On a Class of Unitary Representations of the Braid Groups $B_3$ and $B_4$

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Abstract

We describe a class of irreducible unitary representations of the braid group $B_3$ in every dimension. Moreover using Burau unitarisable representation, we present a class of nontrivial unitary representations also for the braid group $B_4$ in the case where the dimension of the space is a multiple of 3.

Key words: Braid group, unitary representation, tensor product, conjugate class
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1 Introduction

In this paper we shall work with Artin’s braid groups $B_k$, $k \in \mathbb{N}$. It has a standard presentation in generators and relations which first appeared in [2]:

$$B_k = \langle \sigma_1, \ldots, \sigma_{k-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i = 1, \ldots, k-2, \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| > 1 \rangle$$

There are various representations of $B_k$ (see, for example, [4]). The description of finite dimensional unitary representations connects with constrains on the spectra of unitary matrices. Among the results let us mention the finding
all such representations for $B_3$ in small dimensions [13], and the study of representations with a small number of points in the spectrum of the matrix corresponding to $\sigma_1$ (see, for example, [7] and [10]). Also let us mention several interesting connections have been found between the representations of the braid groups and representations of other objects such as quantum groups and Kac-Moody algebras as well as with $R$-matrices and solutions of quantum Yang-Baxter equations, see e.g., [3].

The braid group $B_3$ can be generated by two elements $S$ and $J$ and the relation $S^2 = J^3$ (see Preliminaries for precise formulas). For an irreducible unitary representation $\pi$ of $B_3$, this means that $\pi(S)$ can be defined by one of its eigenvalues and the corresponding eigenspace projection and $\pi(J)$ can be defined by two of its eigenvalues and the two corresponding eigenspaces projections. So the unitary representations of $B_3$ are directly connected with $\ast$-representations of the algebra generated by the three projections $q_1$, $q_2$ and $q_3$ and the relation $q_2q_3 = q_3q_2 = 0$. It was shown in [9] that such an algebra has as many $\ast$-representations as some matrix algebra over the free group with two generators. Therefore the classification of all unitary representations of $B_3$ up to unitary equivalence seems to be a problem which is too complicated to solve by the pure means (see also [8]). Nevertheless in view of certain applications it is interesting to find various classes of non equivalent irreducible representations of $B_3$. Recently it has been found a class of representations of $B_3$ in every dimension $n$ depending on $n$ parameters [1]. The authors use a deformation of Pascal’s triangle connected with $q$-shifted factorials to obtain the representations, and thus generalize the results from [14] and some results from [6].

In section 3 we present a class of unitary irreducible representations of $B_3$ by $n \times n$ matrices for every $n \geq 3$. Also, using a tensor product of the reduced Burau representation [7] and the unitary representations of $B_3$, we find a class of nontrivial irreducible unitary representations for $B_4$, see the section 4. In the last section we describe a procedure for obtaining polynomials in $2n$ variables, each of them being an invariant of a class of conjugate elements of $B_3$ or $B_4$.

2 Preliminaries

Consider the product of generators of $B_n$:

$$J = \sigma_1\sigma_2\ldots\sigma_{k-1}.$$  

It was proved in [2] that $J^k$ lies in the center $Z(B_k)$ of $B_k$. It is generated by the only one generator $J^k$. Using the diagrams for braid groups, it is easy to
show that

\[ \sigma_i = J\sigma_{i-1}J^{-1} = J^{i-1}\sigma_{i-1}J^{1-i}, \ i < k. \]  

(1)

Moreover, following [2], if \( S := \sigma_1 J \), then

\[ S^{k-1} = J^k. \]  

(2)

Thus \( S^{k-1} \) is also a generator of \( Z(B_k) \). Note that the group \( B_k \) can be generated by \( \sigma_1 \) and \( J \) as well as by \( J \) and \( S \). It follows from the braid relation that

\[ \sigma_1 J\sigma_1 J^{-1} \sigma_1 = J\sigma_1 J^{-1}\sigma_1 J^{1-1} \]

or, in \( S \) and \( J \) terms, after multiplying both sides of the above equation by \( J \):

\[ S^2 J^{-2} S = JSJ^{-2} S^2 J^{-2}. \]  

