AdS/CFT Correspondence in the Euclidean Context

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Abstract

We study two possible prescriptions for AdS/CFT correspondence by means of functional integrals. The considerations are non-perturbative and reveal certain divergencies which turn out to be harmless, in the sense that reflection-positivity and conformal invariance are not destroyed.

1 Introduction

In this article we investigate the AdS/CFT correspondence for scalar fields within the Euclidean approach. Originally this conjecture was formulated within the string theoretic context [12]. Soon afterwards it was discovered that it makes perfect sense in a purely quantum field theoretic setting [20]. This conjecture states that a quantum field theory (QFT) on AdS space gives rise to a conformal QFT on its boundary and vice versa. Within the algebraic approach to QFT this correspondence can be made precise, see [17]. We hope that this work can contribute to the recent discussion on the mathematical status of non-algebraic AdS/CFT.

We are interested in the passage from AdS-QFT to CFT by means of functional integrals. Without taking recourse to perturbative arguments we succeed in constructing functional integrals within the infinite dimensional setting. The Euclidean field theory of an interacting QFT is described through a probability measure \( d\mu = e^{-V}d\mu_C / \int e^{-V}d\mu_C \), defined on an appropriate distribution space on the Riemannian counterparts of AdS spaces, which are hyperbolic spaces. The Gaussian measure \( d\mu_C \) with covariance \( C \) specifies the underlying free theory and the density \( e^{-V} \) accounts for the interaction. The measure \( d\mu \) should satisfy the Osterwalder-Schrader axioms in order to make a passage from hyperbolic to AdS-spaces possible [3].

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On hyperbolic spaces, there are two choices of invariant covariance operators, denoted $G_{\pm}$, since there are two linearly independent fundamental solutions to the equation

$(-\Delta + m^2)G(\bar{x}, \bar{x}') = \delta(\bar{x}, \bar{x}')$.

This follows from the fact that, due to invariance, $G$ has to be a function of the geodesic distance $d(\bar{x}, \bar{x}')$, therefore the resulting equation for $G(d)$ involves only the radial part of the Laplacian which can be transformed to a hypergeometric equation possessing two linearly independent solutions. This work is inspired by the ideas in [5] where two natural prescriptions for the AdS/CFT correspondence are compared and shown to essentially agree. One way is to define a Laplace transform where the source term is restricted to the boundary, i.e.

$$\tilde{Z}(f)/\tilde{Z}(0) = \int e^{-V(\phi)} e^{\partial \phi(f)} d\mu_C(\phi)/\tilde{Z}(0).$$ (1)

At this place $\partial \phi$ means the restriction of the bulk field to the boundary. Below we shall see how to make this definition rigorous using a proper scaling. It turns out that in general nontrivial results for (1) can be obtained only through the multiplication with a regularizing factor which nonetheless doesn’t destroy reflection positivity and conformal invariance. Another possibility is to fix the values of the bulk field on the boundary by insertion of a delta function, so that heuristically we set

$$Z(f)/Z(0) = \int e^{-V(\phi)} \delta(\partial \phi - f) d\mu_C(\phi)/Z(0).$$ (2)

It will turn out that the correct choice for $C$ is to take $G_+$ in case (2) and $G_-$ for (1). The construction makes it also explicit that the two functionals agree up to the multiplication of test functions with a constant factor when both are defined.

In section 2 we introduce various propagators which serve as building blocks for the functional integrals. In section 3 we show how to give a rigorous meaning to expressions (2) and (1). Then in section 4 we go over to discuss two basic axiomatic properties.

## 2 Propagators on the hyperbolic space

There are various propagators needed for the definition of AdS/CFT functional integrals, which we introduce in this section. Let us consider the upper half-space model of the $(d + 1)$-dimensional hyperbolic space $\mathbb{H}^{d+1} := \{(z, x) \in \mathbb{R}^{d+1} : z > 0\}$,
equipped with the Riemannian metric \( 1/z^2(dz^2 + dx_1^2 + \cdots + dx_d^2) \). The Green’s functions \( G_\pm \) are explicitly given by

\[
G_\pm(z, x; z', x') = \gamma_\pm(2u)^{-\Delta_\pm} F(\Delta_\pm, \Delta_\pm + 1 - d; -2u^{-1})
\]

where \( u = \frac{(z-z')^2 + (x-x')^2}{2z^2} \), \( \Delta_\pm = \frac{d}{2} \pm \frac{1}{2} \sqrt{d^2 + 4m^2} =: \frac{d}{2} \pm \nu, \nu > 0 \) and \( \gamma_\pm = \Gamma(\Delta_\pm) \frac{\pi^{d/2}}{2^{d/2} \Gamma(\Delta_\pm + 1 - \frac{d}{2})} \). \( F \) is the hypergeometric function which for \( \zeta \in \mathbb{C} \) with \( |\zeta| < 1 \) is given by the absolutely convergent series

\[
F(a, b; c; \zeta) = 1 + \frac{ab}{c} \zeta + \frac{a(a+1)b(b+1)}{c(c+1)} \zeta^2 + \cdots
\]

Its analytic continuation to \( \mathbb{C}\setminus[1, \infty) \) is given by the integral representation

\[
F(a, b; c; \zeta) = \frac{\Gamma(c) \Gamma(c-b)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-\zeta t)^{-a} dt, \quad \text{if } \text{Re} c > \text{Re} b > 0.
\]

It should be noted that \( G_+ \) is the integral kernel of the inverse \( (-\Delta + m^2)^{-1} \).

We would like to obtain a conformal theory on the boundary at infinity \( (z \to 0) \). On the level of propagators this is achieved by taking appropriate scaled limits. From (3) we get as pointwise limits the bulk-to-boundary propagators

\[
H_\pm(z, x; x') = \lim_{z' \to 0} z'^{-\Delta_\pm} G_\pm(z, x; z', x') = \gamma_\pm \left( \frac{z}{z^2 + (x-x')^2} \right)^{\Delta_\pm}
\]

and the boundary propagators

\[
\alpha_\pm(x, x') = \lim_{z \to 0} z^{-\Delta_\pm} H_\pm(z, x; x') = \gamma_\pm (x-x')^{-2\Delta_\pm}.
\]

Since \( 2\Delta_+ \geq d \), the kernels \( \alpha_+ \) have a non-integrable singularity. They will be understood to be regularized by analytic continuation to values \( \nu \neq 0, 1, 2, \ldots \), see [8]. Hence, whenever \( \alpha_+ \) is involved in some argument, statements will hold with the exception of singular points.

**Notation.** The Fourier transform is defined as \( \hat{f}(k) = 1/(2\pi)^{d/2} \int_{\mathbb{R}^d} f(x)^{-ik \cdot x} dx \). We use the notation \( |\zeta| \) for the absolute value of an complex number \( \zeta \), as well as \( |k| \) for the Euclidean norm of a vector \( k \in \mathbb{R}^d \). Tuples \( (z, x) \) will also be denoted by \( z \).

