Asymptotic Expansion of Perturbative Chern-Simons Theory via Wiener Space

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no. 322

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereiches 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

Bonn, Februar 2007
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Abstract

The Chern-Simons integral is divided into a sum of finitely many resp: infinitely many contributions. A mathematical meaning is given to the "finite part" and an asymptotic estimate of the other part is given, using the abstract Wiener space setting. The latter takes the form of an asymptotic expansion in powers of a charge, using the infinite dimensional Malliavin-Taniguchi formula for a change of variables.

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1 Introduction

After seminal work by Schwarz [39] and Witten [42], various papers have been published concerning the mathematical establishment of relations between Chern-Simons integrals, topological invariants of 3-manifolds and knot invariants ([1, 2, 3, 6, 7, 9, 10, 11, 12, 13, 14, 20, 23, 28]).

2000 Mathematics Subject Classification. Primary 57R56; Secondary 28C70, 57M27.

Key words and phrases. Chern-Simons integral, asymptotic expansion, abstract Wiener space, Malliavin-Taniguchi formula, linking number.

* Partly supported by the Grant-in-Aid for Scientific Research (C) of the Japan Society for the Promotion of Science, No. 17540124.
The first author and Schäfer have constructed the integral and the corresponding invariants (linking numbers) for the abelian case by the technique of infinite dimensional Fresnel integrals [1]. A similar representation was given subsequently by Leukert and Schäfer, using methods of white noise analysis. The extension to the representation to the non-abelian case for \( R^3 \) (resp. \( S^2 \times R \)) has been provided by using again methods of white noise analysis by the first author and Sengupta [2], (who used a setting similar to Fröhlich-King [20]). Recently, along this line, Hahn has defined rigorously and computed the Wilson loop variables for Non-Abelian Chern-Simons theory [24, 25]. See also the survey papers [7] and [5]. Hahn subsequently managed to extend the results to the case of \( S^2 \times S^1 \), using “torus gauge” [26]. In the abelian case, some alternative methods for defining rigorously the integrals, via determinants and limits of simplicial approximation [1, 37] have been developed.

Guadagnini, Martellini, Mintchev and Bar-Natan [23, 11, 13] (see also, e.g., [6, 9, 10, 12, 14] for further development) developed heuristically an asymptotic expansion of the Chern-Simons integral in powers of the relevant charge parameter, getting the relations with other topological invariants (Vassiliev invariants). First rigorous mathematical results are in [33, 34], who used methods of stochastic analysis, in particular a formula by Malliavin-Taniguchi [32] on the change of variables in infinite dimensional analysis. The present paper also uses such methods and an idea of Itô concerning Feynman path integrals [27].

Let \( M \) be a compact oriented 3-dimensional manifold, \( \mathcal{G} \) a skew symmetric matrix algebra, which is a linear space with the inner product \((X, Y) = \text{Tr} XY^* = -\text{Tr} XY\), where \( \text{Tr} \) denotes the trace. Let \( A \) be the collection of all \( \mathcal{G} \)-valued 1-forms. Let \( F \) be a complex-valued map on \( A \). Then the Chern-Simons integral of \( F(A) \) is heuristically equal to

\[
\int_A F(A) e^{L(A)} D(A),
\]

where

\[
L(A) = -\frac{ik}{4\pi} \int_M \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}.
\]

The parameter \( k \), called charge, is a positive integer. \( D(A) \) is a heuristic “flat measure”.

Let us expand heuristically in powers of the \( \frac{2}{3} \)-part under above integral, separating it in a “finite part” which consists of the terms up to order \( N \) in \( k \) and the rest:

\[
\frac{1}{Z} \int_A F(A) e^{L(A)} D(A)
= \frac{1}{Z} \sum_{r=0}^N \int_A F(A) \exp \left[ -\frac{ik}{4\pi} \int_M \text{Tr} A \wedge dA \right] \left\{ (-\frac{ik}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A)^r / r! \right\} D(A)
+ \frac{1}{Z} \int_A F(A) \exp \left[ -\frac{ik}{4\pi} \int_M \text{Tr} A \wedge dA \right] \left\{ \sum_{r=N+1}^{\infty} (-\frac{ik}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A)^r / r! \right\} D(A).
\]

\( Z \) denotes the heuristic normalization constant i.e.

\[
Z = \int_A \exp \left[ -\frac{ik}{4\pi} \int_M \text{Tr} A \wedge dA \right] D(A).
\]
Based on the perturbative formulation of the Chern-Simons integral [9, 13, 23] and the idea of Itô [27] for the Feynman path integral, we will give a meaning to the finite part and an asymptotic estimate of the “infinite part”, letting \( k \) tend to \(+\infty\), in an abstract Wiener space setting by using a formula for a change of variables in such a space (Theorems 1 and 2). We shall give the concrete calculation of the coefficients of the expansion in a further paper.

Throughout this paper, as integrand we take the typical example of gauge invariant observables (holonomy operators) such that \( F(A) = \prod_{j=1}^{s} \text{Tr} \ e^{i \gamma_j} A \), where \( \gamma_j, j = 1, 2, ..., s \) are closed oriented loops, (these are called Wilson loop variables in the physics literature).

In Section 2, we shall give the definitions and results. To derive rigorously the invariants first derived heuristically by Witten, we consider the \( \epsilon \)-regularization of the Wilson line, following [42]. In doing this, we also need a regularization of a relevant determinant and this uses methods of the theory of super fields.

In Section 3, we give the proofs of the results stated in Section 2. In these proofs an intermediate \( S \)-operator enters, following a technique by [27].

In the sequel \( \sqrt{z} \) denotes the branch of the root of \( z \in \mathbb{C} \) where \(-\pi < \arg \sqrt{z} < \pi\).

2 Definitions and Results

We consider the perturbative formulation of the Chern-Simons integral [9, 13, 23] and follow the method of super fields.

Let \( A_0 \) be a critical point of the “Chern-Simons action” \( L(A) \), i.e.

\[
dA_0 + A_0 \wedge A_0 = 0.
\]

Let us recall the relations between external fields and the BRS operator \( \delta \) (see e.g. [9, 13, 23] for background concepts). Let \( \phi \) be a bosonic 3-form, \( c \) a fermionic 0-form and \( \hat{c} \) a fermionic 3-form. The BRS operator laws are

\[
\delta A = -D_A c, \quad \delta c = \frac{1}{2} [c, c],
\]

\[
\delta \hat{c} = i \phi, \quad \delta \phi = 0,
\]

where \( D_A = d + [A, \cdot] \) and \( d_{A_0} \) is the covariant exterior derivative such that \( d_{A_0} = d + [A_0, \cdot] \). Let \( E_u, u = 1, 2, ..., d \) be the orthonormal basis of \( \mathcal{G} \), \( A = \sum_{u=1}^{d} A^u \cdot E_u \) and \( B = \sum_{u=1}^{d} B^u \cdot E_u \). We set \( [A, B] = \sum_{i,j} [E_i E_j - E_j E_i] A^i \wedge B^j \).

For simplicity, we assume

**Assumption 1.** \( A_0 \) is isolated and irreducible, i.e. the whole de Rham cohomology vanishes, i.e.

\[
H^*(M, d_{A_0}) = 0.
\]

(see [9, 13]).
For the rest of this paper we will fix an auxiliary Riemannian metric $g$ on $M$. By $\ast$ we will denote the corresponding Hodge $\ast$-operator and by $(d_{A_0})^\ast$ the adjoint operator of $d_{A_0}$ (with respect to the scalar product on $\mathcal{A}$ which is induced by $g$).

Then we fix the Lorentz gauge such that $(d_{A_0})^\ast A = 0$. Define

$$V(A) = \frac{k}{2\pi} \int_M \text{Tr} \hat{c}^\ast d_{A_0} \ast A,$$

Let $c' = \sqrt{k/2\pi}c$ and $\hat{c}' = \ast \sqrt{k/2\pi}\hat{c}$. (see [23, p. 578]). Since

$$L(A_0 + A) = L(A_0) - \frac{ik}{4\pi} \int_M \text{Tr} \left\{ A \wedge d_{A_0} A + \frac{2}{3} A \wedge A \wedge A \right\}$$

and

$$\delta V(A) = \frac{k}{2\pi} \int_M \text{Tr} \left\{ i\phi \ast d_{A_0} \ast A - \hat{c} \ast d_{A_0} \ast DA\hat{c}' \right\},$$

the Lorentz gauge fixed path integral form of the heuristic Chern-Simons integral is given by

$$\frac{1}{Z_k} \int_A \int_{\Phi} \int_{\hat{C}'} \int_{C'} D(A)D(\phi)D(\hat{c}')D(\hat{c}')F(A_0 + A) \exp \left[ L(A_0 + A) - \delta V(A) \right]$$

$$= \frac{1}{Z_k} \int_A \int_{\Phi} \int_{\hat{C}'} \int_{C'} D(A)D(\phi)D(\hat{c}')D(\hat{c}')F(A_0 + A) \exp \left[ L(A_0) \right]$$

$$\times \exp \left[ ik((A, \phi), Q_{A_0}(A, \phi))_+ - \frac{ik}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A + \int_M \text{Tr} \hat{c}' d_{A_0} \ast DA\hat{c}' \right],$$

where heuristically

$$Z_k = \int_A \int_{\Phi} \int_{\hat{C}'} \int_{C'} D(A)D(\phi)D(\hat{c}')D(\hat{c}')$$

$$\exp \left[ L(A_0) + ik((A, \phi), Q_{A_0}(A, \phi))_+ + \int_M \text{Tr} \hat{c}' d_{A_0} \ast d_{A_0} \hat{c}' \right].$$

Here $\Omega^n$ is the space of $G$-valued n-forms with the inner product

$$(\omega_1, \omega_2)_0 = \int_M - \text{Tr} \omega_1 \wedge \ast \omega_2$$

and

$$( (A, \phi), (B, \varphi))_+ = (A, B)_0 + (\phi, \varphi)_0$$

denotes the inner product of the Hilbert space $L^2(\Omega^+) = L^2(\Omega^1 \oplus \Omega^3)$.

We shall set

$$\| \cdot \|_0 = \sqrt{\langle \cdot, \cdot \rangle_0} \quad \text{and} \quad \| \cdot \|_+ = \sqrt{\langle \cdot, \cdot \rangle_+}.$$
Further $Q_{A_0} = \frac{1}{4\pi}(\ast d_{A_0} + d_{A_0} \ast)J$, where $J\phi = -\phi$ if $\phi$ is a zero or a three form and $J\phi = \phi$ if $\phi$ is a one or a two form.

We note that the benefit of the above formulation is that the Lorentz gauge constraint

$(d_{A_0})^* A = 0$ is released in the domain of the path integral representation.

Set $\Delta_0 = \ast d_{A_0} \ast d_{A_0}$, i.e. $\Delta_0$ is the Laplacian in $L^2(\Omega^0)$. Let \{\nu_j, \xi_j, j = 1, 2, \cdots \} be the eigensystem of $\Delta_0$ in $L^2(\Omega^0)$,

\[ \hat{c'} = \sum_{j=1}^{\infty} \xi_j \hat{c}'_j \quad \text{and} \quad c' = \sum_{j=1}^{\infty} \xi_j c'_j. \]

Since

\[
\int_M \text{Tr} \circlearrowleft c' d_{A_0} \ast D_A c' = -(\hat{c'}, \ast d_{A_0} \ast D_A c')_0
\]
\[
= -(\hat{c'}, \ast d_{A_0} \ast d_{A_0} c')_0 - (d_{A_0} \hat{c'}, [A, c'])_0,
\]

we have heuristically

\[
\int_{C'} \int_{C'} D(\hat{c'})D(c') \exp \left[ \int_M \text{Tr} \circlearrowleft c' d_{A_0} \ast D_A c' \right] = \det \ast d_{A_0} \ast D_A
\]

\[
= \prod_{j=1}^{\infty} \nu_j \cdot \begin{vmatrix}
\nu_1 & a_{12} & a_{13} & \cdots & a_{1n} \\
\nu_2 & a_{22} & a_{23} & \cdots & a_{2n} \\
\nu_3 & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\nu_n & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{vmatrix}
\]

\[
= \det \Delta_0 \det R \ast d_{A_0} \ast D_A,
\]

where

\[ a_{ii} = 1 - \frac{1}{\nu_i} \int_M \text{Tr} d_{A_0} \xi_i \wedge [A, \xi_i] \quad , i = 1, 2, \cdots \]

and

\[ a_{ij} = -\frac{1}{\nu_i} \int_M \text{Tr} d_{A_0} \xi_i \wedge [A, \xi_j] \quad , i, j = 1, 2, \cdots \]

Then considering

\[
\int_{C'} \int_{C'} D(\hat{c'})D(c') \exp \left[ \int_M \text{Tr} \circlearrowleft c' d_{A_0} \ast d_{A_0} c' \right] = \det \Delta_0,
\]

we arrive at the perturbative heuristic formulation of the normalized Chern-Simons integral (2.2) such that

\[
\frac{1}{Z_{1,k}} \int_A \int_{\mathcal{D}} D(A)D(\phi) F(A_0 + A)
\]
\[
\times \det R \ast d_{A_0} \ast D_A \exp \left[ ik((A, \phi), Q_{A_0}(A, \phi))_+ - \frac{ik}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A \right],
\]

(2.4)
where heuristically

\[(2.5) \quad Z_{1,k} = \int_A \int_{\Phi} \mathcal{D}(A) \mathcal{D}(\phi) \exp \left[ ik((A, \phi), Q_{A_0}(A, \phi))_+ \right]. \]

2.1 Regularized Determinant

To provide mathematical meaning to this, we have first of all to regularize the formal determinant \( \det_R \ast d_{A_0} \ast D_A \).

Since \( Q_{A_0} \) is self-adjoint and elliptic, \( Q_{A_0} \) has pure point spectrum \([8]\). Let \( \mu_i, e_i = (e_i^A = \sum_{u=1}^d c_{i,u}^A \cdot E_u, e_i^\phi = \sum_{u=1}^d c_{i,u}^\phi \cdot E_u), i = 1, 2, \cdots \) be the eigenvalues resp. eigenvectors of \( Q_{A_0} \) in \( L^2(\Omega_+) \). By the Assumption 1, the eigenvectors form a CONS (Complete Orthonormal System) of \( L^2(\Omega_+) \).

Define

\[ a_{R,ij}^\ell = -\frac{1}{\nu_i} \int_M \text{Tr} r_i d_{A_0} \xi_i \wedge [e_i^A, r_j \xi_j]. \]

Since \( M \) is compact, we can choose appropriate real numbers \( r_j > 0, j = 1, 2, \ldots \) such that

\[(2.6) \quad \sum_{i,j} | a_{R,ij}^\ell | < +\infty, \]

and

\[ \sum_{j=1}^{\infty} | r_j | < +\infty. \]

Then for smooth \( A \in \mathcal{A} \), we consider instead of \( \det_R \ast d_{A_0} \ast D_A \), the regularized determinant defined by

\[ \det_{Reg}(A) = \lim_{m \to \infty} \det_{m,R} \ast d_{A_0} \ast D_A, \]

where

\[
\det_{m,R} \ast d_{A_0} \ast D_A = \begin{vmatrix}
    a_{11}^R(A) & a_{12}^R(A) & \cdots & a_{1m}^R(A) \\
    a_{21}^R(A) & a_{22}^R(A) & \cdots & a_{2m}^R(A) \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1}^R(A) & a_{m2}^R(A) & \cdots & a_{mm}^R(A)
\end{vmatrix},
\]

\[ a_{ii}^R(A) = 1 + \sum_{\ell=1}^{\infty} ((A, \phi), e_\ell)_+ a_{R,ii}^\ell \]

and
\[ a_{ij}^R(A) = \sum_{\ell=1}^{\infty} ((A, \phi), \epsilon\ell) + a_{R,ij}^\ell. \]

We call \((A, \phi)\) smooth if for every natural number \(b\),
\[
\| (A, \phi) \|_b^2 = ((A, \phi), Q_{A_0}^{2b}(A, \phi))_+ < +\infty.
\]

Then for smooth \((A, \phi)\), these series converge and the regularized determinant is well defined since
\[
\sum_{i,j} \sum_{\ell=1}^{\infty} |((A, \phi), \epsilon\ell) + a_{R,ij}^\ell| \leq \sum_{\ell=1}^{\infty} \| (A, \phi) \|_{N_0} \frac{1}{\mu_\ell} \sum_{i,j} |a_{R,ij}^\ell| < +\infty,
\]
if we take some natural number \(N_0\) such that
\[
\sum_{\ell=1}^{\infty} \frac{1}{\mu_\ell} \sum_{i,j} |a_{R,ij}^\ell| < +\infty,
\]
which is guaranteed by the increasing rates of eigenvalues of \(Q_{A_0}\) (c) of Lemma 1.6.3 in [22] and [30]).

