Droplet Spreading Under Weak Slippage:
A Basic Result on Finite Speed of Propagation

Günther Grün

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DROPLET SPREADING UNDER WEAK SLIPPAGE: A BASIC RESULT ON FINITE SPEED OF PROPAGATION

GÜNTHER GRÜN

ABSTRACT. We prove a new qualitative result on finite speed of propagation for the thin film equation subjected to Navier slippage or even weaker slip conditions. Our approach works in multiple space dimensions and is based on a novel technique which combines recently established weighted energy estimates with an Hardy-type inequality and with Stampacchia's iteration lemma. It can be adapted to degenerate parabolic equations of different order than four as well. (to appear in SIAM J. Math. Anal.)

1. INTRODUCTION

In this paper, we present a new method for the proof of qualitative results on finite speed of propagation for degenerate parabolic equations. It is inspired by the idea, Dal Passo, Giacomelli, and the author developed in [10] to show the occurrence of a waiting time phenomenon for the thin film equation. We apply the method to establish new results on finite speed of propagation for the thin film equation

\[ u_t + \text{div}(|u|^n \nabla u) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \]
\[ u(\cdot,0) = u_0 \quad \text{on } \mathbb{R}^N, \] (1.1)

in the parameter range \( n \in [2, 3) \) in space dimensions \( N < 4 \). For the ease of presentation, we confine ourselves to a mobility \( m(u) := |u|^n \). However, modifications of (1.1) replacing the term \(|u|^n\) by more general functions \( m \in C^2(\mathbb{R}; \mathbb{R}_0^+) \), \( m(0) = 0 \), could be handled as well.

Let us make a few comments on the physical background. In the course of lubrication approximation (see e.g. Bernis [2] or Oron, Bankoff, and Davis [29]), the equation

\[ h_t + \frac{\sigma}{3\eta} \text{div} (m(h) \nabla h) = 0 \] (1.2)

is derived to describe the surface tension driven evolution of the thickness \( h \) of a thin film of viscous liquid spreading on a horizontal surface. Here, \( \eta \) is the viscosity of the liquid and \( \sigma \) denotes surface tension. The explicit form of the mobility \( m(\cdot) \) is determined by the flow condition at the liquid-solid interface. In case of a no-slip condition we get

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$m(h) = h^3$, whereas a generic slip condition

$$\mathbf{v}_{\text{hor}} \bigg|_{z=0} = \beta h^{n-2} \cdot \frac{\partial \mathbf{v}_{\text{hor}}}{\partial z} \bigg|_{z=0}$$

entails

$$m(h) = h^3 + \beta h^n.$$  

Here, $\mathbf{v}_{\text{hor}}$ denotes the horizontal component of the fluid velocity field, $\beta$ is a positive parameter and $z$ stands for the vertical coordinate. For $n = 2$, the classical slip condition of Navier is recovered, but in the classical literature (cf. e.g. [15], [16], or [27]) different parameters of $n \in (0, 3)$ were suggested as well to model effects of stronger ($n < 2$) or weaker ($n \in (2, 3)$) slippage.

Analytically, the qualitative behaviour of solutions is governed by the smoothness of $m(\cdot)$ in its point of degeneracy, i.e. in $h_0 = 0$. For this reason, in the mathematical literature usually the model problem (1.1) is studied as it exhibits already all the essential mathematical difficulties. So let us emphasize that the physically important case of surface tension driven thin film flow subjected to Navier’s slip condition or to even weaker slip conditions corresponds in the framework of the model problem (1.1) to the choices $n = 2$ or $2 < n < 3$, respectively. Hence, it is covered by the results to be presented in the present paper.