Then the Fourier transforms of \( H_\pm(z, x; x') \) and \( \alpha_\pm(0, x') \) with respect to \( x' \in \mathbb{R}^d \) read

\[
\hat{H}_\pm(z, k) = \frac{1}{(2\pi)^{d/2} \Gamma(1 \pm \nu)} e^{ik \cdot z} \left( \frac{|k|}{2} \right)^{\pm \nu} z^\frac{d}{2} K_\nu(|k| z),
\]

where \( K_\nu \) is the modified Bessel function of the second kind.
and
\[ \hat{\alpha}_\pm(k) = \frac{\Gamma(\mp \nu)}{2(2\pi)^{\frac{d}{2}} \Gamma(1 \pm \nu)} \left( \frac{|k|}{2} \right)^{\pm 2\nu} C_{-\nu} \left( \frac{|k|}{2} \right)^{\pm 2\nu}, \tag{7} \]
where \( K_\nu \) is the modified Bessel function of the second kind which is given by
\[ K_\nu(\zeta) = \frac{1}{2} \left( \frac{\zeta}{2} \right)^\nu \int_{-\infty}^{\infty} e^{-t-\frac{z^2}{4t}} dt, \quad |\arg\zeta| < \frac{\pi}{2}, \quad \text{Re}\zeta > 0. \]

Lemma 1 With \( c := 2\nu \) we have
\[
G_-(x, x') = G_+(x, x') + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H_+(x, y) c^2 \alpha_- (y, y') H_+(x', y') dy dy'. \tag{8}
\]

Proof. In [16] it was shown that
\[
G_-(x, x') = G_+(x, x') + c \int_{\mathbb{R}^d} H_+(x, y) H_-(x', y) dy. \tag{9}
\]
Let \( \alpha_- (\cdot) \) be the function \( y' \to \alpha_- (0, y') \) then for \( \nu > 0 \)
\[
\int_{\mathbb{R}^d} \alpha_- (y, y') H_+(x', y') dy' = (\alpha_- (\cdot) * H_+(z', x'; \cdot))(y), \tag{10}
\]
where * means convolution. Therefore,
\[
(\alpha_- (\cdot) * H_+(z', x'; \cdot))(y) = \int_{\mathbb{R}^d} e^{iky} \hat{\alpha}_- (k) \hat{H}_+(z', x'; k) dk
\]
\[= \frac{1}{c(2\pi)^d \Gamma(1-\nu)} \int_{\mathbb{R}^d} e^{ik(x'+y)} \left( \frac{|k|}{2} \right)^{-\nu} z'^{\frac{d}{2}} K_\nu(|k|z') dk. \tag{11}\]
On the other hand
\[
\frac{1}{c} H_-(x', y) = \frac{1}{c(2\pi)^d} \int_{\mathbb{R}^d} e^{iky} \hat{H}_-(x'; k) dk = (11),
\]
and the result follows. \( \square \)
Construction and definition of functional integrals

First we try to give a meaningful to the functional integral (2). For $2\nu < d$, $\alpha_-$ is a positive covariance and in this parameter range the splitting given in (8) entails the corresponding splitting for the random fields,

$$\phi_-(x) = \phi_+(x) + cH_+\phi_+(x),$$

where $H_+\phi_+(x) := \int_{\mathbb{R}^d} H_+(x,y)\phi_+(y)dy$, and $\phi_-, \phi_+, \phi_\alpha$ are the random fields with covariances $G_-, G_+$ and $\alpha_-$ respectively. More precisely, $\phi_+, \phi_\alpha$ have to be understood as the first and second component of the following product measure space

$$(D(\mathbb{H}^{d+1})' \times S(\mathbb{R}^d)', B(D(\mathbb{H}^{d+1})') \otimes B(S(\mathbb{R}^d)'), \mu_{G_+} \otimes \mu_{\alpha_-}).$$

$D(\mathbb{H}^{d+1})$ stands for the space of infinitely differentiable real-valued functions with compact support on $\mathbb{H}^{d+1}$ and $S(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decreasing real-valued functions on $\mathbb{R}^d$. The primes indicate the topological duals, or distribution spaces. Finally, $B$ stands for the Borel $\sigma$-algebras obtained from the respective weak-* topologies. Then we have

$$\mathbb{E}_{\mu_{G_+} \otimes \mu_{\alpha_-}}[(\phi_+(f) + cH_+\phi_+(f))(\phi_+(g) + cH_+\phi_+(g))]
= \mathbb{E}_{\mu_{G_+}}[\phi_+(f)\phi_+(g)] + c^2\mathbb{E}_{\mu_{\alpha_-}}[H_+\phi_+(f)H_+\phi_+(g)],$$

because the other terms vanish due to the product measure and the fact that the expectations of the fields vanish. But the last line is just the splitting (8). For this reason we may write for $\nu < \frac{d}{2}$

$$\int_{D'} F(\phi_-)d\mu_{G_-}(\phi_-) = \int_{D' \times S'} F(\phi_+ + cH_+\phi_\alpha)d(\mu_{G_+} \otimes \mu_\alpha)(\phi_+, \phi_\alpha). \quad (12)$$

**Remark.** The bound $-\nu > -\frac{d}{2}$ is dictated by the positivity of $\alpha_-$. For $d = 1$ this is larger than the unitary bound $-\nu > -1$ (in our notation). The same bound is needed when $\alpha_-$ is asked to be reflection-positive, see [10, Theorem 6.2.4]. For $d \geq 2$ reflection-positivity imposes the usual unitary bound $-\nu > -1$.

In order to cope with the delta function we shall boil down things to a finite dimensional approximation for the boundary field, insert the delta function in this case, perform integration over the (finite-dimensional) boundary field and then remove the approximation again. This is done in two steps.
Step 1. We approximate the boundary covariance operator $\alpha_-$ by covariance operators which possess bounded inverses in $L^2(\mathbb{R}^d)$. First we note that

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \alpha_-(x, y) f(y) dxdy = C_\nu \int_{\mathbb{R}^d} |k|^{-2\nu} |\hat{f}|^2 dk, \quad f \in \mathcal{S}(\mathbb{R}^d). \quad (13)
$$

From (13) we see that the bounded approximations can be defined as follows

$$(f, \alpha_n f) := C_\nu \int_{\mathbb{R}^d} \chi_n(|k|) |\hat{f}|^2 dk, \quad (n \in \mathbb{N}),$$

where

$$\chi_n(|k|) := \begin{cases} \frac{1}{n} |k|^{-2\nu}, & \text{for } |k| \leq \frac{1}{n}, \\ |k|^{-2\nu}, & \text{for } \frac{1}{n} < |k| \leq n, \\ |n|^{-2\nu}, & \text{for } |k| > n. \end{cases}$$

Obviously, for their inverses we obtain

$$(f, (\alpha_n^\nu)^{-1} f) = (C_\nu)^{-1} \int_{\mathbb{R}^d} (\chi_n(|k|))^{-1} |\hat{f}|^2 dk.$$