(3)

Now let \( k = 3 \). Then \( S^2 = J^3 \) commutes with every element of \( B_k \) and the equation (3) turns into an identity. This means that \( B_3 \) is generated by two elements \( J \) and \( S \) and only one relation \( S^2 = J^3 \). (If \( k > 3 \), then beside the equation 3 there exist also relations deriving from the commutation relation between the generators of \( B_k \)).

3 Irreducible representations of \( B_3 \)

In this section we shall use the notations \( I_n \) and \( 0_n \) for the \( n \times n \) identity and zero matrices and the notation \( E_{i,j} \) for the matrix unit with 1 in the \((i,j)\) position and 0 in the other positions. A diagonal matrix with entries \( a_1, a_2, \ldots, a_m \) will be denoted by \( \text{diag} (a_1, a_2, \ldots, a_m) \).

Suppose we have an irreducible unitary representation \( \pi \) of \( B_k \) by \( n \times n \) matrices. Then the image of \( Z(B_k) \) is the set of all scalar unitary matrices. So

\[ \pi(S^2) = \pi(J^3) = uI, \ \ u\bar{u} = 1 \text{ with } u \in \mathbb{C}, \ \bar{u} \text{ the conjugator of } u. \]

To describe the representation it suffices to consider the case \( u = 1 \). Note that every such representation will be a unitary representation of the unimodular group \( PSL_2(\mathbb{Z}) \) (see [12]).
Let \( U \) and \( V \) be \( 2n + m \times 2n + m \) block matrices of the form.

\[
U = 2 \begin{pmatrix}
A - I_n & B & C \\
B^* & B^*A^{-1}B & B^*A^{-1}C \\
C^* & C^*A^{-1}B & C^*A^{-1}C - I_m
\end{pmatrix},
\]

\[
V = \text{diag}(I_n, \beta I_n, \beta^2 I_m),
\]

where \( \beta = \sqrt[3]{1} \) is a primitive root, \( 1 \leq m \leq n \), \( A \) and \( B \) are \( n \times n \) matrices and \( C \) is an \( n \times m \) matrix.

**Theorem 1** For appropriate \( A, B \) and \( C \) the representation \( \pi, \pi(S) = U, \pi(J) = V \)

defines an irreducible unitary representation of \( B_3 \).

**Proof.** We have \( V^3 = I_{2n+m} \). At the same time, if \( A^* = A \) and \( BB^* + CC^* = A - A^2 \), then \( U^* = U \) and \( U^2 = I_{2n+m} \). So \( \pi \) is a unitary representation of \( B_3 \). If in addition \( A, B \) are invertible, rank \( C = m, B^*B \) is a diagonal matrix with simple spectrum and every entry \( a_{i,i+1} \) of \( A \) is nonzero for \( i = 1, \ldots, n \), then \( \pi \) will be irreducible. We shall show this using the irreducibility of the corresponding \( \star \)-representation \( \tilde{\pi} \) of the group algebra \( \mathbb{C} \langle B_3 \rangle \) defined on the generators \( S \) and \( J \) by the formulas \( \tilde{\pi}(S) = U, \tilde{\pi}(J) = V \). Let us denote by \( \mathcal{A} \) the algebra \( \tilde{\pi}(\mathbb{C} \langle B_3 \rangle) \). Our goal is to show that \( E_{i,j} \in \mathcal{A} \) for every \( i, j \). Since \( V \) is a block diagonal matrix with different scalar matrices in the blocks, there exists a polynomial \( P_{11} \) such that

\[
E_{11}(I_n) = \text{diag}(I_n, 0_n, 0_n) = P_{11}(V)
\]

