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RECONSTRUCTION OF RADIAL DIRAC OPERATORS

S. ALBEVERIO∗ R. HRYNIV† YA. MYKYTYUK†

Abstract. We study the inverse spectral problem of reconstructing the potential of radial Dirac operators acting in the unit ball of $\mathbb{R}^3$. For each one-dimensional partial Dirac operator corresponding to a nonzero angular momentum, we give a complete description of the spectral data (eigenvalues and suitably defined norming constants), prove existence and uniqueness of solutions to the inverse problem, and present the reconstruction algorithm.

1. Introduction

It is well known [12] that a three-dimensional radial Dirac operator with potential

$$V(r) = v_{\text{el}}(r)1 + v_{\text{am}}(r)\sigma_1 + (m + v_{\text{sc}}(r))\sigma_3$$

can be decomposed into the direct sum of Dirac operators in $L_2(\mathbb{R}_+; \mathbb{C}^2)$ generated by the differential expressions

$$\ell(\kappa, V) := \sigma_2 \frac{1}{i} \frac{d}{dr} + \frac{\kappa}{r} \sigma_1 + V(r).$$

Here $\kappa \in \mathbb{Z}$ is the angular momentum, $1$ is the identity matrix in $\mathbb{C}^2$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices, $m$ is the mass of the particle, and $v_{\text{sc}}, v_{\text{el}},$ and $v_{\text{am}}$ are respectively the scalar potential, electrostatic potential, and anomalous magnetic moment, see [12, Ch. 4]. If the potential $V$ is of bounded support—e.g. contained in $[0, 1]$—then it is natural to consider $\ell(\kappa, V)$ on $(0, 1)$ and to incorporate the mass $m$ into the scalar potential $v_{\text{sc}}$. The differential expression $\ell(\kappa, V)$ considered on functions $u = (u_1, u_2)^\top$ satisfying the boundary condition

$$u_1(1) \sin \theta - u_2(1) \cos \theta = 0, \quad \theta \in [0, \pi),$$

(and a similar one for $x = 0$ if $\kappa = 0$) generates a Dirac operator $\mathcal{H}(\theta, \kappa, V)$ which is self-adjoint in $\mathbb{H} := L_2((0, 1); \mathbb{C}^2)$ and has a simple discrete spectrum tending to $\pm \infty$.

The main aim of the present paper is to give a complete solution to the inverse spectral problem of the reconstruction of the potential $V$ from the spectrum and the corresponding norming constants for the Dirac operator $\mathcal{H}(\theta, \kappa, V)$ at the nonzero

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angular momentum $\kappa$. It is well known \cite{7} that even in the regular case $\kappa = 0$ there are unitary gauge transformations between Dirac operators so that the inverse problem is ill posed in this setting. To avoid the ambiguity, we restrict ourselves to potentials $V$ with zero electric potential,

\begin{equation}
V(r) = v_{\text{am}}(r)\sigma_1 + v_{\text{sc}}(r)\sigma_3,
\end{equation}

giving the Ablowitz–Kaup–Newell–Segur (AKNS) normal form \cite{1}.

For $\kappa = 0$ the inverse problem for Dirac operators was studied by several authors, cf. \cite{2} and the references therein. Recently, Serier \cite{9} considered the general case $\kappa \in \mathbb{N}$ and established a local diffeomorphism between the spectral data and potentials that are in $L_2(0, 1)$ entrywise, paralleling the analysis of \cite{8} for a regular Sturm–Liouville case and of \cite{10} for the case of radial Schrödinger operators.

Our approach is based on the double commutation method. We show how the inverse problem for $\mathcal{H}(\theta, \kappa, V)$ can be reduced to that for a regular Dirac operator with $\kappa = 0$. This allows us to give a complete description of the spectral data for the class of the Dirac operators with potentials in $L_p(0, 1)$ entrywise, where $p$ is an arbitrary number in $[1, \infty)$. We present a reconstruction algorithm and prove existence and uniqueness of solutions to the inverse problem for radial Dirac operators in the AKNS normal form. Our method suggests that most of the results on the spectral properties of the regular ($\kappa = 0$) Dirac operators are easily extended to those with nonzero integer angular momenta. In particular, it follows that the spectral data determine a global diffeomorphism on the set of AKNS potentials of (1.1) in the suitably defined topologies, cf. \cite{9}. Also our reconstruction algorithm can easily be implemented numerically.

The paper is organized as follows. In the next section we introduce the double commutation transformations between different Dirac operators and in Section 3 we choose those of them that map almost isospectrally between Dirac operators with different angular momenta (in the sense that they preserve all but one eigenvalue and corresponding norming constant). This allows us to complete the direct and inverse spectral analysis in Section 4. Finally, Appendices A and B establish some properties of solutions of the Dirac equation near the origin.

Throughout the paper, capital letters usually represent $2 \times 2$ matrices or matrix-valued functions, while vectors and vector-valued functions are denoted by little bold letters. We abbreviate $L_p((a, b); \mathbb{C}^2)$ to $L_p(a, b)$ and say that a potential $V = V(v_{\text{am}}, v_{\text{sc}})$ in the AKNS form (1.1) belongs to $L_p(0, 1)$ if both $v_{\text{am}}$ and $v_{\text{sc}}$ are in $L_p(0, 1)$. For a vector $u = (u_1 u_2)^T$, we use the notations $u^\top$ and $u^*$ for the row-vectors $(u_1, u_2)$ and $(\bar{u}_1, \bar{u}_2)$, and $|u|$ for the Euclidean norm $(|u_1|^2 + |u_2|^2)^{1/2}$. For a continuous function $f$ on $(0, 1)$, we shall write $f \sim x^\alpha$ as $x \to 0+$ if the function $x^{-\alpha}f(x)$ has a finite nonzero limit as $x \to 0+$.

2. DOUBLE COMMUTATION METHOD AND TRANSFORMATION OPERATORS

In this section we introduce transformations that appear in the double commutation method and are sometimes called multiple Darboux transformations; see \cite{3, 11} and also \cite{4, 5} for the case of Sturm–Liouville operators, as well as the references cited therein.