Recall that equation (1.1) admits globally non-negative solutions (cf., e.g. Bernis and Friedman [5] or Grün [19]) and that it implicitly defines a free boundary problem where the free boundary at time $T \geq 0$ is given by $\partial \{\text{supp}(u(\cdot, T))\}$. It is one of the striking features of equation (1.1) that the qualitative behaviour of solutions is sensitive to the mobility growth exponent $n > 0$. Despite the fact that scaling invariances of equation (1.1) (for details cf. Giacomelli and Otto [14]) suggest the existence of compactly supported self-similar solutions for arbitrary $n > 0$, these solutions only exist for $0 < n < 3$ (cf. Bernis, Peletier, Williams [6] and Ferreira, Bernis [13]). Moreover, for $n > 4$ the support of arbitrary solutions is constant in time, as it was proven by Beretta, Bertsch, and Dal Passo [1]. And asymptotic analysis suggests that this behavior holds true for all $n \geq 3$.

This is in good agreement with the spreading paradoxon observed by Dussan and Davis [12] in the framework of Navier-Stokes equations. It says that a no-slip condition at the liquid-solid interface (which entails $n = 3$) implies infinite energy dissipation at the liquid-solid-gas contact line in case of moving droplets. On the other hand, it is well known (see Bernis, Friedman [5], Beretta, Bertsch, Dal Passo [1], Bertozzi, Pugh [7] in space dimension $N = 1$, Dal Passo, Garcke, Grün [9] and Grün [18] in the multi-dimensional case) that for $n \in (0, 3)$ in space dimensions $N < 4$ so called strong solutions exist. They exhibit a zero contact angle at the free boundary, and for $t \to \infty$ their support tends to cover the spatial domain entirely.

The existence of compactly supported self-similar solutions indicates that solutions to equation (1.1) have the property of finite speed of propagation. In Bernis [3], [4], and in
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Bertsch, Dal Passo, Garcke, Grün [8], this could be confirmed rigorously for \( n \in (0, 2) \) in space dimensions \( N < 4 \) and for \( n \in [2, 3) \) in space dimension \( N = 1 \). Surprisingly, the case \( n \in [2, 3), N > 1 \), remained open for rather a long time. Only recently, the author succeeded to prove in his habilitation thesis [21] a first result on finite speed of propagation in the higher dimensional setting for that critical parameter regime.

While the result of [21] took advantage of Bernis’ technique of higher order differential inequalities, the new approach to be presented here essentially uses an iteration lemma due to Stampacchia, and it seems to be technically less involved. Let us emphasize that this method also applies to degenerate parabolic equations of sixth or even higher order or of second order like the porous-media-equation or like doubly degenerate parabolic equations (cf. e.g. Vazquez [30] or Ivanov [26]).

Before describing the outline of the present paper, let us recall the peculiarities of the regime \( n \in [2, 3) \) which seem to exclude an applicability of the techniques used for \( n \in (0, 2) \). As equation (1.1) is fourth order parabolic, comparison principles do not hold and the argumentation has to be based solely on integral estimates. Up to now, there are basically two types of integral estimates known, first the energy estimate

\[
\frac{1}{2} \int_{\Omega} |\nabla u(\cdot, T)|^2 + \int_0^T \int_{\Omega} u^n |\nabla \Delta u|^2 \leq \frac{1}{2} \int_{\Omega} |\nabla u_0|^2
\]

and secondly the \( \alpha \)-entropy estimate

\[
\frac{1}{\alpha (\alpha + 1)} \int_{\Omega} u^{\alpha + 1}(\cdot, T) + C^{-1} \int_0^T \int_{\Omega} \left\{ |\nabla u|^{\frac{\alpha + n + 1}{2}} + |D^2 u|^{\frac{\alpha + n + 1}{2}} \right\} \leq \frac{1}{\alpha (\alpha + 1)} \int_{\Omega} u_0^{\alpha + 1}
\]

which is valid for

\[
\alpha \in \left( \frac{1}{2} - n, 2 - n \right) \setminus \{-1, 0\}.
\]

Note that for compactly supported initial data, the global version of estimate (1.4) is valid only for \( n \in (0, 3) \). Observe moreover that condition (1.5) does not permit the parameter \( \alpha \) to be chosen positive in the parameter regime \( n \in [2, 3) \). As a consequence, in that regime the entropy \( u^{\alpha + 1}(T) \) can no longer be controlled in terms of the initial entropy. This is the reason why analytical approaches based on the entropy estimate are restricted to the interval \( n \in (0, 2) \).

