g)^2:] \\
= & \frac{\lambda^2}{2} \int_{\mathbf{R}^d} f(x)dx E^{\mu_0}[\phi_x \phi(g)]^2 \\
= & \frac{\lambda^2}{2} \int_{\mathbf{R}^d} f(x)dx (\int_{\mathbf{R}^d} g(y)dy E^{\mu_0}[\phi_x\phi_y])^2
\end{aligned}$$

(where in the last but one equality we have used $E^{\mu_0}[:\phi_x^2::\phi(g)^2:] = (E^{\mu_0}[\phi_x\phi(g)])^2$, a simple consequence of the definition of $: \cdot :$).

Therefore, it is sufficient for the left hand side to be finite that $\int_{\mathbf{R}^d} f(x)dx(\int_{\mathbf{R}^d} g(y)dy E^{\mu_0}[\phi_x\phi_y])^2 < \infty$. On the other hand, this is easily seen from the well-known fact that $E^{\mu_0}[\phi_x\phi_y] \sim \frac{1}{|x-y|^{d-2}}$ as $|x-y| \to 0$, the fact that $\sup_{x\in\mathbf{R}^d} \int_{|y-x|<1} \frac{1}{|x-y|^{d-2}} dy < \infty$, and that $f, g \in L^1(\mathbf{R}^d; dx)$. Hence we have proven that $:\phi^2:(f) \equiv \int :\phi_x^2: f(x)dx$ is a Hida distribution.

Remark 3 *By a similar calculation, we have that* $:\phi^n:(f)$ *is a Hida distribution for any* $n \in \mathbf{N}$ *and any dimension* d.

The following follows easily from Proposition 3.2.

PROPOSITION 3.3 $\langle\langle 1, :\phi^2:(f_1)\cdots:\phi^2:(f_n)\rangle\rangle = S_n(f_1,\cdots,f_n)$.

∎

As a immediate result of Proposition 3.3, we have the following

COROLLARY 3.4 *We can identify* $:\phi^2:(f)$ *with* $:\phi_E^2:(f)$.

4 $:\phi^2:(f)$ is not a random variable, for $d \geq 4$

First, we make the observation that $\int_{\mathbf{R}^d} :\phi_x^2: f(x)dx$ can not be in $L^2(\mu_0)$. Heuristically in fact

$$\begin{aligned} & E[\Big(\int_{\mathbf{R}^d} :\phi_x^2: f(x)dx\Big)^2] \\ = & \int_{\mathbf{R}^d} f(x)dx \int_{\mathbf{R}^d} f(y)dy E[:\phi_x^2::\phi_y^2] \\ = & \int_{\mathbf{R}^d} f(x)dx \int_{\mathbf{R}^d} f(y)dy E[\phi_x\phi_y]^2 \end{aligned}$$

is infinite for $d \geq 4$, since $E[\phi_x\phi_y] \sim \frac{1}{|x-y|^{d-2}}$ for $x, y \in \mathbf{R}^d$ as $|x-y| \to 0$. This heuristic argument can be made rigorous by replacing $:\phi_x^2:$ by a regularized version in x, repeating above computation, using Fubini theorem and removing the regularization at the end.

We next discuss the non-existence of the characteristic function of $:\phi^n:(f)$. Indeed, if $:\phi^2:(f)$ were a random variable, then we should be able to consider its characteristic function $E[e^{i\alpha:\phi^2:(f)}]$, $\alpha \in \mathbf{R}$.

As in [3] and [11], by the definition of Gaussian measure μ_0, we first have the following heuristic expression:

$$E[e^{i\alpha:\phi^2:(f)}] = \exp(-\frac{1}{2}Tr\{\ln[1 - i2\alpha f(-\Delta + m_0^2)^{-1}] + i2\alpha f(-\Delta + m_0^2)^{-1}\}). \tag{4.1}$$