The double commutation method rests of the following observation, cf. \cite[Theorem 3.2]{11}.
Lemma 2.1. Let $V_0$ be locally summable on $(0,1]$ and let, for all $\lambda \in \mathbb{C}$, $u(\cdot, \lambda)$ be the solution of the equation
\[
\left( \sigma_2 \frac{1}{i} \frac{d}{dx} + V_0 \right) u = \lambda u
\]
satisfying the terminal condition $u(1) = (\cos \theta, \sin \theta)^\top$. Put
\[
v(x, \lambda) := u(x, \lambda) - \frac{w(x, \lambda)}{1 + w(x, \lambda)} u(x, \lambda_s),
\]
where $w(x, \lambda) := \alpha_s \int_x^1 u^\top(s, \lambda_s)u(s, \lambda)\,ds$ and $\lambda_s$, $\alpha_s$ are complex numbers such that $1 + w(x, \lambda_s)$ does not vanish on $(0,1)$. Then, for all $\lambda \in \mathbb{C}$,
\[
\left( \sigma_2 \frac{1}{i} \frac{d}{dx} + V_0 + V_s \right) v(x, \lambda) = \lambda v(x, \lambda)
\]
with
\[
V_s(x) := \frac{\alpha_s}{1 + w(x, \lambda_s)} [u(x, \lambda_s)u^\top(x, \lambda_s)\sigma_2 \frac{1}{i} - \sigma_2 \frac{1}{i} u(x, \lambda_s)u^\top(x, \lambda_s)].
\]

Proof. Straightforward calculations give
\[
\left( \sigma_2 \frac{1}{i} \frac{d}{dx} + V_0 \right) v(x, \lambda) = \left( \sigma_2 \frac{1}{i} \frac{d}{dx} + V_0 \right) u(x, \lambda)
- \frac{w(x, \lambda)}{1 + w(x, \lambda)} \left( \sigma_2 \frac{1}{i} \frac{d}{dx} + V_0 \right) u(x, \lambda_s)
+ \frac{\alpha_s}{1 + w(x, \lambda_s)} u^\top(x, \lambda_s)u(x, \lambda)\sigma_2 \frac{1}{i} u(x, \lambda_s)
- \frac{w(x, \lambda)}{1 + w(x, \lambda_s)} u^\top(x, \lambda_s)u(x, \lambda_s)\sigma_2 \frac{1}{i} u(x, \lambda_s)
= \lambda u(x, \lambda) - \lambda_s \frac{w(x, \lambda)}{1 + w(x, \lambda_s)} u(x, \lambda_s)
+ \frac{\alpha_s}{1 + w(x, \lambda_s)} \sigma_2 \frac{1}{i} u(x, \lambda_s)u^\top(x, \lambda_s)v(x, \lambda)
= \lambda v(x, \lambda) + (\lambda - \lambda_s) \frac{w(x, \lambda)}{1 + w(x, \lambda_s)} u(x, \lambda_s)
+ \frac{\alpha_s}{1 + w(x, \lambda_s)} \sigma_2 \frac{1}{i} u(x, \lambda_s)u^\top(x, \lambda_s)v(x, \lambda).
\]

The Lagrange-type identity reads
\[
(\lambda - \lambda_s) \int_x^1 u^\top(s, \lambda_s)u(s, \lambda)\,ds
= \int_x^1 u^\top(s, \lambda_s)\sigma_2 \frac{1}{i} u'(s, \lambda)\,ds - \int_x^1 \left[ \sigma_2 \frac{1}{i} u'(s, \lambda_s) \right]^\top u(s, \lambda)\,ds
= \int_x^1 \frac{d}{ds} u^\top(s, \lambda_s)\sigma_2 \frac{1}{i} u(s, \lambda)\,ds = -u^\top(x, \lambda_s)\sigma_2 \frac{1}{i} u(x, \lambda);
\]
in particular,
\[
(2.1) \quad u^\top(x, \lambda_s)\sigma_2 \frac{1}{i} u(x, \lambda) = 0.
\]
Combining the above two relations, we arrive at
\[
(\lambda - \lambda_*) \frac{w(x, \lambda)}{1 + w(x, \lambda_*)} u(x, \lambda_*)
= - \frac{\alpha_*}{1 + w(x, \lambda_*)} u^\top(x, \lambda_*) \sigma_2 \frac{1}{i} u(x, \lambda) u(x, \lambda_*)
= - \frac{\alpha_*}{1 + w(x, \lambda_*)} u(x, \lambda_*) u^\top(x, \lambda_*) \sigma_2 \frac{1}{i} v(x, \lambda),
\]
so that
\[
\left( \frac{\sigma_2}{i} \frac{d}{dx} + V_0 \right) v(x, \lambda) = \lambda v(x, \lambda) - V_*(x) v(x, \lambda)
\]
as claimed. \qed

**Remark 2.2.** Direct calculations show that (cf. [11, Sec. 3])
\[
(2.2) \quad V_*(x) = - \frac{\alpha_* [u_2^2(x, \lambda_*) - u_2^2(x, \lambda_*)]}{1 + w(x, \lambda_*)} \sigma_1 + \frac{2 \alpha_* u_1(x, \lambda_*) u_2(x, \lambda_*)}{1 + w(x, \lambda_*)} \sigma_3
\]
so that the electric potential of the transformed Dirac expression does not change (in particular, an AKNS system remains such), while the anomalous magnetic and scalar potentials gain the quantities equal to the coefficients at \( \sigma_1 \) and \( \sigma_3 \) respectively in the above representation of \( V_* \).

Before proceeding, we recall some other properties of the mapping \( u(\cdot, \lambda) \mapsto v(\cdot, \lambda) \) of Lemma 2.1. In fact, we can introduce an abstract analogue of this transformation in the following manner. Assume that \( f \in \mathbb{L}_{2, \text{loc}}(0, 1) \) and that a nonzero \( \alpha \in \mathbb{R} \) is chosen so that \( \alpha \int_x^1 |f(s)|^2 \, ds > -1 \) for all \( x \in (0, 1] \); then \( \mathcal{U} = \mathcal{U}(f, \alpha) \) is an operator in \( \mathbb{L}_{2, \text{loc}}(0, 1) \) given by (cf. [11])
\[
(\mathcal{U} g)(x) = g(x) - \frac{\alpha f(x)}{1 + \alpha \int_x^1 |f(s)|^2 \, ds} \int_x^1 f^*(s) g(s) \, ds;
\]
in particular,
\[
(\mathcal{U} f)(x) := \frac{f(x)}{1 + \alpha \int_x^1 |f(s)|^2 \, ds}.
\]
The inverse transformation is easily seen to be [5]
\[
(\mathcal{U}^{-1} g)(x) = g(x) + \alpha f(x) \int_x^1 (\mathcal{U} f)^*(s) g(s) \, ds.
\]
Direct calculations (involving integration by parts and simple algebra) give
\[
(2.3) \quad \int_x^1 |(\mathcal{U} g)(s)|^2 \, ds = \int_x^1 |g(s)|^2 \, ds - \alpha \left[ \int_x^1 f^*(s) g(s) \, ds \right]^2 \frac{1}{1 + \alpha \int_x^1 |f(s)|^2 \, ds};
\]
in particular,
\[
(2.4) \quad \int_x^1 |(\mathcal{U} f)(s)|^2 \, ds = \frac{1}{\alpha} \left[ 1 - \left( 1 + \alpha \int_x^1 |f(s)|^2 \, ds \right)^{-1} \right],
\]
so that \( \mathcal{U} f \in \mathbb{H} \) if and only if either \( f \not\in \mathbb{H} \) or \( f \in \mathbb{H} \) and \( 1 + \alpha ||f||^2 > 0 \).
Combining the above properties of the mapping \( \mathcal{U} \), we conclude the following (see [11, Lemma 2.1]).