Step 2. Next we consider finite dimensional approximations for the boundary field $\phi_{\alpha_n}$ ($n$ arbitrary) with covariance $\alpha_n^-$. This approximation is performed by $p_m \phi_{\alpha_n}$, where $p_m$ is the projection on the subspace spanned by the first $m$ basis elements of a Hilbert space basis $(e_i)_{i \geq 1}$ of $L^2(\mathbb{R}^d)$. In order that the matrix elements $(e_i, \alpha_n^\nu e_j)$ be defined, we choose the basis elements to be Schwartz functions, which is possible, since Schwartz spaces are separable. Making in addition the identification $\eta : p_m \phi_{\alpha} \rightarrow \psi_{\alpha} = (\phi_{\alpha})(e_1), \ldots, (\phi_{\alpha})(e_m))^t \in \mathbb{R}^m$, we see that the integral (12) takes the form

$$C_{A_-} \int_{\mathbb{R}^m} \int_{D'} F(\phi_+ + cH_+ (\eta^{-1} \psi_{\alpha})) d\mu_{\phi_+} (\phi_+) e^{-\frac{i}{2} (\psi_{\alpha}, A^- \psi_{\alpha})} d\psi_{\alpha},$$

where $A_- := (\eta p_m \alpha_n^\nu p_m \eta^{-1})^{-1} = \eta (p_m \alpha_n^\nu p_m)^{-1} \eta^{-1}$ and $C_{A_-} = \frac{|\det A_-|^2}{(2\pi)^{d/2}}$. Now it is possible to insert the delta function and we get

$$C_{A_-} \int_{\mathbb{R}^m} \int_{D'} \delta(\psi_{\alpha} - \eta p_m f) F(\phi_+ + cH_+ (\eta^{-1} \psi_{\alpha})) d\mu_{\phi_+} (\phi_+) e^{-\frac{i}{2} (\psi_{\alpha}, A^- \psi_{\alpha})} d\psi_{\alpha} = C_{A_-} e^{-\frac{i}{2} (f, (p_m \alpha_n^\nu p_m)^{-1} f)} \int_{D'} F(\phi_+ + cH_+ (p_m f)) d\mu_{\phi_+} (\phi_+) =: Z_{m,n}(f). \quad (14)$$
We notice that in the quotient $Z_{m,n}(f)/Z_{m,n}(0)$ the constant $C_{A_+}$ drops out. The uniform convergence $p_m \to 1$ leads to $(p_m \alpha^n p_m)^{-1} f \to (\alpha^n)^{-1} f$, due to the boundedness of operators, see [19, Theorem 5.11]. Let $H_+(z; \cdot )$ denote the function $x \to H(z, x; 0)$ then we have \( \| (H_+ p_m f)(z, \cdot ) \|_2 = \| H_+(z; \cdot ) * p_m f \|_2 \leq \| H_+(z; \cdot ) \|_1 \| p_m f \|_2 \) by Young’s inequality. Moreover, $z \to \| H_+(z; \cdot ) \|_1$ remains bounded if $z$ varies in a compact subset of $(0, \infty)$. Therefore, under the assumption that

\[
\| F(\cdot + c H_+(p_m f) - F(\cdot + c H_+(p_n f)) \|_{L^1(\mu_{G_+})} \leq \text{const} \| c(H_+(p_m f - p_n f)) \|_2, \tag{15}
\]

with a compact $\Lambda \subset \mathbb{H}^d$ we get convergence of the integral (14) as $p_m \to 1$. Finally we take the limit $n \to \infty$. From the definitions it is clear that $(f, \alpha^n f) \to (f, \alpha f)$ and $(f, (\alpha^n)^{-1} f) \to (f, \alpha^{-1} f)$. These considerations justify the following rigorous definition of the generating functional (2)

\[
Z(f)/Z(0) := e^{-\frac{1}{2}(f,\alpha^{-1} f)} \int_{D'} F_+(\phi_+ + c H_+ f) d\mu_{G_+} + (\phi_+)/Z(0). \tag{16}
\]

We now come to a second possible prescription for the AdS/CFT-correspondence. This time we take $G_+$ as covariance for the free theory given by the field $\phi \in \mathcal{D}(\mathbb{H}^{d+1})'$. Let us define

\[
\tilde{Z}(f)/\tilde{Z}(0) = \lim_{z \to 0} (\tilde{Y}(f)/\tilde{Y}(0))_z := \lim_{z \to 0} \int_{D'} e^{\theta(z^{-\Delta_+ (\delta z \otimes f))} F(\phi) d\mu_{G_+} (\phi)/\tilde{Z}(0), \tag{17}
\]

where $\delta z \otimes f \in H^{-1}$ is the distribution defined by

\[
(\delta z \otimes f)(g) = \int_{\mathbb{R}^d} f(x)g(z, x)dx, \quad f \in C_0^\infty(\mathbb{R}^d), \quad g \in C_0^\infty(\mathbb{R}_{>0} \times \mathbb{R}^d).
\]

We would like to compare functional (17) with the one found in (16). To this end we rewrite (17) a little bit using the quasiinvariance of Gaussian measures with respect to shifts by elements from $H^1$. Applying the general result on quasiinvariance proven in [1], we thus get with $f_z := z^{-\Delta_+ (\delta z \otimes f)}$

\[
d\mu_{G_+} (\cdot - G_+ f_z) = e^{\theta((-\Delta+ m^2)G_+ f_z)} e^{-\frac{1}{2}(G_+ f_z, (\Delta+ m^2)G_+ f_z)} d\mu_{G_+} (\cdot ) = e^{\theta(f_z)} e^{-\frac{1}{2}(G_+ f_z, f_z)} d\mu_{G_+} (\cdot ).
\]

It should be noted that the random field $\phi(f)$ can be extended to all $f \in H^{-1}$. Using this in (17) we arrive at the following expression

\[
(\tilde{Y}(f)/\tilde{Y}(0))_z = e^{\frac{1}{2}(G_+ f_z, f_z)} \int_{D'} F(\phi + G_+ f_z) d\mu_{G_+} (\phi)/\tilde{Y}(0). \tag{18}
\]
Before being able to perform the limit $z \to 0$ we have to take a closer look at the behavior of the term $(G_z + f)_z$. In appendix A it is shown that in this limit we have to subtract certain divergent terms, more precisely,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \alpha_+(x, y) f(x) f(y) dx dy =$$

$$\lim_\limits{z \to 0} z^{-d-2\nu} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z, x; z, y) f(x) f(y) dx dy -$$

$$\frac{1}{(2\pi)^\frac{d}{2}} \left( \frac{2^{1-\nu}}{\sqrt{\pi \Gamma(\nu + \frac{1}{2})}} \right)^2 \sum_\limits{j=0}^\nu z^{-2(\nu-j)} (-1)^j a_j \int_{\mathbb{R}^d} |\hat{f}(k)|^2 |k|^{2j} dk.$$

$$=: \lim_\limits{z \to 0} z^{-d-2\nu} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z, x; z, y) f(x) f(y) dx dy - (\text{Corr}(z) f, f).$$

In order to get nontrivial results in the limit we have to regularize the exponential prefactor in (18) by multiplying it with $\exp - (\text{Corr}(z) f, f)$. From (3), (4) it is readily seen that, as $z \to 0$, $G_z f_z$ converges to $H_z f$ uniformly on every compact subset. Hence assuming that for some compact $\Lambda$

$$\| F(\cdot + G_z f_z) - F(\cdot + H_z f) \|_{L^1(\mu_{G_z})} \leq \text{const} \| (G_z f_z - H_z f) \|_{L^p},$$

(19)

for some $p$, we see that the integral in (18) converges, which shows that the correct definition for (17) reads

$$\hat{Z}(f)/\hat{Z}(0) = \lim_\limits{z \to 0} e^{-(\text{Corr}(z) f, f)} (\hat{Y}(f)/\hat{Y}(0))$$

$$= e^{\frac{1}{2}(\alpha_f + f)} \int_{D'} F(\phi + H_z f) d\mu_{G_z}(\phi)/\hat{Y}(0).$$

(20)

We note that in the considerations above we have $\hat{Z}(0) = \hat{Y}(0)$.