### 2.2 Holonomy and the Poincaré Dual

To handle the Chern-Simons integral in an abstract Wiener space setting, we need to extend the holonomy
\[ Pe^{\int_{\gamma} A}, \]
from smooth \(A\) to the rough \(A\), which arises in the abstract Wiener space setting. We now regularize the holonomy like in Albeverio and Schäfer [1], (see also the relations to Witten’s considerations [42]).

First we begin with introducing briefly Section 2 of Mitoma-Nishikawa [35]. As in the previous section, let \(M\) be a compact smooth oriented three-manifold and \(A\) the space of \(G\)-valued smooth 1-forms on \(M\). Let \(\gamma : t \in [0,1] \mapsto \gamma(t) \in M\) be a smooth closed curve in \(M\) and set \(\gamma[s,t] = \{\gamma(\tau) \mid s \leq \tau \leq t\}\). We regard \(\gamma[s,t]\) as a linear functional
\[
(\gamma[s,t])[A] = \int_{\gamma[s,t]} A = \int_s^t A(\dot{\gamma}(\tau))d\tau, \quad A \in A
\]
defined on the vector space \(A\). Then \(\gamma[s,t]\) is continuous in the sense of distributions and hence defines a \((G\text{-valued})\) de Rham current of degree two.

To recall the regularization of currents, we first consider the case where \(\gamma\) is a smooth closed curve in \(R^3\) and \(A\) is a \(G\)-valued smooth 1-form with compact support defined on
Let $\phi$ be a nonnegative smooth function on $\mathbb{R}^3$ such that the support of $\phi$ is contained in the unit ball $B^3$ with center $0 \in \mathbb{R}^3$ and

$$\int_{\mathbb{R}^3} \phi(x) dx = 1,$$

and define $\phi_\epsilon(x) = \epsilon^{-3} \phi(x/\epsilon)$ for each $\epsilon > 0$. If we write

$$A = \sum_u A^u \cdot E_u = \sum_{i,u} A_i^u \, dx^i \cdot E_u, \quad \dot{\gamma}(\tau) = \sum_i \dot{\gamma}^i(\tau) \left( \frac{\partial}{\partial x^i} \right) \gamma(\tau)$$

for given $A$ and $\gamma$, then we have

$$\lim_{\epsilon \to 0} \sup_{s \leq \tau \leq t} \left| \int_{\mathbb{R}^3} A_i^u(x) \phi_\epsilon(x - \gamma(\tau)) dx - A_i^u(\gamma(\tau)) \right| = 0,$$

and

$$\left| \sum_i 3 \int_s^t \left( \int_{\mathbb{R}^3} A_i^u(x) \phi_\epsilon(x - \gamma(\tau)) dx \right) \dot{\gamma}^i(\tau) d\tau \right| \leq c_1(\epsilon) \|A^u\|_{L^2(\mathbb{R}^3)} |t - s|.$$

Here and in what follows, we denote by $c_k(\star)$ a constant depending on the quantity $\star$ and simply write $c_k$ whenever no confusion may occur.

Now, according to de Rham [18], the regulator of the current $\gamma[s,t]$ is defined by

$$(R_\epsilon \gamma[s,t])[A] = (\gamma[s,t])[R_\epsilon^* A]$$

$$= \sum_{i=1}^3 \int_s^t \left( \int_{\mathbb{R}^3} A_i^u(\gamma(\tau) + ty) \phi_\epsilon(y) dy \right) \dot{\gamma}(\tau) d\tau \cdot E_u$$

$$= \sum_{i=1}^3 \int_s^t \left( \int_{\mathbb{R}^3} A_i^u(x) \phi_\epsilon(x - \gamma(\tau)) dx \right) \dot{\gamma}(\tau) d\tau \cdot E_u,$$

to which there is associated an operator defined by

$$(A_\epsilon \gamma[s,t])[B] = (\gamma[s,t])[A_\epsilon^* B]$$

$$= \sum_{i,j=1}^3 \int_s^t \left\{ \int_{\mathbb{R}^3} \left( \int_0^1 y^j B_{ij}^u(\gamma(\tau) + ty) dt \right) \phi_\epsilon(y) dy \right\} \dot{\gamma}(\tau) d\tau \cdot E_u,$$

where $B = \sum B_{ij}^u \, dx^i \wedge dx^j \cdot E_u$ is a $\mathcal{G}$-valued smooth two-form with compact support on $\mathbb{R}^3$. Then we have the following relation which shows that $A_\epsilon \gamma[s,t]$ gives a chain homotopy between $R_\epsilon \gamma[s,t]$ and $\gamma[s,t]$ (see [18, p. 65] for the proof).

**Proposition 1.** For each $\epsilon > 0$, $R_\epsilon \gamma[s,t]$ and $A_\epsilon \gamma[s,t]$ are smooth currents whose supports are contained in the $\epsilon$ tubular neighborhood of $\gamma[s,t]$, and satisfy

$$R_\epsilon \gamma[s,t] - \gamma[s,t] = \partial A_\epsilon \gamma[s,t] + A_\epsilon \partial \gamma[s,t],$$

where $\partial$ is the boundary operator of currents.
As in [18], the above construction of regularization generalizes to our case in the following manner. First take a diffeomorphism \( h \) of \( R^3 \) onto the unit ball \( B^3 \) with center 0 which coincides with the identity on the ball of radius 1/3 with center 0. Denote by \( s_y \) the translation \( s_y(x) = x + y \) and let \( s_y \) be the map of \( R^3 \) onto itself which coincides with \( h \circ s_y \circ h^{-1} \) on \( B^3 \) and with the identity at all other points, that is,

\[
s_y(x) = \begin{cases} h \circ s_y \circ h^{-1}(x) & \text{if } x \in B^3, \\ x & \text{if } x \not\in B^3. \end{cases}
\]

Note that with a suitable choice of \( h \) we make \( s_y \) to be a diffeomorphism. Then define \( R_{\epsilon} \gamma[s, t] \) and \( A_{\epsilon} \gamma[s, t] \) by the same equations above, but now by using \( s_y \) and \( s_y \).

Now, let \( \{U_i\} \) be a finite covering of \( M \) such that each \( U_i \) is diffeomorphic to the unit ball \( B^3 \) via a diffeomorphism \( h_i \) which can be extended to some neighborhoods of their closures. Using these diffeomorphisms, we transport the transformed operators \( R_{\epsilon} \) and \( A_{\epsilon} \) defined on \( R^3 \) to \( M \). In fact, let \( f \) be a cutoff function which has its support in the neighborhood of the closure of \( U_i \) and is equal to 1 on \( U_i \). Set \( T = \gamma[s, t] \) for simplicity. Then \( T' = fT \) is a current which has its support contained in a neighborhood of the closure of \( U_i \), and \( h_i T' \) is a current which has its support contained in the neighborhood of the closure of \( B^3 \). Note that the support of \( T'' = T - T' \) does not meet the closure of \( U_i \). We define

\[
R_{\epsilon}^i T = h_i^{-1} \circ R_{\epsilon} \circ h_i T' + T'', \quad A_{\epsilon}^i T = h_i^{-1} \circ A_{\epsilon} \circ h_i T'.
\]

and set inductively

\[
R_{\epsilon}^{(k)} T = R_{\epsilon}^1 \circ R_{\epsilon} 2 \circ \cdots \circ R_{\epsilon}^k T, \quad A_{\epsilon}^{(k)} T = R_{\epsilon}^1 \circ A_{\epsilon} 2 \circ \cdots \circ A_{\epsilon}^{k-1} \circ A_{\epsilon}^k T.
\]

Then \( R_{\epsilon} T \) and \( A_{\epsilon} T \) are obtained to be

\[
R_{\epsilon} T = R_{\epsilon}^{(K)} T, \quad A_{\epsilon} T = \sum_{k=1}^{K} A_{\epsilon}^{(k)} T,
\]

where \( K \) is the number of coverings \( \{U_i\} \).

The construction of the operators \( R_{\epsilon} \) and \( A_{\epsilon} \) are easily generalized to any current \( T \) defined on a compact smooth manifold of arbitrary dimension, and we have the following properties for regularizations of currents.

**Proposition 2** ([18]). Let \( M \) be a compact smooth manifold. Then for each \( \epsilon > 0 \) there exist linear operators \( R_{\epsilon} \) and \( A_{\epsilon} \) acting on the space of de Rham currents with the following properties:

1. If \( T \) is a current, then \( R_{\epsilon} T \) and \( A_{\epsilon} T \) are also currents and satisfy

\[
R_{\epsilon} T - T = \partial A_{\epsilon} T + A_{\epsilon} \partial T.
\]

2. The supports of \( R_{\epsilon} T \) and \( A_{\epsilon} T \) are contained in an arbitrary given neighborhood of the support of \( T \) provided that \( \epsilon \) is sufficiently small.

3. \( R_{\epsilon} T \) is a smooth form.

4. For all smooth forms \( \phi \) we have

\[
R_{\epsilon} T[\phi] \to T[\phi] \quad \text{and} \quad A_{\epsilon} T[\phi] \to 0
\]

as \( \epsilon \to 0 \).

**Perturbative Expansion of Chern-Simons Theory**
Given a smooth closed curve \( \gamma : [0, 1] \to M \) in \( M \), let \( U_\gamma \) be a tubular neighborhood of \( \gamma[0, 1] \) in \( M \) and \( j : U_\gamma \to M \) denote the inclusion. For each \( t \in [0, 1] \) and sufficiently small \( \epsilon > 0 \) we set
\[
C_\epsilon^u(t) = j^*(R_\epsilon \gamma[0, t]),
\]
which we know is a smooth two-form on \( U_\gamma \) and has a compact support in \( U_\gamma \) from Proposition 2. In particular, for \( t = 1 \), [35] showed the following observation, since
\[
dC_\epsilon^1(1) = dj^*(R_\epsilon \gamma[0, 1]) = j^*d(R_\epsilon \gamma[0, 1]) = -j^*R_\epsilon \partial(\gamma[0, 1]) = 0.
\]

**Proposition 3 ([1]).** \( [C_\epsilon^u(1)] = [j^*(R_\epsilon \gamma[0, 1])] \in H^2_c(U_\gamma) \) is the compact Poincaré dual of \( \gamma \) in \( U_\gamma \), where \( H^2_c(U_\gamma) \) denotes the second de Rham cohomology of \( U_\gamma \) with compact supports.

Let \( A \in \mathcal{A} \) be a \( G \)-valued one-form on \( M \) and express \( A \) as in (2.7). Denote the \( u \)-component of \( *C_\epsilon^u(t) \) by
\[
C_\epsilon^u(t) = C_\epsilon^u(t)_\gamma = *C_\epsilon^u(t)_u \cdot E_u.
\]
Then, recalling the construction of regulators of currents and noting (2.8) and (2.9), it is not hard to see that we have
\[
\lim_{\epsilon \to 0} \sup_{0 \leq t \leq 1} \left\| (A, C_\epsilon^u(t))_0 - \int_{\gamma[0, t]} A^u \right\| = 0,
\]
\[
\left\| \int_{\gamma[0, t]} A^u - \int_{\gamma[0, s]} A^u \right\| \leq c_2(A)|t - s|
\]
and
\[
\|C_\epsilon^u(t) - C_\epsilon^u(s)\|_0 \leq c_1(\epsilon)|t - s|,
\]
where \( \| \cdot \|_0 \) is the norm defined in (2.3).

Let us define
\[
A_\epsilon^u(t) = \sum_{u=1}^d (A, C_\epsilon^u(t))_0 \cdot E_u
\]
and
\[
\bar{A}(t) = \int_{\gamma[0, t]} A.
\]

By Chen’s iterated integrals (Theorem 4.3 in § 1.4 of [19], p31, see also [16, 40]), we have
\[
P_ε\int_{\gamma} A_0 + \bar{A} \]
\[
= I + \sum_{r=1}^{\infty} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} d(\bar{A}_0 + \bar{A})(t_1)d(\bar{A}_0 + \bar{A})(t_2)\cdots d(\bar{A}_0 + \bar{A})(t_r).
\]
Then, as [1], we define the $\epsilon$-regularizations of the holonomy and the Wilson line as follows:

\begin{equation}
W^\gamma_\epsilon [A] = I + \sum_{r=1}^{\infty} W^\gamma_{\epsilon,r} [A],
\end{equation}

where

\begin{equation}
W^\gamma_{\epsilon,r} [A] = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} d(A_0 + A^\epsilon)^{(t_1)} d(A_0 + A^\epsilon)^{(t_2)} \cdots d(A_0 + A^\epsilon)^{(t_r)}
\end{equation}

and

\begin{equation}
F^\epsilon (A_0 + A) = \prod_{j=1}^s \text{Tr} \ W^\epsilon_\gamma [A].
\end{equation}

Setting

\[ \tilde{F}^\epsilon_{A_0} (A) = F^\epsilon (A_0 + A) \det_{\text{Reg}} (A) \]

and $x = (A, \phi) = (x_A, x_\phi)$, we arrive at a regularized form of (2.4):

\begin{equation}
\frac{1}{Z_{2,k}} \int_X \tilde{F}^\epsilon_{A_0} (x_A) \exp \left[ ik(x, Q_{A_0} x)_+ - \frac{ik}{4\pi} \int_M \frac{2}{3} x_A \wedge x_A \wedge x_A \right] D(x),
\end{equation}

where $X$ is the space of all $x$ and

\[ Z_{2,k} = \int_X \exp \left[ ik(x, Q_{A_0} x)_+ \right] D(x). \]

By expanding heuristically, for any natural number $N$, we have that (2.17) is equal to $I(k) + II(k)$, with

\begin{align*}
I(k) &= \frac{1}{Z_{2,k}} \sum_{r=0}^{N} \int_X \tilde{F}^\epsilon_{A_0} (x_A) \left\{ \left( -\frac{ik}{4\pi} \int_M \frac{2}{3} x_A \wedge x_A \wedge x_A \right)^r / r! \right\} \exp \left[ ik(x, Q_{A_0} x)_+ \right] D(x) \\
II(k) &= \frac{1}{Z_{2,k}} \int_X \tilde{F}^\epsilon_{A_0} (x_A) \left\{ \sum_{r=N+1}^{\infty} \left( -\frac{ik}{4\pi} \int_M \frac{2}{3} x_A \wedge x_A \wedge x_A \right)^r / r! \right\} \exp \left[ ik(x, Q_{A_0} x)_+ \right] D(x).
\end{align*}
2.3 Definitions of “finite part” and “asymptotic estimate”

Now we proceed to give a meaning to $I(k)$ and to derive an asymptotic estimate for $II(k)$ in an abstract Wiener space setting, with $k$ tending to $\infty$. We will first define the real Hilbert space $H$ of the abstract Wiener space $(X, H, \mu)$ that will be used later.

