Geometric Inequalities and Generalized Ricci Bounds in the Heisenberg Group

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Abstract

In this paper, we prove that no curvature-dimension bound $CD(K,N)$ holds in any Heisenberg group $\mathbb{H}^n$. On the contrary the measure contraction property $MCP(0,2n+3)$ holds and is optimal for the dimension $2n+3$. For the non-existence of a curvature-dimension bound, we prove that the generalized "geodesic" Brunn-Minkowski inequality is false in $\mathbb{H}^n$. We also show in a new and direct way, (and for all $n \in \mathbb{N}\backslash\{0\}$) that the general "multiplicative" Brunn-Minkowski inequality with dimension $N > 2n+1$ is false.

Introduction

The geometric curvature-dimension bound $CD(K,N)$ is a property of some metric measure spaces (X,d,μ) that extends the property for a Riemannian manifold of having a Ricci curvature bound from below. This theory has been recently developed by Sturm (see [St1], [St2]) and independently in a close way by Lott and Villani (see [LV1], [LV2]). Take care of the fact that this concept is different from the curvature-dimension condition by Bakry and Émery (see [BE]). As a basic principle, Lott, Sturm and Villani observed that the optimal mass transportation takes place in a particular way in the Riemannian manifold with a Ricci curvature bound below. Namely the relative entropy of the measure being transported has different levels of convexity that are connected with the curvature bound. Optimal mass transportation in metric measure spaces is a theory about the problem of transporting a probability measure into a second one by minimizing a quadratic transport cost (it is the Monge-Kantorovich problem). Two very good books on this topic are those of Villani (see [Vi1],[Vi2]).

The measure contraction property $MCP(K,N)$ is another geometrical property that also involves curves in the space of measures of a given metric measure space. For the MCP one of the measure at the end is a Dirac mass. The other measure is contracted on it and this contraction reveals some geometrical aspects of the space. The measure contraction property is also seen as a generalization of the Ricci lower bound of a Riemannian manifold. Sturm proved in [St2] (see also [LV2]) that the measure contraction property is a consequence of a geometric curvature-dimension bound CD in the case when there is almost surely a unique geodesic between two points of X.

A weaker and little older setting is given by a local Poincaré inequality and a doubling measure property. It has proved to be very efficient as a minimal framework permitting the use of a lot of analysis tools. This framework has been introduced with the idea to extend the successful concept of analysis on Carnot groups (the Heisenberg group $\mathbb{H}^n$ is the simplest representant of this class of metric measure spaces) to a more general class of metric measure spaces ; see the book by Heinonen [He] and the references therein. The setting permits to extend many concepts of classical analysis that concern first derivatives but does not allow to deal with notions concerning derivatives of higher orders (what CD and MCP in some sense do). If we assume the same as in the last paragraph (almost surely a unique geodesic between two points), the local Poincaré inequality and the doubling measure property follow from an MCP relation (see [vR],[LV2]) and consequently of a curvature-dimension bound CD.

The inequality that we will now present has a longer history because it has been initiated by Brunn in 1887. The Brunn-Minkowski inequality in $\mathbb{R}^N$ estimates the measure of the sum of two sets with the measures of these sets. In our general setting of metric measure spaces, it is interpreted as an estimate of the t-intermediate set built with the point standing at a $t \in [0,1]$ time position on some geodesic from one of the two sets to the other. The property that we denote by $BM(K,N)$ is a consequence of $CD(K,N)$ but not a priori of any measure contraction property.

For a Riemannian manifold of dimension N and with Riemannian metric g, the terminology is coherent because it is equivalent to satisfy the curvature-dimension $CD(K,N)$, to have a measure contraction property $MCP(K,N)$ or to have a Ricci curvature greater than Kg. Another very satisfying point of the theory of Lott-Sturm-Villani is the stability of the curvature-dimension bounds under Gromov-Hausdorff like convergence. The Heisenberg group is a metric measure space that can be obtained as the limit of manifolds. Unfortunately it does not mean that it has a curvature-dimension bound. Namely the convergence theorem is given for bounded spaces (not like $\mathbb{H}^n$) and more essentially the Ricci curvature of the approximating sequence is going to $-\infty$ which eliminates the hope to obtain a curvature-dimension condition in this way. However the Heisenberg group $\mathbb{H}^n$ is a very good candidate in a theory that is missing examples: as we already said, it has been known for a long time that this space verifies a local Poincaré inequality and a doubling measure property (see [Va]).

In this paper we prove the following theorem:

Theorem 0.1. *Let n be a non negative integer. We consider $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$, the n-th Heisenberg group with its Carnot-Carathéodory distance and the Lebesgue measure of $\mathbb{R}^{2n+1}$. We have:*

- *For every $N \in [1,+\infty[$ and every $K \in \mathbb{R}$, the geometric curvature-dimension bound $CD(K,N)$ does* not *hold in $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$.*
- *For $(N,K) \in [1,+\infty[\times\mathbb{R}$, the measure contraction property $MCP(K,N)$ holds in $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$ if and only if $N \geq 2n+3$ and $K \leq 0$.*

We prove the first point in Theorem 3.3 (with Remark 3.4) by showing that there is no Brunn-Minkowski inequality $BM(0,N)$ in $\mathbb{H}^n$. The positive result on the measure contraction property is proved in Theorem 2.3 (also with Remark 3.4).

In the first part of this paper, we give a short presentation of the Heisenberg group $\mathbb{H}^n$ ($n \in \mathbb{N}\backslash\{0\}$) and of its geodesics. We also introduce two maps that will be helpful in the next sections: the time-inversion map $\mathcal{I}$ and the intermediate-points map $\mathcal{M}$. At the begin of the second part we give the definition of $CD(K, N)$ and $MCP(K, N)$ for the case of interest $K = 0$. Then we prove that $MCP(0, 2n + 3)$ holds. The last part is devoted to the proof of the fact that there is no Brunn-Minkowski inequality in the Heisenberg group (and consequently no curvature-dimension bound). In this section we also treat of MCP and CD in the case of non-zero curvature parameter. Additionally we mention the multiplicative Brunn-Minkowski inequality and sketch the fact that this inequality does not hold for a dimension strictly greater than the topological dimension (i.e. $2n + 1$).

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1 The Heisenberg group and its geodesics

1.1 The Heisenberg group

Let n be a non-negative integer. In this section we give a short presentation of the Heisenberg group $\mathbb{H}^n$ as a metric measure space equipped with the Lebesgue measure $\mathcal{L}^{2n+1}$ and the Carnot-Carathéodory metric d_{CC}. As a set $\mathbb{H}^n$ can be defined as $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ and an element of $\mathbb{H}^1$ can also be written $(z, \alpha) = (z_1, \cdots, z_n, \alpha)$ where $z_k := x_k + iy_k \in \mathbb{C}^n$ for $1 \leq k \leq n$ and $\alpha \in \mathbb{R}$. The group law of $\mathbb{H}^n$ is given by:

$$(z_1, \cdots, z_n, \alpha)\cdot(z'_1, \cdots, z'_n, \alpha') = \left(z_1 + z'_1, \cdots, z_n + z'_n, \alpha + \alpha' + 2\sum_{k=1}^{n} \Im(z_k \overline{z'_k}) \right)$$

where $\Im$ denotes the imaginary part of a complex number. Endowed with this law, $\mathbb{H}^n$ is a Lie group with neutral element $0_{\mathbb{H}^n} := (0, 0)$. The inverse element of (z, α) is $(-z, -\alpha)$. The set $L = \{(z, \alpha) \in \mathbb{H}^n \mid z = 0\}$ is the center of the group and will play an important role. Throughout this paper, $\tau_A : \mathbb{H}^n \to \mathbb{H}^n$ will be the left translation

$$\tau_A(B) = A \cdot B.$$

It is affine and the vectorial part of this map has the determinant equal to 1. Hence the Haar measure of $\mathbb{H}^n$ is the Lebesgue measure $\mathcal{L}^{2n+1}$ of $\mathbb{R}^{2n+1}$ which is left (and actually also right) invariant. For $\lambda > 0$, we denote by δ_λ the dilatation

$$\delta_\lambda(z, \alpha) = (\lambda z, \lambda^2 \alpha)$$

where $A, B \in \mathbb{H}^n$ and $\lambda \geq 0$. The dilatation has also a good behavior with the measure because

$$\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E)$$

if $\lambda \geq 0$ and E is a measurable set. The exponent $2n + 2$ is actually the Hausdorff dimension of $\mathbb{H}^n$ for its subriemannian structure given by the Carnot-Carathéodory metric. It is not equal to the topological dimension $2n+1$, which is a basic difference from the Riemannian geometry where both dimensions are the same.

