Some Properties of Dirichlet L-Functions Associated with their Nontrivial Zeros II

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SOME PROPERTIES OF DIRICHLET L-FUNCTIONS ASSOCIATED WITH THEIR NONTRIVIAL ZEROS II

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ABSTRACT. An asymptotic formula is derived for the even derivatives of the function
\[ \xi(s, \chi) = \left( \frac{\pi}{k} \right)^{-\frac{s+\delta}{2}} L(s, \chi) \]
at the point \( s = \frac{1}{2} \), where \( \Gamma(s) \) is the gamma function, \( \chi(n) \) is a real primitive character modulo \( k \), \( \delta = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1 \end{cases} \). \( L(s, \chi) \) is the Dirichlet function corresponding to \( \chi \). On the basis of this formula a sufficient condition for refutation of the Extended Riemann Hypothesis for \( L(s, \chi) \) is given.

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1. Introduction

The present work is the second part and the continuation of the article [1], devoted to the new properties of Dirichlet $L$-functions associated with their nontrivial zeros. We derive an asymptotic formula for the even derivatives of the function

$$
\xi(s, \chi) = \left( \frac{\pi}{k} \right)^{\frac{s+\delta}{2}} \Gamma \left( s + \frac{\delta}{2} \right) L(s, \chi)
$$

at the point $s = \frac{1}{2}$ as the order of the derivatives tends to infinity. $\Gamma(s)$ is the gamma function, $\chi = \chi(n)$ is a real primitive character modulo $k$, $\delta = \begin{cases} 
0, & \text{if } \chi(-1) = 1 \\
1, & \text{if } \chi(-1) = -1
\end{cases}$, $L(s, \chi)$ — a Dirichlet $L$-function. We assume, that the inequality

(1.1) 

$$L\left(\frac{1}{2}, \chi\right) \neq 0$$

holds. It is satisfied, e.g., when $\chi(n)$ is the Legendre symbol $\left( \frac{a}{p} \right)$ modulo prime number $p < 500000$. By virtue of [1] (Theorem 4, section 3) it follows from (1.1) that for all $s$ the equality $\xi(s, \chi) = \xi(1 - s, \chi)$ holds, and hence, all odd derivatives of the function $\xi(s, \chi)$ at the point $s = \frac{1}{2}$ are equal to zero. Finding an explicit asymptotic expression for the even derivatives $\frac{d^m}{ds^m} \xi\left(\frac{1}{2}, \chi\right)$ of $\xi(s, \chi)$ ($m$ is even) as $m \to \infty$ is of interest both in itself and in relation to the Extended Riemann Hypothesis [1] (Theorem 4, section 3). The proof is an analogue of that given for Riemann zeta function in [2].

In subsections 2.1 and 2.2 of section 2 we formulate and prove the main result for an even character, and in section 3 we make the same for an odd character. In section 4, on the basis of these results, a sufficient condition for refutation of the Extended Riemann Hypothesis is given.
2. AN ASYMPTOTIC FORMULA FOR DERIVATIVES OF THE FUNCTION $\xi(s, \chi)$ WITH EVEN CHARACTER $\chi$

2.1. Formulation of Theorem 1.

**Theorem 1.** Assume that $\chi(n)$ is an even real primitive character modulo $k \geq 2$ (that is $\chi(-1) = 1$), and $m$ is an even natural number. Then we have the following asymptotic expression as $m \to \infty$:

$$
\frac{d^m \xi}{ds^m}(\frac{1}{2}, \chi) \sim \left(\frac{1}{2}\right)^{m-1} \left(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + \beta\right)^m \exp \left(-\frac{mk}{\ln \frac{mk}{\pi}} e^\beta\right)
$$

$$
\times \left(\frac{mk}{\pi \ln \frac{mk}{\pi}}\right)^{\frac{1}{4}} \frac{\sqrt{\pi}}{\sqrt{m \left(\frac{1}{2(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi})^2} + \frac{1}{2(\ln \frac{mk}{\pi})}\right)}}
$$

where the function $\beta = \beta(m)$ satisfies the condition $\lim_{m \to \infty} \beta(m) = 0$, and we use the notation $a_m \sim b_m$ for $\lim_{n \to \infty} \frac{a_m}{b_m} = 1$.