For the sake of simplicity, we take $\alpha = \frac{1}{2}$ from now on, and consider

$$\begin{aligned} L(f) &\equiv E[e^{\frac{i}{2}:\phi^2:(f)}] \\ &= \exp(-\frac{1}{2}Tr\{\ln[1 - if(-\Delta + m_0^2)^{-1}] + if(-\Delta + m_0^2)^{-1}\}) \\ &= \exp(-\frac{1}{2}Tr\{\ln(-\Delta + m_0^2 - if) - \ln(-\Delta + m_0^2) + if(-\Delta + m_0^2)^{-1}\}). \end{aligned}$$

As in [3], by using the expressions $-\ln z = \int_0^\infty \frac{ds}{s}(e^{-sz} - e^{-s})$ and $z^{-1} = \int_0^\infty dse^{-sz}$, we get

$$L(f) = \exp(\frac{1}{2}\int_0^\infty \frac{ds}{s}Tr(e^{-s(-\Delta+m_0^2)+isf} - (1 + isf)e^{-s(-\Delta+m_0^2)})).$$

Let

$$M(f) \equiv \int_0^\infty \frac{ds}{s}Tr(e^{-s(-\Delta+m_0^2)+isf} - (1 + isf)e^{-s(-\Delta+m_0^2)}), \tag{4.2}$$

and we will show that this is not well-defined for $f \not\equiv 0$.

Before calculating it rigorously, let us first consider the order for s small of the integrand heuristically. We re-write (4.2) as

$$\int_0^\infty \frac{ds}{s}Tr(e^{-s(-\Delta+m_0^2)}(e^{isf} - 1 - isf)).$$

For s small, we have that $e^{isf} - 1 - isf$ has order s^2 for $f \not\equiv 0$, and $e^{-m_0^2 s}$ has order 1. We next discuss the order of $Tr(e^{s\Delta})$. It is well-known that in dimension 1, Δ has eigenvalues $\{-k^2\}_{k\geq 0}$ (with corresponding eigenfunctions $\sin kx$), so in dimension 1, $Tr(e^{s\Delta}) = \sum_{k=0}^\infty e^{-k^2 s}$, which has order $\frac{1}{\sqrt{s}}$. Therefore, for genernal dimension d, we have that $Tr(e^{s\Delta})$ has order $s^{-d/2}$ for s small. Put all of these together, and we get that the integrand of (4.2) has order $\frac{1}{s}s^{-d/2}s^2$ near $s = 0$, which is not integrable if $d \geq 4$.

Finally, as in [11], by using Feynman-Kac formula, we have that

$$M(f) = \int_0^\infty \frac{ds}{s}e^{-m_0^2 s}\int d^d x dW_{(x,x)}(x(\cdot))\Big\{\exp(i\int_0^s f(x(\tau))d\tau) - 1 - i\int_0^s f(x(\tau))d\tau\Big\}. \tag{4.3}$$

Here $dW_{(x,x)}(x(\cdot))$ means the Wiener measure over closed paths, which start in x at $\tau = 0$ and end at x at $\tau = s$ ("Brownian loop" measure).

This expression is well-defined for $d = 2, 3$ (see [11]). However, this is not the case for $d \geq 4$. In the following, we take the example $d = 4$ for simplicity and show rigorously that the expression (4.3) is NOT well-defined.

PROPOSITION 4.1 *Let $d = 4$. Then for any $f \in \mathcal{S}(\mathbf{R}^d)$ strictly positive, we have that $M(f) = -\infty$.*

Proof. Without loss of generality, we may and do assume that there exists a $r > 0$ such that $f \geq 1_{B(0,r)}$, where $B(0, r)$ stands for the ball in $\mathbf{R}^4$ with center 0 and radius r.

First, by using the same method as below, it is easy to see that the integral of the third remainder of the Taylor expansion (of $\exp(i\int_0^s f(x(\tau))d\tau)$ in power of $\int_0^s f(x(\tau))d\tau$) is finite. Therefore, we only need to show that the quadratic term is infinite, *i.e.*, to show that

$$\int_0^\infty \frac{ds}{s} e^{-m_0^2 s} \int d^d x dW_{(x,x)}(x(\cdot)) \Big[\int_0^s f(x(\tau))d\tau \Big]^2 = \infty.$$

We shall now prove this.