**Proposition 2.3.** The operator \( \mathcal{U} \) performs a unitary equivalence of \( \mathbb{H} \oplus f \) and \( \mathbb{H} \oplus \mathcal{U}f \) (where \( \mathbb{H} \oplus f = \mathbb{H} \) if \( f \) is not in \( \mathbb{H} \), and similarly for \( \mathbb{H} \oplus \mathcal{U}f \)).

3. Spectral transformations of radial Dirac operators

In this section we show that the transformations of the previous section can be chosen so that they map between Dirac operators with different angular momenta almost isospectrally, i.e., they keep unchanged all but one eigenvalue and corresponding norming constant.

Assume that \( \theta \in [0, \pi) \), \( \kappa \in \mathbb{Z} \setminus \{0\} \) and that \( V \in L^p(0,1) \) is an AKNS potential, and denote by \( \lambda_n \), \( n \in \mathbb{Z} \), the eigenvalues of the operator \( \mathcal{H}(\theta, \kappa, V) \). Denoting by \( u(\cdot, \lambda) = (u_1(\cdot, \lambda), u_2(\cdot, \lambda))^\top \) a solution of the equation \( \ell(\kappa, V)u = \lambda u \) satisfying the terminal condition \( u(1) = (\cos \theta, \sin \theta)^\top \), we see that \( u(\cdot, \lambda_k) \) is an eigenvector corresponding to the eigenvalue \( \lambda_n \). The number

\[
\alpha_n := \frac{\|u(\cdot, \lambda_n)\|^2 - \left( \int_0^1 |u(x, \lambda_n)|^2 \, dx \right)^{-1}}{1 + w(x, \lambda_n)}
\]

is called the norming constant corresponding to the eigenvalue \( \lambda_n \). We recall (see Appendix A) that, for \( \kappa > 0 \), the function \( u_1(\cdot, \lambda) \) has at \( x = 0 \) a zero of order \( \kappa + 1 \) if \( \lambda \) is an eigenvalue and a pole of order \( \kappa \) otherwise; the orders of zero and pole of \( u_2(\cdot, \lambda) \) at \( x = 0 \) are \( \kappa \) and \( \kappa - 1 \) respectively. For \( \kappa < 0 \) similar statements hold with \( u_1 \) and \( u_2 \) interchanged and with \( \kappa \) replaced by \( -\kappa \).

3.1. Removing an eigenvalue. Applying Lemma 2.1 to the potential \( V_0 := \frac{\sigma_1}{x} + V \), \( \lambda_s := \lambda_k \), and \( \alpha_s := -\alpha_k \), we find that the function

\[
v(\cdot, \lambda) = \mathcal{U}(u(\cdot, \lambda_k), -\alpha_k u(\cdot, \lambda))
\]

satisfies the equation

\[
\left( \sigma_2 \frac{1}{x} \frac{d}{dx} + \frac{\kappa}{x} \sigma_1 + V + V_s \right)v(x, \lambda) = \lambda v(x, \lambda),
\]

where \( V_s \) is given by (2.2), with \( w(\cdot, \lambda) \) therein equal to

\[
w(x, \lambda) := -\alpha_k \int_x^1 u^\top(s, \lambda_k)u(s, \lambda) \, ds.
\]

We shall further analyse the structure of the added potential \( V_s \) and consider separately the cases \( \kappa > 0 \), \( \kappa < 0 \), and \( \kappa = 0 \).

**Case 1: \( \kappa > 0 \).** In this case \( u_1(x, \lambda_k) \sim x^{\kappa+1} \) and \( u_2(x, \lambda_k) \sim x^\kappa \) as \( x \to 0+ \). Since \( \alpha_k \) is the norming constant corresponding to the eigenvalue \( \lambda_k \), it follows that

\[
1 + w(x, \lambda_k) = \alpha_k \int_0^x |u(s, \lambda_k)|^2 \, ds
\]

has a zero at \( x = 0 \) of order \( 2\kappa + 1 \), i.e., that

\[
1 + w(x, \lambda_k) = x^{2\kappa+1} \hat{w}(x), \quad \text{where } \hat{w} \text{ is positive on } [0, 1].
\]

Moreover, \( \hat{w}(x, \lambda_k) \in W^1_p(0,1) \) by Lemma B.1. We rewrite \( V_s \) as

\[
V_s(x) = -\frac{w'(x, \lambda_k)}{1 + w(x, \lambda_k)} \sigma_1 + \frac{2\alpha_k u_1^2(x, \lambda_k)}{1 + w(x, \lambda_k)} \sigma_1 - \frac{2\alpha_k u_1(x, \lambda_k) u_2(x, \lambda_k)}{1 + w(x, \lambda_k)} \sigma_3
\]

and observe that

\[
\frac{w'(x, \lambda_k)}{1 + w(x, \lambda_k)} = d \frac{\log w(x, \lambda_k)}{x} = \frac{2\kappa + 1}{x} + \frac{\hat{w}'(x)}{\hat{w}(x)},
\]
with \( \tilde{w}'/\tilde{w} \in L_p(0, 1) \). As the second and the third summands in the above representation of \( V_* \) have no singularities at the origin, we conclude that

\[
V_* = -\frac{2\kappa + 1}{x} \sigma_1 + \tilde{V}_*,
\]

where \( \tilde{V}_* \) belongs to \( L_p(0, 1) \). Setting \( \tilde{V} := V + \tilde{V}_* \), we see that

\[
\sigma_2 \frac{d}{dx} + \frac{\kappa}{x} + V + V_* = \sigma_2 \frac{d}{dx} - \frac{\kappa + 1}{x} \sigma_1 + \tilde{V}.
\]

**Case 2:** \( \kappa < 0 \). In this case \( u_1(x, \lambda_k) \sim x^{-\kappa} \) and \( u_2(x, \lambda_k) \sim x^{1-\kappa} \) as \( x \to 0^+ \), and \( 1 + w(x, \lambda_k) \) has zero at \( x = 0 \) of order \(-2\kappa + 1\). We represent \( V_* \) as

\[
V_*(x) = \frac{w'(x, \lambda_k)}{1 + w(x, \lambda_k)} \sigma_1 - \frac{2\alpha_k w_2(x, \lambda_k)}{1 + w(x, \lambda_k)} \sigma_1 - \frac{2\alpha_k u_1(x, \lambda_k) u_2(x, \lambda_k)}{1 + w(x, \lambda_k)} \sigma_3
\]

and, repeating the arguments for Case 1, we find \( \tilde{V} \in L_p(0, 1) \) such that the function \( v(\cdot, \lambda) \) solves the equation \( \ell(1 - \kappa, \tilde{V}) = \lambda v \) for every \( \lambda \in \mathbb{C} \).