**Corollary 1.** Under the assumption of Theorem 1 there exists $m_0 = m_0(k)$ depending on $k$ such that if $m \geq m_0$ then the inequality

$$
\frac{d^m \xi}{ds^m}(\frac{1}{2}, \chi) > 0
$$

holds.

2.2. Proof of Theorem 1.

The proof of the Theorem 1 is carried out in the five subsections below.

1. It follows from the formulas (4.5),(4.3) of [1] (section 4) that the equality

$$
\frac{d^m \xi}{ds^m}(\frac{1}{2}, \chi) = \left(\frac{1}{2}\right)^m \int_1^{\infty} (\ln^m x) x^{-\frac{3}{2}} \theta(x, \chi) dx
$$

holds, where

$$
\theta(x, \chi) = 2e^{-\frac{\pi x}{2}} + 2 \sum_{n=2}^{\infty} \chi(n)e^{-\frac{n^2 x}{4}}.
$$
Let \( x^* = \frac{k}{15} \) and consider the integral

\[
I'_m = \int_{x^*}^\infty (\ln^m x)x^{-\frac{3}{4}}\theta(x, \chi)dx .
\]

Because of Lemma 4 from [1] (section 4) the inequality \( \theta(x, \chi) > 0 \) holds. Then introducing the variable \( u = \ln x \) and using (2.2) we transform \( I'_m \) as follows:

\[
I'_m = \int_{u^*}^\infty u^m e^{\frac{\pi}{k}u}e\theta(e^u, \chi)du = \int_{u^*}^\infty e^{F(u)}du ,
\]

where \( u^* = \ln x^* \),

\[
F(u) = m\ln u + \frac{u}{4} + (\ln 2 - \frac{\pi}{k}e^u) + \ln(1 + \Psi(e^u)) ,
\]

\[
\Psi(e^u) = \sum_{n=2}^\infty \chi(n)e^{-\frac{\pi}{k}u(n^2-1)} .
\]

Differentiation of (2.5) yields the relations

\[
\frac{dF}{du}(u) = \frac{m}{u} + \frac{1}{4} - \frac{\pi}{k}e^u + \frac{d\ln(1 + \Psi(e^u))}{du} ,
\]

\[
\frac{d^2F}{du^2}(u) = -\frac{m}{u^2} - \frac{\pi}{k}e^u + \frac{d^2\ln(1 + \Psi(e^u))}{du^2} .
\]

We consider the equation

\[
\frac{dF}{du}(u) = 0
\]

in the domain \( u \geq u^* \).

Since \( x \geq 1 \), by (2.6)-(2.8), there exists \( m^* = m^*(k) \geq 2 \) such that, if \( m \geq m^* \) then

\[
\frac{d^2F}{du^2}(u) < 0
\]

for \( u \geq u^* \). Hence, if a solution of the equation (2.9) exists in the domain \( u \geq u^* \), then it is unique. We consider the approximate equation

\[
\frac{m}{u} - \frac{\pi}{k}e^u = 0 .
\]

By Lemma 1.2 in [2](§1) its solution \( u = \hat{u} \) has the form

\[
\hat{u} = \ln \frac{mk}{\pi} - \ln\ln \frac{mk}{\pi} + c_1 ,
\]
where the function \( c_1 = c_1(m) \) satisfies the condition \( \lim_{m \to \infty} c_1(m) = 0 \). Therefore, by (2.7), (2.9) and (2.12), the solution \( u = \tilde{u} \) of equation (2.9) can be written as

\[
\tilde{u} = \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2,
\]

where the function \( c_2 = c_2(m) \) satisfies the same condition as \( c_1 = c_1(m) \), that is,

\[
\lim_{m \to \infty} c_2(m) = 0.
\]

Therefore, \( \lim_{m \to \infty} \tilde{u}(m) = \infty \). We take a sufficiently large number \( m^* \) such that the inequality \( \tilde{u}(m) > u^* = \ln \frac{k}{15} \) holds for \( m \geq m^* \). Let \( \varepsilon_m \) be a constant such that

\[
0 < \varepsilon_m < \tilde{u} - u^* = \tilde{u} - \ln \frac{k}{15}.
\]