Since the Wiener measure is translation invariant, for $x(\cdot)$ under $W_{(x,x)}$, we can write $x(t) = x + \omega(t)$, where $\omega(\cdot)$ is under $W_{(0,0)}$, *i.e.*, the path pinned up at 0. Also, we remark that there exists a constant $C_r > 0$ such that

$$\int_{\mathbf{R}^4} d^4 x 1_{B(0,r)}(x + y_1) 1_{B(0,r)}(x + y_2) \geq C_r 1_{B(0,r/2)}(y_1) 1_{B(0,r/2)}(y_2)$$

for any $y_1, y_2 \in \mathbf{R}^4$.

Therefore,

$$\begin{aligned}
& \int_0^\infty \frac{ds}{s} e^{-m_0^2 s} \int d^d x dW_{(x,x)}(x(\cdot)) \Big[\int_0^s f(x(\tau))d\tau \Big]^2 \\
= & \int_0^\infty \frac{ds}{s} e^{-m_0^2 s} \int d^d x dW_{(0,0)}(\omega(\cdot)) \Big[\int_0^s f(x + \omega(\tau))d\tau \Big]^2 \\
\geq & \; 2 \int_0^\infty \frac{ds}{s} e^{-m_0^2 s} \int d^d x dW_{(0,0)}(\omega(\cdot)) \int_0^s d\tau_1 \int_0^s d\tau_2 1_{B(0,r)}(x + \omega(\tau_1)) 1_{B(0,r)}(x + \omega(\tau_2)) \\
= & \; 2 \int_0^\infty \frac{ds}{s} e^{-m_0^2 s} \int_0^s d\tau_1 \int_0^s d\tau_2 dW_{(0,0)}(\omega(\cdot)) \int d^d x 1_{B(0,r)}(x + \omega(\tau_1)) 1_{B(0,r)}(x + \omega(\tau_2)) \\
= & \; 2C_r \int_0^\infty \frac{ds}{s} e^{-m_0^2 s} \int_0^s d\tau_1 \int_0^s d\tau_2 dW_{(0,0)}(\omega(\cdot)) 1_{B(0,r/2)}(\omega(\tau_1)) 1_{B(0,r/2)}(\omega(\tau_2)). \qquad (4.4)
\end{aligned}$$

For any $s > 0$ and any τ_2, τ_1 satisfying $0 < \tau_2 < \tau_1 < s$, let

$$A_{\tau_1,\tau_2} \equiv \int dW_{(0,0)}(\omega(\cdot)) 1_{B(0,r/2)}(\omega(\tau_1)) 1_{B(0,r/2)}(\omega(\tau_2)).$$

Then by the definition of Wiener measure over closed paths,

$$\begin{aligned}
A_{\tau_1,\tau_2} &= \int\int_{y_1,y_2\in B(0,r/2)} (\frac{1}{\sqrt{2\pi\tau_2}})^4 e^{-\frac{y_1^2}{2\tau_2}} \\
&\qquad \times (\frac{1}{\sqrt{2\pi(\tau_1-\tau_2)}})^4 e^{-\frac{(y_1-y_2)^2}{2(\tau_1-\tau_2)}} (\frac{1}{\sqrt{2\pi(s-\tau_1)}})^4 e^{-\frac{y_2^2}{2(s-\tau_1)}} dy_1 dy_2 \\
&= \int_{z_1\in B(0,\frac{r}{2\sqrt{\tau_2}})} dz_1 \int_{z_2\in B(0,\frac{r}{2\sqrt{s-\tau_1}})} dz_2 (\frac{1}{\sqrt{2\pi}})^4 e^{-\frac{z_1^2}{2}} (\frac{1}{\sqrt{2\pi}})^4 e^{-\frac{z_2^2}{2}} \\
&\qquad \times (\frac{1}{\sqrt{2\pi(\tau_1-\tau_2)}})^4 e^{-\frac{1}{2(\tau_1-\tau_2)}(\sqrt{\tau_2}z_1-\sqrt{s-\tau_1}z_2)^2}.
\end{aligned}$$