We recall that $L^2(\Omega^+)$ was the Hilbert space with the inner product

\[(2.18) \quad ((A, \phi), (B, \varphi))_0 = (A, B)_0 + (\phi, \varphi)_0 \]

and the norm

\[(2.19) \quad \| (A, \phi) \|^2_+ = (A, A)_0 + (\phi, \phi)_0 = \|A\|^2_0 + \|\phi\|^2_0.\]

Let $\hat{\Omega}^n$ be the set of all real valued $n$-forms and for any $\alpha, \beta \in \hat{\Omega}^n$,

\[(\alpha, \beta)_{R_0} = \int_M \alpha \wedge^* \beta.\]

Noticing that $x_A = \sum_{u=1}^d x^u_A E_u$ and $x_\phi = \sum_{u=1}^d x^u_\phi E_u$, we define for $\hat{x} = (x^1, x^2, \ldots, x^d)$, $x^u = (x^u_A, x^u_\phi)$ and $\hat{y} = (y^1, y^2, \ldots, y^d)$, $y^u = (y^u_B, y^u_\phi)$,

\[\hat{(x, y)} = \sum_{u=1}^d (x^u, y^u)\]

and

\[\hat{(x^u, y^u)} = (x^u_A, x^u_\phi)_{R_0} + (x^u_B, y^u_\phi)_{R_0}.\]

Then the real Hilbert space of $\hat{x}$ corresponding to $L^2(\Omega^+)$ is denoted by $H$ with the inner product

\[(\hat{x}, \hat{x}) = \sum_{j=1}^\infty (\hat{x}, \hat{e}_j)^2\]

and the corresponding norm such that

\[\| \cdot \|^2 = (\cdot, \cdot),\]

where $\hat{e}_j = ((e^1_A, e^1_\phi, e^2_A, e^2_\phi, \ldots, e^d_A, e^d_\phi))$.

Then $H$ is isometric to $L^2(\Omega^+)$ such that

\[(2.20) \quad (\hat{x}, \hat{y}) = (x, y)_+ \quad \text{and} \quad \|\hat{x}\| = \|x\|_+.\]

For smooth $A$, by (2.18) and (2.20), we have

\[(2.21) \quad (A, C^\phi_u(t))_0 = ((A, \phi), (C^\phi_u(t), 0))_+ = (\hat{x}, (C^\phi_u(t), 0)),\]
Perturbative Expansion of Chern-Simons Theory

where

\( (\overline{C^\epsilon(t)}, 0) = ((0, 0), (0, 0), \cdots, (\ast C^\epsilon(t)^u, 0), \cdots, (0, 0)) \).

Set

\( \hat{x}_\gamma(t) = \sum_{u=1}^{d} (\hat{x}, (\overline{C^\epsilon(t)}, 0)) \cdot E_u. \)

Then replacing \( A_\gamma(t) \) by \( \hat{x}_\gamma(t) \) in the definitions (2.15) and (2.16), we have

\[
W^\epsilon_\gamma(\hat{x}) = I + \sum_{r=1}^{\infty} W^\epsilon_{\gamma,r}(\hat{x}).
\]

and

\[
F^\epsilon_{A_0}(\hat{x}) = \prod_{j=1}^{s} \text{Tr} W^\epsilon_{\gamma,j}(\hat{x}).
\]

The convergence in \( H \) and the well definedness of \( W^\epsilon_\gamma(\hat{x}) \) is guaranteed by (2.13).

We have

\[
(x, Q_{A_0}x)_+ = \sum_{j=1}^{\infty} \mu_j (\hat{x}, \hat{e}_j)^2, \quad x_A = \sum_{j=1}^{\infty} (\hat{x}, \hat{e}_j) e_j^A,
\]

\[
F^\epsilon(A_0 + x_A) = F^\epsilon_{A_0}(\hat{x}) = \prod_{j=1}^{s} \text{Tr} W^\epsilon_{\gamma,j}(\hat{x}),
\]

\[
a^R_{ii}(x_A) = 1 + \sum_{\ell=1}^{\infty} (\hat{x}, \hat{e}_\ell) a_{R,ii}^\ell, \quad a^R_{ij}(x_A) = \sum_{\ell=1}^{\infty} (\hat{x}, \hat{e}_\ell) a_{R,ij}^\ell,
\]

and

\[
(2.22) \quad \frac{-1}{4\pi} \int_M \text{Tr} \frac{2}{3} x_A \wedge x_A \wedge x_A = \sum_{p,j,\ell=1}^{\infty} (\hat{x}, \hat{e}_p)(\hat{x}, \hat{e}_j)(\hat{x}, \hat{e}_\ell) \beta^{pj\ell},
\]

where

\[
\beta^{pj\ell} = -\int_M \text{Tr} \frac{1}{6\pi} e_p^A \wedge e_j^A \wedge e_\ell^A.
\]

Itô technique [27] to handle oscillatory integrals is to introduce a Gaussian kernel with a suitable covariance operator. Thus for the m-dimensional approximation of \( I(k) \), we take the covariance operator \( S_m \) with eigenvalues \( \{\lambda_j > 0, j = 1, 2, \cdots, m\} \) and consider
\[
\lim_{n \to \infty} \lim_{m \to \infty} \frac{V_{\lambda}^{m,n}}{Z_{m,n}} \sum_{r=0}^{N} \int_{\mathbb{R}^m} \tilde{F}_{A_0}^\infty(x^m) \left\{ (ik \sum_{p,j,\ell=1}^{m} x_p x_\ell \beta_{pj\ell})^r / r! \right\} \times \exp \left[ ik(x^m, Q_{A_0} x^m)_+ - \frac{(S_{m}^{-1} x^m, x^m)_+}{2n} \right] \frac{\mu_m(dx)}{(\sqrt{2\pi})^m \sqrt{\det(nS_m)}},
\]

where \( \mu_m \) is the \( m \)-dimensional Lebesgue measure,

\[
x^m = \sum_{j=1}^{m} x_j \hat{e}_j, \quad V_{\lambda}^{m,n} = \prod_{j=1}^{m} \sqrt{1 + n \lambda_j},
\]

\[
(S_{m}^{-1} x^m, x^m)_+ = \sum_{j=1}^{m} \lambda_j^{-1} x_j^2
\]

and

\[
Z_{m,n} = V_{m,n}^\lambda \int_{\mathbb{R}^m} \exp \left[ ik(x^m, Q_{A_0} x^m)_+ - \frac{(S_{m}^{-1} x^m, x^m)_+}{2n} \right] \frac{\mu_m(dx)}{(\sqrt{2\pi})^m \sqrt{\det(nS_m)}}.
\]

In the sequel we will show that it is possible to find a realization of the expression (2.23) using a suitable abstract Wiener space \((X, H, \mu)\). The real Hilbert space \( H \) was already defined above. Now we will choose \( X \) and a Gaussian measure \( \mu \) such that \( H \) is densely and continuously embedded into \( X \) and

\[
\int_X e^{i \langle x, \xi \rangle} d\mu(x) = e^{-\frac{||\xi||^2}{2}},
\]

holds, where \( \langle x, \xi \rangle \) denotes the canonical bilinear form on \( X \times X^* \). We denote \((x^1, x^2, \cdots, x^d) \in X \) by \( x \) and \((\xi^1, \xi^2, \cdots, \xi^d) \in X^* \) by \( \xi \). In the usual manner, we can extend the bilinear form to any element \( h \in H \), then we denote it again by the same symbol \( \langle x, h \rangle, h \in H \) such that for any \( x, h \in H, \langle x, h \rangle = (x, h) \). Since \((\hat{e}_i, i = 1, 2, \cdots)\) forms a complete orthonormal system of the Hilbert space \( H \), \( Q_{A_0} \) induces naturally a linear operator defined in \( H \) having the same spectrum as \( Q_{A_0} \). We denote this linear operator in \( H \) again by the same symbol \( Q_{A_0} \). Using the above CONS of \( H \), we can construct a positive definite and self-adjoint operator \( S \) with eigenvalues \( \{ 1 \geq \lambda_j > 0, j = 1, 2, \cdots \} \) on the Hilbert space \( H \) such that

\[
(S.1) \quad \sum_{j=1}^{\infty} \sqrt{\lambda_j} | \mu_j | < +\infty,
\]

\[
(S.2) \quad \sum_{j=1}^{\infty} \sqrt{\lambda_j} < +\infty
\]
and

\[ \sup_{\ell} \sqrt[\ell]{\lambda} \sum_{i,j} |a_{R,ij}| < +\infty. \]

(\[38, 43\]).

For any \( \epsilon > 0 \) and \( x \in X \), we define

\[ x_\gamma^\epsilon(t) = \sum_{u=1}^{\tilde{d}} < x, (C_\gamma^\epsilon(t), 0) > E_u. \]

Briefly we denote

\[ < x, (C_\gamma^\epsilon(t), 0) > \text{ by } x_\gamma^{\epsilon,u}(t). \]

Then by (2.19) and (2.20), we remark that \( x_\gamma^{\epsilon,u}(t) \) is a Gaussian random variable such that

\[ E\left[ (x_\gamma^{\epsilon,u}(t))^2 \right] = \| (C_\gamma^\epsilon(t), 0) \|^2 = \| C_\gamma^\epsilon(t) \|^2. \]

Since (2.13) yields

\[ E\left[ |x_\gamma^{\epsilon,u}(t) - x_\gamma^{\epsilon,u}(s)|^2 \right] \leq c_1(\epsilon)^2 |t - s|^2, \]

\( x_\gamma^{\epsilon,u}(t) \) has a continuous version in \( t \), by the Kolmogorov-Totoki criterion \([41]\). Henceforth we denote this continuous version, again by the same symbol \( x_\gamma^{\epsilon,u}(t) \).

For any natural number \( n \), set

\[ T_n = \sum_{j=1}^{2^n} |x_\gamma^{\epsilon,u}\left(\frac{j}{2^n}\right) - x_\gamma^{\epsilon,u}\left(\frac{j-1}{2^n}\right)|. \]

Since \( T_n \leq T_{n+1} \), we have

\[ E\left[ \lim_{n \to \infty} T_n \right] = \lim_{n \to \infty} E[T_n] \]

\[ = \lim_{n \to \infty} E\left[ \sum_{j=1}^{2^n} \left| x_\gamma^{\epsilon,u}\left(\frac{j}{2^n}\right) - x_\gamma^{\epsilon,u}\left(\frac{j-1}{2^n}\right) \right| \right] \]

\[ \leq \lim_{n \to \infty} \sum_{j=1}^{2^n} E\left[ \left| x_\gamma^{\epsilon,u}\left(\frac{j}{2^n}\right) - x_\gamma^{\epsilon,u}\left(\frac{j-1}{2^n}\right) \right|^2 \right]^{\frac{1}{2}} \]

\[ \leq \lim_{n \to \infty} \sum_{j=1}^{2^n} c_1(\epsilon) \left| \frac{j}{2^n} - \frac{j-1}{2^n} \right| \]

\[ \leq 1, \]
which implies

\[(2.27) \quad \lim_{n \to \infty} T_n < +\infty \quad \text{almost surely with respect to } \mu.\]

Noticing that \(x^u_\gamma(t)\) is continuous in \(t\) almost surely, by (2.27), we obtain, for almost all \(x \in X\) with respect to \(\mu\), that

\[x^u_\gamma(t) \quad \text{is of bounded variation.}\]

Therefore the Lebesgue-Stieltjes integral

\[\int_0^t dx^u_\gamma(t_1) \int_0^{t_1} dx^v_\gamma(t_2)\]

is well defined and continuous in \(t\) almost surely with respect to \(\mu\) and is equal to

\[\sum_{a=1}^d \sum_{v=1}^d \int_0^t dx^u_\gamma(t_1) \int_0^{t_1} dx^v_\gamma(t_2) \cdot E_a E_v.\]

Repeating such arguments, we arrive at the well definedness of the iterated integrals and in a similar manner as for the definitions of \(W^\epsilon_\gamma(\hat{x})\) and \(F^\epsilon_{A_0}(\hat{x})\), we define

\[F^\epsilon_{A_0}(x) = \prod_{j=1}^s \text{Tr} W^\epsilon_{\gamma_j}(x),\]

where the well definiteness is guaranteed by (1) in Lemma 1 which will be stated below.

We call, for \(\epsilon > 0\), \(F^\epsilon_{A_0}(x)\) the \(\epsilon\)-regularization of the Wilson loop variable [1].

In the definitions of \(\det_{n,R} d_{A_0} * D_A\) and \(\det_{\text{Reg}}(A)\), we replace \(a^R_{ii}(A)\) by

\[a^R_{ii}(\sqrt{nS}x) = 1 + \sum_{\ell=1}^\infty \sqrt{n\lambda_\ell} < x, \hat{e}_\ell > a^\ell_{R,ii}\]

and \(a^R_{ij}(A)\) by

\[a^R_{ij}(\sqrt{nS}x) = \sum_{\ell=1}^\infty \sqrt{n\lambda_\ell} < x, \hat{e}_\ell > a^\ell_{R,ij},\]

which are \(\det_{n,R} d_{A_0} * D_{\sqrt{nS}x}\) and \(\det_{\text{Reg}}(\sqrt{nS}x)\). The convergence of the series is guaranteed by (S.2) and (S.3), since

\[\left| \sum_{\ell=1}^\infty \sqrt{n\lambda_\ell} < x, \hat{e}_\ell > a^\ell_{R,ij} \right|\]

\[\leq \sum_{\ell=1}^\infty \sqrt{n\lambda_\ell} < x, \hat{e}_\ell > |\sup_{\ell} \sqrt{\lambda_\ell a^\ell_{R,ij}}|.

Let us set
We shall provide the definition of $I(k)$ in the abstract Wiener space by setting it equal to

**Definition 1.**

\[
\limsup_{n \to \infty} \frac{1}{Z_{n,k}} \sum_{r=0}^{N} \int_{X} \tilde{F}_{A_0}^{r} (\sqrt{nSx}) \exp \left[ ikQ(\sqrt{nSx}) \right] \times \left\{ (ik \sum_{p,j,\ell=1}^{\infty} \bar{e}_p \bar{e}_j \bar{e}_\ell > \beta_{p,j,\ell}^{r}/r! \right\} \mu(dx),
\]

where

\[
Z_{n,k} = \int_{X} \exp \left[ ikQ(\sqrt{nSx}) \right] \mu(dx).
\]

We use the notations for real $x_n, y_n$ :

\[
\limsup_{n \to \infty} (x_n + iy_n) = \limsup_{n \to \infty} x_n + i \limsup_{n \to \infty} y_n
\]

and by above convergence assumptions (S.1) and (S.2), we get

\[
Q(\sqrt{nSx}) = \sum_{j=1}^{\infty} n\lambda_j \mu_j < x, \hat{e}_j > ^2 + \infty
\]

(cf. [32]).