We shall now address the existence of $P_2(\phi)$ models with interaction restricted to some bounded region $\Lambda$. We look at $F_\Lambda(\phi + G_z f_z) = e^{-V_\Lambda(\phi + G_z f_z)}$ with potentials

$$V_\Lambda(\phi) = \int_{\Lambda} \sum_{j=0}^n : \phi_j(x) : G_z f_j(x) dx \equiv \sum_{j=0}^n : \phi_j : (f_j 1_\Lambda) =: \left( \sum_{j=0}^n : \phi_j : (f_j) \right)(1_\Lambda),$$

where $: : G_z$ denotes Wick-ordering with respect to $G_z$.

Since $:(\phi + f)^n G_z (g) = \sum_{j=0}^n \binom{n}{j} : \phi_j G_z (g f^{n-j})$, a polynomial interaction $V_\Lambda$ is transformed into such under shifts and we may study the polynomial interactions itself.
Proposition 1 Let $V_{\Lambda}$ be a polynomial interaction as above with $n$ even and let $f_i$ be radial $L^2$-functions. Then

$$\begin{align*}
(a) \quad & \|V_{\Lambda}\|_{L^p(\mu_{G_+})} \leq \text{const}(p, n) \sum_{i=0}^{n} \|f_i\|_2, \quad 1 \leq p < \infty, \\
\text{and} \quad & \int e^{-V_{\Lambda}(\phi)} d\mu_{G_+}(\phi) \leq e^{\text{const}([N(f)+\ln(M(f)+1)])^\frac{2}{P}},
\end{align*}$$

where $N(f) = \sum_{i=0}^{n} \|f_i/f_n\|_{n/(n-i)}$, $M(f) = \sum_{i=1}^{n} \|f_i\|_{n/(n-i)}$.

$$\begin{align*}
(b) \quad & \lim_{\varepsilon \to 0} \|e^{-V_{\Lambda}(\cdot+G_{+}f_\varepsilon)} - e^{-V_{\Lambda}(\cdot+H_{+}f_\varepsilon)}\|_{L^p(\mu_{G_+})} = 0, \quad 1 \leq p < \infty.
\end{align*}$$

Proof. The proof is just an adaption of the arguments given in [6] and [10, Chapter 8] in that Fourier transformation on $\mathbb{H}^2$ is used, see Appendix B. Here we repeat the main steps. Let us consider the expression

$$R_\varepsilon(n, w) = \int \prod_{\mu=1}^{N} \phi_\varepsilon(y_\mu)^{n_\mu} : w(y_1, \ldots, y_N) dy, \quad (n = (n_1, \ldots, n_N) \in \mathbb{N}_0^N), \quad (21)$$

where $\phi_\varepsilon(y) := (\phi \ast \chi_\varepsilon)(y)$ and $\chi_\varepsilon(y) := a(\varepsilon)\chi(\varepsilon y)$, $\varepsilon > 0$, is an approximate unity with $\chi \in C_0^\infty(\mathbb{H}^2)$ being a radial function with support in the unit ball and the factors $a(\varepsilon)$ are chosen such that $2\pi \int_{\mathbb{H}^2} \chi_\varepsilon(r) \sinh rdr = 1$ for all $\varepsilon$. We assume that the support of $w$ is contained in $B_1 \times \cdots \times B_N$, where the $B_i$ are balls in $\mathbb{H}^2$. The integral of (21) with respect to $d\mu_{G_+}$ can be calculated as a sum of graphs. The graphs in the present case are built as follows: Consider $N$ vertices each having $n_\mu$ ($1 \leq \mu \leq N$) legs and combine arbitrary pairs of legs from different vertices to lines obtaining in this manner a possible graph in the sum. Denoting $[I]$ the set of all legs and $\Gamma_0(I)$ the set of all graphs obtained by the above construction the integral of (21) can be estimated as ($0 \leq \rho \leq 1$)

$$\left| \int R(w, n) d\mu_{G_+} \right| \leq \text{M}(\rho, n, G_+) \|w\|_2 \prod_{(\mu, k) \in [I]} \|\chi_{\mu, k}\|_1^{-\rho} \|\hat{\chi}_{\mu, k}(m^2 + \frac{1}{4} + \lambda^2)^{-\delta/4}\|_{\infty}^\rho, \quad (22)$$

where $\chi_{\mu, k} = \chi_\varepsilon$ and the constant is given as ($n_* = \sup_\mu n_\mu$, $p' = \frac{P}{p-1}$, $\delta \leq 2$)

$$\begin{align*}
M(\rho, n, G_+) = |B| \sum_{G \in \Gamma_0(I)} \prod_{l \in G} \|G_l\|_{2n_*} \|G_+\|_{2n_*}^{1-\rho} \|G_{2n_*} \ast G_+\|_{B_{(2n_*+\delta)}}^{p'}, \quad (23)
\end{align*}$$

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where $\zeta_{\mu,k} = \zeta_\mu$ is any radial $C_0^\infty(\mathbb{H}^2)$ function which is identically one on $\{x: \text{dist}(x, B_\mu) \leq 1\}$. In (23) we have used the norm

$$
\|\zeta_\psi\|_{B_{r,\delta}} := \|(1 + |\lambda|^2)^{\frac{r}{2}}(\zeta_\psi')\|_{L^r}.
$$

Using the coordinate characterization of Sobolev spaces we see that $G_+ \in H^{-1} \times H^{-1}$ implies $\zeta G_+ \in H^{-1} \times H^{-1}$. The Fourier space characterization of Sobolev spaces then shows that the norms $\|G_+ \otimes G_+\|_{B_{2n,\delta}}$ are finite. Moreover, using (22) and noting that $\|\hat{\chi}_\varepsilon(m^2 + \frac{1}{4} + \lambda^2)^{-\delta/4}\|_{\infty} \leq \text{const} \|\hat{\chi}_\varepsilon\|_1$ and $\|((\hat{\chi}_\varepsilon - \hat{\chi}_{\varepsilon'})(m^2 + \frac{1}{4} + \lambda^2)^{-\delta/4}\|_{\infty} \leq O(1)(\varepsilon \wedge \varepsilon')^{-\delta/2}$, see Appendix B, one derives

$$
\|R_{\varepsilon}(w, n)\|_{L^p(\mu_{G_+})} \leq \text{const}(p, n)\|w\|_2 
$$

(24)

and

$$
\|R_{\varepsilon}(w, n) - R_{\varepsilon'}(w, n)\|_{L^p(\mu_{G_+})} \leq \text{const}(p, n)(\varepsilon \wedge \varepsilon')^{-\delta/2}\|w\|_2.
$$