Notice that if $\tau_2 \in (0, \frac{s}{4})$ and $\tau_1 \in (\frac{3}{4}s, s)$, then $\tau_1 - \tau_2 > \frac{s}{2}$, so we have

$$\frac{(\sqrt{\tau_2}z_1 - \sqrt{s-\tau_1}z_2)^2}{\tau_1-\tau_2} < \frac{\frac{s}{4}(|z_1|+|z_2|)^2}{\frac{s}{2}} = \frac{1}{2}(|z_1|+|z_2|)^2$$

for any $z_1, z_2 \in \mathbf{R}^4$. Let $C \equiv (\int_{|z|\le 1}(\frac{1}{\sqrt{2\pi}})^4 e^{-\frac{z^2}{2}} dz)^2 e^{-1} \cdot \frac{1}{\pi^2}$. Then as long as $\frac{r}{\sqrt{s}} \ge 1$, we have that

$$\begin{aligned}
A_{\tau_1,\tau_2} &\ge \int_{|z_1|\le 1}\int_{|z_2|\le 1} (\frac{1}{\sqrt{2\pi}})^4 e^{-\frac{z_1^2}{2}} (\frac{1}{\sqrt{2\pi}})^4 e^{-\frac{z_2^2}{2}} dz_1 dz_2 e^{-\frac{1}{4}(1+1)^2} \cdot (\frac{1}{\sqrt{2\pi\cdot\frac{s}{2}}})^4 \\
&= C \cdot \frac{1}{s^2}, \qquad \text{for any } \tau_2 \in (0, \frac{s}{4}), \tau_1 \in (\frac{3}{4}s, s).
\end{aligned}$$

Substituting this into (4.4), since

$$\int dW_{(0,0)}(\omega(\cdot)) = (\frac{1}{\sqrt{2\pi s}})^4,$$

we get that

$$\begin{aligned}
&\int_0^\infty \frac{ds}{s} e^{-m_0^2 s} \int d^d x dW_{(x,x)}(x(\cdot)) \Big[\int_0^s f(x(\tau))d\tau\Big]^2 \\
\ge\ & 2C_r \int_0^{r^2} \frac{ds}{s} e^{-m_0^2 s} \int_{\frac{3}{4}s}^s d\tau_1 \int_0^{\frac{s}{4}} d\tau_2 C \cdot \frac{1}{s^2} \\
=\ & \tilde{C} \int_0^{r^2} \frac{ds}{s} e^{-m_0^2 s},
\end{aligned}$$

which is infinity.

This completes the proof of our assertion. ∎

Now, we are ready to proof our main result.

THEOREM 4.2 *Let $d = 4$. Then for any $f \in \mathcal{S}(\mathbf{R}^d)$ strictly positive, we have that* $:\phi^2:(f)$ *is not a random variable.*

Proof. For any $N \in \mathbf{N}$, define $\phi_N(x) = \int_{|p|\leq N} e^{ipx}\widetilde{\phi}(p)dp$, and let $:\phi_N^2:(f) = \int :\phi_N^2:(x)f(x)dx$, where $\tilde{}$ denotes the Fourier transform. Write the characteristic function of $:\phi_N^2:(f)$ at $\frac{1}{2}$ as

$$E[e^{\frac{i}{2}:\phi_N^2:(f)}] = e^{\frac{1}{2}M_n(f)}$$

(notice that this is rigorously the characteristic function at $\frac{1}{2}$ of the random variable $:\phi_N^2:(f)$ for any $N \in \mathbf{N}$). If $:\phi^2:(f)$ were a random variable, then $:\phi_N^2:(f)$ would converge weakly to $:\phi^2:(f)$, hence the corresponding characteristic functions would also converge, therefore, we would have $M_n(f) \to M(f)$, with $e^{\frac{1}{2}M(f)}$ the value of a characteristic function at $\frac{1}{2}$. This contradicts Proposition 4.1. Therefore, $:\phi^2:(f)$ is not a random variable. ∎

Remark 4 *Following the ideas of the proof of Proposition 4.1 and Theorem 4.2, one can see that a corresponding result holds for all $d \geq 5$.*

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