In order to define the Carnot-Carathéodory metric, we consider the Lie Algebra associated to $\mathbb{H}^n$. This is the vector space of left-invariant vector fields. A basis for this vector space is given by $\left(\overrightarrow{X}_1, \cdots, \overrightarrow{X}_n, \overrightarrow{Y}_1, \cdots, \overrightarrow{Y}_n, \overrightarrow{T}\right)$ where

$$\begin{aligned}\overrightarrow{X}_k &= \partial_{x_k} + 2y_k\partial_t\\ \overrightarrow{Y}_k &= \partial_{y_k} - 2x_k\partial_t\\ \overrightarrow{T} &= \partial_t.\end{aligned}$$

Roughly speaking, the Carnot-Carathéodory distance between two points A and B is the infimum of the lengths of the horizontal curves connecting A and B. We call horizontal curve an absolutely continuous curve $\gamma : [0, r] \to \mathbb{H}^n$ whose derivative γ' is spanned by $(\overrightarrow{X}_1, \cdots, \overrightarrow{X}_n, \overrightarrow{Y}_1, \cdots, \overrightarrow{Y}_n)$ in each point $\gamma(t)$. The length of this curve is then

$$\mathrm{length}(\gamma) = \int_0^r \|\gamma'(t)\|\, dt$$

where $\| \sum_{k=1}^n (a_k\overrightarrow{X}_k + b_k\overrightarrow{Y}_k)\|^2 = \sum_{k=1}^n (a_k^2 + b_k^2)$. The value of the Carnot-Carathéodory distance between A and B is then

$$d_{CC}(A, B) := \inf \int_0^r \|\gamma'(t)\|\, dt \tag{1}$$

where the infimum is taken upon all horizontal curve connecting A and B. The fact that this set is not empty is ensured by the Chow Theorem (see for example [Mt]). The Carnot-Carathéodory metric has (like the Lebegue measure) a good behavior with the translation τ_A and the dilatation δ_λ. It is left-invariant:

$$d_{CC}(\tau_A B, \tau_A C) = d_{CC}(B, C)$$

and we also have

$$d_{CC}(\delta_\lambda(B), \delta_\lambda(C)) = \lambda d_{CC}(B, C)$$

for $\lambda > 0$.

The metric space $(\mathbb{H}^n, d_{CC})$ is complete and separable. An other essential fact is that the topology given by d_{CC} is the usual topology of $\mathbb{R}^{2n+1}$.

1.2 A geodesic space

Let us first give the terminology that we will use in this paper.

Definition 1.1. Let (X, d) be a metric space. Let M_0 and M_1 be two points of this set. We call t-intermediate point from M_0 to M_1 a point M_t such that

$$\begin{aligned} d(M_0, M_t) &= td(M_0, M_1) \quad \text{and}\\ d(M_t, M_1) &= (1-t)d(M_0, M_1).\end{aligned}$$

We call *geodesic* from M_0 to M_1 a curve γ defined on a segment $[a, b]$ (with $a < b$) such that for every $c \in [a, b]$, the point$\gamma(c)$ is a $\frac{c-a}{b-a}$-intermediate point from M_0 to M_1. We call *normal geodesic* a geodesic defined on $[0,1]$ (here $t=c$). We call *local geodesic* a curve γ defined on a non-trivial interval I, such that for any point t in the interior of I there is an $\varepsilon > 0$ with $[t-\varepsilon, t+\varepsilon] \subset I$ and $\gamma\mid_{[t-\varepsilon,t+\varepsilon]}$ is a geodesic. The metric space (X,d) is said to be a *geodesic space* if there is a geodesic between each two points of X.

We now come to the geodesics of $\mathbb{H}^n$. A first fact is that the infimum in (1) is actually a minimum. We have the following proposition which is for example explained in [AR]:

Proposition 1.2. *The metric space $(\mathbb{H}^n, d_{CC})$ is a geodesic space. Moreover every geodesic between two points A and B of $\mathbb{H}^n$ is horizontal and has length $d_{CC}(A,B)$.*

The equations of the local geodesics of $\mathbb{H}^n$ have been known since the paper of Gaveau (see [Ga]). In [Mo1] for example, there is an explicit computation of these equations. Ambrosio and Rigot gave in [AR] the cut locus of the local geodesic going through $0_{\mathbb{H}}$. In this paper, we will investigate how the measure is transported along the geodesics: that requires to know their equations. Because the Carnot-Carathéodory distance and consequently the geodesics are left-invariant, it is sufficient to know the equation of the geodesics going through $0_{\mathbb{H}}$. Let (z,φ) be in $\mathbb{C}^n \times \mathbb{R}$. We call curve with parameter (z,φ) the curve $\gamma_{z,\varphi}$ defined on $\mathbb{R}$ by

$$\gamma_{z,\varphi}(t) = \begin{cases} \left(i\frac{e^{-i\varphi t}-1}{\varphi}z, 2|z|^2\frac{\varphi t-\sin(\varphi t)}{\varphi^2}\right) \in \mathbb{C}^n\times\mathbb{R} & \text{if } \varphi \neq 0 \\ (tz, 0) & \text{if } \varphi = 0. \end{cases} \tag{2}$$

Here $|z|$ is $\sqrt{|z_1|^2+\cdots+|z_n|^2}$. Obviously, the map $(z,\varphi,t) \to \gamma_{z,\varphi}(t))$ is real analytic on $\mathbb{C}^n\times\mathbb{R}\times\mathbb{R}$ so that all its partial derivatives are well defined and continuous. The curve $\gamma_{z,\varphi}$ is horizontal and has the length between a and b is equal to $|z|(b-a)$. In this paper, we call Γ_t the map

$$\Gamma_t(z,\varphi) := \gamma_{z,\varphi}(t)$$

and we will particulary use it for $t=1$. Let us notice that $\Gamma_t(z,\varphi)$ is $\Gamma_1(tz,t\varphi)$. The next proposition is proved in the paper by Ambrosio and Rigot, (see [AR]) and stated in almost the same formulation. We just adapted it to our notations. Let us recall that L is the set $\{(z,t)\in\mathbb{H}^n \mid z\neq 0\}$.

Proposition 1.3 (Parametrization of normal geodesics)**.** *The normal geodesic starting from $0_{\mathbb{H}}$ are the restrictions to $[0,1]$ of curves with parameter (z,φ) for $(z,\varphi)\in\mathbb{C}^n\times[-2\pi,2\pi]$. In particular restrictions on $[0,1]$ of curves with parameter (z,φ) with $|\varphi| > 2\pi$ are not normal geodesics. Conversely any restriction on $[0,1]$ of a curve with parameter $(z,\varphi)\in\mathbb{C}^n\times[-2\pi,2\pi]$ is a normal geodesic starting from $0_{\mathbb{H}}$. Moreover we have the following more precise description:*

- *For any $A=(0,\alpha)\in L^*$, normal geodesics from $0_{\mathbb{H}}$ to A are exactly the restrictions to $[0,1]$ of the curves with parameter $(z, \frac{\alpha}{|\alpha|}2\pi)$ where z is varying on the sphere of the vectors with norm $\sqrt{\pi|\alpha|}$.*
- *For any $A\in\mathbb{H}^n\backslash L$ there exists a* unique *normal geodesic connecting $0_{\mathbb{H}}$ and A. This is the restriction to $[0,1]$ of a curve of parameter (z,φ) where $|\varphi|<2\pi$.*

Remark 1.4. The curves with parameter $(a + ib, v, r)$ from [AR] (with $|a|^2 + |b|^2 = 1$) have constant speed equal to one and are in fact the curves with parameter $(a + ib, v)$ restricted on $[0, r]$. The restrictions to $[0, 1]$ of the curves with parameter (z, φ) have length $|z|$ and are in fact the curves $t \in [0, 1] \to \exp_{\mathbb{H}}(tz, t\varphi/4)$ where $\exp_{\mathbb{H}}$ is the Heisenberg-exponential map from the end of [AR].

Remark 1.5. The map $t \to \delta_t(A)$ is not a geodesic.

We give a corollary of Proposition 1.3 for the local geodesics.