**Case 3:** \( \kappa = 0 \). Then a boundary condition must be specified at \( x = 0 \) in order for \( \mathcal{H}(\theta, 0, V) \) to be self-adjoint. Of interest to us will be two boundary conditions, \( u_1(0) = 0 \) and \( u_2(0) = 0 \), for functions \( \mathbf{u} = (u_1, u_2)^\top \) in the domain, and the corresponding self-adjoint restrictions of \( \mathcal{H}(\theta, 0, V) \) will be denoted by \( \mathcal{H}_1(\theta, 0, V) \) and \( \mathcal{H}_2(\theta, 0, V) \) respectively. The analysis of the operator obtained after the transformation of \( \mathcal{H}_1(\theta, 0, V) \) is similar to that for the case \( \kappa > 0 \), and the transformation of \( \mathcal{H}_2(\theta, 0, V) \) is similar to that of \( \mathcal{H}(\theta, \kappa, V) \) with \( \kappa < 0 \).

In the following theorem, we shall formally write \( \mathcal{H}(\theta, \kappa, V + V_*) \) for one of the operators \( \mathcal{H}(\theta, -\kappa - 1, \tilde{V}) \) or \( \mathcal{H}(\theta, 1 - \kappa, \tilde{V}) \) obtained after the above-described transformation of the operator \( \mathcal{H}(\theta, \kappa, V) \) according to Cases 1–3.

**Theorem 3.1.** The spectrum of the operator \( \mathcal{H}(\theta, \kappa, V + V_*) \) consists of the eigenvalues \( \lambda_n \), \( n \in \mathbb{Z} \setminus \{k\} \), the corresponding norming constants being \( \alpha_n \).

**Proof.** For the sake of definiteness, we shall assume that the Case 1 holds, i.e., that \( \mathcal{H}(\theta, \kappa, V + V_*) = \mathcal{H}(\theta, -\kappa - 1, \tilde{V}) \) and only remark that the proof is also valid for the other two cases.

The function \( v(\cdot, \lambda) \) solves the equation \( \ell(-\kappa - 1, \tilde{V}) = \lambda v \) and satisfies the terminal condition \( v(1) = (\cos \theta, \sin \theta)^\top \). It remains to prove that \( v(\cdot, \lambda) \) is in \( \overline{H} \) if and only if \( \lambda \) equals \( \lambda_n \) for \( n \neq k \).

The considerations preceding Proposition 2.3 show that the function \( v(\cdot, \lambda_n) \) does not belong to \( \overline{H} \). Since the operator \( \mathcal{U}(\mathbf{u}(\cdot, \lambda_k), -\alpha_k) \) performs the unitary equivalence of \( \overline{H} \ominus \mathbf{u}(\cdot, \lambda_k) \) and \( \overline{H} \), it follows that \( v(\cdot, \lambda_n) \) are in \( \overline{H} \) for every integer \( n \neq k \) and, moreover, \( \|v(\cdot, \lambda_n)\| = \|u(\cdot, \lambda_n)\| \) for such \( n \).

If there were \( \lambda \in \mathbb{R} \setminus \{\lambda_n\}_{n \in \mathbb{Z}} \) in the spectrum of \( \mathcal{H}(\theta, -\kappa - 1, \tilde{V}) \), then \( v(\cdot, \lambda) \) would be the corresponding eigenfunction and, in particular, \( v(\cdot, \lambda) \) would vanish at the origin. Therefore the function \( v(x, \lambda_n)^*v(x, \lambda) \) would be integrable at the origin, and

\[
\mathbf{u}(x, \lambda) = \mathcal{U}^{-1} v(x, \lambda) = v(x, \lambda) - \alpha_k \mathbf{u}(x, \lambda_k) \int_x^1 v(s, \lambda_n)^*v(s, \lambda) \, ds
\]

would have the same behaviour at the origin as \( \mathbf{u}(\cdot, \lambda_k) \). Thus \( \mathbf{u}(\cdot, \lambda) \) would be an eigenfunction of the operator \( \mathcal{H}(\theta, \kappa, V) \) corresponding to the eigenvalue \( \lambda \), which is
impossible. This contradiction shows that \( \mathcal{H}(\theta, -\kappa - 1, \hat{V}) \) has no other eigenvalues. The proof is complete. \( \square \)

3.2. Adding an eigenvalue. We assume now that \( \kappa \neq 0 \), take a real \( \lambda_* \) not in the spectrum of \( \mathcal{H}(\theta, \kappa, V) \) and a positive \( \alpha_* > 0 \), and apply Lemma 2.1 to the potential \( V_0 := \xi \sigma_1 + V \). We find then that the function
\[
v(\cdot, \lambda) = \mathcal{W}(u(\cdot, \lambda_*), \alpha_*) u(\cdot, \lambda)
\]
satisfies the equation
\[
(\sigma_2 \frac{1}{i} \frac{d}{dx} + \frac{\kappa}{x} \sigma_1 + V + V_*) v(x, \lambda) = \lambda v(x, \lambda),
\]
where \( V_* \) is given by (2.2). Again the cases \( \kappa > 0 \) and \( \kappa < 0 \) must be considered separately.

If \( \kappa > 0 \), then \( u_1(x, \lambda_*) \sim x^{-\kappa} \) and \( u_2(x, \lambda_*) \sim x^{1-\kappa} \) as \( x \to 0^+ \), so that \( w(\cdot, \lambda_*) \) has a pole of order \( 2\kappa - 1 \) at \( x = 0 \). We rewrite \( V_* \) as
\[
V_*(x) = \frac{w'(x, \lambda_*)}{1 + w(x, \lambda_*)} \sigma_1 - \frac{2\alpha_* u_2^*(x, \lambda_*)}{1 + w(x, \lambda_*)} \sigma_1 - \frac{2\alpha_* u_1(x, \lambda_*) u_2(x, \lambda_*)}{1 + w(x, \lambda_*)} \sigma_3
\]
and observe that the last two summands have no singularities at the origin, while
\[
\frac{w'(x, \lambda_*)}{1 + w(x, \lambda_*)} = \frac{d}{dx} \log w(x, \lambda_*) = \frac{-2\kappa + 1}{x} + \frac{\hat{w}'(x)}{\hat{w}(x)},
\]
with \( \hat{w}'/\hat{w} \in L_p(0, 1) \), see Appendix B. Thus
\[
V_* = -\frac{2\kappa - 1}{x} \sigma_1 + \tilde{V}_*,
\]
where \( \tilde{V}_* \) belongs to \( L_p(0, 1) \), and
\[
\sigma_2 \frac{1}{i} \frac{d}{dx} + \frac{\kappa}{x} \sigma_1 + V + V_* = \sigma_2 \frac{1}{i} \frac{d}{dx} - \frac{\kappa - 1}{x} \sigma_1 + \hat{V}.
\]
for \( \hat{V} := V + \tilde{V}_* \in L_p(0, 1) \).