which yields that \( E_{11}(I_n) \) belongs to \( \mathcal{A} \). Similarly we see that \( E_{22}(I_n) \in \mathcal{A} \) and \( E_{33}(I_m) \in \mathcal{A} \). The matrix \( BB^* \) is diagonal with simple spectrum, so similar arguments for the diagonal matrix \( E_{11}(I_n)UE_{22}(I_n)UE_{11}(I_n) \) lead to \( E_{i,i} \in \mathcal{A} \) for \( i = 1, \ldots, n \). On the other hand, \( a_{i,i+1} \neq 0 \). So \( E_{i,i}UE_{i+1,i+1} \in \mathcal{A} \) and \( E_{i+1,i+1} \in \mathcal{A} \). Recalling that \( \tilde{\pi} \) is a star representation, we have \( E_{i,i+1}^* \in \mathcal{A} \). It is now easy to show that \( \text{diag}(W_1, 0_n, 0_n) \in \mathcal{A} \) for every \( n \times n \) matrix \( W_1 \). Putting \( \tilde{B} = \text{diag}(B^{-1}, 0_n, 0_n) \), we obtain

\[
E_{12}(I_n) = \begin{pmatrix}
0_n & I_n & 0_{nm} \\
0_n & 0_n & 0_{nm} \\
0_{mm} & 0_{mn} & 0_m
\end{pmatrix} = \tilde{B}UE_{2,2}(I_n) \in \mathcal{A}, \text{ hence } E_{21}(I_n) \in \mathcal{A},
\]
where $0_{nm}$ is an $n \times m$ zero matrix. So

$$\text{diag} \left( W_2, 0_m \right) \in A$$

for every $2n \times 2n$ matrix $W_2$. At the same time, the matrix $C$ has rank $m$. Therefore there exists an $n \times n$ matrix $\hat{C}$ such that we have for the transposed matrix

$$D^T = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 3 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & m & 0 & \ldots & 0 \end{pmatrix} = (\hat{C}C)^T.$$}

This leads to the inclusion

$$\tilde{D} = \begin{pmatrix} 0_n & 0_n & D \\ 0_n & 0_n & 0_{nm} \\ 0_{mn} & 0_{mn} & 0_m \end{pmatrix} = \begin{pmatrix} \hat{C} & 0_n & 0_{nm} \\ 0_n & 0_n & 0_{nm} \\ 0_{mn} & 0_{mn} & 0_m \end{pmatrix} U E_{33}(I_m) \in A.$$}

Using once again the fact that $A$ is a $*$-algebra, we obtain that

$$\tilde{D}^* \tilde{D} = \text{diag} \left( 0_n, 0_n, 1, 4, \ldots, m^2 \right) \in A, \text{ hence } E_{ii} \in A$$

for $2n < i \leq 2n + m$. Thus every $E_{ii} \in A$, $1 \leq i \leq 2n + m$. Beside this, we have

$$\begin{pmatrix} 1 & 1/2 & \ldots & 1/m & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix} \tilde{D} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 & \ldots & 1 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}.$$
Now (4) yields

\[
T = \begin{pmatrix}
1 & \ldots & 1 \\
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix} \in \mathcal{A}
\]

and we achieve our goal to show \( E_{i,j} \in \mathcal{A} \) from the equalities \( E_{i,i} = TE_{i,i} \) and \( E_{i,j} = T^*E_{1,i}E_{1,j} \). Therefore \( \mathcal{A} \) contains every \( E_{i,j} \), \( i, j = 1, 2, \ldots, 3n \), and as a result \( \mathcal{A} \) is the matrix algebra of all \( 2n + m \times 2n + m \) complex matrices. This proves the irreducibility of the representation \( \tilde{\pi} \). Since a reducible representation of a group leads to a reducible representation of the corresponding group algebra, we conclude that \( \pi \) is an irreducible unitary representation of \( B_3 \). \( \square \)

**Corollary 2** If one consider two unitary irreducible representations \( \pi_1 \) and \( \pi_2 \) from Theorem 1 with matrices \( A_1, B_1 \) and \( C_1 \) and \( A_2, B_2 \) and \( C_2 \) respectively, then if \( A_1 \) is not similar to \( A_2 \) or \( B_1B_i^* \) is not similar to \( B_2B_i^* \), then the representations \( \pi_1 \) and \( \pi_2 \) are not equivalent.