On the other hand, the energy estimate requires additional analytical tools in order to become accessible to Gagliardo-Nirenberg-type arguments. To this end, it would be desirable to estimate the dissipated energy \( \int |\nabla u|^6 \) from below by derivatives of certain powers of \( u \). In the multi-dimensional case, this goal was achieved only recently by virtue of the interpolation inequality

\[
\int_{\Omega} |\nabla u|^{\frac{\alpha + 6}{2}} + \int_{\Omega} |\nabla \Delta u|^{\frac{\alpha + 6}{2}} \leq C(n, N) \int_{\Omega} u^n |\nabla u|^2
\]
which was proven in [21] (see also the recently published paper [20]) and which holds on convex domains \( \Omega \) for positive functions of class \( H^2 \) having zero normal derivatives on the boundary.

This result was the key observation to establish in [21] on bounded convex domains \( \Omega \) the existence of strong solutions to the thin film equation associated with compactly supported, non-negative initial data which satisfy besides an \( \alpha \)-entropy estimate the following energy-type estimate:

\[
\int_{\Omega} |\nabla u(\cdot, T)|^2 + C(n, N) \left\{ \int_0^T \int_{\Omega} |\nabla u^{\frac{n+2}{n}}|^6 + \int_0^T \int_{\Omega} |\nabla \Delta u^{\frac{n+2}{n}}|^2 \right\} \leq \int_{\Omega} |\nabla u_0|^2. \tag{1.7}
\]

By virtue of appropriate weighted versions of that estimate, first existence results for the Cauchy problem in the parameter regime \( n \in (2 - \sqrt{1 - \frac{N}{8}}, 3) \) could be established as well, see [21].

In the present paper, we will prove that these solutions have the property of finite speed of propagation in the following sense.

**Definition 1.1.** Let \( v : \mathbb{R}^N \times [0, \infty) \to \mathbb{R} \) be a non-negative function and assume that \( v(\cdot, 0) \) has compact support in \( \mathbb{R}^N \). We say that \( v \) has finite speed of propagation iff for each ball \( B_{R_0}(x_0) \), \( x_0 \in \mathbb{R}^N \), \( R_0 > 0 \), that contains \( \text{supp} v(\cdot, 0) \), a continuous, monotonically increasing function \( R : [0, \infty) \to \mathbb{R}^*_0 \), \( R(0) = 0 \), exists such that

\[
\text{supp} v(\cdot, t) \subset \overline{B_{R_0+R(t)}(x_0)}.
\]

In Section 2, we will recall the properties of strong solutions to the Cauchy problem as constructed in [21]. Section 3 is devoted to the proof of a Hardy-type inequality on exterior domains. Combining that Hardy-type inequality with the aforementioned weighted version of the energy estimate, we may formulate an integral estimate which will serve as the key ingredient for the subsequent proof of finite speed of propagation which follows in Section 4. The idea of proof is to derive via appropriate interpolation arguments an recursive inequality for the function

\[
G_T(R) := \int_0^T \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{n}}
\]

which permits an application of Stampacchia’s iteration lemma (see Lemma 4.3). This way, we deduce for fixed \( T > 0 \) the existence of a number \( 0 < R(T) < \infty \) such that \( G_T(R(T)) = 0 \). As a consequence, it becomes evident that \( \text{supp}(u(\cdot, t)) \subset B_{R(T)}(0) \) for all \( 0 \leq t < T \). Furthermore, \( R(T) \) continuously depends on \( T \) which gives the result.