Substitution of (2.13) in (2.5) gives

\[
e^{F(\tilde{u})} = 2 \left( \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2 \right)^m \exp \left( -\frac{mk}{\ln \frac{mk}{\pi}} c_2 \right) \]

\[
\times \left( \frac{mk}{\pi \ln \frac{mk}{\pi} c_2} \right)^{\frac{1}{4}} \left( 1 + \Psi \left( \frac{mk}{\pi \ln \frac{mk}{\pi} c_2} \right) \right).
\]

Next, by (2.6) and (2.14), the asymptotic relations

\[
\left( \frac{mk}{\pi \ln \frac{mk}{\pi} c_2} \right)^{\frac{1}{4}} \sim \left( \frac{mk}{\pi \ln \frac{mk}{\pi}} \right)^{\frac{1}{4}},
\]

\[
\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2 \sim \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi},
\]

and the equality \( \lim_{m \to \infty} \Psi \left( \frac{mk}{\pi \ln \frac{mk}{\pi} c_2} \right) = 0 \) hold as \( m \to \infty \). Therefore, by (2.16), we obtain that the asymptotic formula

\[
e^{F(\tilde{u})} = 2 \left( \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2 \right)^m \exp \left( -\frac{mk}{\ln \frac{mk}{\pi}} c_2 \right) \left( \frac{mk}{\pi \ln \frac{mk}{\pi}} \right)^{\frac{1}{4}}
\]

holds as \( m \to \infty \). Substitution of (2.13) in (2.8) yields

\[
\frac{d^2}{du^2} F(\tilde{u}) = -\frac{m}{(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)^2} - \frac{\pi}{k} \exp \left( \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2 \right)
\]

\[
+ \frac{d^2 \ln (1 + \Psi(1 + \exp(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)))}{du^2}
\]

\[
= -\frac{m}{(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)^2} - \frac{m}{\ln \frac{mk}{\pi}} + o(m)
\]

\[
< -c_3 \frac{m}{\ln \frac{mk}{\pi}},
\]
where $c_3 > 0$ does not depend on $m$ and $\lim_{m \to \infty} \frac{\sigma(m)}{m} = 0$. Let us estimate $\left| \frac{d^3 F(u)}{du^3} \right|$ for $\check{u} - \varepsilon_m \leq u \leq \check{u} + \varepsilon_m$, where $\varepsilon_m$ is a constant satisfying (2.15). Differentiating (2.8), we obtain

$$\frac{d^3}{du^3} F(u) = \frac{2m}{u^3} - \frac{\pi}{k} e^u + \frac{d^3 \ln(1 + \Psi(e^u))}{du^3}.$$ 

Therefore, by (2.13),

$$\sup_{\check{u} - \varepsilon_m \leq u \leq \check{u} + \varepsilon_m} \left| \frac{d^3 F(u)}{du^3} \right| < c_4 \left( \frac{m}{(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2 - \varepsilon_m)^3} + \frac{mke^{\varepsilon_m}}{\ln \frac{mk}{\pi}} \right),$$

where $c_4 > 0$ is a constant not depending on $m$. For $\check{u} - \varepsilon_m \leq u \leq \check{u} + \varepsilon_m$, the equality

$$(2.20) \quad F(u) = \tilde{F}(u) + \check{F}(u)$$

holds, where

$$(2.21) \quad \tilde{F}(u) = F(\check{u}) + \frac{1}{2} \frac{d^2 F(\check{u})}{du^2} (u - \check{u})^2, \quad \check{F}(u) = \frac{1}{6} \frac{d^3 F(\theta)}{du^3} (u - \check{u})^3,$$

and $\check{u} - \varepsilon_m \leq u \leq \check{u} + \varepsilon_m$. Applying Lemma 2.1 in [2](§2) and formula (2.18), we obtain

$$\int_{\check{u} - \varepsilon_m}^{\check{u} + \varepsilon_m} \exp \left( \frac{1}{2} \frac{d^2 F(\check{u})}{du^2} (u - \check{u})^2 \right) du = \frac{\sqrt{\pi}}{\sqrt{\left| \frac{1}{2} \frac{d^2 F(\check{u})}{du^2} \right|}} (1 + R_{\varepsilon_m}),$$

where

$$|R_{\varepsilon_m}| < \frac{\exp(\frac{1}{2} \frac{d^2 F(\check{u})}{du^2} \varepsilon_m^2)}{1 + \sqrt{1 - \exp(\frac{1}{2} \frac{d^2 F(\check{u})}{du^2} \varepsilon_m^2)}}.$$