(25)

The latter inequalities show that $R_{\varepsilon}$ is a Cauchy-sequence in $L^p(\mu_{G_+})$ with limit $R(w, n)$ obeying the bound (24). Applying this to the special case $R(w, n) = V_\Lambda$ we get statement (a). In 2 dimensions there is just a logarithmic singularity $G_+(x, x') \sim \text{const} |\ln d(x, x')|$ for small distances. With the aid of (24) and (25) by employing the arguments given in [10, Theorem 8.6.2] we see that also (b) holds true. In order to prove (c), we write

$$
e^{-V(\cdot + G_+ f_\varepsilon)(1_\Lambda)} - e^{-V(\cdot + H_+ f)(1_\Lambda))}
$$

$$
= \int_0^1 e^{-V(\cdot + G_+ f_\varepsilon)(s\Lambda)} V(\cdot + H_+ f)(1_\Lambda) - V(\cdot + G_+ f_\varepsilon)(1_\Lambda)) e^{-V(\cdot + H_+ f)((1-s)(1_\Lambda))} ds.
$$

The $L^p(\mu_{G_+})$-norm of the latter integral can be estimated as

$$
\sup_{0 \leq s \leq 1} \left( \|e^{-V(\cdot + G_+ f_\varepsilon)(s\Lambda)}\|_{L^p(\mu_{G_+})} \|e^{-V(\cdot + H_+ f)((1-s)(1_\Lambda))}\|_{L^p(\mu_{G_+})} \right)
$$

$$
\times \|V(\cdot + G_+ f_\varepsilon)(1_\Lambda) - V(\cdot + H_+ f)(1_\Lambda))\|_{L^p(\mu_{G_+})},
$$

which by (a) and (b) proves the assertion. $\square$

From (7) we see that $\alpha_-^{-1} = -c^2\alpha_+$ and therefore $Z(f)/Z(0) = \tilde{Z}(cf)/\tilde{Z}(0)$ when $\nu < \frac{d}{2}$.
4 Reflection positivity and invariance

In this section we probe the functional integrals for reflection positivity and conformal invariance. These two properties are essential to qualify them as providing us a conformal field theory on the boundary. The following considerations are valid for \( d \geq 1 \), if we assume that the objects exist for bounded \( \Lambda \) and for \( \Lambda \nearrow \mathbb{H}^{d+1} \). The existence and related questions of uniqueness are left to future work.

For simplicity let us consider the reflection \( \theta \) with respect to coordinate \( x_1 \) of \( \mathbb{R}^d \) and let \( \Lambda \) be reflection-symmetric. We want to verify that the integrals \( (\tilde{Y}(f)/\tilde{Y}(0))_z \) are reflection positive. For local interactions the latter are restrictions of a reflection-positive generating functional, see [10], [4], in the sense that

\[
(\tilde{Y}(f)/\tilde{Y}(0))_z = \lim_{n \to \infty} \int_{D'} e^{\phi(\mathbb{R}^d)} = \int_{D'} e^{\phi(z-\Delta + gn)} F_{\Lambda}(\phi) d\mu_G(\phi)/\tilde{Z}(0),
\]

for a sequence \( g_n \) converging to \( \delta_z \otimes f \) in \( H^{-1} \). In order that \( \tilde{Z}(f)/\tilde{Z}(0) \) be reflection positive, we need that the correcting factor \( \exp(-\text{Corr}(z)f,f) \) be reflection positive. But the covariances in \( \text{Corr}(z) \) are given by the inverse Fourier transforms of \( |k|^{2j} \) which equal \( \text{const}(j)(-1)^{2j}\delta^{(2j)}(|x|) \), see [8]. According to [10, Theorem 6.2.2] the generating functional of a Gaussian measure \( d\mu_C \) is positive if the covariance satisfies \( (\theta f,Cf) \geq 0 \) for all \( f \) supported at positive \( x_1 \). Using an approximation argument it is sufficient to check this property for functions of the form \( f = f_r f_{\varphi} \), with \( f_r \in C^\infty(\mathbb{R}_0) \) and \( f_{\varphi} \in C^\infty(S^{d-1}_+), \) where \( S^{d-1}_+ = \{ x \in \mathbb{R}^d : |x| = 1, x_1 > 0 \} \). In this case \( (\theta f_r f_{\varphi},\delta^{(2j)}(f_r f_{\varphi})) = ((\theta f_r)^{(j)} f_{\varphi}, f_r^{(j)} f_{\varphi}) = 0 \), hence the claim. It follows that reflection positivity holds also for \( \Lambda \nearrow \mathbb{H}^{d+1} \).

The basic implication of the AdS/CFT correspondence is that covariance of the bulk functional integral translates into a conformal invariance on the boundary. On geometrical grounds the orientation preserving isometry group \( \text{Iso}_+(\mathbb{H}^{d+1}) \) acts by conformal transformations on the boundary, see [13]. This means in particular that

\[
g^*dx = \det \left( \frac{\partial g(x)}{\partial x} \right) dx,
\]

where \( dx \) is the volume form on \( \mathbb{R}^d \) and \( \partial g(x)/\partial x \) denotes the Jacobi matrix. In order to take into consideration the transformations (26), we regard our functionals as functions of \( d \)-forms \( \omega \) with compact support, i.e., \( \omega = f dx \) with \( f \in C_0^\infty(\mathbb{R}^d) \). Suppose that \( \tilde{Z}_{\lim}(\omega) : = \lim_{\Lambda \nearrow \mathbb{H}^{d+1}} \tilde{Z}(f)/\tilde{Z}(0) \) exists uniquely, then conformal invariance means the property that

\[
\tilde{Z}_{\lim}(g\omega) = \tilde{Z}_{\lim}(\lambda_g \cdot \omega),
\]

(27)
with action $g \omega := g^{-1} \omega$ and density $\lambda_g(x) = \left( \det \left( \frac{\partial g(x)}{\partial x} \right) \right)^{-\Delta_+}$. For (27) to hold the bulk-to-boundary propagator has to fulfill the following intertwining property.