Throughout the paper, we use the usual notation for Sobolev and Lebesgue spaces. We write \( \|u\|_p \) for \( (\int |u|^p)^{1/p} \) also in the case \( 0 < p < 1 \). Finally, \( B_R(x) \) denotes the ball with radius \( R \) and center \( x \in \mathbb{R}^N \), and \( [u > 0]_T \) stands for the set \( \{(x, t) \in \mathbb{R}^N \times (0, T) \mid u(x, t) > 0 \} \).
2. Preliminaries

In this section, we will summarize recent results on strong solutions for the Cauchy problem in the multi-dimensional case. In addition, we will formulate a version of Gagliardo-Nirenberg's inequality to be used in the sequel.

**Theorem 2.1.** Let \( n \in (2 - \sqrt{1 - \frac{N}{8 + N}}, 3) \), \( N < 4 \), and assume \( u_0 \in H^1(\mathbb{R}^N) \) to be non-negative with compact support in the sense that \( u_0(x) = 0 \) almost everywhere on \( \mathbb{R}^N \setminus B_{R_0}(0) \) for a positive number \( R_0 \). Then, a non-negative function \( u \) exists that has the following properties:

i) **Regularity:**

\[
\begin{align*}
    & u_t \in L^2(\mathbb{R}^+; (W^{1,r}(\Omega))') \text{ for } p > \frac{4N}{2N + n(2-N)} \text{ and any } \Omega \subset \subset \mathbb{R}^N, \quad (2.1) \\
    & u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^N)), \quad (2.2) \\
    & D^2u \frac{a+n+1}{2} \in L^2(\mathbb{R}^N \times \mathbb{R}^+) \text{ for any } \alpha \in (\max\{-1,1/2 - n\}, 2 - n), \quad (2.3) \\
    & \nabla u \frac{a+n+1}{4} \in L^4(\mathbb{R}^N \times \mathbb{R}^+) \text{ for any } \alpha \in (\max\{-1,1/2 - n\}, 2 - n), \quad (2.4) \\
    & J = \left\{ \begin{array}{ll}
        u^n \nabla \Delta u & \text{on } [u > 0]_T \\
        0 & \text{on } [u = 0]_T
    \end{array} \right. \in L^2(\mathbb{R}^+; L^q(\mathbb{R}^N)) \quad (2.5)
\end{align*}
\]

for any \( 1 < q < \frac{4N}{2N + n(N - 2)} \).

ii) \( u \) is a solution to the Cauchy problem in the sense that

\[
\int_0^T \langle u_t, \phi \rangle_{W^{1,r}(B(0))'} \times W^{1,r}(B(0)) - \int_{[u > 0]_T} u^n \nabla \Delta u \nabla \phi = 0 \quad (2.6)
\]

for \( p > \frac{4N}{2N + n(2-N)} \), arbitrary \( T > 0 \) and for all test functions \( \phi \) contained in \( L^2((0,T); W^{1,\infty}(\mathbb{R}^N)) \) such that \( \bigcup_{t \in [0,T]} \text{supp}(\phi(\cdot, t)) \subset B(0) \), where \( B(0) \) is an arbitrary ball centered in the origin \( 0 \in \mathbb{R}^N \).

iii) The solution \( u \) attains initial data \( u_0 \) in the sense that

\[
\lim_{t \searrow 0} u(\cdot, t) = u_0(\cdot) \quad \text{in } L^\beta_{\text{loc}}(\mathbb{R}^N) \quad (2.7)
\]

for arbitrary \( 1 \leq \beta < \frac{2N}{N-2} \).

iv) The solution \( u \) is element of \( L^\infty(\mathbb{R}^+; L^\beta(\mathbb{R}^N)) \) for \( 1 < \beta < \frac{2N}{N-2} \). More precisely, a positive constant \( C = C(\beta, N) \) exists such that the following estimate holds:

\[
\sup_{t \in \mathbb{R}^+} \int_{\mathbb{R}^N} u^\beta(x,t)dx \leq C(\beta, N) \left\{ \left( \int_{\mathbb{R}^N} |\nabla u_0|^\beta \right)^{1/2} + \int_{\mathbb{R}^N} u_0^\beta \right\}, \quad (2.8)
\]
v) $u$ satisfies for arbitrary $T > 0$ the basic energy estimate

$$
\int_{\mathbb{R}^N} |\nabla u(\cdot, T)|^2 + C_0^{-1} \int_0^T \int_{\mathbb{R}^N} \left\{ |\nabla u^{\frac{n+2}{2}}| + |\nabla \Delta u^{\frac{n+2}{2}}| \right\} \leq \int_{\mathbb{R}^N} |\nabla u_0|^2.
$$