Corollary 1.6. *The curve with parameter (z, φ) is a local geodesic. More precisely the restriction of $\gamma_{z,\varphi}$ to $[a, b]$ is a geodesic if and only if $(b - a)|\varphi| \leq 2\pi$. Moreover this is the unique geodesic defined on $[a, b]$ if and only if $(b-a)|\varphi| < 2\pi$.*

Proof. The case $a = 0$ and $\gamma_{z,\varphi}(a) = 0_{\mathbb{H}}$ is included in Proposition 1.3. To complete the proof we compute the left translation of the curve mapping $\gamma_{,\varphi}(a)$ on $0_{\mathbb{H}}$ and obtain

$$\gamma_{z,\varphi}(a)^{-1} \cdot \gamma_{z,\varphi}(a + t) = \gamma_{z',\varphi}(t)$$

with $z' = e^{-i\varphi a}z$. The proposition is a direct consequence of this equation and of the fact that d_{CC} is left-invariant. □

We denote $D_1 := (\mathbb{C}^n \backslash 0) \times]-2\pi, 2\pi[$ and similarly $D_t := (\mathbb{C}^n \backslash 0) \times]-2t\pi, 2t\pi[$ for $t \in [0, 1]$. A second and important corollary of Proposition 1.3 is the following proposition:

Proposition 1.7. *The map Γ_1 is a C^∞-diffeomorphism from D_1 to $\mathbb{H}^n \backslash L$. Similarly for $t \in [-1, 0[\cup]0, 1[$, the map Γ_t is a C^∞-diffeomorphism from D_1 to $\Gamma_1(D_{|t|})$.*

Proof. For a general t the assertion is a direct consequence of the case $t = 1$ and of the relation $\Gamma_t(z, \varphi) = \Gamma_1(tz, t\varphi)$. With Proposition 1.3, it is clear that Γ_1 is one-to-one on D_1 and it is C^∞-differentiable because it is real analytic. We postpone the proof that its Jacobian determinant does not vanish to Proposition 1.12 at the end of this section. □

We now introduce two helpful maps: the intermediate-points map $\mathcal{M}$ and inversion-time map $\mathcal{I}$. The left-invariance of the Carnot-Carathéodory metric tells us whether there is a unique or several normal geodesics between two given points. If $A = (z, \alpha)$ and $B = (z', \alpha')$, the isometry $\tau_{A^{-1}}$ maps A to $0_{\mathbb{H}}$ and B to $A^{-1} \cdot B = (z - z', \alpha'')$ for some α'' in $\mathbb{R}$. It follows from Proposition 1.3 that there is a unique normal geodesic from A to B if and only if $z \neq z'$ or $A = B$. We will denote the open set $\{(A, B) \in (\mathbb{H}^n)^2 \mid z_A \neq z_B\} = \{(A, B) \in (\mathbb{H}^n)^2 \mid A^{-1} \cdot B \notin L\}$ by U. With this set we define our first map.

Definition 1.8. We define the *intermediate-points map* $\mathcal{M}$ from the set $U \times [0, 1]$ to $\mathbb{H}^n$ by

$$\mathcal{M}(A, B, t) = \tau_A \circ \Gamma_t \circ \Gamma_1^{-1} \circ \tau_{A^{-1}}(B).$$

The point $\mathcal{M}(A, B, t)$ is actually the unique t-intermediate point between A and B. It is really a t-intermediate point when $A = 0_{\mathbb{H}}$ because $\Gamma_t \circ \Gamma_1^{-1}(\gamma_{z,\varphi}(1))$ equals $\gamma_{z,\varphi}(t)$ for $(z, \varphi) \in D_1$. The general case follows from the left-invariance of the Carnot-Carathéodory metric. Moreover $\mathcal{M}(A, B, t)$ is the unique t-intermediate point between A and B because there is a unique normal geodesic from A to B (the couple (A, B) is in U) and because the t-intermediate points in a geodesic space are on the geodesics connecting two points.

In the next sections, we will extend $\mathcal{M}$ in (two) different manners to $(\mathbb{H}^n)^2 \times [0,1]$. Using the proposition 1.7 and recalling that τ_A is affine, we have this regularity lemma:

Lemma 1.9. *The map $\mathcal{M}$ is measurable, it is continuous and C^∞ on $U \times]0,1[$. The curve $t \in [0,1] \to \mathcal{M}(A, B, t)$ is the unique normal geodesic from A to B.*

Let us now introduce the inversion-time map $\mathcal{I}$.

Definition 1.10. We define the *inversion-time map* $\mathcal{I}$ on $\mathbb{H}^n \backslash L$ by $\mathcal{I}(A) = \Gamma_{-1} \circ \Gamma_1^{-1}(A)$.

The name comes from the fact that for $(z, \varphi) \in D_1$ and $t \in [-1, 1]$ we have with Proposition 1.7:

$$\begin{aligned}
\mathcal{I}(\gamma_{z,\varphi}(t)) &= \mathcal{I}(\gamma_{tz,t\varphi}(1)) \\
&= \Gamma_{-1} \circ \Gamma_1^{-1}(\Gamma_1(tz, t\varphi)) \\
&= \Gamma_{-1}(tz, t\varphi) \\
&= \gamma_{z,\varphi}(-t).
\end{aligned}$$

A consequence is that $\mathcal{I} \circ \mathcal{I}$ is the identity on $\mathbb{H}^n \backslash L$. That is why for $A \in \mathbb{H}^n$ we will call $(A, \mathcal{I}(A))$ a couple of time-conjugate points. We now establish the connection between $\mathcal{M}$ and $\mathcal{I}$.

Lemma 1.11. *Let A be in $\mathbb{H}^n \backslash L$. Then $\mathcal{M}(\mathcal{I}(A), A, 1/2)$ is well defined and equals $0_{\mathbb{H}}$ if and only if the φ-coordinate of $\Gamma_1^{-1}(A)$ verifies $|\varphi| < \pi$ that is when $A \in \Gamma_1(D_{1/2})$.*

Proof. Proposition 1.3 tell us that $A = \Gamma_1(z, \varphi)$ for some $|\varphi| < 2\pi$. Additionally the definition of $\mathcal{I}$ give us $\mathcal{I}(A) = \Gamma_{-1}(z, \varphi)$. Therefore we have to say when $\mathcal{M}(\gamma_{z,\varphi}(-1), \gamma_{z,\varphi}(1), 1/2)$ exists and if it is $0_{\mathbb{H}}$.

A consequence of the equation (2) is that the z-coordinates of $\gamma_{z,\varphi}(-1)$ and $\gamma_{z,\varphi}(1)$ are equal if and only if $|\varphi| = \pi$. Therefore $(\gamma_{z,\varphi}(-1), \gamma_{z,\varphi}(1)) \in U$ if and only if $|\varphi| \neq \pi$. In this case there is a unique geodesic δ on $[-1, 1]$ between the two points and we can define the midpoint

$$\delta(0) = \mathcal{M}(\delta(-1), \delta(1), 1/2) = \mathcal{M}(\gamma_{z,\varphi}(-1), \gamma_{z,\varphi}(1), 1/2).$$

If $|\varphi| < \pi$ we have also $2|\varphi| < 2\pi$. In this case the curve δ is the restriction of $\gamma_{z,\varphi}$ to $[-1, 1]$ because with Corollary 1.6, both map are the unique geodesic defined on $[-1, 1]$ that goes from $\mathcal{I}(A)$ to A. The midpoint is then $\delta(0) = \gamma_{z,\varphi}(0) = 0_{\mathbb{H}}$.

If $\pi < |\varphi| < 2\pi$, let us assume to the contrary that $\delta(0) = 0_{\mathbb{H}}$. Then because of Proposition 1.3, the curve $\delta \mid_{[0,1]}$ is the unique normal geodesic from $0_{\mathbb{H}}$ to A and $t \in [0, 1] \to \delta(-t)$ is the unique normal geodesic between $0_{\mathbb{H}}$ and $\mathcal{I}(A)$. It follows that δ equals $\gamma_{z,\varphi}$ on these two segments contradicting the fact that $|\varphi| > \pi$. Namely for $2|\varphi| > 2\pi$, Corollary 1.6 shows that the restriction on $[-1, 1]$ of $\gamma_{z,\varphi}$ is not a geodesic and consequently can not be δ. Hence $\mathcal{M}(A, \mathcal{I}(A), 1/2)$ is not $0_{\mathbb{H}}$. $\square$

As announced before, we present the computation of the Jacobian determinant. For the proof of the proposition 1.7, we just have to prove that the Jacobian of Γ_1 does not vanish. This fact is mentioned in [AR] where the authors state that Γ_1 is a diffeomorphism (in fact in this paper z is given by its polar coordinates $(|z|, \frac{z}{|z|})$. The result of the calculation is given for $\mathbb{H}^1$ in the paper of Monti (see [Mo1]). We show the complete computation for every $n \in \mathbb{N}\backslash\{0\}$ because we do not only need the fact that the Jacobian determinant does not vanish, but also the exact expression.