**Lemma 2** For $g \in \text{Iso}_+^{+}(\mathbb{H}^{d+1})$ let $g(z, x) = (z_g(z, x), x_g(z, x)) \equiv (z_g, x_g)$ denote the action of $g$. Then with $g(x) = \lim_{z \to 0} x_g(z, x)$ we have

$$H_+(g(z, x); x') = \left( \det \left( \frac{\partial g^{-1}(x')}{\partial x'} \right) \right)^{-\Delta_+} H_+(z, x; g^{-1}(x')).$$

**Proof.** We note that $H_+(z, x; x') = \lim_{z' \to 0} z' - \Delta + \left( \frac{z z'}{\left( z - z' \right)^2 + \left( x - x' \right)^2} \right)^{\Delta_+}$ Now, $rac{z z'}{(z-z')^2+(x-x')^2}$ is invariant with respect to isometries and therefore

$$H_+(g(z, x); x') = \lim_{z' \to 0} z' - \Delta + \left( \frac{z z'}{\left( z - z' \right)^2 + \left( x - x' \right)^2} \right)^{\Delta_+}.$$

In order to see the effect of the transformation $g^{-1}$ we use its action on the isometric model of $\mathbb{H}^{d+1}$ given by

$$L^{d+1} := \{ \zeta \in \mathbb{M}^{d+1,1} | \zeta_1^2 + \cdots + \zeta_{d+1}^2 - \zeta_{d+2}^2 = -1, \zeta_{d+2} > 0 \},$$

equipped with the metric induced from Minkowski space $\mathbb{M}^{d+1,1}$ with metric $d\zeta_1^2 + \cdots + d\zeta_{d+1}^2 - d\zeta_{d+2}^2$. For $L^{d+1}$ the isometry group is $SO_0(d+1, 1)$ and the isometry map $\vartheta : \mathbb{H}^{d+1} \to L^{d+1}$ is given by

$$\zeta_i = \frac{x_i}{z}, \quad 1 \leq i \leq d,$$

$$\zeta_{d+1} = -\frac{1}{2z}(z^2 + x^2 - 1), \quad \zeta_{d+2} = \frac{1}{2z}(z^2 + x^2 + 1).$$

with inverse

$$z = \frac{1}{\zeta_{d+1} + \zeta_{d+2}}, \quad x_i = \frac{\zeta_i}{\zeta_{d+1} + \zeta_{d+2}}.$$ 

Thus, an arbitrary isometry on $\mathbb{H}^{d+1}$ can be cast in the form

$$\vartheta^{-1} \circ g \circ \vartheta, \quad g \in SO_0(d+1, 1).$$
Using this fact, one easily shows that $\frac{\partial g^{-1}}{\partial x'}$ and $\frac{\partial g^{-1}}{\partial z'}$ tend to zero as $z' \to 0$, whereas $\frac{\partial g^{-1}}{\partial x'} \to \partial g^{-1}(x')/\partial x'$ and $\partial g^{-1} / \partial z' \sim z'_1$. Moreover, invariance of the volume measure $z^{-d-1} \, dz \, dx$ implies

$$z'^{d-1} = (z'_g)^{d-1} \det \left( \frac{\partial g^{-1}(z', x')}{\partial (z', x')} \right).$$

Combining all this, gives

$$\lim_{z' \to 0} \left( \frac{z'_g}{z} \right) = \left( \det \left( \frac{\partial g^{-1}(x')}{\partial x'} \right) \right)^{-\frac{1}{d}},$$

which shows the statement of the lemma. □

We have thus completed the proof of AdS/CFT for Euclidean quantum fields up to the infra-red problem $\Lambda \nearrow H^{d+1}$. Due to the different nature of source terms, which include bulk to boundary propagators of rather slow decay, this infra-red problem is different from, and probably much harder as, the related one in [10] where sources are rapidly decaying. We will come back to this point elsewhere.

### A Divergencies in $\lim_{z \to 0} z^{-\Delta_+} (G_{+f_z}, f_z)$

In investigating this limit we shall use the following integral representation, see [2], [14],

$$G(z, x; z', y) = (zz')^{d/2} \frac{1}{(2\pi)^{d/2}} \int_0^\infty \int_{\mathbb{R}^d} \frac{1}{\omega^2 + |k|^2} e^{ik(x-y)} dk J_\nu(z\omega) J_\nu(z'\omega) \omega d\omega$$

$$= (zz')^{d/2} \frac{1}{(2\pi)^{d/2}} \int_0^\infty C_\omega (x-y) J_\nu(z\omega) J_\nu(z'\omega) \omega d\omega,$$

where $C_\omega$ is the integral kernel of $(-\Delta + \omega^2)^{-1}$ in $\mathbb{R}^d$. In addition, for $\text{Re} \nu > -\frac{1}{2}$, $J_\nu$ can be represented as

$$J_\nu(u) = \frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} u^\nu \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(ut) dt.$$

Then with $f \in \mathcal{S}(\mathbb{R}^d)$ we get

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z, x; z, y) f(x) f(y) \, dx \, dy = \frac{z^{d+2\nu}}{(2\pi)^{d/2}} \left( \frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \right)^2 \times$$
\[
\int_0^\infty \int_{\mathbb{R}^d} \frac{|\hat{f}(k)|^2}{\omega^2 + |k|^2} dk \left( \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(\omega t) dt \right)^2 \omega^{2\nu + 1} d\omega.
\]

Employing the geometric series expansion

\[
\frac{1}{\omega^2 + |k|^2} = \frac{1}{\omega^2} \left( \frac{1}{1 + |k|^2/\omega^2} \right) = \sum_{j=0}^{[\nu]} (-1)^j \frac{|k|^{2j}}{\omega^{2j+2}} + \frac{(-|k|^2/\omega^2)^{[\nu]+1}}{\omega^2 + |k|^2}
\]

we obtain

\[
z^{-d-2\nu} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z, x; z, y) f(x) f(y) dx dy = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \right)^2 \times
\]

\[
\left\{ \sum_{j=0}^{[\nu]} (-1)^j \int_0^{\infty} \omega^{2(\nu-j)-1} \left( \int_0^1 \cos(\omega t)(1 - t^2)^{\nu - \frac{1}{2}} dt \right)^2 d\omega \int_{\mathbb{R}^d} |\hat{f}(k)|^2 |k|^{2j} dk
\]

\[
+ (-1)^{[\nu]+1} \int_0^{\infty} \int_{\mathbb{R}^d} \frac{\omega^2 + |k|^2}{|k|^2}\omega^{2(\nu-[\nu]-1)} \left( \int_0^1 \cos(\omega t)(1 - t^2)^{\nu - \frac{1}{2}} dt \right)^2 d\omega \right\}.
\]

On the one hand, the terms

\[
\int_0^{\infty} \omega^{2(\nu-j)-1} \left( \int_0^1 \cos(\omega t)(1 - t^2)^{\nu - \frac{1}{2}} dt \right)^2 d\omega
\]

\[
= z^{-2(\nu-j)} \int_0^{\infty} \left( \int_0^1 \cos(\omega t)(1 - t^2)^{\nu - \frac{1}{2}} dt \right)^2 \omega^{2(\nu-j)-1} d\omega =: z^{-2(\nu-j)} a_j
\]

diverge as \(z \to 0\). On the other hand, using dominated convergence, one can show that the last term in (29), for \(z \to 0\) converges to constant times

\[
\left( \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} dt \right)^2 \int_{\mathbb{R}^d} \frac{|\hat{f}(k)|^2 |k|^{2[\nu]+2} d\omega}{|k|^2 + \omega^2} \omega^{2(\nu-[\nu])-1} d\omega.
\]