If \( \kappa < 0 \), then \( u_1(x, \lambda_*) \sim x^{\kappa+1} \) and \( u_2(x, \lambda_*) \sim x^\kappa \) as \( x \to 0^+ \). Thus \( w(x, \lambda_*) \) has a pole of order \( -2\kappa - 1 \) at \( x = 0 \), and the representation of \( V_* \) of (3.1) yields a \( \hat{V} \in L_p \) such that
\[
\sigma_2 \frac{1}{i} \frac{d}{dx} + \frac{\kappa}{x} \sigma_1 + V + V_* = \sigma_2 \frac{1}{i} \frac{d}{dx} - \frac{\kappa + 1}{x} \sigma_1 + \hat{V}.
\]

We use a formal shorthand notation \( \mathcal{H}(\theta, \kappa, V + V_*) \) for the above-constructed Dirac operator, which will stand for the operator \( \mathcal{H}(\theta, -\kappa + 1, \hat{V}) \) if \( \kappa > 1 \), \( \mathcal{H}_2(\theta, -\kappa + 1, \hat{V}) \) if \( \kappa = 1 \), \( \mathcal{H}_3(\theta, -\kappa - 1, \hat{V}) \) if \( \kappa = -1 \), and \( \mathcal{H}(\theta, -\kappa - 1, \hat{V}) \) if \( \kappa < -1 \); here \( \hat{V} \) is the potential of the transformed Dirac expression introduced above.

**Theorem 3.2.** Under the above notations, the spectrum of the operator \( \mathcal{H}(\theta, \kappa, V + V_*) \) consists of eigenvalues \( \lambda_n, n \in \mathbb{Z} \), and \( \lambda_* \), with the corresponding norming constants \( \alpha_n \) and \( \alpha_* \) respectively.

**Proof.** We shall only consider in detail the case \( \kappa > 0 \) as the case \( \kappa < 0 \) is completely analogous.

If \( \kappa > 1 \), it again suffices to prove that \( v(\cdot, \lambda) \) belongs to \( \mathbb{H} \) if and only if \( \lambda \) is either \( \lambda_* \) or one of \( \lambda_n, n \in \mathbb{Z} \). For \( \kappa = 1 \), we in addition have to show that the functions
\[
v(\cdot, \lambda) = (v_1(\cdot, \lambda), v_2(\cdot, \lambda))^T
\]
satisfy for such \( \lambda \) the initial condition \( v_2(0, \lambda) = 0 \).
Using the properties of the mapping \( \mathcal{U} := \mathcal{U}(u(\cdot, \lambda), \alpha) \), we conclude from relation (2.4) that the function \( v(\cdot, \lambda) = \mathcal{U}u(\cdot, \lambda) \) belongs to \( \mathbb{H} \) and has norm \( \alpha^{-1/2} \). Since \( \mathcal{U} \) performs the unitary equivalence of \( \mathbb{H} \) and \( \mathbb{H} \oplus v(\cdot, \lambda) \), the function \( v(\cdot, \lambda_n) \) belongs to \( \mathbb{H} \oplus v(\cdot, \lambda) \) for every \( n \in \mathbb{Z} \) and \( \| v(\cdot, \lambda_n) \| = \| u(\cdot, \lambda_n) \| = \alpha_n^{-1/2} \).

If the operator \( \mathcal{H}(\theta, -\kappa + 1, \hat{V}) \) (or \( \mathcal{A}_\ell(\theta, 0, \hat{V}) \) if \( \kappa = 1 \) had an eigenvalue \( \lambda \in \mathbb{R} \) different from \( \lambda \) and all \( \lambda_n, n \in \mathbb{Z} \), then \( v(\cdot, \lambda) \) would be an eigenfunction orthogonal to \( v(\cdot, \lambda) \). However, then \( u(\cdot, \lambda) \) would belong to \( \mathbb{H} \) and \( \lambda \) would be an eigenvalue of the operator \( \mathcal{H}(\theta, \kappa, V) \), which is impossible.

For \( \kappa > 1 \) this proves that the spectrum of the operator \( \mathcal{H}(\theta, -\kappa + 1, \hat{V}) \) coincides with the set \( \{ \lambda_n \}_{n \in \mathbb{Z}} \cup \{ \lambda_0 \} \) and that the norming constants are as stated. For \( \kappa = 1 \), we have to verify that the second component of the functions \( v(\cdot, \lambda) \) for \( \lambda = \lambda^\ast \) and \( \lambda = \lambda_n, n \in \mathbb{Z} \), vanishes at \( x = 0 \). Indeed, for \( \kappa = 1 \) we have \( \int_x^1 |u(s, \lambda_n)|^2 ds \sim x^{-1} \) as \( x \to 0^+ \), and therefore the limit

\[
v(0, \lambda) = \lim_{x \to 0^+} \frac{u(x, \lambda)}{1 + \alpha_s \int_x^1 |u(s, \lambda_n)|^2 ds}
\]

exists and has the second component equal to zero. As the expression \( u(x, \lambda)^\ast u(x, \lambda_n) \) is integrable at the origin, we find now that the second component of the function

\[
v(x, \lambda_n) = u(x, \lambda) - \alpha_s v(x, \lambda_n) \int_x^1 u(s, \lambda_n)^\ast u(s, \lambda_n) ds
\]

vanishes at \( x = 0 \) for all \( n \in \mathbb{Z} \). \( \square \)

4. DIRECT AND INVERSE SPECTRAL ANALYSIS

4.1. Spectral data. Starting with the operator \( \mathcal{H}(\theta, \kappa, V) \) with \( \kappa \in \mathbb{Z} \setminus \{0\} \) and \( V \in \mathcal{L}_p(0, 1) \), we apply \( |\kappa| \) times the transformations of Subsection 3.2 to arrive at the operator \( \mathcal{H}_j(\theta, 0, V_0) \) with \( V_0 \in \mathcal{L}_p(0, 1) \) and \( j = 1 \) for \( \kappa = -1, 2, -3, \ldots \) and \( j = 2 \) for \( \kappa = 1, -2, 3, \ldots \), whose spectrum has \( |\kappa| \) extra eigenvalues, and the norming constants for the common eigenvalues coincide.

Recalling the description of the spectral data for regular Dirac operators given in [2], we get the following result. We denote by \( e_n(f) \) the \( n \)-th Fourier coefficient of a function \( f \in \mathcal{L}_1(0, 1) \) and by \( \{ a \} \) the fractional part of a number \( a \).