To derive an asymptotic estimate for $II(k)$ as $k \to \infty$, let us exploit compensations in the numerator and denominator. The m-dimensional approximation of $II(k)$ is given by

\[
\frac{1}{mZ_{2,k}} \int_{R^m} \tilde{F}_{A_0}^{r}(x^m) \left\{ \sum_{r=N+1}^{\infty} (ik \sum_{p,j,\ell=1}^{m} x_{p,j,\ell} (\beta_{p,j,\ell}^{r}/r!) \right\} \exp \left[ ik(x^m, Q_{A_0}x^m) + \right] \frac{\mu_m(dx)}{(\sqrt{2\pi})^m},
\]

where

\[
mZ_{2,k} = \int_{R^m} \exp \left[ ik(x^m, Q_{A_0}x^m) + \right] \frac{\mu_m(dx)}{(\sqrt{2\pi})^m}.
\]

Set $\sqrt{k}x_i = y_i$, then by the formula of changing variables of up and down integral, we obtain (2.29)
\[
= \frac{1}{Z_{3,k}^m} \int_{R^m} \tilde{F}_{A_0} \left( \frac{1}{\sqrt{k}} y^m \right) \left\{ \sum_{r=N+1}^{\infty} \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{m} y_p y_j y_\ell (\beta^{p j \ell})^r / r! \right\} \exp \left[ i (y^m, Q_{A_0} y^m) + \frac{\mu_m(dy)}{(\sqrt{2\pi})^m} \right],
\]

where
\[
Z_{3,k}^m = \int_{R^m} \exp \left[ i (y^m, Q_{A_0} y^m) + \frac{\mu_m(dy)}{(\sqrt{2\pi})^m} \right].
\]

To discuss the asymptotic estimate of \( II(k) \), we may modify the Itô technique by introducing the Gaussian kernel with parameter
\[
k \geq \frac{1}{2}, 0 < \eta < \frac{1}{12}
\]
and consider
\[
\lim_{k \to \infty} \sqrt{k}^{N+1} II(k)
\]

\[
(2.31) \quad II(k) = \lim_{k \to \infty} \sqrt{k}^{N+1} \lim_{m \to \infty} \frac{1}{Z_{4,k}^m} \int_{R^m} \tilde{F}_{A_0} \left( \frac{1}{\sqrt{k}} x^m \right) \left\{ \sum_{r=N+1}^{\infty} \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{m} x_p x_j x_\ell (\beta^{p j \ell})^r / r! \right\} \exp \left[ i (x^m, Q_{A_0} x^m) + \frac{(S^{-1} x^m, x^m)}{2k^{2\eta}} \right] \frac{\mu_m(dx)}{\sqrt{2\pi}^m \sqrt{\det(k^{2\eta} S_m)}},
\]

where
\[
Z_{4,k}^m = \int_{R^m} \exp \left[ i (x^m, Q_{A_0} x^m) + \frac{(S^{-1} x^m, x^m)}{2k^{2\eta}} \right] \frac{\mu_m(dx)}{\sqrt{2\pi}^m \sqrt{\det(k^{2\eta} S_m)}}.
\]

We can find a realization of the expression (2.31) in the abstract Wiener space setting as equal to

**Definition 2.**

\[
(2.32) \quad \lim_{k \to \infty} \sqrt{k}^{N+1} \frac{1}{Z_{2,k}^N} \int_X \tilde{F}_{A_0} \left( \frac{\sqrt{k^{2\eta} S}}{\sqrt{k}} x \right) \exp \left[ ik^{2\eta} Q(\sqrt{k^{2\eta} S} x) \right] \mu(dx),
\]

where
\[
\hat{Z}_{2,k} = \int_X \exp \left[ ik^{2\eta} Q(\sqrt{k^{2\eta} S} x) \right] \mu(dx).
\]
Here we add a further assumption to the operator $S$ such that for some $\delta < \eta$ and a natural number $M$ such that $M\delta > 2\eta$,

$$\sum_{j=1}^{\infty} M\sqrt{\lambda_j |\mu_j|} = C \leq c_3 < +\infty.$$  \hspace{1cm} (S.4)

### 2.4 Integrabilities and Theorems

Before stating Theorems, we shall discuss the integrability of the Wilson lines operator and of the determinant arising below.

Set

$$Y_{1,0}^1(0) = I,$$

$$Y_{1,r}^1(i) = \sum_{1 \leq l_1 < l_2 < \cdots < l_r \leq i} \int_0^{t_{l_1}} \cdots \int_0^{t_{l_r-1}} dx_1^\gamma(t_{l_1}) \cdots dx_r^\gamma(t_{l_r})$$

and

$$Y_{1,r}^1(i) = \sum_{r=i}^{\infty} Y_{1,r}^1(i).$$

Since

$$W_{\gamma}^{r}(x) = \int_0^{t_r} \cdots \int_0^{t_{r-1}} d(A_0 + x\gamma_1^\gamma(t_1)) \cdots d(A_0 + x\gamma_r^\gamma(t_r))$$

$$= \sum_{i=0}^{\infty} Y_{1,r}^1(i),$$

we have

$$W_{\gamma}^{r}(x) = I + \sum_{r=1}^{\infty} W_{\gamma}^{r}(x) = \sum_{i=0}^{\infty} Y_{1,r}^1(i).$$

To proceed further, we need several notations. Let us set

$$Y_{A_0}^{1,m}(x) = \sum_{i_1 + i_2 + \cdots + i_s = m} \prod_{j=1}^{s} \text{Tr} Y_{1,j}^1(i_j).$$  \hspace{1cm} (2.33)

Furthermore, let us define $\bar{a}_i^R(x) = a_i^R(x) - 1$ and denote by $\mathcal{S}_m$ the collection of all permutations of $\{1, 2, \cdots, m\}$ and by $\mathcal{S}_{m,t}$ the collection of all $\sigma \in \mathcal{S}_m$ such that $\sigma(i_j) \neq i_j, j = 1, 2, \cdots, t$ and otherwise $\sigma(i) = i$. Set
\[
Y^{2,r} = \lim_{m \to \infty} \left( \sum_{i=1}^{r} \sum_{\sigma \in \mathfrak{S}_{m,i}} \text{sign} \sigma a_{i_1 \sigma(i_1)}^R(x) a_{i_2 \sigma(i_2)}^R(x) \cdots a_{i_t \sigma(i_t)}^R(x) \right)
\times \left\{ \sum_{1 \leq j_1 < j_2 \cdots < j_{r-t} \leq m, j \notin \{i_1, i_2, \ldots, i_t\}} \hat{a}_{j_1 j_1}^R(x) \hat{a}_{j_2 j_2}^R(x) \cdots \hat{a}_{j_{r-t} j_{r-t}}^R(x) \right\}
+ \sum_{1 \leq j_1 < j_2 \cdots < j_t \leq m} \hat{a}_{j_1 j_1}^R(x) \hat{a}_{j_2 j_2}^R(x) \cdots \hat{a}_{j_t j_t}^R(x)
\]

and

\[
Y^{2,0} = 1.
\]

Set

\[
\bar{F}_m(x) = \sum_{r+r'=m} Y_{A_0}^{1,r}(x) Y^{2,r'}(x).
\]

Define

\[
R_{n,k} = \frac{\sqrt{nS}}{\sqrt{1 - 2ik \sqrt{nSQ_{A_0}}} \sqrt{nS}}.
\]

For an \(n \times n\) matrix \(E = (a_{ij})\), define

\[
\|E\| = \sum_{i,j=1}^{n} |a_{ij}|.
\]

We define a regularization of \(Q_{A_0}\) such that the eigenvalues \(\mu_j\) of \(Q_{A_0}\) satisfy the "renormalization condition":

**Renormalization.**

\[
\sum_{\ell=1}^{\infty} \frac{1}{|\mu_\ell|} (1 + \sum_{i,j} |a^\ell_{R,i,j}|) < +\infty.
\]

We also assume

**Assumption 2.**

\[
|\beta^{\rho j \ell}| \leq \beta < +\infty.
\]

**Remark 1.** We expect that in the case where \(M\) is compact, this assumption is automatically satisfied.

The following Lemma will be proved later in Section 3.
Lemma 1. Let us denote any natural number by \( b \) and let \( E[\cdot] \) the integral with respect to \( \mu \) on \( X \).

1. For any \( \epsilon > 0 \),
\[
E \left[ \| W_\gamma(x) \|^2b \right] < +\infty.
\]

2. For any fixed \( \epsilon > 0 \), there exists a constant \( c_4(\epsilon) \) independent of \( n \) such that
\[
E \left[ \| W_\gamma^b(R_{n,k}x) \|^2b \right] \leq c_4(\epsilon) < +\infty.
\]

3. For any fixed \( \epsilon > 0 \), there exists a constant \( c_4(\epsilon) \) independent of \( n \) such that
\[
E \left[ \sum_{m=N}^{\infty} Y_{A_0}^{1,m}(R_{n,k}x) \right] = o\left( \frac{1}{\sqrt{k}} \right)^N,
\]
where \( o\left( \frac{1}{\sqrt{k}} \right)^N \) means
\[
\lim_{k \to +\infty} \sqrt{k}^N \left| o\left( \frac{1}{\sqrt{k}} \right)^N \right| < +\infty.
\]

4. Under the Assumption 2,
\[
E \left[ \| \det_{\text{Reg}}(\sqrt{nS}x) \|^2b \right] < +\infty.
\]

5. Under the Renormalization assumption on \( Q_{A_0} \) and the Assumption 2, there exists a constant \( c_5(b) \) independent of \( n \) such that
\[
E \left[ \| R_{n,k}x, \hat{e}_p \| \sqrt{nS}x, \hat{e}_j \| \beta^{pj\ell} \|^2b \right] \leq c_5(b) \left( \frac{1}{\sqrt{k}} \right)^{2b} \left( \sum_{j=1}^{\infty} \frac{1}{|R_j|} \right)^{6b}.
\]

6. Under the Assumption 2,
\[
E \left[ \| \det_{\text{Reg}}(\sqrt{nS}x) \|^2b \right] < +\infty.
\]

7. Under the same assumptions as in (5), there exists a constant \( c_6(b) \) independent of \( n \) such that
\[
E \left[ \| \det_{\text{Reg}}(R_{n,k}x) \|^2b \right] \leq c_6(b) < +\infty.
\]

8. Under the same assumptions as in (5),
\[
E \left[ \sum_{r=N}^{\infty} Y^{2r}(R_{n,k}x) \right] = o\left( \frac{1}{\sqrt{k}} \right)^N.
\]

By Lemma 1, the Cauchy integral theorem in finite dimensions and a limiting argument, we have
Theorem 1. Under the Assumption 2,

\[ I(k) = \limsup_{n \to \infty} \sum_{r=0}^{N} \int_{X} \tilde{F}_{A_{0}}(R_{n,k}x) \times \left\{ (ik \sum_{p,j,\ell=1}^{\infty} < R_{n,k}x, \hat{e}_{p} > < R_{n,k}x, \hat{e}_{j} > < R_{n,k}x, \hat{e}_{\ell} > \beta^{p\ell})^r \right\} \mu(dx) \]

is well-defined.

Under the Renormalization assumption on \( Q_{A_{0}} \) and the Assumption 2, the right hand side of the above equality is equal to:

\[ \sum_{m+m' \leq N} \int_{X} \tilde{F}_{m}(R_{k}x) \times \left\{ (ik \sum_{p,j,\ell=1}^{\infty} < R_{k}x, \hat{e}_{p} > < R_{k}x, \hat{e}_{j} > < R_{k}x, \hat{e}_{\ell} > \beta^{p\ell})^{m'/m!} \right\} \mu(dx) \]  

\[ + o((1/\sqrt{k})^{N+1}) \]

\[ = \sum_{m+m' \leq N} (1/\sqrt{k})^{m+m'} J_{CS}^{m+m'} + o((1/\sqrt{k})^{N+1}), \]

where

\[ (1/\sqrt{k})^{m+m'} J_{CS}^{m+m'} = \int_{X} \tilde{F}_{m}(R_{k}x) \times \left\{ (ik \sum_{p,j,\ell=1}^{\infty} < R_{k}x, \hat{e}_{p} > < R_{k}x, \hat{e}_{j} > < R_{k}x, \hat{e}_{\ell} > \beta^{p\ell})^{m'/m!} \right\} \mu(dx) \]

and

\[ R_{k} = \frac{1}{\sqrt{-2i k Q_{A_{0}}}}. \]

Now we will proceed to the asymptotic estimate of \( II(k) \), as \( k \to \infty \).

For \( 0 < \delta < \eta \), set

\[ D = \left\{ x \in X; \sum_{j=1}^{\infty} |< \sqrt{S} x \sqrt{1 - 2i k^{2\eta} \sqrt{S} Q_{A_{0}} \sqrt{S}} , \hat{e}_{j} > | \leq 3k^{\delta} \right\} \]

and

\[ V_{k} = \frac{\sqrt{k^{2\eta} S}}{\sqrt{1 - 2i k^{2\eta} Q_{A_{0}} \sqrt{k^{2\eta} S}}}. \]
Theorem 2. Under the Assumption 2,

\begin{equation}
\frac{1}{Z_{2,k}} \int_X \tilde{F}_{A_0}(\sqrt{k^2\eta} S x) \exp \left[ i k^2 Q(\sqrt{S} x) \right] \times \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \right)^r \mu \left( \sum_{p,j,\ell=1}^{\infty} \sqrt{k^2\eta} x, \hat{e}_p, \hat{e}_j, \hat{e}_\ell > \beta_{p,j,\ell}^r / r! \right) \right\} \mu(dx)
\end{equation}

is well-defined and equal to

\begin{equation}
\int_D \tilde{F}_{A_0}(V_k x) \times \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \right)^r \mu \left( \sum_{p,j,\ell=1}^{\infty} V_k x, \hat{e}_p, \hat{e}_j, \hat{e}_\ell > \beta_{p,j,\ell}^r / r! \right) \right\} \mu(dx)
+ R(k)
\end{equation}

where \( R(k) \) satisfies

\begin{equation}
R(k) \leq c_7 e^{-k^\delta/2}
\end{equation}

for sufficiently large \( k \), so that under the Renormalization assumption on \( Q_{A_0} \) and the Assumption 2,

\[ \lim_{k \to \infty} \sqrt{k}^{N+1} |II(k)| < +\infty. \]

Remark 2. If we make a stronger renormalization on \( Q_{A_0} \) such that

\[ \sum_{p,j,\ell=1}^{\infty} \left( \frac{\beta_{p,j,\ell}^r}{\mu_p \mu_j \mu_\ell} \right)^2 < +\infty \]

and impose appropriate conditions on the eigenvalues of \( S \), we can release Assumption 2. As remarked before, Assumption 2 is expected to be automatically satisfied when \( M \) is compact.

3 Proofs of Lemmas and Theorems

First we remark that \( \mu \)-almost surely,

\[ \sum_{u=1}^{d} < R_{n,t} (\{C^u_v(t)\}, 0) >> E_u \]

\[ = \sum_{u=1}^{d} \sum_{r=1}^{\infty} < x, \hat{e}_r > (\{C^u_v(t)\}, \hat{e}_r) \frac{\sqrt{n \lambda_r}}{\sqrt{1 - 2 i n k \lambda_r \mu_r}} E_u. \]
Since
\[
\sqrt{n\lambda_r} \frac{1}{\sqrt{1 - 2mk\lambda_r\mu_r}} = \sqrt{n\lambda_r} \frac{1}{\sqrt{1 + 4(nk\lambda_r\mu_r)^2}} e^{i\theta}
\]
\[= \alpha_{n,k}^{n,k} + i\beta_{n,k}^{n,k}, \quad \text{where} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},\]
set
\[
R_{n,k}^x(t) = \sum_{u=1}^d < R_{n,k}x, (\hat{C}_u^x(t), 0) > E_u,
\]
\[
R_{n,k}^x(t)^u = < R_{n,k}x, (\hat{C}_u^x(t), 0) >,
\]
\[
R_{n,k}^x(t)^u = \sum_{r=1}^\infty \alpha_{r}^{n,k} < x, \hat{e}_r > ((\hat{C}_u^x(t), 0), \hat{e}_r),
\]
and
\[
R_{n,k}^x(t)^u = \sum_{r=1}^\infty \beta_{r}^{n,k} < x, \hat{e}_r > ((\hat{C}_u^x(t), 0), \hat{e}_r).
\]
Define
\[
c_E = \max_{u=1}^d \| E_u \|.
\]

### 3.1 Proofs of Lemmas

Before we proceed to the proof of Lemma 1, we remark the following lemma used several times below.