Therefore, by (2.18)

$$\int_{\check{u} - \varepsilon_m}^{\check{u} + \varepsilon_m} \exp \left( \frac{1}{2} \frac{d^2 F(\check{u})}{du^2} (u - \check{u})^2 \right) du = \frac{\sqrt{\pi}}{\sqrt{m \left( \frac{1}{2(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)}^2 + \varepsilon_m^2 + o(m) \right)}} (1 + R_{\varepsilon_m}),$$

where

$$|R_{\varepsilon_m}| < \frac{\exp \left( - \frac{me^{2\varepsilon_m}}{2(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)^2} - \frac{me^{2\varepsilon_m}}{2\ln \frac{mk}{\pi} + \varepsilon_m^2} + o(m) \right)}{1 + \sqrt{1 - \exp \left( - \frac{me^{2\varepsilon_m}}{2(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)^2} - \frac{me^{2\varepsilon_m}}{2\ln \frac{mk}{\pi} + \varepsilon_m^2} + o(m) \right)}}.$$


By (2.20), we have
\[ e^{F(u)} = e^{\tilde{F}(u)} + e^{\hat{F}(u)} \left( e^{\hat{F}(u)} - 1 \right). \]

Consequently,
\[ \int_{\tilde{u} - \varepsilon_m}^{\tilde{u} + \varepsilon_m} e^{F(u)} \, du = \int_{\tilde{u} - \varepsilon_m}^{\tilde{u} + \varepsilon_m} e^{\tilde{F}(u)} \, du + R'_{\varepsilon_m}, \]
where, according to (2.21)
\[ |R'_{\varepsilon_m}| < e^{F(\tilde{u})} \sup_{\tilde{u} - \varepsilon_m \leq u \leq \tilde{u} + \varepsilon_m} |e^{\hat{F}(u)} - 1| \int_{\tilde{u} - \varepsilon_m}^{\tilde{u} + \varepsilon_m} \exp \left( \frac{1}{2} \frac{d^2 F(\tilde{u})}{du^2} (u - \tilde{u})^2 \right) \, du. \]

Applying the relation for \( \hat{F}(u) \) in (2.21) and inequality (2.19), and assuming that \( \varepsilon_m^3 \) does not exceed a constant not depending on \( m \), we derive the inequality
\[ \sup_{\tilde{u} - \varepsilon_m \leq u \leq \tilde{u} + \varepsilon_m} |e^{\hat{F}(u)} - 1| < c_5 \varepsilon_m^3 \frac{mk}{\ln \frac{mk}{\pi}}, \]
where \( c_5 \) is a constant not depending on \( m \). But if the relation
\[ \lim_{m \to \infty} \varepsilon_m^3 m = 0 \]
holds, then the previous inequality and (2.25) imply that
\[ |R'_{\varepsilon_m}| = e^{F(\tilde{u})} \int_{\tilde{u} - \varepsilon_m}^{\tilde{u} + \varepsilon_m} \exp \left( \frac{1}{2} \frac{d^2 F(\tilde{u})}{du^2} (u - \tilde{u})^2 \right) \, du \cdot o'(m) \]
where \( \lim_{m \to \infty} o'(m) = 0 \). Therefore by (2.24) the asymptotic relation
\[ \int_{\tilde{u} - \varepsilon_m}^{\tilde{u} + \varepsilon_m} e^{F(u)} \, du \sim e^{\tilde{F}(\tilde{u})} \int_{\tilde{u} - \varepsilon_m}^{\tilde{u} + \varepsilon_m} \exp \left( \frac{1}{2} \frac{d^2 F(\tilde{u})}{du^2} (u - \tilde{u})^2 \right) \, du \]
holds as \( m \to \infty \). Now we apply (2.17), (2.22), (2.23) and (2.27) and, assuming (2.26) and the relation
\[ \lim_{m \to \infty} \varepsilon_m^2 m = \infty, \]
obtain the asymptotic expression
\[ \int_{\tilde{u} - \varepsilon_m}^{\tilde{u} + \varepsilon_m} e^{F(u)} \, du \sim 2(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)^m \exp \left( -\frac{mk}{\ln \frac{mk}{\pi}} e^{c_2} \right) \]
\[ \times \left( \frac{mk}{\pi \ln \frac{mk}{\pi}} \right)^4 \sqrt{\frac{\pi}{m \left( 2(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi})^2 + \frac{1}{2} \ln \frac{mk}{\pi} \right)}}, \]
as \( m \to \infty \), where the function \( c_2 = c_2(m) \) satisfies inequality (2.14).