**PROOF.** It follows from the proof of the Theorem that there exist polynomials \( P_{11} \) and \( P_{22} \) such that \( P_{ii}(\tilde{\pi}_1(J)) = P_{ii}(\tilde{\pi}_2(J)) = E_{ii}(I_n) \). Also direct calculations show that

\[
E_{11}(I_n)\tilde{\pi}_1(S)E_{11}(I_n) = \begin{pmatrix}
A_i & 0_n & 0_{nm} \\
0_n & 0_n & 0_{nm} \\
0_{mn} & 0_{mn} & 0_m
\end{pmatrix}
\]

and

\[
E_{11}(I_n)\tilde{\pi}_1(S)E_{22}(I_n)\tilde{\pi}_1(S)E_{11}(I_n) = \begin{pmatrix}
B_iB_i^* & 0_n & 0_{nm} \\
0_n & 0_n & 0_{nm} \\
0_{mn} & 0_{mn} & 0_m
\end{pmatrix}, \quad i = 1, 2
\]

So, defining equivalence of representations by \( \sim \) and similarity of matrices by \( \approx \), if \( \pi_1 \sim \pi_2 \), then both \( A_1 \approx A_2 \) and \( B_1B_i^* \approx B_2B_i^* \). \( \square \).

**Remark 3** At the begin of the proof of Theorem 1, instead of the set \( \Omega = \{(i, i + 1) \mid i = 1, \ldots, n\} \) for which \( a_{i,j} \neq 0 \), \( (i, j) \in \Omega \), one can choose
any set of indices $\tilde{\Omega}$ having the property that the set of the matrix units $E_{i,1}, \ldots, E_{n,n}, E_{i,j}, E_{j,i}, (i, j) \in \tilde{\Omega}$ generates the algebra of all $n \times n$ complex matrices.

4 Irreducible representations of $B_4$

To construct nontrivial irreducible representations of $B_4$ we use the notion of tensor products of matrices. Let $D$ and $G$ be $d \times d$ and $l \times l$ matrices respectively. Then the $ld \times ld$ matrix

$$\text{diag}(D, D, \ldots, D) = \begin{pmatrix}
g_{11}I_d & \cdots & g_{1l}I_d \\
\vdots & \ddots & \vdots \\
g_{l1}I_d & \cdots & g_{ll}I_d
\end{pmatrix}$$

will be the tensor product $D \otimes G$ of $D$ and $G$.

Suppose we have two irreducible unitary representations $\pi_1$ and $\pi_2$ of $B_4$. Then $\pi_1 \otimes \pi_2$ is a unitary representation of $B_4$ too. Now putting $\pi_1(\sigma_1) = \pi_1(\sigma_3) = \pi(\sigma_1)$ and $\pi_1(\sigma_2) = \pi(\sigma_2)$ for the representation $\pi$ from the previous section, we obtain trivially a representation of $B_4$. For the representation $\pi_2$ we use reduced Burau representation (see [7]) written in the base where every matrix $\pi_2(\sigma_i)$ is unitary:

$$\pi_2(\sigma_1) = \text{diag}(u, 1, 1), \quad \pi_2(\sigma_2) = \begin{pmatrix}
(u-1)\alpha_1 + 1 & (u-1)\sqrt{\alpha_1 - \alpha_1^2} & 0 \\
(u-1)\sqrt{\alpha_1 - \alpha_2^2} & (1-u)\alpha_1 + u & 0 \\
0 & 0 & 1
\end{pmatrix}$$

$$\pi_2(\sigma_3) = \begin{pmatrix}
1 & 0 & 0 \\
0 & (u-1)\alpha_2 + 1 & (u-1)\sqrt{\alpha_2 - \alpha_2^2} \\
0 & (u-1)\sqrt{\alpha_2 - \alpha_2^2} & (1-u)\alpha_2 + u
\end{pmatrix},$$

where $u\bar{u} = 1, \alpha_1 = -u/(u-1)^2, \alpha_2 = \alpha_1/(1-\alpha_1)$. We remark that since both numbers $\alpha_1$ and $\alpha_2$ have to be positive and less than 1, we have to assume that the real part of $u$ is less than 0.