**Theorem 4.1.** The eigenvalues \( \lambda_n, n \in \mathbb{Z} \), of the operator \( \mathcal{H}(\theta, \kappa, V) \) can be labelled so that \( \lambda_n \) strictly increase with \( n \) and for some \( g_1 \in \mathcal{L}_p(0, 1) \) satisfy the relation

\[
\lambda_n = \begin{cases} 
\pi(n + \{ \frac{\pi}{2} \} + \text{sign}(n)\{\frac{n}{2}\}) + \theta + e_n(g_1) & \text{if } \kappa = 1, -2, 3, \ldots, \\
\pi(n + \{ \frac{\kappa+1}{2} \} + \text{sign}(n)\{\frac{n}{2}\}) + \theta + e_n(g_1) & \text{if } \kappa = -1, 2, -3, \ldots,
\end{cases}
\]

The norming constants \( \alpha_n \) are positive and, for some \( g_2 \in \mathcal{L}_p(0, 1) \), satisfy the relation

\[
\alpha_n = 1 + e_n(g_2).
\]

4.2. Existence and the reconstruction algorithm. Now we demonstrate that any two sequences as in the above theorem are indeed spectral data for some radial Dirac operator.

**Theorem 4.2.** Assume that two sequences \( (\lambda_n)_{n \in \mathbb{Z}} \) and \( (\alpha_n)_{n \in \mathbb{Z}} \) satisfy the conditions of the above theorem. Then there exists a \( V \in \mathcal{L}_p(0, 1) \) such that \( (\lambda_n)_{n \in \mathbb{Z}} \) is the sequence...
of eigenvalues of the Dirac operator \( \mathcal{H}(\theta, \kappa, V) \) and \((\alpha_n)_{n \in \mathbb{Z}}\) are the corresponding norming constants.

We prove the theorem by presenting the reconstruction algorithm and then justifying that it gives an operator searched for.

**The reconstruction algorithm:**

1. Choose \(|\kappa|\) pairwise distinct real points \(\mu_1, \ldots, \mu_{|\kappa|}\) not belonging to the set \(\{\lambda_n\}_{n \in \mathbb{Z}}\) and \(|\kappa|\) positive numbers \(\beta_1, \ldots, \beta_{|\kappa|}\).
2. Find a unique regular Dirac operator \(\mathcal{H}_j(\theta, 0, V_0)\), with \(V_0 \in \mathbb{L}_p(0, 1)\) and \(j = 1\) for \(\kappa = -1, 2, -3, \ldots\) and \(j = 2\) for \(\kappa = 1, -2, 3, \ldots\), for which the sets \(\{\mu_1, \ldots, \mu_{|\kappa|}\} \cup \{\lambda_n\}_{n \in \mathbb{Z}}\) and \(\{\beta_1, \ldots, \beta_{|\kappa|}\} \cup \{\alpha_n\}_{n \in \mathbb{Z}}\) are sets of eigenvalues and norming constants.
3. Remove \(|\kappa|\) eigenvalues \(\mu_1, \ldots, \mu_{|\kappa|}\) from the spectrum of the regular Dirac operator \(\mathcal{H}_j(\theta, 0, V_0)\) applying the procedure of Subsection 3.1. We end up with a Dirac operator \(\mathcal{H}(\theta, \kappa, V_\kappa)\), with some \(V_\kappa \in \mathbb{L}_p(0, 1)\).

That the sets \(\{\mu_1, \ldots, \mu_{|\kappa|}\} \cup \{\lambda_n\}_{n \in \mathbb{Z}}\) and \(\{\beta_1, \ldots, \beta_{|\kappa|}\} \cup \{\alpha_n\}_{n \in \mathbb{Z}}\) form spectral data for some regular Dirac operator \(\mathcal{H}_j(\theta, 0, V_0)\) as claimed in Step 2 follows from the properties of \(\lambda_n\) and \(\alpha_n\), stated in Theorem 4.1. The procedure for finding this regular Dirac operator is given in, e.g., [2]. The fact that the constructed Dirac operator \(\mathcal{H}(\theta, \kappa, V_\kappa)\) has the spectrum \(\{\lambda_n\}_{n \in \mathbb{Z}}\) and the norming constants \(\{\alpha_n\}_{n \in \mathbb{Z}}\) follows from Theorem 3.1.

4.3. **Uniqueness.** Formally, the above reconstruction algorithm depends on the choice of the added eigenvalues \(\mu_1, \ldots, \mu_{|\kappa|}\) and their norming constants—e.g., the auxiliary potential \(V_0\) clearly depends on these quantities. We prove next that the reconstructed operator \(\mathcal{H}(\theta, \kappa, V_\kappa)\) is unique.

**Theorem 4.3.** The radial Dirac operator is uniquely determined by the given spectral data.

**Proof.** The resolution of identity for \(\mathcal{H}(\theta, \kappa, V_\kappa)\) reads

\[
\mathcal{I} = \lim_{N \to \infty} \sum_{n=-N}^{N} \alpha_n \langle \cdot, u_n \rangle u_n,
\]

where \(u_n := u(\cdot, \lambda_n)\) being the corresponding eigenfunction. Restricting this equality to the interval \((\varepsilon, 1)\), we get a similar relation

\[
\mathcal{I}_\varepsilon = \lim_{N \to \infty} \sum_{n=-N}^{N} \alpha_n \langle \cdot, u_n \rangle _\varepsilon u_n,
\]

\(\mathcal{I}_\varepsilon\) denoting the identity operator in \(\mathbb{L}_2(\varepsilon, 1)\) and \(\langle \cdot, \cdot \rangle _\varepsilon\) the scalar product therein.

Since the total potential \((\kappa/x)\sigma_1 + V_\kappa\) belongs to \(\mathbb{L}_p(\varepsilon, 1)\), there exists a transformation operator \(\mathcal{I}_\varepsilon + \mathcal{K}_\varepsilon\) that maps the functions \(c(\cdot, \lambda) = (\cos(\lambda x - \lambda + \theta), \sin(\lambda x - \lambda + \theta))^\top\) on \((\varepsilon, 1)\) into the solution \(u(\cdot, \lambda)\) of the equation \(\ell(\kappa, V_\kappa)u = \lambda u\) on the interval \((\varepsilon, 1)\). The operator \(\mathcal{K}_\varepsilon\) is an integral operator with an upper-triangular kernel and belongs to the algebra \(\mathcal{G}_p(\varepsilon, 1)\) consisting of all integral operators in \(\mathbb{L}_2(\varepsilon, 1)\) whose kernels \(K\) have the property that the mappings

\[
x \mapsto K(x, \cdot), \quad x \mapsto K(\cdot, x)
\]
are continuous from $[0, 1]$ to $L_p(\varepsilon; 1); \mathbb{C}$). Therefore the above identity can be recast in the form
\[
J_\varepsilon = (J_\varepsilon + K_\varepsilon) \left[ \lim_{N \to \infty} \sum_{n=-N}^{N} \alpha_n (\cdot, \varepsilon) c_n \right] (J_\varepsilon + K_\varepsilon^*),
\]
where $c_n := c(\cdot; \lambda_n)$. In other words, the operator
\[
J_\varepsilon := \lim_{N \to \infty} \sum_{n=-N}^{N} \alpha_n (\cdot, \varepsilon) c_n
\]
can be factorised as
\begin{equation}
(4.1) \quad J_\varepsilon = (J_\varepsilon + K_\varepsilon)^{-1}(J_\varepsilon + K_\varepsilon^*)^{-1}.
\end{equation}