Using the formula

\[
\int_0^1 t^{2a+1} (1 - t^2)^b dt = \frac{1}{2} \left( \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \right)
\]
with \(a = -\frac{1}{2}, b = \nu - \frac{1}{2}\) we thus get

\[
\lim_{z \to 0} \left\{ z^{-d-2\nu} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z, x; z, y) f(x) f(y) dx dy - \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{2^{1-\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \right)^2 \sum_{j=0}^{[\nu]} z^{-2(\nu-j)} (-1)^j a_j \int_{\mathbb{R}^d} |\hat{f}(k)|^2 |k|^{2j} dk \right\} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{2^{-\nu} \Gamma(\frac{1}{2})}{\sqrt{\pi} \Gamma(\nu + 1)} \right)^2 \left( -1 \right)^{[\nu]+1} \int_{0}^{\infty} \int_{\mathbb{R}^d} \frac{|\hat{f}(k)|^2}{\omega^2 + |k|^2} |k|^{2[\nu]+2} dk \omega^{2(\nu-[\nu])-1} d\omega. \tag{30}
\]

Let us perform the \(\omega\)-integration in (30) first. With the aid of

\[
\int_{0}^{\infty} \frac{x^{a-1}}{1 + x^b} dx = \frac{\pi}{b \sin(a\pi/b)}, \quad 0 < a < b,
\]

where \(a = 2(\nu - [\nu]), b = 2\), we get for the integral

\[
\int_{\mathbb{R}^d} \frac{\pi |k|^{2(\nu-[\nu])-2}}{2 \sin \left( \frac{2(\nu-[\nu])\pi}{2} \right)} |\hat{f}(k)|^2 |k|^{2\nu+2} dk = (-1)^{[\nu]} \frac{\pi}{2 \sin \nu \pi} \int_{\mathbb{R}^d} |\hat{f}(k)||k|^{2\nu} dk,
\]

and therefore (30) simplifies to

\[
- \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{\pi}{2 \sin(\nu \pi)} \left( \frac{1}{2 \nu \Gamma(\nu + 1)} \right)^2 \int_{\mathbb{R}^d} |\hat{f}(k)|^2 |k|^{2\nu} dk. \tag{31}
\]

Comparing (31) with (7) and exploiting relations \(\Gamma(\nu) \Gamma(1 - \nu) = \pi / \sin(\nu \pi)\) and \(\Gamma(1 - \nu) = -\nu \Gamma(-\nu)\) we see that the latter expression equals \((f, \alpha_+)\).

**B Fourier and spherical Fourier transform on \(\mathbb{H}^d\)**

Hyperbolic spaces belong to the class of Riemannian symmetric spaces which can be represented in the form \(X = G/K\) with \(G\) a noncompact semisimple Lie group and \(K\) a maximal compact subgroup, i.e. \(\mathbb{H}^{d+1} \simeq SO_0(d+1,1)/SO_0(d+1)\). For these type of spaces there is an analogue of the Fourier transform in \(\mathbb{R}^d\). Let \(g = \mathfrak{g} \oplus \mathfrak{p}\) be the Cartan decomposition of the Lie algebra \(g\) of \(G\). Then we have the following Iwasawa decomposition \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}\), where \(\mathfrak{a}\) is a maximal abelian subspace of \(\mathfrak{p}\), \(\mathfrak{n} := \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha\) with \(\Sigma_+\) being a choice of positive roots with respect to \((\mathfrak{g}, \mathfrak{a})\).
The norm induced from the Killing-form on \( p \) will be denoted by \( \| \cdot \| \). There is a corresponding Iwasawa decomposition for the Lie group \( G = KAN = NAK \) and every \( g \in G \) can be written as \( g = k(g) \exp H(g)n(g) \) with unique elements \( k(g) \in K, H(g) \in a, n(g) \in N \). Let \( M \) denote the centralizer of \( A \) in \( K \), \( B := K/M \) and let \( A(x,b) \in a \) be the vector \( A(x,b) := A(k^{-1}g) \) for \( x = gK \in X \) and \( b = kM \in B \). The Fourier transform of a function \( f \in C_0^\infty(X) \) is now defined as

\[
\hat{f}(\lambda, b) := \int_X f(x) e^{(-i\lambda + \frac{1}{2})(x,b)} dx, \quad \lambda \in \mathfrak{a}_C^*, b \in B,
\]

where \( \rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha, m_\alpha = \text{dim} \mathfrak{a}_\alpha \). Let us have a closer look at the space \( \mathbb{H}^2 \) which can be represented as the open disk \( D := \{ w \in \mathbb{C} : |w| < 1 \} \) equipped with the Riemannian metric \( g_D = 4(1 - |w|^2)^{-2}(dw_1^2 + dw_2^2) \), which in turn is diffeomorphic to the homogenous space \( G/K \) where the Lie group

\[
G = SU(1,1) = \left\{ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}
\]

acts on \( D \) by

\[
g \cdot w = \frac{aw + b}{bw + \bar{a}}
\]

and the isotropy group of 0 is \( K = SO(2) \). In this picture the Fourier transform of a function on \( D \) is given by

\[
\hat{f}(\lambda, b) = \int_D f(w) e^{(-i\lambda + \frac{1}{2})(w,b)} d\sigma(w), \quad \lambda \in \mathbb{C}, b \in \partial D = B,
\]

where \( d\sigma \) is the volume form related to \( g_D \) and \( (w,b) \) denotes the geodesic distance from 0 to the circle which passes through \( w \), and at \( b \), is tangential to the boundary \( \partial D \) of \( D \). The spherical Fourier transform is defined by

\[
\hat{f}(\lambda) = \int_D f(w) \phi_{-\lambda}(w) d\sigma(w),
\]

where \( \phi_{\lambda} \) is the spherical function

\[
\phi_{\lambda}(w) = \int_{\partial D} e^{(i\lambda + \frac{1}{2})(w,b)} db.
\]

In the general case spherical functions are given by \( \phi_{\lambda}(g) = \int_K e^{(i\lambda + \rho)A(k^{-1}g)} dk \) and obey \( \phi_{\lambda}(e) = 1 \) and \( -\Delta \phi_{\lambda} = (\|\lambda\|^2 + \|\rho\|^2) \phi_{\lambda} \). We notice that for radial functions
f, i.e. \( f(w) = f(|w|) \), the transform (33) may be written as

\[
\hat{f}(\lambda) = 2\pi \int_0^\infty f(\tanh \frac{r}{2}) \phi_\lambda(\tanh \frac{r}{2}) \sinh r dr, \quad (r = d(0, w)),
\]

Moreover, since \( e^{(w, b)} = \frac{1-|w|^2}{|w-b|^2} \), with the substitutions \( w = \tanh \frac{r}{2}, b = e^{i\theta} \) we may write

\[
\phi_\lambda(\tanh \frac{r}{2}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cosh r - \sinh r \cos \theta)^{(i\lambda+\frac{1}{2})} d\theta,
\]

and setting further \( u = \tanh \frac{1}{2} \theta, \frac{1}{2} d\theta = (1 + u^2)^{-1} du \), we get

\[
\phi_\lambda(\tanh \frac{r}{2}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \cosh r + \sinh r \frac{1-u^2}{1+u^2} \right)^{(i\lambda-\frac{1}{2})} \frac{du}{1+u^2}.
\]

(34)