Theorem 4 The representation $\pi_1 \otimes \pi_2$ is an irreducible unitary representation of $B_4$ by $3n \times 3n$ matrices and the natural restriction of it to a representation of $B_3$ is reducible.
PROOF. To minimize the number of additional symbols we shall use the same notation $\pi_i$ for the $*$-representation of the group algebra $\mathbb{C}\langle B_4 \rangle$. Let $\mathcal{A} = \pi_1 \otimes \pi_2(\mathbb{C}\langle B_4 \rangle)$. As in the proof of Theorem 1 we shall show that $E_{m,k} \otimes E_{i,j} \in \mathcal{A}$. We remark that

$$[\pi_1(\sigma_1) \otimes \pi_2(\sigma_1) \cdot \pi_1(\sigma_2) \otimes \pi_2(\sigma_2)]^3 = I \otimes \begin{pmatrix} u^3 & 0 & 0 \\ 0 & u^3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

This implies that

$$P = I \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}.$$ 

Consider the element

$$(\pi_1(\sigma_1) \otimes \pi_2(\sigma_1))^* = \pi_1(\sigma_1^{-1}) \otimes \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

It lies in $\mathcal{A}$ and we can multiply it with another element of $\mathcal{A}$, $\pi_1(\sigma_3) \otimes \pi_2(\sigma_3) = \pi_1(\sigma_1) \otimes \pi_2(\sigma_3)$ from the right. Using also the projection $P$ constructed above, one has

$$P(\pi_1(\sigma_1) \otimes \pi_2(\sigma_1))^* \pi_1(\sigma_3) \otimes \pi_2(\sigma_3)P = I \otimes \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & (u-1)\alpha_2+1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}.$$ 

Since $\bar{u} \neq (u-1)\alpha_2+1$ by the definitions of $u$ and $\alpha_2$, we have $I \otimes E_{i,i} \in \mathcal{A}$. Note that by construction of $\pi_2$ every entry of $\pi_2(\sigma_2)$ and $\pi_2(\sigma_3)$ which depends on $u$, $\alpha_1$ or $\alpha_2$ is different from zero. Therefore, $I \otimes E_{1,2}$ is a scalar multiple of

$$I \otimes E_{1,1} \cdot \pi_1(\sigma_2^*) \otimes \pi_2(\sigma_2^*) \cdot I \otimes E_{1,1} \cdot \pi_1(\sigma_2) \otimes \pi_2(\sigma_2) \cdot I \otimes E_{2,2}$$

and $I \otimes E_{2,3}$ is a scalar multiple of

$$I \otimes E_{2,2} \cdot \pi_1(\sigma_3^*) \otimes \pi_2(\sigma_3^*) \cdot I \otimes E_{2,2} \cdot \pi_1(\sigma_3) \otimes \pi_2(\sigma_3) \cdot I \otimes E_{3,3}.$$
Thus since $\mathcal{A}$ is a $*$-algebra, $I \otimes E_{i,j} \in A$ for every $i, j$. This leads to $I \otimes \pi_2(\sigma^*_i) \in A$ and $I \otimes \pi_2(\sigma^*_i) \cdot \pi_1(\sigma_i) \otimes \pi_2(\sigma_i) = \pi_1(\sigma_i) \otimes I_3 \in A$ for every $i$. Using the irreducibility of $\pi_1$, we conclude that $E_{m,k} \otimes I_3 \in A$ and, hence $E_{m,k} \otimes E_{i,j} \in A$.