2. We write \( u_+ = \tilde{u} + \varepsilon_m \), \( u_- = \tilde{u} - \varepsilon_m \). Since \( u = \tilde{u} \) is a solution of equation (2.9), we have

\[
F(u_+) = F(\tilde{u}) + \frac{\varepsilon_m^2}{2} \frac{d^2 F(\zeta_+)}{du^2}, \quad F(u_-) = F(\tilde{u}) + \frac{\varepsilon_m^2}{2} \frac{d^2 F(\zeta_-)}{du^2},
\]

where the numbers \( \zeta_+ \) and \( \zeta_- \) satisfy the inequalities

\[
\tilde{u} \leq \zeta_+ \leq \tilde{u} + \varepsilon_m, \quad \tilde{u} - \varepsilon_m \leq \zeta_- \leq \tilde{u}.
\]

The application of (2.8), (2.13) and (2.31) results in

\[
\max \left( \frac{d^2 F(\zeta_-)}{du^2}, \frac{d^2 F(\zeta_+)}{du^2} \right) < -\frac{m}{(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2 + \varepsilon_m)^2} - \frac{mke^{c_2-\varepsilon_m}}{\ln \frac{mk}{\pi}} + c_6,
\]

where \( c_6 \) is a constant not depending on \( m \). Now let

\[
\varepsilon_m = m^{-\left(\frac{1}{2} - \delta\right)}, \quad 0 < \delta < \frac{1}{6}.
\]

Then (2.26) and (2.28) hold and, by (2.30) and (2.32), we have the inequalities

\[
\begin{cases}
F(\tilde{u}) - F(u_+) > c_7 m^\delta, & F(\tilde{u}) - F(u_-) > c_7 m^\delta, \\
e^{F(u_+)} = e^{F(\tilde{u}) - (F(\tilde{u}) - F(u_+))} < \frac{e^{F(\tilde{u})}}{e^{c_7 m^\delta}}, \\
e^{F(u_-)} = e^{F(\tilde{u}) - (F(\tilde{u}) - F(u_-))} < \frac{e^{F(\tilde{u})}}{e^{c_7 m^\delta}},
\end{cases}
\]

where \( c_7 > 0 \) is a constant not depending on \( m \). By (2.8), the inequality \( \frac{d^2 F(u)}{du^2} < 0 \) holds for large \( m \) and \( u^* \leq u \leq \tilde{u} \), and since \( \frac{dF(\tilde{u})}{du} = 0 \) then \( F(u) \) is a monotone increasing function as \( u \) grows for \( u^* \leq u \leq \tilde{u} \). Therefore formulae (2.13) and (2.34) imply the inequality