We notice that the operator $J_\varepsilon$, being a compression to $L_2(\varepsilon, 1)$ of the operator $J + K$ with an integral operator $K$ belonging to $G_p(0, 1)$, see [2], differs from $J_\varepsilon$ by an operator from $G_p(\varepsilon, 1)$. Moreover, $J_\varepsilon$ is positive since $\{c_n\}_{n \in \mathbb{Z}}$ is a complete set in $L_2(\varepsilon, 1)$, for every $\varepsilon > 0$. It follows that $J_\varepsilon$ can be (uniquely) factorised in $G_p(\varepsilon, 1)$, and comparison with (4.1) shows that $(J_\varepsilon + K_\varepsilon)^{-1}$ and its adjoint are the corresponding factors.

It follows that the transformation operator $J_\varepsilon + K_\varepsilon$ on $(\varepsilon, 1)$ is uniquely determined by $J_\varepsilon$, i.e., uniquely determined by the eigenvalues $(\lambda_n)_{n \in \mathbb{Z}}$ and the norming constants $(\alpha_n)_{n \in \mathbb{Z}}$. Next, the potential $(\kappa/x) \sigma_1 + V_\varepsilon$ on $(\varepsilon, 1)$ is related to the kernel $K_\varepsilon(x, t)$ of the operator $K_\varepsilon$ as
\[
\kappa \sigma_1 + V_\varepsilon(x) = \frac{1}{4} \left[ \sigma_2 K_\varepsilon(x, x) - K_\varepsilon(x, x) \sigma_2 \right],
\]
see [7] in the case of continuous potentials and [2] for the modification based on the Krein equation in the case of potentials in $L_p(\varepsilon, 1)$. Since $\varepsilon$ is arbitrary, we conclude that the spectral data determine the potential $(\kappa/x) \sigma_1 + V_\varepsilon$ uniquely. The theorem is proved.

\begin{flushright}
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\end{flushright}

\textbf{Appendix A. Properties of solutions}

If $V = 0$, then the equation $\ell(\kappa, 0)u = \lambda u$ has a fundamental system of solutions
\[
u_1^0(\cdot, \lambda) := (u_{1,1}^0(\cdot, \lambda), u_{2,1}^0(\cdot, \lambda))^\top, \quad \nu_2^0(\cdot, \lambda) := (u_{1,2}^0(\cdot, \lambda), u_{2,2}^0(\cdot, \lambda))^\top
\]
with
\[
u_{1,1}^0(x, \lambda) = \sqrt{\frac{2\lambda \pi}{\kappa}} J_{\kappa + \frac{1}{2}}(\lambda x), \quad \nu_{2,1}^0(x, \lambda) = \sqrt{\frac{2\lambda \pi}{\kappa}} J_{-\kappa - \frac{1}{2}}(\lambda x),
\]
\[
u_{1,2}^0(x, \lambda) = \sqrt{\frac{2\lambda \pi}{\kappa}} Y_{\kappa + \frac{1}{2}}(\lambda x), \quad \nu_{2,2}^0(x, \lambda) = \sqrt{\frac{2\lambda \pi}{\kappa}} Y_{-\kappa - \frac{1}{2}}(\lambda x),
\]
where \( J_\nu \) and \( Y_\nu \) are the Bessel functions of the first and of the second kind and order \( \nu \) respectively. Recalling the behaviour at the origin of the Bessel functions [13, Sect. 3.13], we conclude that the functions \( u_{1,1}^0(\cdot, \lambda) \) and \( u_{2,1}^0(\cdot, \lambda) \) have at \( x = 0 \) zeros of orders \( \kappa + 1 \) and \( \kappa \) respectively, while \( u_{1,2}^0(\cdot, \lambda) \) and \( u_{2,2}^0(\cdot, \lambda) \) have there poles of orders \( -\kappa \) and \( 1 - \kappa \) (here and in what follows, we adopt the convention that a zero of order \( k < 0 \) is a pole of order \( -k \), and similarly for poles of negative orders).

We claim that a fundamental system of solutions with the same behaviour at the origin exists also for an arbitrary \( V \in \mathbb{L}_p(0,1) \). Denote by \( U^0(\cdot, \lambda) = (u_{j,k}^0(\cdot, \lambda)) \) the fundamental matrix for the unperturbed system; then

\[
(A.1) \quad G(x, t, \lambda) := U^0(x, \lambda)[U^0(t, \lambda)]^{-1}\sigma_2 \frac{1}{1} = u_{1,2}^0(x, \lambda)u_{1,1}^0(t, \lambda)^\top - u_{1,1}^0(x, \lambda)u_{2,2}^0(t, \lambda)^\top
\]

is its Green’s function.

We construct a regular solution \( u_1(\cdot, \lambda) \) as a solution of the integral equation

\[
(A.2) \quad u_1(x, \lambda) = u_0^0(x, \lambda) + \int_0^x G(x, t, \lambda)V(t)u_1(t, \lambda) \, dt.
\]

The function \( u_1 \) satisfying (A.2) can be constructed by the Picard’s iteration method, cf. [9] for details, and solves the equation \( \ell(\kappa, V)u = \lambda u \). Put \( A(x) := \text{diag}\{x^{-\kappa-1}, x^{-\kappa}\} \) and \( v_1(x, \lambda) := A(x)u_1(x, \lambda) \); then \( v_1 \) satisfies the equation

\[
\begin{align*}
\quad v_1(x, \lambda) &= v_0^0(x, \lambda) - \int_0^x A(x)G(x, t, \lambda)V(t)A^{-1}(t)v_1(t, \lambda) \, dt, \\
\text{with} \quad v_j^0(x, \lambda) &= A(x)u_j^0(x, \lambda), \quad j = 1, 2.
\end{align*}
\]

We shall show below that

\[
(A.3) \quad \int_0^x \|A(x)G(x, t, \lambda)V(t)A^{-1}(t)\| \, dt \rightarrow 0
\]

as \( x \rightarrow 0+ \); hence the Gronwall inequality gives \( \sup_{x \in [0,1]} |v_1(x, \lambda)| < \infty \) and then we see that

\[
|v_1(x, \lambda) - v_1^0(x, \lambda)| \leq \int_0^x \|A(x)G(x, t, \lambda)V(t)A^{-1}(t)\| |v_1(t, \lambda)| \, dt = o(1)
\]

as \( x \rightarrow 0+ \). This shows that \( v_1(x, \lambda) \) is continuous at \( x = 0 \) and \( v_1(0, \lambda) \) has nonzero components. In particular, we conclude that the components \( u_{1,1}(\cdot, \lambda) \) and \( u_{1,1}(\cdot, \lambda) \) of the function \( u_1(\cdot, \lambda) \) satisfy the relations \( u_{1,1}(x, \lambda) \sim x^{\kappa+1} \) and \( u_{1,1}(x, \lambda) \sim x^\kappa \) as \( x \rightarrow 0+ \).