Proposition 1.12. *The Jacobian determinant of* Γ_1 *is*

$$\mathrm{Jac}(\Gamma_1)(z,\varphi) = \begin{cases} 2^{2n+1}|z|^2 \left(\frac{1-\cos(\varphi)}{\varphi^2}\right)^{n-1} \frac{2(1-\cos(\varphi))-\varphi\sin(\varphi)}{\varphi^4} & \textit{for } \varphi \neq 0, \\ |z|^2/3 & \textit{else.} \end{cases}$$

It does not vanish on D_1.

Proof. Let us first write what is precisely Γ_1:

$$\Gamma_1(z,\varphi) = \begin{cases} \left(i\frac{e^{-i\varphi}-1}{\varphi}z_1, \cdots, i\frac{e^{-i\varphi}-1}{\varphi}z_n, 2|z|^2\frac{\varphi-\sin(\varphi)}{\varphi^2}\right) & \text{if } \varphi \neq 0, \\ (z,0) & \text{else} \end{cases}$$

where $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. We begin to make the calculation of $\mathrm{Jac}(\Gamma_1) = \det(D\Gamma_1)$ for $\varphi \neq 0$. The case $\varphi = 0$ is obtained as a limit.

We first have to compute the real derivative of Γ_1, that is the derivative of Γ_1 as a map from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+1}$. However we write $D\Gamma_1$ as a matrix $\left(\begin{smallmatrix} P & C \\ R & s \end{smallmatrix}\right)$ where the block P is made of the $2n$ first rows and columns. If we identify complex numbers with 2×2 matrices ($a+ib$ is $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$), we can write P as the $n\times n$ complex matrix $i\frac{e^{-i\varphi}-1}{\varphi}I_n$ where I_n is the identity matrix of $M_n(\mathbb{C})$. The column C is $(\frac{e^{-i\varphi}}{\varphi} - i\frac{e^{-i\varphi}-1}{\varphi^2})z$ seen as a $\mathbb{R}^{2n}$ vector, the row R is $(4x_1\frac{\varphi-\sin(\varphi)}{\varphi^2}, 4y_1\frac{\varphi-\sin(\varphi)}{\varphi^2}, \cdots, 4x_n\frac{\varphi-\sin(\varphi)}{\varphi^2}, 4y_n\frac{\varphi-\sin(\varphi)}{\varphi^2})$, and the real number s is $2|z|^2\left(\frac{2sin(\varphi)}{\varphi^3} - \frac{1+cos(\varphi)}{\varphi^2}\right)$.

It is difficult to compute directly the determinant of $\left(\begin{smallmatrix} P & C \\ R & s \end{smallmatrix}\right)$ in any point. Because of this we now prove that if $|z| = |z'|$, then $\mathrm{Jac}(\Gamma_1)(z,\varphi) = \mathrm{Jac}(\Gamma_1)(z',\varphi)$. Let T be a unitary $\mathbb{C}$-linear map so that $T(z) = z'$. Consider now T' defined by $T'(z,\varphi) = (T(z),\varphi)$. Then it is not difficult to see that $\Gamma_1 \circ T' = T' \circ \Gamma_1$. It follows that $(\mathrm{Jac}(\Gamma_1)\circ T)\cdot \det_{\Re}(T') = \det_{\Re}(T')\cdot \mathrm{Jac}(\Gamma_1)$ and consequently we have $\mathrm{Jac}(\Gamma_1)(z,\varphi) = \mathrm{Jac}(\Gamma_1)(z',\varphi)$. We use this relation and simplify the computation and by choosing $z' = (0,\cdots,0,|z|)$. With this new vector z', the matrixes C and R have most of the entries equal to zero, which permit to make a block calculation of the determinant of $D\Gamma_1 = \left(\begin{smallmatrix} P & C \\ R & s \end{smallmatrix}\right)$. We obtain that $\mathrm{Jac}(\Gamma_1)(z,\varphi)$ is the product of

$$\begin{vmatrix} \sin(\varphi)/\varphi & (1-\cos(\varphi))/\varphi \\ (\cos(\varphi)-1)/\varphi & \sin(\varphi)/\varphi \end{vmatrix}^{n-1}$$

with

$$\begin{vmatrix} \sin(\varphi)/\varphi & (1-\cos(\varphi))/\varphi & |z|(\frac{\cos(\varphi)}{\varphi} - \frac{\sin(\varphi)}{\varphi^2}) \\ (\cos(\varphi)-1)/\varphi & \sin(\varphi)/\varphi & |z|(\frac{-\sin(\varphi)}{\varphi} - \frac{\cos(\varphi)-1}{\varphi^2}) \\ 4|z|\frac{\varphi-\sin(\varphi)}{\varphi^2} & 0 & 2|z|^2\left(\frac{2\sin(\varphi)}{\varphi^3} - \frac{1+\cos(\varphi)}{\varphi^2}\right) \end{vmatrix}.$$

It simply equals

$$2^n|z|^2\left(\frac{1-\cos(\varphi)}{\varphi^2}\right)^{n-1}\begin{vmatrix} \sin(\varphi)/\varphi & (1-\cos(\varphi))/\varphi & \frac{\cos(\varphi)}{\varphi} \\ (\cos(\varphi)-1)/\varphi & \sin(\varphi)/\varphi & \frac{-\sin(\varphi)}{\varphi} \\ 2\frac{\varphi-\sin(\varphi)}{\varphi^2} & 0 & \frac{1-\cos(\varphi)}{\varphi^2} \end{vmatrix}$$

and finally

$$2^{n+1}|z|^2\left(\frac{1-\cos(\varphi)}{\varphi^2}\right)^{n-1}\frac{2(1-\cos(\varphi))-\varphi\sin(\varphi)}{\varphi^4}.$$

The continuously corresponding value for $\varphi=0$ is $|z|^2/3$.

It remains to show that $\mathrm{Jac}(\Gamma_1)$ does not vanish on D_1. It is clear for $\varphi=0$. In the other case it has to be shown that the even function $f(u):=2(1-\cos(u))-u\sin u$ does not vanish for $u\in]0,2\pi[$. The values in 0 and 2π are 0. The first derivative of f is the map $f'(u)=\sin u-u\cos u$ that also vanishes in 0 and the second derivative is $f''(u)=u\sin u$. The function f'' is non-negative on $]0,\pi[$ and non-positive on $]\pi,2\pi[$. Hence f is convex increasing on $[0,\pi]$, concave on $[\pi,2\pi]$ and ends with the value 0. It follows that f does not vanish on $]0,2\pi[$. □

We recall that for $0<|t|\le 1$ we have $\Gamma_t(z,\varphi)=\Gamma_1(tz,t\varphi)$ and obtain the following corollary.

Corollary 1.13. *Let $0<|t|\le 1$. The Jacobian determinant of Γ_t on D_1 is*

$$\mathrm{Jac}(\Gamma_t)(z,\varphi)=\begin{cases}2^{n+1}t|z|^2\left(\frac{1-\cos(t\varphi)}{\varphi^2}\right)^{n-1}\frac{2(1-\cos(t\varphi))-t\varphi\sin(t\varphi)}{\varphi^4} & \textit{for } \varphi\neq 0,\\ t^{2n+3}|z|^2/3 & \textit{else.}\end{cases}$$

2 Validity of the measure contraction property in the Heisenberg group

In general metric measure spaces, there are two conditions which are regarded as replacements of Ricci curvature bounds of differential geometry: the geometric curvature-dimension $CD(K,N)$ and the measure contraction property $MCP(K,N)$. In our case where the geodesic between two points is almost surely unique, the curvature-dimension $CD(K,N)$ is more restrictive than the measure contraction property $MCP(K,N)$ although it was not clear for a long time whether the two properties are equivalent. Moreover in the same framework (almost surely a unique normal geodesic between two points), the measure contraction property implies a local Poincaré inequality and the doubling property for metric measure space. This is shown in [vR] and [LV2]. Metric measure spaces verifying a local Poincaré inequality and the doubling property have proved to be a very ideal setting for doing analysis. A good reference on this new theory is the book by Heinonen (see [He]). It is possible to install a differentiable structure on these space as it is proved in the paper by Cheeger (see [Ch]) or to define Sobolev spaces with interesting properties (see [Ch],[HK] and [Sh]). An other application domain of the local Poincaré inequality is conformal geometry where it permits to analyze the quasi-conformal maps between metric spaces (see the survey [BP2]). Some of the more famous examples of doubling metric measure spaces with a local Poincaré inequality are the boundary of hyperbolic buildings (see [BP1]), some Cantor-like set with worm-hole (see [La]) and the Heisenberg group (see [Va]).