(2.9)

Moreover, for arbitrary $R_0 \geq 0$ and arbitrary $T > 0$ the following weighted version of the energy estimate holds with a positive constant $C_1$ that only depends on $n$ and $N$:

$$
\int_{\mathbb{R}^N \setminus B_{R_0}(0)} (|x| - R_0)^6 |\nabla u(\cdot, T)|^2 \, dx
\quad + C_1^{-1} \int_0^T \int_{\mathbb{R}^N \setminus B_{R_0}(0)} (|x| - R_0)^6 \left\{ |\nabla u^{\frac{n+2}{6}}|^6 + |\nabla \Delta u^{\frac{n+2}{2}}|^2 \right\} \leq \int_{\mathbb{R}^N \setminus B_{R_0}(0)} (|x| - R_0)^6 \cdot |\nabla u_0|^2 \, dx
\quad + C_1 \int_0^T \int_{\mathbb{R}^N \setminus B_{R_0}(0)} u^{n+2}.
$$

(2.10)

Remark: For $n \in (\frac{1}{2}, 2 - \sqrt{1 - \frac{N}{8+N}}]$, an existence result for the Cauchy problem can be found in Bertsch, Dal Passo, Garcke, Grün [8]. However, the solution concept applied in that paper is technically much more involved since third order derivatives are not controlled.

In the course of proof of the result on finite speed of propagation, we need a homogeneous version of Gagliardo-Nirenberg’s inequality valid on the complement of balls in $\mathbb{R}^N$. It reads as follows.

**Lemma 2.2.** Let $1 \leq r < \infty$, $0 < q < p$, $m \in \mathbb{N}_+$ such that

$$
\frac{1}{r} - \frac{m}{N} < \frac{1}{p}.
$$

Assume $w$ to be contained in $W^{m,r}(\mathbb{R}^N \setminus \overline{B_R(0)}) \cap L^q(\mathbb{R}^N \setminus \overline{B_R(0)})$. There is a positive constant $K_1 = K_1(N, m, p, q, r)$ such that:

$$
\|w\|_{p, \mathbb{R}^N \setminus \overline{B_R(0)}} \leq K_1 \cdot \|D^m w\|_{r, \mathbb{R}^N \setminus \overline{B_R(0)}} \cdot \|w\|_{q, \mathbb{R}^N \setminus \overline{B_R(0)}}^{1-a}.
$$

(2.11)

Here, $a = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{m} + \frac{1}{q} - \frac{1}{r}}$.

Remark: With a slight misuse of notation, we write $\|u\|_p$ for $(\int |u|^p)^{1/p}$ also in the case $0 < p < 1$. 
Before giving the proof, let us state Gagliardo-Nirenberg’s inequality in the following form (see Dal Passo, Giacomelli, Shishkov [11]).

**Lemma 2.3.** Let $1 \leq r \leq \infty$, $0 < q < p$, $m \in \mathbb{N}_+$ such that

$$
\frac{1}{r} - \frac{m}{N} < \frac{1}{p}.
$$

If $\Omega \subset \mathbb{R}^N$ is bounded with piecewise smooth boundary, then positive constants $c_1$ and $c_2$ depending only on $\Omega, r, p, m$ and $q$ exist such that for any $u \in L^q(\Omega)$ satisfying $D^m u \in L^r(\Omega)$, the following inequality holds:

$$
\|u\|_p \leq c_1 \|D^m u\|_p \|u\|^{1-a}_q + c_2 \|u\|_q
$$

(2.12)

where $a = \frac{1 - \frac{1}{p}}{\frac{1}{q} - \frac{m}{N}}$.