To prove (A.3) we use formula (A.1) for the Green function to see that

\[
A(x)G(x, t, \lambda)V(t)A^{-1}(t) = v_2^0(x, \lambda)u_0^0(t, \lambda)^\top V(t)A^{-1}(t) - v_1^0(x, \lambda)u_2^0(t, \lambda)^\top V(t)A^{-1}(t).
\]

Direct calculations show that

\[
u_0^0(t, \lambda)^\top V(t)A^{-1}(t) = t^{2k+1}\tilde{V}(t)
\]

for some summable matrix-valued function \( \tilde{V} \), and thus, for \( 0 < t \leq x \),

\[
\|A(x)u_2^0(x, \lambda)u_0^0(t, \lambda)^\top V(t)A^{-1}(t)\| \leq C\|\tilde{V}(t)\|
\]

with some \( C \) independent of \( x \). Similar estimates apply to the second summand, thus yielding (A.3).
A singular solution $u_2(\cdot, \lambda)$ of the equation $\ell(\kappa, V)u = \lambda u$ can be constructed as a solution of the integral equation

$$u_2(x, \lambda) = u_0^0(x, \lambda) - \int_x^1 G(x, t, \lambda)V(t)u_2(t, \lambda) \, dt. \tag{A.4}$$

Similar arguments justify the singular behaviour of $u_2(\cdot, \lambda)$ at the origin, namely that $u_{2,1}(x, \lambda) \sim x^{-\kappa}$ and $u_{2,2}(x, \lambda) \sim x^{1+\kappa}$ as $x \to 0+$.

The above results imply that every solution of the equation $\ell(\kappa, V)u = \lambda u$ has a similar behaviour at the origin, namely:

**Lemma A.1.** Assume that $\kappa \in \mathbb{Z}$, $V \in \mathbb{L}_p(0,1)$, $\lambda \in \mathbb{R}$, and that $u(x) = (u_1(x), u_2(x))^T$ is a solution of the equation $\ell(\kappa, V)u = \lambda u$. Then, as $x \to 0+$, either $u_1 \sim x^{\kappa+1}$ and $u_2 \sim x^\kappa$ or $u_1 \sim x^{-\kappa}$ and $u_2 \sim x^{1-\kappa}$.

**Appendix B. Properties of the function $\tilde{w}$**

We justify here the following statement:

**Lemma B.1.** In the notations of Subsection 3.1, we have

$$1 + w(x, \lambda_k) = x^{2\kappa+1}\tilde{w}(x),$$

where $\tilde{w}$ belongs to the Sobolev space $W^{1}_p(0,1)$.

We recall that the Sobolev space $W^{1}_p(0,1)$ consists of the absolutely continuous functions $f$ whose derivative $f'$ belongs to $L_p(0,1)$, the norm being $\|f\|_{L_p} + \|f'\|_{L_p}$.

**Proof.** We represent the function $\tilde{w}$ in the form

$$\tilde{w}(x) = \frac{\alpha_k}{x^{2\kappa+1}} \int_0^x s^{2\kappa} \frac{|u(s, \lambda_k)|^2}{s^{2\kappa}} \, ds$$

and prove first that the function $f(s) := s^{-2\kappa}|u(s, \lambda_k)|^2$ belongs to $W^{1}_p(0,1)$ and then that the operator

$$T : g \mapsto \frac{1}{x^{2\kappa+1}} \int_0^x s^{2\kappa} g(s) \, ds$$

is continuous in $W^{1}_p(0,1)$.

For the first statement, we notice that the function $f$ is absolutely continuous on $[0,1]$ and that

$$f'(s) = -\frac{2\kappa}{s^{2\kappa+1}} [u_1^2(s, \lambda_k) + u_2^2(s, \lambda_k)] + \frac{2}{s^{2\kappa}} [u_1'(s, \lambda_k) u_1(s, \lambda_k) + u_2'(s, \lambda_k) u_2(s, \lambda_k)]$$

Using the Dirac equation, we substitute for $u_1'$ and $u_2'$ above to obtain

$$f'(s) = -\frac{4\kappa u_1^2(s, \lambda_k)}{s^{2\kappa+1}} + \frac{2}{s^{2\kappa}} \sum_{1 \leq i \leq j \leq 2} v_{ij} u_i(s, \lambda_k) u_j(s, \lambda_k)$$

with some functions $v_{ij}$ belonging to $L_p(0,1)$. Since $x^{-\kappa-1}u_1(x, \lambda_k)$ and $x^{-\kappa}u_2(x, \lambda_k)$ are continuous functions on $[0,1]$, we conclude that $f'$ belongs to $L_p(0,1)$ and thus $f \in W^{1}_p(0,1)$.

To show that $T$ is bounded in $W^{1}_p(0,1)$, we first observe that

$$(Tg)(x) = \frac{g(x)}{2\kappa + 1} - \frac{1}{2\kappa + 1} \frac{1}{x^{2\kappa+1}} \int_0^x s^{2\kappa+1} g'(s) \, ds.$$
The function \( h(x) := x^{-2\kappa-1} \int_0^x s^{2\kappa+1} g'(s) \, ds \) is clearly continuous, and \( |h(x)| \leq \|g'\|_{L_p} \); moreover,

\[
h'(x) = g'(x) - \frac{2\kappa + 1}{x^{2\kappa+2}} \int_0^x s^{2\kappa+1} g'(s) \, ds.
\]

Since the Hardy operator

\[
f \mapsto \frac{2\kappa + 1}{x^{2\kappa+2}} \int_0^x s^{2\kappa+1} f(s) \, ds
\]

is bounded in \( L_p(0,1) \) [6, Sect. 9.9], we conclude that \( h' \in L_p(0,1) \) and \( \|h'\|_{L_p} \leq C\|g'\|_{L_p} \) for some constant \( C \) independent of \( g \). The lemma is proved. \( \square \)

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