**Remark 5** One can consider any irreducible unitary representation $\hat{\pi}$ of $B_4$ instead of $\pi_1$ in the Theorem with the restriction of $\hat{\pi}$ to $B_3$ generated by $\sigma_1$ and $\sigma_2$ being irreducible. The proof is similar if one defines an orthogonal projection $P$ as in the proof of the Theorem 4 and uses the inclusion

$$(P\pi_1(\sigma_3) \otimes \pi_2(\sigma_3)P)^* \pi_1(\sigma_3) \otimes \pi_2(\sigma_3)P \in A$$

to show that $I \otimes E_{i,i} \in A$.

**Remark 6** Although every Burau representation is unitary with respect to some Hermitian form[11] it is a curious fact that there are some restrictions on the spectra of unitary dilations to be the images of the standard generators of $B_n$ for an irreducible unitary representation of $B_n$. More discussions on this theme can be found in [15].

5 Invariant polynomials of conjugacy classes in $B_3$ and $B_4$

The results obtained in the previous sections provide many representations of $B_3$ and $B_4$. For example, for $m = n$, one can find no less than $n(n-1)/8 - 1$ real parameters $c_i, c_i \in [a_i, b_i], a_i \neq b_i \in \mathbb{R}$ yielding non equivalent representations of $B_3$ by $3n \times 3n$ matrices. For the purpose of applications, it is convenient to suggest that some parameters are simply fixed real numbers. Let us consider one such construction. Let $t_0, t_1, \ldots, t_n, t \in \mathbb{R}, 0 = t_0 < t_1 < \ldots < t_n < t < 1/4$. Suppose $x_i$ is a variable taking values in $(t_{i-1}, t_i)$. The matrices $B$ and $C$ are defined by the following formulas

$$B = \text{diag} \left( x_1, \ldots, x_n \right), \quad C = \text{diag} \left( \sqrt{t - x_1^2}, \ldots, \sqrt{t - x_n^2} \right).$$

Note that

$$BB^* + CC^* = \text{diag} \left( t, \ldots, t \right).$$
Let \( \alpha \) and \( \beta \) be the roots of the equation \( x - x^2 - t = 0 \) and consider the idempotent \( n \times n \) matrix

\[
Q = \frac{1}{n} \begin{pmatrix}
1 & \ldots & 1 \\
& \ddots & \vdots \\
& & 1 & \ldots & 1
\end{pmatrix},
\]

We then define

\[
A = \alpha Q + \beta I_n.
\]

This leads to the simple form for \( A^{-1} \):

\[
A^{-1} = ((\alpha + \beta)^{-1} - \beta^{-1})Q + \beta^{-1}I_n.
\]

Thus by Theorem 1, we have a simple construction of representations \( \pi = \pi(x_1, \ldots, x_n) \) of the group \( B_3 \) which are continuous in \( x_1, \ldots, x_n \).

Let \( a \in B_3 \). Set \( P(x_1, \ldots, x_n) = \text{tr} \pi(x_1, \ldots, x_n)(a) \). This is a polynomial in the variables \( x_i \) and \( \sqrt{t - x_i^2} \). Since similar matrices have the same trace, we obtain the following proposition.

**Proposition 7** If the elements \( a \) and \( b \) are conjugate in \( B_3 \), then

\[
P(x_1, \ldots, x_n)(a) = P(x_1, \ldots, x_n)(b).
\]

**Remark 8** In the above construction one can replace the numerical matrix \( Q \) with any orthogonal projection depending on some variables \( c_i \). But not every such representation will be irreducible.

**Remark 9** Polynomials similar to the ones in Proposition 7 can be constructed for elements of \( B_4 \) using the representation described in section 4.

We hope that the obtained representations will help in the solutions of the word and conjugacy problems in \( B_k \) [5] and in discussing knot theoretical problems [7].

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