We now give the definition of the curvature-dimension $CD(0, N)$ and of the measure contraction property $MCP(0, N)$. Let us recall that our aim is to prove that the first property does not hold for any N whereas the Heisenberg group $\mathbb{H}^n$ verifies $MCP(0, 2n+3)$ with $2n+3$ as sharp dimension. The case where $K \neq 0$ in not really interesting in the Heisenberg group. We will see why and which properties hold in Remark 3.4.

Let (X, d, μ) be a metric measure space. We assume moreover that this space is separable and complete (Polish space). The curvature-dimension condition $CD(K, N)$ is a geometric condition on the optimal transportation of mass between any couple of absolutely continuous probability measures on (X, d, μ). For $N \geq 1$, the curvature-dimension condition $CD(0, N)$ simply states that the functional $S_N(\cdot \mid \mu)$ is convex on the L^2-Wasserstein space $\mathcal{P}_2$. In the last sentence $S_N(\cdot \mid \mu)$ is the relative Rényi entropy functional defined for a measure m with density $\rho_m \in L^1(\mu)$ by:

$$S_N(m \mid \mu) = -\int_X \rho_m^{1-1/N} d\mu.$$

The basic quantity for the optimal transportation theory is the so-called L_2-Wasserstein distance between two probability measures m_0 and m_1. It is defined as

$$d_W(m_0, m_1) = \inf_q \left(\int_{X \times X} d^2(x, y) dq(x, y) \right)$$

where the infimum is taken over all couplings q of m_0 and m_1. In a Polish space like X, there is a coupling that attains the infimum. It is said to be an optimal coupling. Let $\mathcal{P}_2(X)$ be the space of probability measures m on X with second moment (i.e. $\int_X d(x_0, x)^2 dm(x) < +\infty$ for an $x_0 \in X$). With the distance d_W, the space $\mathcal{P}_2(X)$ is also a complete and separable metric space. Thus it is possible to speak about geodesics in $(\mathcal{P}_2(X), d_W)$. For a detailed presentation, and more about optimal transportation, refer to [Vi1] or [Vi2]. We now give the definition of $CD(0, N)$. It is a specific case of the curvature-dimension condition introduced by Sturm in [St2].

Definition 2.1. Let $N \geq 1$. We say that the curvature-dimension condition $CD(0, N)$ holds in (X, d, μ) if for each couple (m_0, m_1) of absolutely continuous measure of $\mathcal{P}_2(X)$, there is a geodesic $(m_t)_{t \in [0,1]}$ connecting m_0 and m_1 and consisting of absolutely continuous measures m_t that verifies the following condition:

$$S_N(m_t \mid \mu) \leq (1-t) S_N(m_0 \mid \mu) + t S_N(m_1 \mid \mu). \tag{3}$$

We will see in the next section (Theorem 3.3) that this property does not hold in the Heisenberg group.

Remark 2.2. In the paper by Ambrosio and Rigot (see [AR]), the authors prove that there is a unique normal geodesic between two measures of $\mathcal{P}_2(\mathbb{H}^n)$. But it is still an open problem if the intermediate measures are absolutely continuous (see [AR]).

The measure contraction property $MCP(0,N)$ (see [St2], [LV2], [Oh]) is a condition on metric measure spaces (X,μ,d). Its formulation is much simpler if there exists a measurable map

$$\mathcal{N} : (x,y,t) \in X \times X \times [0,1] \to X$$

such that for each $x \in X$ and μ-a.e $y \in X$, the curve $t \in [0,1] \to \mathcal{N}(x,y,t)$ is the unique normal geodesic from x to y. Then the space (X,d,μ) satisfies $MCP(0,N)$ if and only if for each $x \in X$ and $t \in [0,1]$ and each μ-measurable set E we have

$$t^N \mu(\mathcal{N}_{x,t}^{-1}(E)) \leq \mu(E) \tag{4}$$

where $\mathcal{N}_{x,t}(y) := \mathcal{N}(x,y,t)$.

In Definition 1.8, we defined the map $\mathcal{M}$ on $U \times [0,1]$. We now extend it to $(\mathbb{H}^n)^2 \times [0,1]$ by $\mathcal{M}(A,B,t) = A$ if $(A,B) \notin U$ (We will use another extension in the next section). With Lemma 1.9, we see that $\mathcal{M}$ verifies the conditions of $\mathcal{N}$ on measurability and almost surely uniqueness of a geodesic. We state in the next theorem that the property (4) additionally holds for $\mathcal{N} = \mathcal{M}$.

Theorem 2.3. *The measure contraction property $MCP(0,N)$ holds in $\mathbb{H}^n$ if and only if $N \geq 2n+3$.*

Proof. We first prove that $MCP(0,N)$ holds for $N \geq 2n+3$ and will then show that $MCP(0,N)$ does not hold for $N < 2n+3$.

Let then N be greater than $2n+3$. Because the Lebesgue measure, the Carnot-Carathéodory distance and the geodesics are left-invariant, we obtain also that $\mathcal{M}(\tau_A B, \tau_A C, t) = \tau_A \circ \mathcal{M}(B,C,t)$ for A, B and C in $\mathbb{H}^n$. Because all these objects are left-invariant, we can reduce the inequality (4) to be proved in the case $x = 0_{\mathbb{H}}$. Let E be a μ-measurable set with non-zero measure and $t \in]0,1[$. The map $\mathcal{M}_{0_{\mathbb{H}},t} := \mathcal{M}(0_{\mathbb{H}},\cdot,t)$ maps the line L on $0_{\mathbb{H}}$ (because of the definition of our extension) but is one-to-one on $\mathbb{H}^n \backslash L$ where it equals $\Gamma_t \circ \Gamma_1^{-1}$. If we denote $F := \mathcal{M}_{0_{\mathbb{H}},t}^{-1}(E)$, as a result of Proposition 1.7 and because the measure of L is 0, we have:

$$\mathcal{L}^{2n+1}(E) \geq \int_{F\backslash L} \mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},t})(Q) d\mathcal{L}^{2n+1}(Q). \tag{5}$$

With the expression of $\mathcal{M}_{0_{\mathbb{H}},t}$ on $\mathbb{H}^n \backslash L$ we find $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},t}) = \frac{\mathrm{Jac}(\Gamma_t)}{\mathrm{Jac}(\Gamma_1)} \circ \Gamma_1^{-1}$. But we know the expression of these Jacobian determinants from Proposition 1.12 and Corollary 1.13. Hence in order to state the relation (4) it suffices to establish that

$$\frac{\mathrm{Jac}(\Gamma_t)}{\mathrm{Jac}(\Gamma_1)}(z,\varphi) = t \left(\frac{1-\cos t\varphi}{1-\cos\varphi} \right)^{n-1} \left(\frac{2(1-\cos(t\varphi)) - t\varphi \sin(t\varphi)}{2(1-\cos(\varphi)) - \varphi\sin(\varphi)} \right) \geq t^N \tag{6}$$

happens for $(z,\varphi) \in D_1$ (in the case $\varphi \neq 0$). For $\varphi = 0$ this relation must be changed into

$$\frac{\mathrm{Jac}(\Gamma_t)}{\mathrm{Jac}(\Gamma_1)}(z,0) = t^{2n+3} \geq t^N \tag{7}$$

that is obviously true. Both members of inequality (6) have the value 0 in 0 and 1 in 1. It is the same if we raise these expressions to the power of $1/N$. Hence, we want to prove that $t \to \left(\frac{\mathrm{Jac}(\Gamma_t)}{\mathrm{Jac}(\Gamma_1)}\right)^{1/N}(z,\varphi)$ is above the cord between $(0,0)$ and $(1,1)$. That is in particular true if this function is concave for each $(z,\varphi)\in D_1$. This is equivalent to the $1/N$-concavity on $]0,2\pi[$ of the even function g_n defined by $g_n(u)=u[2(1-\cos u)-u\sin u](1-\cos u)^{n-1}$ (that means g_n is positive and $g_n^{1/N}$ is concave). For the limit case $N=2n+3$ we have the following lemma:

Lemma 2.4. *The function g_n is $(2n+3)^{-1}$-concave on $[0,2\pi]$.*

Proof. We begin to prove that g_1 is $1/5$-concave. For simplicity we will denote $g=g_1$. This function is positive because it equals $uf(u)$ where f appears in the proof of Proposition 1.12. A classic study shows that it is increasing on $[0,\beta_2]$ and decreasing on $[\beta_2,2\pi]$, convex on $[0,\beta_1]$ and concave on $[\beta_1,2\pi]$ with $\pi<\beta_1<3.84<4<\beta_2<2\pi$. The $1/5$-concavity is equivalent to the negativity of $(g''g-g'^2)+\frac{1}{5}g'^2$. A first step is to prove the weaker relation $g''g-g'^2\le 0$ which is equivalent to the log-concavity (that means g is positive and $\log(g)$ is concave). We write $g(u)=[4u][\sin(u/2)][\sin(u/2)-(u/2)\cos(u/2)]$ and call the three factors respectively $p(u)$, $q(u)$ and $r(u)$. Each one is log-concave because p and q are concave and

$$r''r-r'^2=\frac{1}{4}\left[\sin^2\frac{u}{2}-\left(\frac{u}{2}\right)^2\right]\le 0.$$

It follows that g is log-concave. Alternatively we can write

$$g''g-g'^2=(pq)^2(r''r-r'^2)+(pr)^2(q''q-q'^2)+(qr)^2(p''p-p'^2)$$

where the three terms of the sum are non-positive. For the $1/5$-concavity, the task is then to prove the negativity of $(g''g-g'^2)+\frac{1}{5}g'^2$ that equals

$$\begin{aligned}
&\left[4u\sin\frac{u}{2}\right]^2\frac{1}{4}\left(\sin^2\frac{u}{2}-\left(\frac{u}{2}\right)^2\right)+\left[4u\left(\sin\frac{u}{2}-\frac{u}{2}\cos\frac{u}{2}\right)\right]^2(-\frac{1}{4})\\
&+\left[\sin\frac{u}{2}\left(\sin\frac{u}{2}-\frac{u}{2}\cos\frac{u}{2}\right)\right]^2(-16)+\frac{1}{5}\left(2-\left(2+u^2\right)\cos u\right)^2\\
=&\frac{1}{5}\left(2-\left(2+u^2\right)\cos u\right)^2-\left(\sin\frac{u}{2}-\frac{u}{2}\cos\frac{u}{2}\right)^2\left(4u^2+16\sin^2\frac{u}{2}\right)\\
&-4\left[u\sin\frac{u}{2}\right]^2\left(\left(\frac{u}{2}\right)^2-\sin^2\frac{u}{2}\right).
\end{aligned}$$

It is quite fastidious to prove that this long expression is negative. For this, we replace cos and sin in each term by the beginning of their Taylor series. We intend to replace the last expression by a pointwise greater polynomial. For instance, we obtain

$$\frac{1}{5}g'^2(u)=\frac{1}{5}\left(2-\left(2+u^2\right)\cos u\right)^2\le\frac{1}{5}\left(2-\left(2+u^2\right)\left(1-\frac{u^2}{2}+\frac{u^4}{24}-\frac{u^6}{720}\right)\right)^2$$

for $u\le\beta_2$. With the same type of calculus we obtain that $(g''g-g'^2)+1/5g'^2$

is smaller than

$$\begin{aligned}
&\frac{1}{5}\left(\frac{5u^4}{12}-\frac{7u^6}{180}+\frac{u^8}{720}\right)^2-\left(\frac{u^3}{24}\right)^2\left(1-\frac{u^2}{32}\right)\left(8u^2-\frac{u^4}{3}+\frac{u^6}{144}\right)\\
&-4\left(u\left(\frac{u}{2}-\frac{u^3}{48}\right)\right)^2\left(\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}-\frac{u^3}{48}+\frac{u^5}{2^5\cdot 5!}\right)^2\right)\\
=&u^{10}\left(\frac{-1}{384}+\frac{30463u^2}{82944000}-\frac{751u^4}{4147000}+\frac{1237u^6}{3538944000}+\frac{u^8}{8493465600}\right)
\end{aligned}$$

if $u \leq 4$. But this polynomial is negative for $u \leq 3.84$. For $u \in [3.84, 2\pi]$, we are on the segment $[\beta_1, 2\pi]$ where g is concave. It follows that $(g''g - g'^2) + 1/5g'^2$ is negative on $[0, 2\pi]$ and we have the $1/5$-concavity of g.

Let us now prove recursively that g_{n+1} is $1/(2n+5)$-concave. For this let us assume that g_n is $1/(2n+3)$-concave and let h be defined on $[0, 2\pi]$ by $h(u) = 1 - \cos u = 2\sin^2(u/2)$. We have now to prove the negativity of

$$\begin{aligned}
&\left((g_nh)''(g_nh)-(g_nh)'^2\right)+\frac{1}{2n+5}(g_nh)'^2\\
=&(g_n''g_n-g_n'^2)h^2+(h''h-h'^2)g_n^2+\frac{1}{2n+5}(g_nh)'^2\\
=&(g_n''g_n-g_n'^2)h^2+(-h)g_n^2+\frac{g_n'^2h^2+2g_ng_n'hh'+g_n^2h'^2}{2n+5}\\
=&h^2\left((g_n''g_n-g_n'^2)+\frac{g_n'^2}{2n+3}\right)+h^2\left(\frac{g_n'^2}{2n+5}-\frac{g_n'^2}{2n+3}\right)\\
&+g_n^2\left(\frac{h'^2}{2n+5}-h\right)+\frac{2g_ng_n'hh'}{2n+5}.
\end{aligned}$$

The first term T_1 in the last sum is negative because of the $1/(2n+3)$-concavity of g_n. The second term T_2 is clearly negative. The third term T_3 is also negative because $C\sin^2(u/2) - \sin^2(u)$ is positive on $[0, 2\pi]$ if and only if $C \geq 4$. It remains just to prove that $|T_4| \leq |T_2| + |T_3|$ where T_4 is the last term. We prove it by comparing $|T_4|^2$ to $(2\sqrt{|T_2||T_3|})^2 \leq (|T_2| + |T_3|)^2$:

$$\begin{aligned}
&(2n+5)^2\left(4|T_2||T_3|-T_4^2\right)\\
=&(2n+5)^2\left(4\left[g_n^2\left(h-\frac{h'^2}{2n+5}\right)\right]\left[h^2\left(\frac{g_n'^2}{2n+3}-\frac{g_n'^2}{2n+5}\right)\right]\right.\\
&\left.-\left[\frac{2g_ng_n'hh'}{2n+5}\right]^2\right)\\
=&4g_n^2\left((2n+5)\,h-h'^2\right)\frac{2h^2g_n'^2}{2n+3}-4g_n^2g_n'^2h^2h'^2\\
=&4g_n^2g_n'^2h^2\left[\frac{2\,(2n+5)\,h}{2n+3}-\frac{(2n+5)\,h'^2}{2n+3}\right]\geq 0
\end{aligned}$$

which ends the proof because $2h - h'^2 = 4\sin^2(u/2) - \sin^2(u)$ is positive on $[0, 2\pi]$. □

For $N \geq 2n+3$, the $1/N$-concavity of g_n is a consequence of the $1/(2n+3)$-concavity. It proves that $MCP(0, N)$ holds in $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$.