\[
\int_{u^*}^{\tilde{u} - \varepsilon_m} e^{F(u)} du < \tilde{u} e^{F(u_-)} < \frac{(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2)e^{F(\tilde{u})}}{e^{c_7 m^\delta}}.
\]
By the equality \( x^* = \frac{k}{15} \) and (2.2) we have
\[
\left| \int_1^{x^*} \left( \ln^{m} x \right) x^{-\frac{4}{3}} \theta(x, \chi) dx \right| < c_8 k \ln^{m} \frac{k}{15},
\]
where \( c_8 \) is a constant not depending on \( m \). The two last inequalities give the following:
\[
(2.35) \left| \int_1^{x^-} \left( \ln^{m} x \right) x^{-\frac{4}{3}} \theta(x, \chi) dx \right| < c_8 \left( \frac{\ln \left( m k \pi \right) - \ln \ln \left( m k \pi \right) + c_2 e^{F(\tilde{u})}}{e^{c \pi^2 m^2}} + k \ln^{m} \frac{k}{15} \right),
\]
where
\[
(2.36) \quad x^- = e^{u^-} = e^{\tilde{u} - \varepsilon m}.
\]

3. Suppose that a positive number \( \tilde{x}_m \) satisfies the equation
\[
(2.37) \ln^{m k} \tilde{x}_m = e^{\pi \delta \tilde{x}_m},
\]
where \( \delta \) is the parameter introduced in (2.33). Then the relations
\[
(2.38) \quad \frac{mk}{\pi \delta} = \frac{\tilde{x}_m}{\ln \ln \tilde{x}_m},
\]
\[
\mu \overset{\text{def}}{=} \ln \frac{mk}{\pi \delta} = \ln \tilde{x}_m - \ln \ln \tilde{x}_m = \tilde{y}_m - \ln \ln \tilde{y}_m \overset{\text{def}}{=} f(\tilde{y}_m)
\]
hold, where \( \tilde{y}_m \) is defined by
\[
(2.39) \quad \tilde{y}_m = \ln \tilde{x}_m.
\]
Let \( \hat{y}_m \) satisfy the equation
\[
(2.40) \quad \mu = \hat{y}_m - \ln \hat{y}_m.
\]
By (2.38) and (2.40), we have
\[
f(\hat{y}_m) > \mu, \quad f(\mu) < \mu,
\]
and, since \( f(y) \) is a monotone increasing function as \( y \) grows, relation (2.38) implies that \( \tilde{y}_m \) satisfies the inequality
\[
(2.41) \quad \mu < \tilde{y}_m < \hat{y}_m.
\]
According to (2.38) and Lemma 1.1 in [2] (§1), the solution $\hat{y}_m$ of the equation (2.40) has the form

$$\hat{y}_m = \ln \frac{mk}{\pi \delta} + \ln \ln \frac{mk}{\pi \delta} + c_9,$$

and the function $c_9 = c_9(m)$ satisfies the condition

$$\lim_{m \to \infty} c_9(m) = 0.$$

Consequently, by (2.39) and (2.41), we have

$$\ln \frac{mk}{\pi \delta} < \ln \tilde{x}_m < \ln \frac{mk}{\pi \delta} + \ln \ln \frac{mk}{\pi \delta} + c_9.$$

Using (2.34), (2.42) and (2.43) for large $m$ we obtain

$$\left| \int_{u_+}^{\ln \tilde{x}_m} e^{F(u)} du \right| < e^{F(u_+)} \left| \ln \tilde{x}_m - u_+ \right|$$

$$< \frac{e^{F(\delta)}}{e^{c_7 m^2}} \left( \ln \frac{mk}{\pi \delta} + \ln \ln \frac{mk}{\pi \delta} + c_9 - u_+ \right).$$

4. We estimate the integral

$$\tilde{I}_m \overset{\text{def}}{=} \int_{\tilde{x}_m}^{\infty} (\ln^m x)x^{-\frac{3}{4}} \theta(x, \chi) dx,$$

where $\tilde{x}_m$ is defined by (2.37). According to (2.37), (2.42), (2.43) and (2.2), we have

$$\left| \tilde{I}_m \right| < 2 \int_{\tilde{x}_m}^{\infty} x^{-\frac{3}{4}} \sum_{n=1}^{\infty} e^{-\frac{mk}{\pi \delta} (n^2-\delta)} dx$$

$$< 2 \int_{\frac{mk}{\pi \delta}}^{\infty} x^{-\frac{3}{4}} \sum_{n=1}^{\infty} e^{-\frac{mk}{\pi \delta} (n^2-\delta)} dx.$$