Especially, if $\Omega$ is an infinite cone, i.e. for given points $x_0, y_0 \in \mathbb{R}^N$, $x_0 \notin B_1(y_0)$ a set

$$
C_{x_0,y_0} := \{z \in \mathbb{R}^N | z = x_0 + \lambda(y - x_0), \ y \in B_1(y_0), \ \lambda > 0 \},
$$

then (2.12) holds with constants $c_1 = c(\|x_0 - y_0\|, r, p, m, q)$ and $c_2 = 0$.

**Proof of Lemma 2.2.** Let us prove the result first for the special case $\Omega = \mathbb{R}^N \setminus \overline{B_1(0)}$. To this purpose, we write

$$
\Omega = \Omega_+ \cup \Omega_-
$$

where $\Omega_+$ and $\Omega_-$ are open sets which are $W^{m,\infty}$-diffeomorphic to the half-space $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N | x_N > 0 \}$. In the case $m = 1$, for instance, we may choose $\Omega_+$ and $\Omega_-$ as the complements in $\mathbb{R}^N$ of the closed sets

$$
A_+ := \overline{B_1(0)} \cup \{x \in \mathbb{R}^N | x_N \geq 0 \}
$$

and

$$
A_- := \overline{B_1(0)} \cup \{x \in \mathbb{R}^N | x_N \leq 0 \},
$$

respectively. For $m > 1$, an appropriate smoothing procedure has to be applied.

By virtue of Lemma 2.3 and a straightforward transformation argument, an Gagliardo-Nirenberg inequality in the spirit of (2.11) holds both on $\Omega_+$ and on $\Omega_-$. This is sufficient
to prove (2.11) for \( \Omega = \mathbb{R}^N \setminus \overline{B_1(0)} \). Indeed,

\[
\int_{\Omega} |w|^p \leq \int_{\Omega_+} |w|^p + \int_{\Omega_+} |w|^p \\
\leq C \left\{ \left( \int_{\Omega_+} |D^m w|^r \right)^{\frac{q}{r}} \cdot \left( \int_{\Omega_+} |w|^q \right)^{\frac{p}{q} (1-a)} \right. \\
\quad + \left. \left( \int_{\Omega_-} |D^m w|^r \right)^{\frac{q}{r}} \cdot \left( \int_{\Omega_-} |w|^q \right)^{\frac{p}{q} (1-a)} \right\} \\
\leq C \left( \int_{\Omega_+} |D^m w|^r + \int_{\Omega_-} |D^m w|^r \right)^{\frac{q}{r}} \left( \int_{\Omega_+} |w|^q + \int_{\Omega_-} |w|^q \right)^{\frac{p}{q} (1-a)} \\
\leq K_1 \left( \int_{\Omega} |D^m w|^r \right)^{\frac{q}{r}} \left( \int_{\Omega} |w|^q \right)^{\frac{p}{q} (1-a)}.
\]

In the second step of this estimate, we used the assumption

\[
\frac{1}{r} - \frac{m}{N} < \frac{1}{p} < \frac{1}{q}
\]

together with the calculus inequality

\[
\sum_{i=1}^{k} a_i \beta_i \leq \left( \sum_{i=1}^{k} a_i \right)^{\alpha} \left( \sum_{i=1}^{k} b_i \right)^{\beta},
\]

which holds for numbers \( \alpha, \beta \) and \( a_i, b_i, \ i = 1, \ldots, k \) that satisfy:

\[
a_i, b_i \geq 0, \\
\alpha, \beta > 0 \quad \text{and} \quad \alpha + \beta \geq 1.
\]

For a proof, see, for instance, Dal Passo, Giacomelli, Shishkov [11]. Finally, (2.11) follows for arbitrary \( R > 0 \) by a straightforward scaling argument. \( \square \)

3. AN APPLICATION OF HARDY’S INEQUALITY

In this section we will prove an Hardy-type estimate on \( \mathbb{R}^N \setminus B_R(0) \) which will be combined with the weighted energy-type estimate (2.10) to yield a key ingredient for the proof of the main result of the paper. The Hardy-type estimate reads as follows.