**Lemma 2.** Let \( b \) be a natural number and \( X_{i,j}, i, j = 1, 2, ... \) be real numbers. Then
\[
\sum_i |\sum_j X_{i,j}|^{2b} \leq \left( \sum_j \left( \sum_i |X_{i,j}|^{2b} \right)^{\frac{1}{2b}} \right)^{2b}
\]
Especially for any random variables \( X_i, i = 1, 2, ... \) on a probability space, we have
\[
E\left[ |\sum_j X_j|^{2b} \right] \leq \left( \sum_j E\left[ |X_j|^{2b} \right]^{\frac{1}{2b}} \right)^{2b}.
\]

**Proof.** Since
\[
\left( \sum_j |X_{i,j}| \right)^{2b} = \sum_{j_1, j_2, \cdots, j_{2b}} |X_{i,j_1}| |X_{i,j_2}| \cdots |X_{i,j_{2b}}|
\]
and by the Hölder’s inequality recursively.
\[ \sum_i |X_{i,j_1}| |X_{i,j_2}| \cdots |X_{i,j_{2b}}| \leq \left[ \sum_i |X_{i,j_1}|^{2b} \right]^{1/2} \left[ \sum_i (|X_{i,j_2}| \cdots |X_{i,j_{2b}}|)^{2b} \right]^{1/2} \]

\[ \leq \left[ \sum_i |X_{i,j_1}|^{2b} \right]^{1/2} \left[ \sum_i |X_{i,j_2}|^{2b} \right]^{1/2} \cdots \left[ \sum_i (|X_{i,j_3}| \cdots |X_{i,j_{2b}}|)^{2b} \right]^{1/2} \]

and so on, we have

\[ \sum_i \sum_j |X_{i,j}|^{2b} \leq \sum_i \left[ \sum_{j_1,j_2,\cdots,j_{2b}} |X_{i,j_1}| |X_{i,j_2}| \cdots |X_{i,j_{2b}}| \right] \]

\[ = \sum_{j_1,j_2,\cdots,j_{2b}} \left[ \sum_i |X_{i,j_1}| |X_{i,j_2}| \cdots |X_{i,j_{2b}}| \right] \]

\[ \leq \sum_{j_1,j_2,\cdots,j_{2b}} \left[ \sum_i |X_{i,j_1}|^{2b} \right]^{1/2} \left[ \sum_i |X_{i,j_2}|^{2b} \right]^{1/2} \cdots \left[ \sum_i (|X_{i,j_3}| \cdots |X_{i,j_{2b}}|)^{2b} \right]^{1/2}, \]

which is equal to the right hand side of the inequality in Lemma 2. This completes the proof. \( \square \)

Now we proceed to the proof of Lemma 1.

Define

\[ \mathcal{A}_0^\mu(t) = \int_{\gamma_0[t]} A_0^\mu. \]

Since

\[ \int_0^1 \int_0^{t_1} \cdots \int_0^{t_r-1} d(A_0 + x_\gamma^\mu)(t_1) d(A_0 + x_\gamma^\mu)(t_2) \cdots d(A_0 + x_\gamma^\mu)(t_r) \]

\[ = \sum_{m=0}^r \sum_{1 \leq t_1 < t_2 < \cdots < t_m \leq t_r} \int_0^1 dA_0(t_1) \cdots \int_0^{t_1-1} dx_\gamma^\mu(t_1) \]

\[ \cdots \int_0^{t_{m-1}} dx_\gamma^\mu(t_m) \cdots \int_0^{t_r-1} dA_0(t_r) \]

\[ = \sum_{m=0}^r \sum_{1 \leq t_1 < t_2 < \cdots < t_m \leq t_r} \sum_{j_1,j_2,\cdots,j_r=1}^d \int_0^1 d\overline{A}_0^\mu(t_1) \cdots \int_0^{t_1-1} dx_\gamma^\mu(t_1)^{j_1} \]

\[ \cdots \int_0^{t_{m-1}} dx_\gamma^\mu(t_m)^{j_m} \cdots \int_0^{t_r-1} d\overline{A}_0^\mu(t_r) E_{j_1} E_{j_2} \cdots E_{j_r} \]
and

\[
\int_0^{t_1} \cdots \int_0^{t_{r-1}} d(A_0 + R_{n,k}x^\gamma_\gamma)(t_1)d(A_0 + R_{n,k}x^\gamma_\gamma)(t_2) \cdots d(A_0 + R_{n,k}x^\gamma_\gamma)(t_r)
= \sum_{m=0}^{r} \sum_{1 \leq l_1 < l_2 < l_3 < \cdots < l_m \leq r} \sum_{j_1, j_2, \ldots, j_r = 1}^{d} \int_0^{t_1} \cdots \int_0^{t_{r-1}} dA_j^{j_1}(t_1) \cdots \int_0^{t_{r-1}} dA_j^{j_m}(t_r)
\]

we have, by Lemma 2,

\[
(3.1)
\]

\[
E \left[ \| W^r_\gamma (x) \|^{2b} \right] 
\leq E \left[ \sum_{r=0}^{\infty} \sum_{m=0}^{r} \sum_{1 \leq l_1 < l_2 < l_3 < \cdots < l_m \leq r} \sum_{j_1, j_2, \ldots, j_r = 1}^{d} c_r^{j} \int_0^{t_1} \cdots \int_0^{t_{r-1}} dx_j^{j_1}(t_1) \cdots dx_j^{j_m}(t_r) \right]^{2b}
\]

and

\[
E \left[ \| W^r_\gamma (R_{n,k}x) \|^{2b} \right] 
\leq E \left[ \sum_{r=0}^{\infty} \sum_{m=0}^{r} \sum_{1 \leq l_1 < l_2 < l_3 < \cdots < l_m \leq r} \sum_{j_1, j_2, \ldots, j_r = 1}^{d} c_r^{j} \int_0^{t_1} \cdots \int_0^{t_{r-1}} dx_j^{j_1}(t_1) \cdots dx_j^{j_m}(t_r) \right]^{2b}
\]

(3.2)
Now we begin by recalling the following well known lemma [17]:

**Lemma 3.** Let $X_i, i = 1, 2, \ldots, 2\ell$ be a mean-zero Gaussian system. Then

$$E[X_1X_2 \cdots X_{2\ell}] = \sum_{\text{comb}(\ell)} E[X_{i_1}\gamma_1] E[X_{i_2}\gamma_2] \cdots E[X_{i_{\ell}}\gamma_{\ell}]$$

(3.3)

$$= \frac{1}{2^{2\ell}} \sum_{\sigma \in \mathfrak{S}_{2\ell}} E[X_{\sigma(1)}X_{\sigma(2)}] E[X_{\sigma(3)}X_{\sigma(4)}] \cdots E[X_{\sigma(2\ell-1)}X_{\sigma(2\ell)}],$$

where $\mathfrak{S}_{2\ell}$ denotes the collection of all permutations of $\{1, 2, \ldots, 2\ell\}$ and $\sum_{\text{comb}(\ell)}$ is the summation taken over all pairs of $\{1, 2, \ldots, 2\ell\}$ such that $i_1 < i_2 < \cdots < i_{\ell}$, $i_k \leq j_k$ for $k = 1, 2, \ldots, \ell$.

On the other hand, let $s_i, i = 0, 1, \ldots, r$ be non-negative integers. Then setting $t_0^{s_0} = 1$ and

$$t_i^{s_i} = \begin{cases} 0 & \text{if } s_i = 0, \\ t_i^{s_i-1} + \frac{t_i^{s_i-1}}{n_i} & \text{if } s_i \geq 1 \end{cases}$$

and

$$A[s_i] = \overline{A}_0^{j_i} (t_i^{s_i+1}) - \overline{A}_0^{j_i} (t_i^{s_i}),$$

$$x[s_i] = x^{\varepsilon_i}(t_i^{s_i+1})^{j_i} - x^{\varepsilon_i}(t_i^{s_i})^{j_i},$$

by (2.12) and (2.13), we have

$$E\left[ \left| \int_0^1 d\overline{A}_0^{j_1}(t_1) \cdots \int_0^{t_{11}-1} dx^{\varepsilon_1}(t_1)^{j_1} \cdots \int_0^{t_{1m}-1} dx^{\varepsilon_1}(t_{1m})^{j_1m} \cdots \int_0^{t_{r1}-1} d\overline{A}_0^{j_1}(t_{r1}) \right|^2 \right]$$

$$= E\left[ \left| \lim_{n_1 \to \infty} \sum_{s_1=0}^{n_1-1} A[s_1] \cdots \lim_{n_{11} \to \infty} \sum_{s_1=0}^{n_{11}-1} x[s_1] \cdots \lim_{n_r \to \infty} \sum_{s_r=0}^{n_r-1} A[s_r] \right|^2 \right]$$

$$\leq E\left[ \left| \lim_{n_1 \to \infty} \cdots \lim_{n_{11} \to \infty} \cdots \lim_{n_r \to \infty} \sum_{s_1=0}^{n_1-1} \sum_{s_1=0}^{n_{11}-1} \sum_{s_1=0}^{n_r-1} A[s_1] \cdots x[s_1] \cdots A[s_r] \right|^2 \right].$$
Thus we obtain that (3.1) is dominated by
\[ c_2(A_0)^{2b(r-m)} \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} E \left[ \sum_{s_1 = 0}^{n_1-1} \sum_{s_2 = 0}^{n_2-1} \sum_{s_3 = 0}^{n_3-1} \sum_{s_4 = 0}^{n_4-1} \right] \]
\[ \sum_{s_r = 0}^{n_r-1} |t_1^{s_1+1} - t_1^{s_1}| \cdots |x[s_1^r \gamma]_1| \cdots |x[s_m^r \gamma]_m| \cdots |t_r^{s_r+1} - t_r^{s_r}|^{2b} \]
\[ \leq c_2(A_0)^{2b(r-m)} \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} \left( \sum_{s_1 = 0}^{n_1-1} \sum_{s_2 = 0}^{n_2-1} \sum_{s_3 = 0}^{n_3-1} \sum_{s_4 = 0}^{n_4-1} \right) \]
\[ \sum_{s_r = 0}^{n_r-1} E \left[ (|x[s_1^r \gamma]_1| \cdots |x[s_m^r \gamma]_m|)^{2b} \right] \leq \frac{(2bm)!c_1(\epsilon)^{2bm}}{2^{bm}(bm)!} \left| t_1^{s_1 + 1} - t_1^{s_1} \right|^{2b} \cdots \left| t_m^{s_m + 1} - t_m^{s_m} \right|^{2b}. \]

By Lemma 3, we have
\[ E \left[ (|x[s_1^r \gamma]_1| \cdots |x[s_m^r \gamma]_m|)^{2b} \right] \leq \frac{(2bm)!c_1(\epsilon)^{2bm}}{2^{bm}(bm)!} \left| t_1^{s_1 + 1} - t_1^{s_1} \right|^{2b} \cdots \left| t_m^{s_m + 1} - t_m^{s_m} \right|^{2b}. \]

Therefore we have
\[ E \left[ \int_0^1 dA_0^b(t_1) \cdots \int_0^{t_1} \int_0^{t_{j_1}} \int_0^{t_{j_1+1}} \int_0^{t_{j_2}} \int_0^{t_{j_2+1}} \cdots \int_0^{t_{j_m}} \int_0^{t_{j_m+1}} \cdots \int_0^{t_{j_r}} \right] \frac{1}{2^{r}} \]
\[ \leq c_2(A_0)^{r-m} \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} \sum_{s_1 = 0}^{n_1-1} \sum_{s_2 = 0}^{n_2-1} \sum_{s_3 = 0}^{n_3-1} \sum_{s_4 = 0}^{n_4-1} \cdots \]
\[ \sum_{s_r = 0}^{n_r-1} \left\{ \frac{(2bm)!c_1(\epsilon)^{2bm}}{2^{bm}(bm)!} \right\} \frac{1}{2^{r}} \left| t_1^{s_1 + 1} - t_1^{s_1} \right| \cdots \left| t_1^{s_m + 1} - t_1^{s_m} \right| \cdots \left| t_r^{s_r + 1} - t_r^{s_r} \right|. \]
\[ \leq c_2(A_0)^{r-m} c_1(\epsilon)^m \left\{ \frac{(2bm)!}{2^{bm}(bm)!} \right\} \frac{1}{2^{r}} \int_0^1 \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{r-1}} dt_1 dt_2 \cdots dt_r. \]

Since \((bm)! \leq (m!b^m)^b\), the last term is less or equal
\[ c_8(A_0)^{r} \left\{ \frac{(m!)^b(1 \cdot 2 \cdots (2bm - 2) \cdot (2bm - 1) \cdot 2bm)^{2bm} b^{bm}}{(r!)^{2b}(2^b \cdot 2 \cdot 2 \cdots 2(2bm - 1) \cdot 2bm)} \right\} \frac{1}{2^{r}} \]
\[ \leq c_8(A_0)^{r} \sqrt{2^b m!} \frac{\sqrt{m!}}{r!}, \]
where \(c_8(A_0) = \max\{c_2(A_0), c_1(\epsilon)\} \).

Thus we obtain that (3.1) is dominated by
\[
\left( \sum_{r=0}^{\infty} (d_{cECS}(A_0))^r \right) \sum_{m=0}^{\infty} r C_m \sqrt{2b^m} \sqrt{m!} \frac{1}{r!} 2^b \leq \left( \sum_{r=0}^{\infty} (d_{cECS}(A_0))^r (1 + \sqrt{2b})^r \frac{1}{\sqrt{r!}} \right) 2^b < +\infty,
\]

which yields the proof of (1) of Lemma 1.

By (2.13), we have for \(\zeta = 1\) or \(2\),

\[
E \left[ |R_{\zeta, k}^c x^\zeta(t) - R_{\zeta, k}^c x^\zeta(s)|^2 \right] \leq \left( \sum_{r=1}^{\infty} |\mu_r|^{-1} ((C_{u}^c(t), 0) - (C_{u}^c(s), 0), \hat{e}_r)^2 \right)^{\frac{1}{2}}
\]

(3.4)

\[
= \left( \sum_{r=1}^{\infty} |\mu_r|^{-1} ((C_{u}^c(t) - C_{u}^c(s), 0), \hat{e}_r)^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2k\rho}} \|C_{u}^c(t) - C_{u}^c(s)\|_0 \leq c_{1}(\epsilon) \frac{1}{\sqrt{2k\rho}} |t - s|,
\]

where

(3.5) \(\min_{r=1,2,\ldots} |\mu_r| = \rho > 0\).

In a manner similar to the proof of the former part, (3.2) is dominated by

\[
\left( \sum_{r=0}^{\infty} (d_{cECS}(A_0))^r \right) \sum_{m=0}^{\infty} r C_m \sqrt{2b^m} \sqrt{m!} \frac{1}{r!} 2^b \leq \left( \sum_{r=0}^{\infty} (d_{cECS}(A_0)(1 + 2\sqrt{b} \sqrt{2k\rho})^r \frac{1}{\sqrt{r!}} \right) 2^b < +\infty,
\]

which provides the proofs of (2) and (3) of Lemma 1.

Now we proceed to the proof of the rest of Lemma 1.