5. Let

$$I_m = \int_{1}^{\infty} (\ln^m x)x^{-\frac{3}{4}} \theta(x, \chi) dx.$$

We have

$$I_m = \int_{1}^{x} (\ln^m x)x^{-\frac{3}{4}} \theta(x, \chi) dx + \int_{u_+}^{x} e^{F(u)} du$$

$$+ \int_{u_+}^{\ln \tilde{x}_m} e^{F(u)} du + \int_{\frac{mk}{\pi \delta}}^{\infty} (\ln^m x)x^{-\frac{3}{4}} \theta(x, \chi) dx.$$
where \( u_+ = \hat{u} - \varepsilon_m, \ u_- = \hat{u} + \varepsilon_m, \ x_+ = e^{u_+} \) (see (2.36)) and \( \tilde{x}_m \) is the solution of equation (2.37). Assuming that (2.33) holds, we obtain, by (2.35),(2.44) and (2.45), the inequality

\[
\left| \int_0^{x_-} (\ln^m x) x^{-\frac{3}{4}} \theta(x, \chi) dx + \int_{x_+}^{\infty} e^{F(u)} du + \int_{\tilde{x}_m}^{x_-} (\ln^m x) x^{-\frac{3}{4}} \theta(x, \chi) dx \right| 
\leq c_{10} \left( \frac{\ln mk}{\pi \delta} + \ln \ln \frac{mk}{\pi \delta} + c_9 \right) + 2 \int_{\tilde{x}_m}^{\infty} \sum_{n=1}^{\infty} e^{-\frac{\pi}{n} (n^2 - \delta)} dx,
\]

where \( c_{10} \) is a constant not depending on \( m \). By (2.22),(2.23) and (2.27), the first and second terms on the right-hand side of (2.48) are \( o(\int_{\hat{u}-\varepsilon_m}^{\hat{u}+\varepsilon_m} e^{F(u)} du) \) as \( m \to \infty \).

Therefore, by (2.47) and (2.48), the asymptotic relation \( I_m \sim \int_{\hat{u}-\varepsilon_m}^{\hat{u}+\varepsilon_m} e^{F(u)} du \) holds as \( m \to \infty \) and, by (2.29), the asymptotic relation

\[
I_m \sim 2 \left( \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + c_2 \right) m \exp \left( -\frac{mk}{\ln \frac{mk}{\pi} e^{c_2}} \right) 
\times \left( \frac{mk}{\pi \ln \frac{mk}{\pi}} \right)^{\frac{1}{4}} \sqrt{\frac{\pi}{m}} \sqrt{m \left( \frac{1}{2(\ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi})^2} + \frac{1}{2 \ln \frac{mk}{\pi}} \right)}
\]

holds as \( m \to \infty \), where the function \( c_2 = c_2(m) \) satisfies (2.14). Since, by (2.1) and (2.46), we have \( \frac{d}{dm} \xi(\frac{1}{2}, \chi) = (\frac{1}{2})^m I_m \), then the relation (2.49) implies Theorem 1 for \( \beta(m) = c_2(m) \). Theorem 1 is proved.

3. An asymptotic formula for the derivatives of the function \( \xi(s, \chi) \) with an odd character \( \chi \)

3.1. Formulation of Theorem 2.

**Theorem 2.** Assume that \( \chi = \chi(n) \) is an odd real primitive character modulo \( k \geq 2 \) (that is \( \chi(-1) = -1 \)), the inequality (1.1) holds and \( m \) is an even natural number.
Then we have the following asymptotic expression as \( m \to \infty \)
\[
\frac{d^m \xi}{ds^m} \left( \frac{1}{2}, \chi \right) \sim \left( \frac{1}{2} \right)^{m-1} \left( \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} + \beta \right)^m
\]
\[
\times \exp \left( - \frac{mk}{\ln \frac{mk}{\pi}} \right) \left( \frac{mk}{\pi \ln \frac{mk}{\pi}} \right)^{\frac{3}{2}}
\]
\[
\times \sqrt{\frac{m \left( \frac{1}{2 \left( \ln \frac{mk}{\pi} - \ln \ln \frac{mk}{\pi} \right)^2} + \frac{1}{2 \ln \frac{mk}{\pi}} \right)}{\sqrt{\pi}}},
\]
where the function \( \beta = \beta(m) \) satisfies the condition \( \lim_{m \to \infty} \beta(m) = 0 \).