Let now N be strictly smaller than $2n+3$, let P be the point $((1,0,\cdots,0),0) = \Gamma_1((1,0,\cdots,0),0)$ and K_r the (Euclidian) ball $\mathcal{B}(P,r)$ with center P and radius r. For an r small enough, it is included in $\mathbb{H}^n\backslash L$. For a fixed t in $]0,1[$, we define now the set E_r by $\mathcal{M}_{0_{\mathbb{H}},t}(K_r)$. Then we have:

$$\mathcal{L}^{2n+1}(E_r) = \int_{K_r} \mathrm{Jac}\,(\mathcal{M}_{0_{\mathbb{H}},t}(Q))\, d\mathcal{L}^{2n+1}(Q).$$

But $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},t})(P) = t^{2n+3} < t^N$ because $\Gamma_1^{-1}(P)$ has the φ-coordinate equal to 0 (see equation (7)). By continuity, we can find a radius $r > 0$ such that $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},t})(Q) < t^N$ holds for every $Q \in K_r$. For this choice, it follows that $\mathcal{L}^{2n+1}(E_r) < t^N \mathcal{L}^{2n+1}(K_r)$, which is contradicting $MCP(0,N)$. □

3 The Brunn-Minkowski inequalities in $\mathbb{H}^n$

The classical Brunn-Minkowski inequality in $\mathbb{R}^{2n+1}$ (see for instance [Fe]) is a very instructive geometric estimate of the Minkowski sum (the usual set sum in $\mathbb{R}^{2n+1}$) of two compact sets in $\mathbb{R}^N$. An equivalent statement for K_0 and K_1 two compact set of $\mathbb{R}^N$ and $t \in [0,1]$ is:

$$(\mathcal{L}^N)^{1/N}(tK_1 + (1-t)K_0) \geq t(\mathcal{L}^N)^{1/N}(K_1) + (1-t)(\mathcal{L}^N)^{1/N}(K_0)$$

with $tK_1 + (1-t)K_0 = \{tk_1 + (1-t)k_0 \in \mathbb{R}^N \mid k_1 \in K_1 \quad k_0 \in K_0\}$. We want to interpret what is $tK_1+(1-t)K_0$ in a geodesic metric space. For this we consider the set of the t-intermediate points from a point k_0 in K_0 to a point k_1 in K_1. We call this set the t-intermediate set and denote it by "$tK_1 + (1-t)K_0$".

Let (X,d,μ) be a metric measure space and N be greater than 1. We say that the *geodesic* Brunn-Minkowski inequality $BM(0,N)$ holds in (X,d,μ) if the inequation

$$\mu^{1/N}(\text{“}tK_1 + (1-t)K_0\text{”}) \geq t\mu^{1/N}(K_1) + (1-t)\mu^{1/N}(K_0) \tag{8}$$

is true for every couple (K_0,K_1) of compact sets (where μ("$tK_1 + (1-t)K_0$") will denote the outer measure of "$tK_1 + (1-t)K_0$" if the latter is not measurable). There is also a "multiplicative" Brunn-Minkowski inequality that has been introduced in the Heisenberg group by Monti in [Mo2] (see also [LM]). We deal with this inequality in Remark 3.5.

Remark 3.1. Let K be a real number and $N \geq 1$. The general definition of $CD(K,N)$ (see [St2]) involves a modification of the geometric inequality (3) by factors depending roughly speaking on the Wasserstein distance between the measures m_0 and m_1. These factors also appear for $MCP(K,N)$ and $CD(K,N)$ in the generalization of the inequalities (4) and (8). For these three geometric properties there is a common hierarchy when K and N vary: the property for (K,N) implies the property for (K',N) where $K' < K$. Similarly for a fixed curvature K, the property (K,N) implies the one for (K,N') where $N' > N$. Nevertheless a priori there is no optimal couple (K,N) when the curvature and the dimension both vary (see [Oh],[St2]).

It is proven in [St2] that the curvature-dimension property $CD(0,N)$ implies $BM(0,N)$. In order to deny $CD(0,N)$ in $\mathbb{H}^n$, we will prove that no geodesic Brunn-Minkowski inequality holds in this space.

In $\mathbb{H}^n$ it will be useful to interpret the t-intermediate set with the intermediate-points map. Thus we extend $\mathcal{M}$ in a different way from the last section. Here $\mathcal{M}$ is no longer a map but a multi-valued map defined on $(\mathbb{H}^n)^2\times]0,1[$ by $\mathcal{M}(A,B,t) = \{M_t \in \mathbb{H}^n \mid d_{CC}(A,M_t) = td_{CC}(A,B) \text{ and } d_{CC}(M_t,B) = (1-t)d_{CC}(A,B)\}$. If (A,B) is in U, we identify the single-valued set $\mathcal{M}(A,B,t)$ with its unique element, which is coherent assignment with Definition 1.8. For more precisions on the value taken by $\mathcal{M}$ on $(\mathbb{H}^n)^2\backslash U\times]0,1[$, we just have to consult Proposition 1.3 and use left translations. We will now prove the following lemma:

Lemma 3.2. *There are two compact sets K and K' such that*

$$\mathcal{L}^{2n+1}(K) = \mathcal{L}^{2n+1}(K') > \mathcal{L}^{2n+1}(\mathcal{M}^{1/2}(K,K'))$$

where $\mathcal{M}^{1/2}(K,K') = \{\mathcal{M}(k,k',1/2) \in \mathbb{H}^n \mid k \in K \text{ and } k' \in K'\}$.

Let N be a dimension greater than 1. We can raise the inequality in Lemma 3.2 to the power of $1/N$ and considering relation (8) we obtain as a corollary the following theorem.

Theorem 3.3. *The geodesic Brunn-Minkowski inequality $BM(0,N)$ and the geometric curvature-dimension $CD(0,N)$ do not hold for any N.*

We now give a proof of Lemma 3.2

Proof. Let us consider a simple geodesic: the curve of parameter $((1,\cdots,0),0)$ on the interval $[-1,1]$. As $2\cdot 0 < 2\pi$ Corollary 1.6 tells us that it is the unique geodesic defined on $[-1,1]$ from $P' = (-1,0,\cdots,0)$ to $P = (1,0,\cdots,0)$: the points P and P' are time-conjugate and have midpoint $0_{\mathbb{H}}$. Actually $\mathcal{M}^{1/2} := \mathcal{M}(\cdot,\cdot,1/2)$ is the expression of the midpoint map. On U this map is univalued and is defined by putting $t = 1/2$ in Definition 1.8:

$$\mathcal{M}^{1/2}(Q',Q) = \tau_{Q'} \circ \Gamma_{1/2} \circ \Gamma_1^{-1} \circ \tau_{Q'^{-1}}(Q). \tag{9}$$

We will now use the time-inversion map introduced in the first section. We recall that Lemma 1.11 exactly says when the midpoint of two time-conjugate points in U is $0_{\mathbb{H}}$. For P and P' it is true so that P and P' are in the open set $\Gamma_1(D_{1/2})$. The counterexample that we want to build consists in taking a small compact ball $K_r := \mathcal{B}(P,r)$ with center P and (Euclidian) radius r and in considering the set of midpoints between K_r and $K'_r = \mathcal{I}(K_r)$. By continuity we can choose r small enough such that $K_r \subset \Gamma_1(D_{1/2})$ and $K_r \times K'_r \subset U$.

We have to show that K'_r has the same measure as K_r and that this measure is greater than the measure of $\mathcal{M}^{1/2}(K_r,K'_r)$. The first fact is actually not mysterious because of this: Γ_1 and Γ_{-1} are diffeomorphisms between the same sets (Proposition 1.7) and have the same Jacobian determinant up to the sign (Corollary 1.13). Hence we have

$$\mathcal{L}^{2n+1}(K'_r) = \mathcal{L}^{2n+1}(\Gamma_{-1}(\Gamma_1^{-1}(K_r))) = \mathcal{L}^{2n+1}(\Gamma_1(\Gamma_1^{-1}(K_r))) = \mathcal{L}^{2n+1}(K_r).$$

The key of the second statement is

$$\mathcal{M}^{1/2}(K'_r,K_r) = \bigcup_{A,B\in K_r} \mathcal{M}^{1/2}(\mathcal{I}(A),B) = \bigcup_{A,B\in K_r} \mathcal{M}^{1/2}(\mathcal{I}(A),A+(B-A)). \tag{10}$$

As $K_r \subset \Gamma_1(D_{1/2})$, Lemma 1.11 shows that if $A \in K_r$, then $\mathcal{M}^{1/2}(\mathcal{I}(A), A) = 0_{\mathbb{H}}$. Therefore the mid-set $\mathcal{M}^{1/2}(K'_r, K_r)$ has a very small measure. We will use differentiation tools in order to quantify this. As a consequence of Lemma 1.9, $\mathcal{M}^{1/2}$ is C^∞-differentiable on U. For any $Q \in \mathbb{H}^n \backslash L$ let $\mathcal{M}_Q^{1/2}$ be the map $\mathcal{M}(Q, \cdot, 1/2)$. We can now write

$$\begin{aligned} &\mathcal{M}^{1/2}(\mathcal{I}(A), A + (B - A)) \\ =&0 + D\mathcal{M}^{1/2}_{\mathcal{I}(A)}(A).(B - A) \\ &+ \left[\mathcal{M}^{1/2}\left(\mathcal{I}(A), A + (B - A)\right) - D\mathcal{M}^{1/2}_{\mathcal{I}(A)}(A).(B - A)\right] \\ =&D\mathcal{M}^{1/2}_{P'}(P).(B - A) + \left[\left(D\mathcal{M}^{1/2}_{\mathcal{I}(A)}(A) - D\mathcal{M}^{1/2}_{P'}(P)\right).(B - A)\right] \\ &+ \left[\mathcal{M}^{1/2}\left(\mathcal{I}(A), A + (B - A)\right) - D\mathcal{M}^{1/2}_{\mathcal{I}(A)}(A).(B - A)\right]. \end{aligned} \tag{11}$$