Since

\[
|R_{n,k,x}, \hat{e}_r| = |\frac{\sqrt{n\lambda_r}}{\sqrt{1 - 2i(nk\lambda_r \mu_r)}}| < x, \hat{e}_r| \leq \frac{1}{\sqrt{2k|\mu_r|}} |< x, \hat{e}_r|,
\]
by Lemmas 2 and 3 and the Assumption 2, we have under the Renormalization assumption,

\[
E \left[ | \sum_{p,j,\ell=1}^{\infty} \beta^p \beta^j \beta^\ell | < R_{n,k} x, \hat{e}_p > < R_{n,k} x, \hat{e}_j > < R_{n,k} x, \hat{e}_\ell > \beta^{p+j+\ell} \right]^{2b}
\]

\[
\leq E \left[ \left( \sum_{p,j,\ell=1}^{\infty} \frac{\beta^p \beta^j \beta^\ell}{\sqrt{2k|\mu_p|} \sqrt{2k|\mu_j|} \sqrt{2k|\mu_\ell|}} \right) | < x, \hat{e}_p > || < x, \hat{e}_j > || < x, \hat{e}_\ell > | \right]^{2b}
\]

\[
\leq \left( \sum_{p,j,\ell=1}^{\infty} \frac{\beta^p \beta^j \beta^\ell}{\sqrt{2k|\mu_p|} \sqrt{2k|\mu_j|} \sqrt{2k|\mu_\ell|}} \right)^{2b}
\]

\[
\leq c_5(b) \left( \sum_{i=1}^{\infty} \frac{1}{\sqrt{\mu_i}} \right)^{6b} < \infty.
\]

The proofs of (4) and (5) of Lemma 1 are obtained by replacing \( R_{n,k} \) by \( \sqrt{nS} \), in effect the latter estimate is easier than the former.

Since

\[
|a_{ij}^R(R_{n,k}x)| \leq |a_{ij}(0)| + \sum_{\ell=1}^{\infty} | < x, \hat{e}_\ell > | \frac{\sqrt{n\lambda}}{\sqrt{1 - 2n\lambda \mu_i \mu_\ell}} a_{R,ij}^\ell
\]

\[
\leq 1 + \sum_{\ell=1}^{\infty} | < x, \hat{e}_\ell > | \frac{1}{\sqrt{2k|\mu_\ell|}} |a_{R,ij}^\ell|,
\]

where

\[
a_{ij}(0) = \delta_{ij}, i, j = 1, 2, \cdots, m,
\]

noticing the Renormalization assumption, we have

\[
E \left[ | \det_{Reg}(R_{n,k}x) |^{2b} \right] \leq \lim_{m \to \infty} E \left[ \left( \sum_{\sigma \in S_m} | a_{1\sigma(1)}^R(R_{n,k}x)a_{2\sigma(2)}^R(R_{n,k}x) \cdots a_{m\sigma(m)}^R(R_{n,k}x) |^{2b} \right]^{2b}
\]

\[
\leq \lim_{m \to \infty} E \left[ \left( \prod_{i,j} (1 + \sum_{\ell=1}^{\infty} | < x, \hat{e}_\ell > | \frac{1}{\sqrt{2k|\mu_\ell|}} |a_{R,ij}^\ell|) \right)^{2b} \right]
\]

\[
\leq \lim_{m \to \infty} E \left[ \exp \left[ 2b \sum_{i,j} \sum_{\ell=1}^{\infty} | < x, \hat{e}_\ell > | \frac{1}{\sqrt{2k|\mu_\ell|}} |a_{R,ij}^\ell| \right] \right]
\]

\[
\leq E \left[ \exp \left[ 2b \sum_{\ell=1}^{\infty} | < x, \hat{e}_\ell > | \frac{1}{\sqrt{2k|\mu_\ell|}} \left( \sum_{i,j=1}^{\infty} |a_{R,ij}^\ell| \right) \right] \right] < \infty, (see \ [29],
\]

which yields the proofs of (6),(7) and (8) of Lemma 1.
3.2 Proof of Theorem 1

Now we proceed to the proof of Theorem 1. Set

\[ x^\epsilon_{L,\gamma}(t) = \sum_{u=1}^{d} \{ \sum_{i=1}^{L} < x, \hat{e}_i > ((C^\epsilon_u(t), 0), \hat{e}_i) \} \cdot E_u \]

and replace \( x^\epsilon_{\gamma}(t) \) by \( x^\epsilon_{L,\gamma}(t) \) in the definition of \( W^\epsilon_r(x) \). Then the object to be obtained is denoted by

\[ W^\epsilon_{L,\gamma}(x). \]

Define

\[ W^\epsilon, J, L(x) = I + \sum_{r=1}^{J} W^\epsilon_{L,\gamma}(x) \]

and

\[ F^\epsilon, J, L,A_0(x) = \prod_{j=1}^{s} Tr W^\epsilon_{\gamma, J, L}(x). \]

In the definition of \( \det_{J,R} * d_{A_0} * D_{\sqrt{nSx}} \), we replace \( a^R_{ij}(\sqrt{nSx}) \) by

\[ 1 + \sum_{\ell=1}^{L} \sqrt{n\lambda_\ell} < x, \hat{e}_\ell > a^\ell_{R,ii} \]

and \( a^R_{ij}(\sqrt{nSx}) \) by

\[ \sum_{\ell=1}^{L} \sqrt{n\lambda_\ell} < x, \hat{e}_\ell > a^\ell_{R,ij}, \]

which is denoted by \( \det_{J,R,L}(x) \).

Setting

\[ \tilde{F}^\epsilon, J, L,A_0(x) = F^\epsilon, J, L,A_0(x) \det_{J,R,L}(x), \]

\[ G_{L,J}(< x, \hat{e}_1 >, < x, \hat{e}_2 >, \ldots, < x, \hat{e}_L >) \]

\[ = \sum_{r=0}^{N} \tilde{F}^\epsilon, J, L,A_0(x) \{ (ik \sum_{p,j,\ell=1}^{L} < \sqrt{nSx}, \hat{e}_p > < \sqrt{nSx}, \hat{e}_j > < \sqrt{nSx}, \hat{e}_\ell > \beta^{pj\ell} r^r / r!) \} \]

and

\[ Q_{L}(x) = \sum_{j=1}^{L} \mu_j < x, \hat{e}_j >^2, \]
then by (1),(4) and (6) of Lemma 1, we have

\[
\sum_{r=0}^{N} \int_{X} \tilde{F}_{A_{0}}^{r}(\sqrt{nS}x) \exp \left[ ikQ(\sqrt{nS}x) \right] \\
\times \left\{ (ik \sum_{p,j,\ell=1}^{\infty} \langle \sqrt{nS}x, \hat{e}_{p} \rangle <_{\sqrt{nS}x} \hat{e}_{j} >_{\sqrt{nS}x} \hat{e}_{\ell} >_{\beta^{pj\ell}r} / r! \right\} \mu(dx)
\]

(3.6)

\[
= \lim_{L \to \infty} \lim_{J \to \infty} \int_{X} G_{L,J}(\langle x, \hat{e}_{1} \rangle, \langle x, \hat{e}_{2} \rangle, \cdots, \langle x, \hat{e}_{L} \rangle) \\
\times \exp \left[ ikQ_{L}(\sqrt{nS}x) \right] \mu(dx)
\]

\[
= \lim_{L \to \infty} \lim_{J \to \infty} \int_{X} G_{L,J}(x_{1}, x_{2}, \cdots, x_{L}) \\
\times \exp \left[ ik \left( \sum_{j=1}^{L} n\lambda_{j}(x_{j}^{2}) \right) \right] \prod_{j=1}^{L} \frac{1}{\sqrt{2\pi}} e^{-x_{j}^{2}/2} dx_{j}.
\]

(3.6)

Setting

\[ z_{j} = 1 - 2ikn\lambda_{j} \mu_{j} = |z_{j}| e^{-i\alpha_{j}}, 0 < (\text{sign } \mu_{j})\alpha_{j} < \frac{\pi}{2}, \] we get by using recursively the Cauchy integral theorem, that (3.6) is equal to

\[
\lim_{L \to \infty} \lim_{J \to \infty} \int \cdots \int_{R^{L}} \exp \left[ \frac{1}{2} \sum_{j=1}^{L} \frac{\alpha_{j}}{2} \right] \\
\times G_{L,J}(e^{i\alpha_{1} x_{1}}, e^{i\alpha_{2} x_{2}}, \cdots, e^{i\alpha_{L} x_{L}}) \prod_{j=1}^{L} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{|z_{j} | x_{j}^{2}}{2} \right] dx_{j}.
\]

(3.7)

Using (2),(5) and (7) of Lemma 1, we see that (3.7) is equal to

\[
\lim_{L \to \infty} \lim_{J \to \infty} \frac{1}{\sqrt{\det(1-2iknSQ_{A_{0}})\sqrt{nS}}} \int_{X} G_{L,J}(\langle R_{n,k}x, \hat{e}_{1} \rangle, \langle R_{n,k}x, \hat{e}_{2} \rangle, \cdots, \langle R_{n,k}x, \hat{e}_{L} \rangle) \mu(dx)
\]

\[
= \frac{1}{\sqrt{\det(1-2iknSQ_{A_{0}})\sqrt{nS}}} \sum_{r=0}^{N} \int_{X} \tilde{F}_{A_{0}}^{r}(R_{n,k}x) \\
\times \left\{ (ik \sum_{p,j,\ell=1}^{\infty} \langle R_{n,k}x, \hat{e}_{p} \rangle <_{R_{n,k}x} \hat{e}_{j} >_{R_{n,k}x} \hat{e}_{\ell} >_{\beta^{pj\ell}r} / r! \right\} \mu(dx).
\]

(3.7)

Therefore
\[ Z_{2,k}^n = \int_X \exp \left[ ikQ(\sqrt{nS}x) \right] \mu(dx) = \frac{1}{\sqrt{\det(1 - 2ik\sqrt{nSQ_A} \sqrt{S})}}, \]

which, together with (3),(5) and (8) of Lemma 1, completes the proof of Theorem 1.

3.3 Proof of Theorem 2

Now we proceed to the proof of Theorem 2. Since

\[ \int_X \exp \left[ ik^{2n}Q(\sqrt{S}x) \right] \mu(dx) = \frac{1}{\sqrt{\det(1 - 2ik^{2n}\sqrt{SQ_A} \sqrt{S})}}, \]

we have that (2.34) is equal to

\[ \sqrt{\det(1 - 2ik^{2n}\sqrt{SQ_A} \sqrt{S})} \int_X \tilde{F}^x_{A_0}(\frac{\sqrt{k^{2n}S}}{\sqrt{k}}x) \exp \left[ ik^{2n}Q(\sqrt{S}x) \right] \]

\[ \left\{ \sum_{r=N+1}^{\infty} \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^\infty < \sqrt{k^{2n}S}x, \hat{e}_p> < \sqrt{k^{2n}S}x, \hat{e}_j> < \sqrt{k^{2n}S}x, \hat{e}_\ell> \beta^{pj\ell}r^r/r! \right\} \mu(dx). \]

By the convergence assumption (S,2), we have

\[ E\left[ \exp \left[ \sum_{i=1}^{\infty} \sqrt{\lambda_i} |< x, \hat{e}_i > | \right] \right] < +\infty, \]

so that by the Chebyshev inequality, we get

\[ \mu\left[ x \in X; \sum_{i=1}^{\infty} \sqrt{\lambda_i} |< x, \hat{e}_i > |> 3k^\delta \right] \leq c_9 e^{-3k^\delta}. \]

Let

\[ X_k = \left\{ x \in X; \sum_{i=1}^{\infty} \sqrt{\lambda_i} |< x, \hat{e}_i >| \leq 3k^\delta \right\}. \]

Then we prove
Lemma 4.

\[
\sqrt{\det(1 - 2ik^{2\eta}\sqrt{SQ_{\alpha_0}})} \int_{X_0} \tilde{F}_{\alpha_0} \left( \frac{\sqrt{k^{2\eta}S}}{\sqrt{k}} \right) \exp \left[ ik^{2\eta}Q(\sqrt{S} x) \right] \\
\times \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \sum_{p, j, \ell = 1}^{\infty} < \sqrt{k^{2\eta}S x}, \hat{e}_p > < \sqrt{k^{2\eta}S x}, \hat{e}_j > < \sqrt{k^{2\eta}S x}, \hat{e}_\ell > \beta^{pj\ell} r / r! \right) \right\} \mu(dx)
\]

\[= o(e^{-k^{3}}).\]

Proof. First we show that for sufficiently large \(k\),

(3.10) \[\lim_{k \to \infty} \sqrt{\det(1 - 2ik^{2\eta}\sqrt{SQ_{\alpha_0}})} e^{-\frac{k^{3}}{2}} = 0.\]

By the convergence assumption (S,4), we have for some natural number \(M\) such that \(M\delta > k^{3}\) and

\[\sum_{j=1}^{\infty} M^{\sqrt{\lambda_j}} | \mu_j | = C \leq c_3 < +\infty.\]

Hence we have

\[\left| \sqrt{\det(1 - 2ik^{2\eta}\sqrt{SQ_{\alpha_0}})} \exp \left[ - k^{3} \sum_{j=1}^{\infty} \frac{M^{\sqrt{\lambda_j}} | \mu_j |}{2C} \right] \right| \]

\[= \prod_{j=1}^{\infty} \sqrt{1 - 2ik^{2\eta} \lambda_j \mu_j} \exp \left[ - k^{3} \frac{M^{\sqrt{\lambda_j}} | \mu_j |}{2C} \right] \]

\[\leq \prod_{j=1}^{\infty} \frac{\sqrt{1 - 2ik^{2\eta} \lambda_j \mu_j}}{\sqrt{1 + \frac{M^{3\sqrt{\lambda_j}} | \mu_j |}{2C}}} \]

which gives (3.10).