**Remark.** By [1] (Theorem 4, section 3) it follows from (1.1) that for all \( s \) the equality \( \xi(s, \chi) = \xi(1 - s, \chi) \) holds, and hence, all odd derivatives of the function \( \xi(s, \chi) \) at the point \( s = \frac{1}{2} \) are equal zero.

**Corollary 2.** Under the assumption of Theorem 2 there exists \( m_0 = m_0(k) \) depending on \( k \) such that if \( m \geq m_0 \) is even then the inequality
\[
(3.1) \quad \frac{d^m \xi}{ds^m} \left( \frac{1}{2}, \chi \right) > 0
\]
holds.

### 3.2. Proof of Theorem 2.

It follows from the formulas (4.9),(4.11) of the paper [1] (section 4) that the equality
\[
(3.2) \quad \frac{d^m \xi}{ds^m} \left( \frac{1}{2}, \chi \right) = \left( \frac{1}{2} \right)^m \int_1^\infty (\ln^m x) x^{-\frac{1}{4}} \theta_1(x, \chi) dx
\]
holds, where
\[
(3.3) \quad \theta_1(x, \chi) = 2e^{-\frac{\pi x}{k}} + 2 \sum_{n=2}^\infty n\chi(n)e^{-\frac{2\pi n x}{k}}.
\]
Let \( x^* = \frac{k}{l} \) and consider the integral
\[
(3.4) \quad I'_m = \int_{x^*}^\infty (\ln^m x) x^{-\frac{1}{4}} \theta_1(x, \chi) dx.
\]
Since according to Lemma 5 from [1] (section 4) the inequality \( \theta_1(x, \chi) > 0 \) holds, then introducing the variable \( u = \ln x \) and using (3.3) we transform \( I_m' \) as follows:

\[
I_m' = \int_{u_*}^{\infty} u^m e^{\frac{3u}{4}} \theta_1(e^u, \chi) du = \int_{u_*}^{\infty} e^{F(u)} du ,
\]

where \( u^* = \ln x^* = \ln \frac{k}{r} \),

\[
F(u) = m \ln u + \frac{3u}{4} + (\ln 2 - \frac{\pi}{k} e^u) + \ln(1 + \Psi(e^u)) \, ,
\]

\[
\Psi(e^u) = \sum_{n=2}^{\infty} n\chi(n)e^{-\frac{\pi u}{k}(n^2-1)} .
\]

Differentiation of (3.6) results in the relations

\[
\frac{d^2F}{du^2}(u) = -\frac{m}{u^2} - \frac{\pi}{k} e^u + \frac{d^2 \ln(1 + \Psi(e^u))}{du^2} .
\]

Now, using equalities (3.2)-(3.9) the proof of Theorem 2 runs in a completely analogue way as the proof of Theorem 1.

4. SUFFICIENT CONDITION FOR REFUTATION OF EXTENDED RIEMANN HYPOTHESIS

**Theorem 3.** Assume that \( \chi \) is a real primitive character such that \( L(\frac{1}{2}, \chi) < 0 \).

Then the Extended Riemann Hypothesis is false: in this case there exists a nontrivial zero of \( L(s, \chi) \) that does not lie on the line \( \text{Res} = \frac{1}{2} \).

**Proof.** Using the equality

\[
\xi(s, \chi) = \left( \frac{\pi}{k} \right)^{-\frac{s+\delta}{2}} \Gamma \left( \frac{s + \delta}{2} \right) L(s, \chi) ,
\]

where \( \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1 \end{cases} \), we obtain that if \( L(\frac{1}{2}, \chi) < 0 \), then the inequality

\[
\xi(\frac{1}{2}, \chi) < 0
\]
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holds. Now, the statement of Theorem 3 follows from the assumption of theorem 3, from (4.1), and Corollaries 1 and 2, and from Theorem 4 as proved in [1](section 3). Theorem 3 is proved.

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References


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