**Lemma 3.1.** Assume that \( w(r) := r^4 \) and \( v(r) := r^6 \) and that \( u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) satisfies the inequalities

\[
\int_{\mathbb{R}^N} v(|x|) \cdot |\nabla u|^2 \, dx < \infty, \quad \text{and} \quad (3.1)
\]

\[
\int_{\partial B_R(0)} u^2 \, d\mathcal{H}^{N-1} \leq C_1 \cdot R^{-1}. \quad \text{(3.2)}
\]
Then a positive constant $C_2$ exists which is independent of $R > 0$ such that
\[ \int_{\mathbb{R}^N \setminus B_R(0)} w(\text{dist}(x, B_R(0))) \cdot u^2 \, dx \leq C_2 \int_{\mathbb{R}^N \setminus B_R(0)} v(\text{dist}(x, B_R(0))) |\nabla u|^2 \, dx. \] (3.3)

Let us first recall the following version of Hardy’s inequality in one space dimension (see [23], [24] and the monograph [28]).

**Lemma 3.2.** Let $a$ be a real number and assume the weight functions $v, w$ to be non-negative and measurable on $(a, \infty)$. Consider for $x \in (a, \infty)$ the quantity
\[ F_{\text{Har}}(x) := \int_a^x w(s) \, ds \cdot \int_x^\infty v^{-1}(s) \, ds. \]

If $\sup_{a < x < \infty} F_{\text{Har}}(x) < \infty$, then there exists a positive constant $C = C(a, v, w)$ such that
\[ \int_a^\infty w(s) \cdot u^2(s) \, ds \leq C \int_a^\infty v(s) \cdot u_x^2(s) \, ds \] (3.4)
for all \( u \in AC_R(a, \infty) := \{ u \in W^{1,1}_{\text{loc}}(a, \infty) : \lim_{x \to \infty} u(x) = 0 \} \).

**Proof of Lemma 3.1:** The strategy is to apply the one-dimensional Hardy-inequality (3.4) to appropriate integrals over spheres of radius $r$. Therefore, let us switch to polar coordinates and let us prove an estimate on the derivative of the $L^2$-norm of $u$ over such spheres. First we need some notation.

\[ \int_{\mathbb{R}^N \setminus B_R(0)} w(\text{dist}(x, B_R(0))) u^2 \, dx = \int_{R + \infty}^\infty w(r - R) u^2 r^{N-1} \, dS^{N-1} \, dr \]
\[ = \int_{R}^\infty w(r - R) U^2(r) r^{N-1} \, dr \] (3.5)

where we defined
\[ U(r) := \left( \int_{S^{N-1}} u^2(r, \theta) \, dS^{N-1} \right)^{1/2}. \]

Note that we use “$u$” both to denote the function $u$ in polar and in Euclidean coordinates. Here, “$\theta$” is an abbreviation of the angular coordinates and “$dS^{N-1}$” stands for the surface element on the unit sphere. We have in particular that
\[ \frac{\partial}{\partial r} U(r) = \left( \int_{S^{N-1}} u^2(r, \theta) \, dS^{N-1} \right)^{-1/2} \cdot \left( \int_{S^{N-1}} u(r, \theta) \cdot u_r(r, \theta) \cdot dS^{N-1} \right) \] (3.6)
\[ \leq \left( \int_{S^{N-1}} u^2 dS^{N-1} \right)^{1/2} \left( \int_{S^{N-1}} u^2 dS^{N-1} \right)^{1/2}. \]
which implies that
\[ \left| \frac{\partial}{\partial r} U(r) \right|^2 \leq \int_{S^{N-1}} u_r^2 dS^{N-1}. \] (3.7)