Since \(3\eta < \frac{1}{12} < \frac{1}{2}\) and

(3.11) \[\tilde{F}_{\alpha_0} \left( \frac{\sqrt{k^{2\eta}S}}{\sqrt{k}} \right) \left\{ \sum_{r=1}^{N} \left( \frac{i}{\sqrt{k}} \sum_{p, j, \ell = 1}^{\infty} < \sqrt{k^{2\eta}S x}, \hat{e}_p > < \sqrt{k^{2\eta}S x}, \hat{e}_j > < \sqrt{k^{2\eta}S x}, \hat{e}_\ell > \beta^{pj\ell} r / r! \right) \right\} \]

\[= \tilde{F}_{\alpha_0} \left( \frac{k^{\eta}}{\sqrt{k}} \sqrt{S} x \right) \left\{ \sum_{r=1}^{N} \left( \frac{k^{3\eta}}{\sqrt{k}} \sum_{p, j, \ell = 1}^{\infty} < \sqrt{S} x, \hat{e}_p > < \sqrt{S} x, \hat{e}_j > < \sqrt{S} x, \hat{e}_\ell > \beta^{pj\ell} r / r! \right) \right\},\]
by the estimations similar to the proof of Lemma 1, we get under Assumption 2,

\[
\int_X \left| \tilde{F}_{A_0} \left( \frac{k\eta}{\sqrt{k}} \sqrt{S}x \right) \right| \times \left\{ \sum_{r=1}^{N} \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^{2\eta}Sx}, \hat{e}_p > < \sqrt{k^{2\eta}Sx}, \hat{e}_j > < \sqrt{k^{2\eta}Sx}, \hat{e}_\ell > \beta_{pjal} \right)^r \right\}^2 d\mu(dx) \\
\leq c_{10}(N) < +\infty,
\]

so that by (3.9) and (3.10) we have

(3.12)

\[
\sqrt{\det(1 - 2ik^{2\eta}\sqrt{S}Q_{A_0}\sqrt{S})} \int_{X_k^\varepsilon} \tilde{F}_{A_0}^{\varepsilon} \left( \frac{\sqrt{k^{2\eta}S}}{\sqrt{k}} x - x \right) \exp \left[ ik^{2\eta}Q(\sqrt{S}x) \right] \times \exp \left[ \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^{2\eta}Sx}, \hat{e}_p > < \sqrt{k^{2\eta}Sx}, \hat{e}_j > < \sqrt{k^{2\eta}Sx}, \hat{e}_\ell > \beta_{pjal} \right] d\mu(dx) = o(e^{-k^4}).
\]

Similarly we get

\[
\int_X | \tilde{F}_{A_0}^{\varepsilon} \left( \frac{\sqrt{k^{2\eta}S}}{\sqrt{k}} x - x \right) |^2 d\mu(dx) \leq c_{11} < +\infty,
\]

so that using that

\[
Q(\sqrt{k^{2\eta}Sx}) + \frac{k^{3\eta}}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{S}x, \hat{e}_p > < \sqrt{S}x, \hat{e}_j > < \sqrt{S}x, \hat{e}_\ell > \beta_{pjal}
\]

is real, by (3.9) and (3.10) we have

(3.13)

\[
\sqrt{\det(1 - 2ik^{2\eta}\sqrt{S}Q_{A_0}\sqrt{S})} \int_{X_k^\varepsilon} \tilde{F}_{A_0}^{\varepsilon} \left( \frac{\sqrt{k^{2\eta}S}}{\sqrt{k}} x - x \right) \exp \left[ ik^{2\eta}Q(\sqrt{S}x) \right] \times \exp \left[ \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^{2\eta}Sx}, \hat{e}_p > < \sqrt{k^{2\eta}Sx}, \hat{e}_j > < \sqrt{k^{2\eta}Sx}, \hat{e}_\ell > \beta_{pjal} \right] d\mu(dx) = o(e^{-k^4}).
\]

Then
\[
\int_{X_k} \tilde{F}_{\delta A_0}(\frac{\sqrt{k^2 \eta S}}{\sqrt{k}} x) \exp \left[ i k^{2n} Q(\sqrt{S} x) \right] \\
\left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^2 \eta S} x, \hat{e}_p > < \sqrt{k^2 \eta S} x, \hat{e}_j > < \sqrt{k^2 \eta S} x, \hat{e}_\ell > \beta^{|p|j|\ell|}/|r|! \right) \right\} \mu(dx)
\]
\[
= \int_{X_k} \tilde{F}_{\delta A_0}(\frac{\sqrt{k^2 \eta S}}{\sqrt{k}} x) \exp \left[ i k^{2n} Q(\sqrt{S} x) \right] \\
\times \left( \exp \left[ \sum_{r=1}^{N} \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^2 \eta S} x, \hat{e}_p > < \sqrt{k^2 \eta S} x, \hat{e}_j > < \sqrt{k^2 \eta S} x, \hat{e}_\ell > \beta^{|p|j|\ell|}/|r|! \right) \right] \\
- \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^2 \eta S} x, \hat{e}_p > < \sqrt{k^2 \eta S} x, \hat{e}_j > < \sqrt{k^2 \eta S} x, \hat{e}_\ell > \beta^{|p|j|\ell|}/|r|! \right) \right\} \mu(dx),
\]
which, together with (3.12) and (3.13), completes the proof of Lemma 4.

By Lemma 4, for sufficiently large \( k \) we have

(3.14)

\[
\sqrt{\det(1-2ik^{2n}\sqrt{S}Q_{A_0}/\sqrt{S})} \int_{X} \tilde{F}_{\delta A_0}(\frac{\sqrt{k^2 \eta S}}{\sqrt{k}} x) \exp \left[ i k^{2n} Q(\sqrt{S} x) \right] \\
\times \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^2 \eta S} x, \hat{e}_p > < \sqrt{k^2 \eta S} x, \hat{e}_j > < \sqrt{k^2 \eta S} x, \hat{e}_\ell > \beta^{|p|j|\ell|}/|r|! \right) \right\} \mu(dx)
\]
\[
= \sqrt{\det(1-2ik^{2n}\sqrt{S}Q_{A_0}/\sqrt{S})} \int_{X_k} \tilde{F}_{\delta A_0}(\frac{\sqrt{k^2 \eta S}}{\sqrt{k}} x) \exp \left[ i k^{2n} Q(\sqrt{S} x) \right] \\
\times \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < \sqrt{k^2 \eta S} x, \hat{e}_p > < \sqrt{k^2 \eta S} x, \hat{e}_j > < \sqrt{k^2 \eta S} x, \hat{e}_\ell > \beta^{|p|j|\ell|}/|r|! \right) \right\} \\
\mu(dx) + o(e^{-k^\delta}).
\]

Set

\[ H_{L,J}(< x, \hat{e}_1 >, < x, \hat{e}_2 >, \cdots, < x, \hat{e}_L >) \]
\[
= \tilde{F}_{L,A_0,J}(\frac{\sqrt{k^2 \eta S}}{\sqrt{k}} x) \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{L} < \sqrt{k^2 \eta S} x, \hat{e}_p > < \sqrt{k^2 \eta S} x, \hat{e}_j > < \sqrt{k^2 \eta S} x, \hat{e}_\ell > \beta^{|p|j|\ell|}/|r|! \right) \right\}
\]
and

\[ X_k^L = \left\{ x \in X; \sum_{j=1}^{L} \{ < x, \hat{e}_j > \leq 3k^\delta \} \right\}.
\]
Further we get by the Cauchy integral theorem,

\[ \int_{X_k} \tilde{F}_{\nu_0} (\frac{\sqrt{k}\eta S}{\sqrt{k}} x) \exp \left( ik^{2\eta} Q(\sqrt{k} x) \right) \times \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \sum_{p,j,l=1}^{L} < \sqrt{k^{2\eta} S x}, \epsilon_p > < \sqrt{k^{2\eta} S x}, \epsilon_j > < \sqrt{k^{2\eta} S x}, \epsilon_l > \beta^{2(\alpha+\epsilon)}_r / r! \right) \right\} \mu(dx) \]

= \lim_{L \to \infty} \lim_{J \to \infty} \int_{X_k} \exp \left( ik^{2\eta} Q_L(\sqrt{S} x) \right) H_{L,J}(x, \epsilon_1, < x, \epsilon_2 >, \cdots, < x, \epsilon_L >) \mu(dx) \]

Since

\[ \int_{X_k} \exp \left( ik^{2\eta} Q_L(\sqrt{S} x) \right) H_{L,J}(x, \epsilon_1, < x, \epsilon_2 >, \cdots, < x, \epsilon_L >) \mu(dx) \]

= \int \cdots \int \{ \sum_{j=1}^{L} \frac{\lambda_j \mu_j x_j^2}{\sqrt{2\pi}} \} H_{L,J}(x_1, x_2, \cdots, x_L) \prod_{j=1}^{L} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}} dx_j \]

= \int_{|x_1| \leq \frac{\lambda_1}{\sqrt{2\lambda_1}} \int_{|x_2| \leq \frac{\lambda_2}{\sqrt{2\lambda_2}} (3\lambda_2 - \sqrt{\lambda_2} |x_1|)} \cdots \int_{|x_L| \leq \frac{\lambda_L}{\sqrt{2\lambda_L}} (3\lambda_L - \sqrt{\lambda_L} |x_{L-1}|)} x_L \prod_{j=1}^{L} \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{1}{2} (1 - 2ik^{2\eta} \lambda_j \mu_j) x_j^2 \right) dx_j, \]

setting

\[ z_j = 1 - 2ik^{2\eta} \lambda_j \mu_j = |z_j| e^{-i\alpha_j}, 0 < \alpha_j < \frac{\pi}{2}, \]

we get by the Cauchy integral theorem,

\[ \int_{|x_L| \leq \frac{\lambda_L}{\sqrt{2\lambda_L}} (3\lambda_L - \sqrt{\lambda_L} |x_{L-1}|)} H_{L,J}(x_1, x_2, \cdots, x_L) \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{1}{2} (1 - 2ik^{2\eta} \lambda_L \mu_L) x_L^2 \right) dx_L \]

= \int_{|x_L| \leq \frac{\lambda_L}{\sqrt{2\lambda_L}} (3\lambda_L - \sqrt{\lambda_L} |x_{L-1}|)} e^{ix_\pi / 2} H_{L,J}(x_1, \cdots, e^{i\pi / 2} x_L) \frac{1}{\sqrt{2\pi}} e^{-\frac{x_L^2}{2}} dx_L \]

+ ( \int_{0}^{\pi} + \int_{\pi}^{\pi+\pi} ) \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{2z_L}{2} \frac{1}{\sqrt{\lambda_L}} (3\lambda_L - \sum_{j=1}^{L-1} \sqrt{\lambda_j} |x_j| e^{i\theta} \right) \]

\times H_{L,J}(x_1, \cdots, \frac{1}{\sqrt{\lambda_L}} (3\lambda_L - \sum_{j=1}^{L-1} \sqrt{\lambda_j} |x_j|) e^{i\theta} - \frac{1}{\sqrt{\lambda_L}} (3\lambda_L - \sum_{j=1}^{L-1} \sqrt{\lambda_j} |x_j|) i e^{i\theta} d\theta.

Further
Using the Cauchy integral theorem, we have

\[
\left| \int \frac{dz_1}{\sqrt{|z_1|}} \cdots \int \frac{dz_{L-1}}{\sqrt{|z_{L-1}|}} \right| \leq \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{e^{i\theta}}{2} \frac{\sqrt{|z_1|}}{\sqrt{|z_1|}} \cdots \frac{1}{\sqrt{L-1}} \frac{\sqrt{|z_{L-1}|}}{\sqrt{|z_{L-1}|}} \right] |H_{L,J}(x_1, \ldots, e^{i\theta})| \prod_{j=1}^{L-1} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} z_j x_j^2 \right] \] 

Since

\[
\int |z_1| \leq \frac{\sqrt{3}}{\sqrt{2\pi}} \int |z_2| \leq \frac{1}{\sqrt{2\pi}} \left(3k^4 - \frac{3}{\sqrt{|z_1|}} |x_1| \right) \cdots \int |z_{L-1}| \leq \frac{1}{\sqrt{2\pi}} \left(3k^4 - \frac{3}{\sqrt{|z_{L-1}|}} |x_{L-1}| \right) e^{\frac{i\theta}{2}} H_{L,J}(x_1, \ldots, e^{i\theta} x_L) 
\]

\[
\prod_{j=1}^{L-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{e^{i\theta}}{2} x_j^2} dx_j \frac{1}{\sqrt{2\pi}} e^{-\frac{|z_j| x_j^2}{2}} dx_j \frac{1}{\sqrt{2\pi}} e^{-\frac{|z_{L-1}} x_{L-1}^2}{2} dx_{L-1},
\]

the right hand side of the above equality is equal to

\[
\int |x_1| \leq \frac{\sqrt{3}}{\sqrt{2\pi}} \int |x_2| \leq \frac{1}{\sqrt{2\pi}} \left(3k^4 - \frac{3}{\sqrt{|z_1|}} |x_1| \right) \cdots \int |x_{L-1}| \leq \frac{1}{\sqrt{2\pi}} \left(3k^4 - \frac{3}{\sqrt{|z_{L-1}|}} |x_{L-1}| \right) e^{i\theta} H_{L,J}(x_1, \ldots, e^{i\theta} x_L) 
\]

\[
\prod_{j=1}^{L-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{e^{i\theta}}{2} x_j^2} dx_j \frac{1}{\sqrt{2\pi}} e^{-\frac{|z_j| x_j^2}{2}} dx_j \frac{1}{\sqrt{2\pi}} e^{-\frac{|z_{L-1}} x_{L-1}^2}{2} dx_{L-1}.
\]

Using the Cauchy integral theorem, we have
\[ \int_{|x_{L-1}| \leq \frac{1}{\sqrt{\lambda_{L-1}}}(3k^3 - \sum_{j \neq L-1, j=1}^{L} \sqrt{\lambda_j}|x_j|)} H_{L,j}(x_1, \ldots, e^{i\frac{\alpha_{L}}{2}}x_L) \frac{1}{\sqrt{2\pi}} e^{-\frac{x_{L-1}^2}{2}} dx_{L-1} = \int_{|x_{L-1}| \leq \frac{1}{\sqrt{\lambda_{L-1}}}(3k^3 - \sum_{j \neq L-1, j=1}^{L} \sqrt{\lambda_j}|x_j|)} e^{i\frac{\alpha_{L-1}}{2}} H_{L,j}(x_1, \ldots, e^{i\frac{\alpha_{L-1}}{2}}x_{L-1}, e^{i\frac{\alpha_{L}}{2}}x_L) \frac{1}{\sqrt{2\pi}} e^{-\frac{|x_{L-1}|^2}{2}} dx_{L-1} + R. \]

\( R \) is a remainder, which is estimated as follows.

\[
| R | \leq \int_0^{\frac{|x_{L-1}|}{x}} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{Re(z_{L-1}e^{2i\theta} )}{2\sqrt{\lambda_{L-1}}} (3k^3 - \sum_{j \neq L-1, j=1}^{L} \sqrt{\lambda_j} |x_j|)^2 \right] \times \sum_{\nu=0}^{1} | H_{L,j}(x_1, \ldots, (\frac{-1)^\nu}{\sqrt{\lambda_{L-1}}} (3k^3 - \sum_{j \neq L-1, j=1}^{L} \sqrt{\lambda_j} |x_j|) e^{i\theta}, e^{i\frac{\alpha_L}{2}}x_L) | \frac{1}{\sqrt{\lambda_{L-1}}} 3k^3 d\theta. \]

By repeating this argument recursively, we have

\[
\int \int \ldots \int \{ \sum_{j=1}^{L} \sqrt{\lambda_j}|x_j| \leq 3k^3 \} \exp [ik^2 \sum_{j=1}^{L} \lambda_j \mu_j x_j^2] H_{L,j}(x_1, x_2, \ldots, x_L) \prod_{j=1}^{L} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}} dx_j = \int \int \ldots \int \{ \sum_{j=1}^{L} \sqrt{\lambda_j}|x_j| \leq 3k^3 \} e^{i \sum_{j=1}^{L} \frac{\alpha_j}{2}} H_{L,j}(e^{i\frac{\alpha_1}{2}}x_1, e^{i\frac{\alpha_2}{2}}x_2, \ldots, e^{i\frac{\alpha_L}{2}}x_L) \prod_{j=1}^{L} \frac{1}{\sqrt{2\pi}} e^{-\frac{|x_j|^2}{2}} dx_j + R_L, \]

where
\[ |R_L| \leq \sum_{m=1}^{L} \left| \int_{|x_1| \leq \frac{3k^\delta}{\sqrt{\lambda_1}} \int_{|x_2| \leq \frac{1}{\sqrt{\lambda_2}} (3k^\delta - \sqrt{\lambda_1} |x_1|)} \ldots \int_{|x_L| \leq \frac{1}{\sqrt{\lambda_L}} (3k^\delta - \sum_{j \neq m, j=1}^{L-1} \sqrt{\lambda_j} |x_j|)} \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{\text{Re}(z_m e^{2i\theta})}{2\sqrt{\lambda_m}} (3k^\delta - \sum_{j \neq m, j=1}^{L} \sqrt{\lambda_j} |x_j|) e^{i\theta} \right] \frac{1}{\sqrt{\lambda_m}} 3k^\delta \right] \]

Now, we prove that for sufficiently large \( k \),

\[ |\det(1 - 2ik^{2\eta} \sqrt{S}Q_{A_0} \sqrt{S})| \sup_{L} |R_L| = o(e^{-L^\delta}). \]

We begin by remarking that if

\[ x \in \left\{ x \in R^L; |x_1| \leq \frac{3k^\delta}{\sqrt{\lambda_1}} |x_2| \leq \frac{1}{\sqrt{\lambda_2}} (3k^\delta - \sqrt{\lambda_1} |x_1|), \ldots, \right\}, \]

\[ |x_{m-1}| \leq \frac{1}{\sqrt{\lambda_{m-1}}} (3k^\delta - \sum_{j=1}^{m-2} \sqrt{\lambda_j} |x_j|), |x_{m+1}| \leq \frac{1}{\sqrt{\lambda_{m+1}}} (3k^\delta - \sum_{j=1}^{m-1} \sqrt{\lambda_j} |x_j|), \ldots, \]

\[ |x_L| \leq \frac{1}{\sqrt{\lambda_L}} (3k^\delta - \sum_{j \neq m, j=1}^{L-1} \sqrt{\lambda_j} |x_j|) \}

the following inequalities hold.