For A and B close to P, the two last terms of the last sum are small and can be estimated using usual definitions and propositions of differential calculus in $\mathbb{R}^{2n+1}$: when r is going to zero, we have

$$\begin{aligned} \sup_{A,B \in K_r} &\Big| \left(D\mathcal{M}^{1/2}_{\mathcal{I}(A)}(A) - D\mathcal{M}^{1/2}_{P'}(P)\right).(B - A) \\ &+ \mathcal{M}^{1/2}(\mathcal{I}(A), A + (B - A)) - D\mathcal{M}^{1/2}_{\mathcal{I}(A)}(A).(B - A)\Big| = o(r). \end{aligned}$$

Therefore because $K_r - K_r = \{Q \in \mathbb{R}^{2n+1} \mid Q = A - B \quad A, B \in \mathcal{B}(P, r)\} = \mathcal{B}(0, 2r)$, the relations (10) and (11) give the following set inclusion

$$\mathcal{M}^{1/2}(K'_r, K_r) \subset D\mathcal{M}^{1/2}_{P'}(P).(\mathcal{B}(0, 2r)) + \mathcal{B}(0, \varepsilon(r)r) \tag{12}$$

where $\varepsilon(r)$ is a non-negative function going to zero when r goes to zero. In the last relation, the measure of the containing set is equivalent to the one of $D\mathcal{M}^{1/2}_{P'}(P).(\mathcal{B}(0, 2r))$. If we now consider relation (9) and recalling that the left-invariant and affine maps $\tau_{P'}$ and $\tau_{P'^{-1}} = \tau_{P'}^{-1}$ have the derivative equal to their linear part, we obtain that $\mathrm{Jac}(\mathcal{M}^{1/2}_{P'})(P)$ has the same value like $\mathrm{Jac}(\Gamma_{1/2} \circ \Gamma_1^{-1})$ taken in the point $P'^{-1} \cdot P = ((2, 0, \cdots, 0), 0) = \Gamma_1((2, 0, \cdots, 0), 0)$. This Jacobian determinant has been computed in the second section (see equations (6) and (7)). In our case as the φ-coordinate of $\Gamma_1^{-1}(P'^{-1} \cdot P)$ is 0, we have to consider equation (7) for $t = 1/2$. Then the value of the Jacobian determinant is $\frac{1}{2^{2n+3}}$. It follows that

$$\mathcal{L}^{2n+1}(D\mathcal{M}^{1/2}_{P'}(P).(\mathcal{B}(0, 2r))) = \frac{2^{2n+1}}{2^{2n+3}}\mathcal{L}^{2n+1}(\mathcal{B}(P, r)) = \frac{1}{4}\mathcal{L}^{2n+1}(K_r).$$

Hence with relation (12) and the commentary that we made after it, we obtain

$$\mathcal{L}^{2n+1}(\mathcal{M}^{1/2}(K'_r, K_r)) \leq \frac{1}{4}\mathcal{L}^{2n+1}(K_r)(1 + o(r))$$

when r goes to zero. We now select an r small enough and the lemma is proved. □

Remark 3.4. (i) The previous result does not only yields that $CD(0, N)$ does not hold. This also implies that $CD(K, N)$ does not hold for any $K > 0$ because this condition is less demanding than $CD(0, N)$. Alternatively, spaces verifying $CD(K, N)$ with $K > 0$ are bounded.

(ii) Also for any $K < 0$, the curvature-dimension bound $CD(K, N)$ does not hold. Assume to the contrary that $CD(K, N)$ holds in the space $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$ for $K < 0$. Therefore the "scaled space" property from [St2] tells us that $(\mathbb{H}^n, \lambda^{-1} d_{CC}, \lambda^{-(2n+2)} \mathcal{L}^{2n+1})$ verifies $CD(\lambda^2 K, N)$for $\lambda > 0$. But with the dilatation δ_λ this last space is exactly isomorphic to our metric measure space. Hence $CD(K', N)$ would hold in $(\mathbb{H}^n, d_{CC}, \mathcal{L}^{2n+1})$ for every non-positive K'. It is proved in [AR] that the optimal transportation between two measures is unique so that inequality (3) defining $CD(0, N)$ is obtained as limit of the corresponding inequalities for $CD(K', N)$, which is contradicting Theorem 3.3. As a consequence, $CD(K, N)$ does not hold in $\mathbb{H}^n$.

(iii) In the same way with the dilatations of $\mathbb{H}^n$, we could have proved directly that $BM(K, N)$ is false for any $K \in \mathbb{R}$. We recover that $CD(K, N)$ does not hold because this last condition implies the other for the same curvature and dimension parameters.

(iv) For every N, the measure contraction property $MCP(K, N)$ is false for $K > 0$. Namely as for CD, spaces verifying such a condition are bounded (see [St2]).

(v) The property $MCP(K, N)$ does also not hold for $N > 2n + 3$ and $K < 0$ because this case is similar to (ii): using dilatations $MCP(K, N)$ implies that $MCP(0, N)$ is true, which is contradicting Theorem 2.3.

Remark 3.5. In [Mo2], Monti compares for two compact sets F and F' their measures to that of $F \cdot F' = \{a \cdot b \in \mathbb{H}^n \mid a \in F \quad b \in F'\}$. He proves that

$$\mathcal{L}^3(F \cdot F')^{1/4} \geq \mathcal{L}^3(F)^{1/4} + \mathcal{L}^3(F')^{1/4}$$

does not hold in $\mathbb{H}^1$ (4 is the Hausdorff dimension in $\mathbb{H}^1$) using an argument of the non-optimality of the unit ball in the isoperimetric inequality of $\mathbb{H}^1$.

Another proof in $\mathbb{H}^n$ with the corresponding dimension $2n+2$ is the following: Take for F the same set K_r as before in this section and denote by F' the set $\{b \in \mathbb{H}^n \mid \exists c \in F, \, c \cdot b = 0_{\mathbb{H}}\}$ of the inverse elements (it is simply $-F$ because $(z, \alpha)^{-1} = (-z, -\alpha)$). Then using the same technic as in this section we find that $F \cdot F'$ is very close to $D\tau_{P'}(P).(\mathcal{B}(0, 2r))$. The measure of this last set is $2^{2n+1}\mathcal{L}(F)$ because, as we said in the first section, $\mathrm{Jac}(\tau_{P'}) = 1$ in every point. As $\mathcal{L}^{2n+1}(F) = \mathcal{L}^{2n+1}(F')$ it follows that for r small enough

$$\mathcal{L}^{2n+1}(F \cdot F')^{\frac{1}{2n+2}} < \mathcal{L}^{2n+1}(F)^{\frac{1}{2n+2}} + \mathcal{L}^{2n+1}(F')^{\frac{1}{2n+2}} \tag{13}$$

and the multiplicative Brunn-Minkowski inequality is false for the Hausdorff dimension (i.e. $2n + 2$). In the paper by Leonardi and Masnou (see [LM]), the authors show that the multiplicative Brunn-Minkowski inequality is true with topological dimension (i.e. $2n+1$). They explain that there could be in principle an $N \in]2n+1, 2n+2[$ such that the multiplicative Brunn-Minkowski inequality holds in $\mathbb{H}^n$: in fact if the equality holds for N, then it holds for $N' < N$. As said in Remark 3.4 and by contrast, $BM(K, N')$ is a consequence of $BM(K, N)$ if $N' > N$.

We proved in relation (13) that the sets F and F' defined in this remark are a counterexample to the multiplicative Brunn-Minkowski inequality with dimension $N = 2n + 2$. They are actually also a counterexample for any $N > 2n+1$. It follows that $2n+1$ is the greater dimension for which the multiplicative Brunn-Minkowski inequality is true.

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