Because of the group structure we may consider the convolution

\[
(f_1 \ast f_2)(g \cdot o) := \int_G f_1(h \cdot o)f_2(h^{-1}g \cdot o)dh, \quad o = eK.
\]

For radial functions \( f_1, f_2 \) one gets

\[
\widehat{(f_1 \ast f_2)}(\lambda) = \hat{f}_1(\lambda)\hat{f}_2(\lambda),
\]

whenever both sides exist. We also need the following estimate

\[
\|((\hat{\chi}_\varepsilon - \hat{\chi}_{\varepsilon'})\lambda^2 + \frac{1}{4} + m^2)^{-\frac{1}{2}}\|_\infty \leq O(1)(\varepsilon \wedge \varepsilon')^{-\frac{1}{2}}.
\]

To see this let us regard \( \hat{\chi}_\varepsilon \) as a function of \( \frac{\lambda}{\varepsilon} \) by setting \( g_\varepsilon(\lambda/\varepsilon) := \hat{\chi}_\varepsilon(\lambda) \). Then

\[
\frac{d}{d(\lambda/\varepsilon)}(g_\varepsilon) = \varepsilon \frac{d}{d\lambda}(\hat{\chi}_\varepsilon) \quad \text{and with the aid of (34) and the substitution } y = \varepsilon r \text{ we get that}
\]

\[
|\hat{\chi}_\varepsilon(\lambda) - \hat{\chi}_{\varepsilon'}(\lambda)| \leq O(1) \left( \frac{|\lambda|}{\varepsilon \wedge \varepsilon'} \right)^{\frac{3}{2}}
\]

for \( |\lambda| \leq \varepsilon \wedge \varepsilon' \) and

\[
|\hat{\chi}_\varepsilon(\lambda) - \hat{\chi}_{\varepsilon'}(\lambda)| \leq O(1)
\]

for \( |\lambda| > \varepsilon \wedge \varepsilon' \).
C Sobolev spaces

In this section we introduce Sobolev spaces on hyperbolic spaces. For \( \beta \geq 0 \) let us define the Sobolev space of order \( \beta \) as in [18]

\[
H^\beta := \left\{ u \in L^2(\mathbb{H}^{d+1}) : u = (-\Delta + m^2)^{-\frac{\beta}{2}} v, \ v \in L^2(\mathbb{H}^{d+1}) \right\}
\]

with \( \| u \|_{H^\beta} := \| v \|_{L^2(\mathbb{H}^{d+1})} \). For \( \beta < 0 \) we define

\[
H^\beta := \left\{ u \in D' : u = (-\Delta + m^2)^k v, \ v \in H^{2k+\beta} \right\}
\]

with \( k \) such that \( 2k + \beta > 0 \), and norm \( \| u \|_{H^\beta} := \| v \|_{H^{2k+\beta}} \).

The spaces \( H^\beta \) can be identified with the completion of \( C_\infty^0(\mathbb{H}^{d+1}) \) in the norm \( \| f \|_{H^\beta} = \| (-\Delta + m^2)^\beta f \|_{L^2(\mathbb{H}^{d+1})} \).

In section 3 we have used the distribution \( f_z = \delta_z \otimes f \) with \( f \in S(\mathbb{R}^d) \). Using the explicit expression (28) and a proper smoothing with an approximate unit one sees that \( f_z \in H^{-1} \).

A second equivalent definition of Sobolev spaces uses local coordinates, see [15]. For this one first considers the space \( H^\beta_0 \) of distributions which are supported in a ball of fixed radius \( r, B(o,r) \), around some fixed point \( o \) equipped with geodesic coordinates and defines the norm \( \| u \|_{H^\beta_0} \) as the pull-back of the \( H^\beta(\mathbb{R}^{d+1}) \) norm in the chosen coordinates. For another \( a \in \mathbb{H}^{d+1} \) and distribution \( f \) supported in \( B(a,r) \) one defines \( \| f \|_{H^\beta_0} := \| f \circ g \|_{H^\beta_0} \) where \( g \) is an isometry with \( g(o) = a \). Then points \( (a_k)_{k \in \mathbb{N}} \) are chosen in order to obtain a locally finite covering by the balls \( B(a_k, r) \). Finally, employing a partition of unity \( (\varphi_k)_{k \in \mathbb{N}} \) w.r.t. the balls \( B(a_k, r) \) one says that \( u \in H^\beta(\mathbb{H}^{d+1}) \) if

\[
\sum_k \| \varphi_k u \|_{H^\beta_{a_k}} < \infty.
\]

A third definition can be given using Fourier transforms, where we follow [7]. For this we define the Schwartz space \( S(\mathbb{H}^{d+1}) = S(X) \), consisting of complex-valued \( C^\infty \)-functions \( f \) on \( X \) satisfying

\[
\tau_{D,m}(f) = \sup_{g \in G} (1 + |g|)^m \phi_0(g)^{-1} |Df(g)| < \infty,
\]

for all \( m \in \mathbb{N}_0 \) and differential operators \( D \) invariant under the left action of \( G \). The norm of \( g \) is defined as \( |g| = |\exp Xk| = \| X \|, \ X \in p, k \in K \). The space \( S(X) \)
becomes a Fréchet space when topologized by means of the seminorms $\tau_{D,m}$. Let $\mathcal{S}(\mathfrak{a}^* \times K/M)$ be the complex-valued $C^\infty$-functions on $\mathfrak{a}^* \times K/M$ such that

$$\nu_{E,u,r}(f) = \sup_{\lambda, kM} (1 + \|\lambda\|^r \|Ef\|_{(\lambda, kM)}) < \infty,$$

for all differential operators $E$ on $\mathfrak{a}^*$, $u$ invariant on $K/M$ and $r \in \mathbb{N}_0$. With these seminorms $\mathcal{S}(\mathfrak{a}^* \times K/M)$ becomes a Fréchet space. The Fourier transform (32) establishes a topological isomorphism between $\mathcal{S}(X)$ and $\mathcal{S}(\mathfrak{a}^* \times B)_W = \mathcal{S}(\mathfrak{a}^*_+ \times B)$, where the subscript $W$ denotes the quotient space under the action of the Weyl group on $\mathfrak{a}^*$. Moreover, the Fourier transform extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}^*_+ \times B, |c(\lambda)|^{-2}d\lambda db)$, where $c(\lambda)$ is the Harish-Chandra $c$-function. From the property $(-\hat{\Delta} f)(\lambda, b) = (\|\lambda\|^2 + \|\rho\|^2)\hat{f}(\lambda, b)$ we see that $H^\beta(X)$ equals the space of $u \in \mathcal{S}(X)'$ such that

$$\int_{\mathfrak{a}^*_+} \int_B |\hat{u}|^2 (\|\lambda\|^2 + \|\rho\|^2 + m^2)^\beta |c(\lambda)|^{-2}d\lambda db < \infty.$$

It should be noted that in the case $\mathbb{H}^2$ we have $\mathfrak{a}^*_+ = \mathbb{R}_+$, $\|\rho\|^2 = \frac{1}{4}$ and $|c(\lambda)|^{-2} = (2\pi)^{-1} \lambda \tanh \pi \lambda$.

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