We have

\[ \left| \sum_{\ell=1}^{m-1} \sqrt{\lambda_\ell} x_\ell a_{R,ij}^\ell + \sqrt{\lambda_m} e^{i\theta} (3k^\delta - \sum_{\ell \neq m, \ell=1}^{L} \sqrt{\lambda_\ell} |x_\ell|) a_{R,ij}^m + \sum_{\ell=1}^{L} \sqrt{\lambda_\ell} e^{i\frac{\theta}{2}} x_\ell a_{R,ij}^\ell \right| \]

\[ \leq \left\{ \sum_{\ell=1}^{m-1} \sqrt{\lambda_\ell} |x_\ell| + \sqrt{\lambda_m} e^{i\theta} (3k^\delta - \sum_{\ell \neq m, \ell=1}^{L} \sqrt{\lambda_\ell} |x_\ell|) |x_\ell| + \sum_{\ell=1}^{L} \sqrt{\lambda_\ell} |x_\ell| \right\} \]

\[ \times \sup_{\ell} \sqrt{\lambda_\ell} \sup_{\ell} a_{R,ij}^\ell \leq 3k^\delta \sup_{\ell} \sqrt{\lambda_\ell} |a_{R,ij}^\ell| \}

Then by (S.3) and \( \eta + \delta \leq 2\eta < \frac{\delta}{12} < \frac{1}{2} \),
| det_{J,R} * dA_0 + D | \frac{k^0}{\sqrt{k}} \left\{ \sum_{m=1}^{m-1} \sqrt{\lambda_m} e_j^m + \sqrt{\lambda_m} e^{i\theta (3k \delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j} | x_j |)} e_m + \sum_{m=m+1}^{L} \sqrt{\lambda_m} e^{i\theta} x_j e_j \right\} | \\
\leq \exp \left[ \frac{k^0}{\sqrt{k}} 3k^3 \sum_{i,j=1}^\infty \sup_{\ell} \sqrt{\lambda_\ell} | a_{R,ij} | \right] \leq e^{c_{12}}.

Since

\left| \sum_{\ell=1}^{m-1} \sqrt{\lambda_\ell} x_\ell ((C_u^0(t), 0), \tilde{c}_\ell) + \sqrt{\lambda_m} e^{i\theta (3k \delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j} | x_j |)} ((C_u^0(t), 0), \tilde{c}_m) \right| \\
+ \sum_{\ell=m+1}^L \sqrt{\lambda_\ell} e^{i\alpha_\ell} x_\ell ((C_u^0(t), 0), \tilde{c}_\ell) | \\
\leq \left\{ \sum_{\ell=1}^{m-1} \sqrt{\lambda_\ell} | x_\ell | + \sqrt{\lambda_m} e^{i\theta (3k \delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j} | x_j |)} \right\} ||C_u^0(t)||_0 \\
\leq 3k^3 ||C_u^0(t)||_0 \\
and

| \tilde{A}_0'(t) | \leq c_2(A_0)t,

similarly we have for sufficiently large k,

| F_{L,A_0}^J (x_1, \ldots, (-1)^\nu \sqrt{\lambda_m} e^{i\theta (3k \delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j} | x_j |)} e^{\frac{m+1}{2} x_{m+1}, \ldots, e^{\alpha_\ell} x_L} | \\
\leq c_{13}(A_0).

By Assumption 2 and since we take \delta < \eta < \frac{1}{12},

\sum_{r=N+1}^{\infty} i \frac{k^3}{\sqrt{k}} \left\{ \sum_{p=1}^{m-1} \sqrt{\lambda_p} x_p + \sqrt{\lambda_m} e^{i\theta (3k \delta - \sum_{p \neq m, p=1}^L \sqrt{\lambda_p} | x_p |)} + \sum_{p=m+1}^L \sqrt{\lambda_p} e^{i\alpha_p} x_p \right\} \\
\times \left\{ \sum_{j=1}^{m-1} \sqrt{\lambda_j} x_j + \sqrt{\lambda_m} e^{i\theta (3k \delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j} | x_j |)} + \sum_{j=m+1}^L \sqrt{\lambda_j} e^{i\alpha_j} x_j \right\} \\
\times \left\{ \sum_{\ell=1}^{m-1} \sqrt{\lambda_\ell} x_\ell + \sqrt{\lambda_m} e^{i\theta (3k \delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j} | x_\ell |)} + \sum_{\ell=m+1}^L \sqrt{\lambda_\ell} e^{i\alpha_\ell} x_\ell \right\} \beta^pL | r | / r! \\
\leq e^{\frac{3}{4k}(3k^3\beta)^3} = c_{14}(\beta).

Then for sufficiently large k,
and noticing (S.1), we have for sufficiently large we get

$$
|\ H_{L,j}(\cdots,(-1)^{\nu}e^{i\theta}3k^{\delta} - \sum_{j \neq m,j=1}^{L} \sqrt{\lambda_j} |x_j|) | e^{i\alpha_{m+1}} x_{m+1},\cdots, e^{i\alpha_L} x_L |
\leq e^{c_{12}c_{13}(A_0)c_{14}(\beta)} = c_{15}(A_0, \beta) < +\infty.
$$

Now first we consider the case where

$$
3k^{\delta} - \sum_{j \neq m,j=1}^{L} \sqrt{\lambda_j} |x_j| > \frac{3k^{\delta}}{2}.
$$

Since

$$
z_m e^{2i\theta} = |z_m| e^{i(-\alpha_m + 2\theta)} \text{ and } |\alpha_m + 2\theta| \leq |\alpha_m|,
$$

we get $Re(z_m e^{2i\theta}) \geq 1$ and further noticing that

$$
|\alpha_m | \leq 2k^{2\eta} \lambda_m | \mu_m |, \quad |z_m | = | 1 - 2\sqrt{-1}k^{2\eta} \lambda_m \mu_m | \geq 1
$$

and noticing (S.1), we have for sufficiently large $k$,

$$
|\sum_{m=1}^{L} \int_{|x_1| \leq \frac{3k^{\delta}}{2\sqrt{\lambda_1}}} \int_{|x_2| \leq \frac{1}{\sqrt{2\pi}}(3k^{\delta} - \sqrt{\lambda_2} |x_2|)} \cdots \int_{|x_L| \leq \frac{1}{\sqrt{2\pi}}(3k^{\delta} - \sum_{j \neq m,j=1}^{L-1} \sqrt{\lambda_j} |x_j|)} \frac{1}{\sqrt{\lambda_m}} |x_m| \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{Re(z_m e^{2i\theta})}{2\sqrt{\lambda_m}}(3k^{\delta} - \sum_{j \neq m,j=1}^{L} \sqrt{\lambda_j} |x_j|)\right]^2 |\frac{1}{\sqrt{\lambda_m}} 3k^{\delta} \right] |\sum_{\nu=0}^{L} H_{L,j}(x_1,\cdots,(-1)^{\nu}e^{i\theta}(3k^{\delta} - \sum_{j \neq m,j=1}^{L} \sqrt{\lambda_j} |x_j|) |e^{i\theta}, e^{i\alpha_{m+1}} x_{m+1},\cdots, e^{i\alpha_L} x_L |
\right|
\times \chi(3k^{\delta} - \sum_{j \neq m,j=1}^{L} \sqrt{\lambda_j} |x_j| > \frac{3k^{\delta}}{2}) \prod_{j=1}^{m-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}} dx_j \prod_{j=m+1}^{L} \frac{1}{\sqrt{2\pi}} e^{-\frac{|x_j|}{2}} dx_j d\theta |
\leq \sum_{m=1}^{L} 2c_{15}(A_0, \beta) \frac{1}{\sqrt{2\pi} \sqrt{\lambda_m}} 3k^{\delta} |\alpha_m | \exp\left[-\frac{(3k^{\delta})^2}{8 \max_{m=1,2,\cdots, L} \sqrt{\lambda_m}} \right]
\leq \sum_{m=1}^{L} \frac{12c_{15}(A_0, \beta)}{\sqrt{2\pi} \sqrt{\lambda_m}} 3k^{\delta} |\mu_m | k^{2\eta+\delta} \exp\left[-\frac{9k^{2\delta}}{8 \max_{m=1,2,\cdots, L} \sqrt{\lambda_m}} \right]
\leq c_{16}e^{-k^{\delta}}.
$$

Secondly we look at the complementary case :

$$
(3.18) \quad 3k^{\delta} - \sum_{j \neq m,j=1}^{L} \sqrt{\lambda_j} |x_j| \leq \frac{3k^{\delta}}{2}.
$$
In this case, dominating
\[
\exp \left[ - \frac{\text{Re}(z_m e^{2i\theta})}{2\sqrt{\lambda_m}} (3k^\delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j | x_j |})^2 \right]
\]
by 1 in the above estimation, we have that the estimation of the right hand side of (3.13) for this case is reduced to
\[
\hat{\mu}(\frac{3k^\delta}{2}) \leq \sum_{j \neq m, j=1}^L \sqrt{\lambda_j | x_j |},
\]
where \( \hat{\mu} = \prod_{j=1}^{m-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}} dx_j \prod_{j=m+1}^L \frac{1}{\sqrt{2\pi}} e^{-\frac{|x_j|^2}{2}} dx_j. \)

Since
\[
\int \cdots \int_{R_{\varepsilon-1}} \exp \left[ \sum_{j \neq m, j=1}^L \sqrt{\lambda_j | x_j |} \prod_{j=1}^{m-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_j^2}{2}} dx_j \prod_{j=m+1}^L \frac{1}{\sqrt{2\pi}} e^{-\frac{|x_j|^2}{2}} dx_j \right] \leq E \left[ \exp \left[ \sum_{j=1}^{\infty} \sqrt{\lambda_j | x_j |} < x, e_j > \right] \right] \leq c_{17},
\]
so that
\[
\hat{\mu}(\frac{3k^\delta}{2}) \leq \sum_{j \neq m, j=1}^L \sqrt{\lambda_j | x_j |} \leq c_{17} e^{-\frac{3k^\delta}{2}}.
\]

Hence in the case of (3.18), we have
\[
| \sum_{m=1}^L \int_{|x_1| \leq \frac{3k^\delta}{2} \sqrt{\lambda_m}} \frac{1}{\sqrt{2\pi}} \int_{|x_2| \leq \frac{1}{\sqrt{2\pi}} (3k^\delta - \sqrt{\lambda_1 | x_1 |})} \cdots \int_{|x_L| \leq \frac{1}{\sqrt{2\pi}} (3k^\delta - \sum_{j \neq m, j=1}^{L-1} \sqrt{\lambda_j | x_j |})} \int_{0}^{\frac{|x_m|}{2k^\delta}} \frac{1}{\sqrt{2\pi}} e^{\frac{3k^\delta}{2}} \exp \left[ - \frac{\text{Re}(z_m e^{2i\theta})}{2\sqrt{\lambda_m}} (3k^\delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j | x_j |})^2 \right] \prod_{j \neq m, j=1}^L \sqrt{\lambda_j | x_j |} dx_j d\theta \]
\[
\times \sum_{\nu=0}^1 \int_{H_{L,n}} \langle x_1, \cdots, \frac{(-1)^\nu}{\sqrt{2\lambda_m}} (3k^\delta - \sum_{j \neq m, j=1}^L \sqrt{\lambda_j | x_j |}) e^{i\theta} e^{\frac{\alpha_m+1}{2} x_{m-1} + \cdots + e^{\alpha_L} x_L} \rangle \]
\[
\leq c_{17} c_{15}(A_0, \beta) \sqrt{\lambda_m}^{m-3} \mu_m^{2n+\delta} e^{-\frac{3k^\delta}{2}} \leq c_{18} e^{-k^\delta}.
\]

This, together with (3.10), yields the completion of the proof of (3.17).

Since we take \( \delta < \eta \), we have for \( x \in D \),
\[
\sum_{j=1}^{\infty} \left| \left< \frac{\sqrt{S} x}{\sqrt{1 - 2ik^{2n} \sqrt{SA_0} \sqrt{S}}} , \hat{e}_j > \right| \leq 3k^n.
\]

Hence by the Assumption 2,

\begin{equation}
(3.19) \quad \left( \frac{1}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} \left| < V_k x, \hat{e}_p > < V_k x, \hat{e}_j > < V_k x, \hat{e}_\ell > \beta^{pj\ell} \right|^r \right) \leq (27\beta^r) \left( \frac{1}{\sqrt{k}} \right)^{6b^r},
\end{equation}

so that \( 0 < 6\eta < \frac{6}{12} = \frac{1}{2} \) gives some natural number \( n(N+1) \) such that

\begin{equation}
(3.20) \quad \lim_{k \to \infty} (\sqrt{k})^{N+1} \left( \frac{6\eta}{\sqrt{k}} \right)^{n(N+1)} = 0.
\end{equation}

By (3.19) and the estimations similar to (1) and (6) of Lemma 1, we have

\[
\frac{1}{\sqrt{\text{det}(1 - 2ik^{2n} \sqrt{SA_0} \sqrt{S})}} \int_D \tilde{F}_A^0 \left( \frac{V_k}{\sqrt{k}} x \right) \times \left\{ \sum_{r=N+1}^{\infty} \left( \frac{i}{\sqrt{k}} \right) \sum_{p,j,\ell=1}^{\infty} < V_k x, \hat{e}_p > < V_k x, \hat{e}_j > < V_k x, \hat{e}_\ell > \beta^{pj\ell} \right\} \mu(dx)
\]

\[
= \lim_{L=\infty} \lim_{J=\infty} \int \cdots \int_{\{ \sum_{j=1}^{L} 4\sqrt{\eta_j x_j} \leq 3k^3 \}} e^{i \sum_{j=1}^{L} \frac{\alpha_j}{2} H_{L,j}(e^{i \alpha_j} x_1, e^{i \alpha_j} x_2, \cdots, e^{i \alpha_j} x_L)} \prod_{j=1}^{L} \frac{1}{\sqrt{2\pi}} e^{-|x_j|^2/2} dx_j,
\]

which together with (3.13), (3.14), (3.15), (3.16) and (3.17), completes the proofs of (2.34) and (2.35).

In a manner similar to the proof of (5) of Lemma 1, under the Renormalization assumption on \( Q_{A_0} \) and the Assumption 2, we further have

\[
E \left[ \left( \frac{i}{\sqrt{k}} \sum_{p,j,\ell=1}^{\infty} < V_k x, \hat{e}_p > < V_k x, \hat{e}_j > < V_k x, \hat{e}_\ell > \beta^{pj\ell} \right)^{2b} \right] \leq c_{19}(b) \frac{1}{\sqrt{k}}^{2b} \left( \sum_{j=1}^{\infty} \frac{1}{\sqrt{\mu_j}} \right)^{6b}.
\]

This and estimations similar to (2) and (7) of Lemma 1, together with (3.19) and (3.20), completes the proof of Theorem 2.
Acknowledgement. Both authors gratefully acknowledge a kind invitation of R. Léandre to attend the workshop held in C.I.R.M. in Luminy in 2004. The second author would like to express his sincere thanks to Prof. A. Hahn for his valuable suggestions.

References


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