On the other hand, (3.2) entails the decay estimate
\[ U(r) \leq C \cdot r^{-\frac{N}{2}} \]
for \( r > 0 \). Altogether,
\[ U(r) \in AC_R(R, \infty). \]

Now assuming that we may apply Hardy’s inequality for the weight functions
\[ \tilde{w}(r) := (r - R)^4 \cdot r^{N-1} \]
and
\[ \tilde{v}(r) := (r - R)^6 \cdot r^{N-1}, \]
we obtain the following result by virtue of Lemma 3.2,
\[
\begin{align*}
\int_{\mathbb{R}^N \backslash B_R(0)} w(\text{dist}(x, B_R(0)))u(x)^2 \, dx &= \int_{R}^{\infty} \tilde{w}(r) \cdot U^2(r) \, dr \\
&\leq C \int_{R}^{\infty} \tilde{v}(r) \cdot U^2(r) \, dr \\
&\leq \int_{\mathbb{R}^N \backslash B_R(0)} v(\text{dist}(x, B_R(0))) |\nabla u|^2 \, dx.
\end{align*}
\] (3.8)

It remains to verify that we were indeed allowed to use Hardy’s inequality. This means we have to convince ourselves that
\[ \sup_{R < x < \infty} \left\{ \int_{R}^{x} \tilde{w}(s) \, ds \cdot \int_{x}^{\infty} \tilde{v}^{-1}(s) \, ds \right\} < \infty. \]

Indeed, we find for arbitrary \( x \in (R, \infty) \) that
\[
\begin{align*}
\int_{R}^{x} (r - R)^4 \cdot r^{N-1} \, dr &\cdot \int_{x}^{\infty} (r - R)^{-6} r^{1-N} \, dr \\
&\leq x^{N-1} \cdot \int_{R}^{x} (r - R)^4 \, dr \cdot x^{1-N} \int_{x}^{\infty} (r - R)^{-6} \, dr \\
&\leq \int_{R}^{x} (r - R)^4 \, dr \cdot \int_{x}^{\infty} (r - R)^{-6} \\
&= \frac{1}{25},
\end{align*}
\]
and the lemma is proved. \( \square \)

Lemma 3.1 can be combined with Theorem 2.1 to establish the following estimate on solutions to the Cauchy problem.
Lemma 3.3. Let $u$ be a solution to the Cauchy problem as constructed in Theorem 2.1. Then a positive constant $C_2$ which is independent of $R_0 \geq 0$ exists such that
\[
\int_{\mathbb{R}^N \setminus B_{R_0}(0)} (|x| - R_0)^4 u(\cdot, T)^2 \, dx
\]
\[+ C_2^{-1} \int_0^T \int_{\mathbb{R}^N \setminus B_{R_0}(0)} (|x| - R_0)^6 \left\{ |\nabla u^{\frac{n+2}{n-2}}|^6 + |\nabla \Delta u^{\frac{n+2}{n-2}}|^2 \right\} \, dx \, dt \leq C_2 \left\{ \int_{\mathbb{R}^N \setminus B_{R_0}(0)} (|x| - R_0)^6 \cdot |\nabla u_0|^2 \, dx + \int_0^T \int_{\mathbb{R}^N \setminus B_{R_0}(0)} u^{n+2} \right\}.\] (3.9)

Proof. The proof will be an immediate consequence of Lemma 3.1 and the weighted energy estimate (2.10), provided we can show a decay estimate like (3.2) for $\int_{\partial B_{R}(0)} u^2 \, d\mathcal{H}^{N-1}$. Note that the weighted energy estimate (2.10) and the global energy estimate
\[
\int_{\mathbb{R}^N} |\nabla u(\cdot, T)|^2 + C_1 \int_0^T \int_{\mathbb{R}^N} \left\{ |\nabla u^{\frac{n+2}{n-2}}|^6 + |\nabla \Delta u^{\frac{n+2}{n-2}}|^2 \right\} \leq \int_{\mathbb{R}^N} |\nabla u_0|^2
\]
imply that
\[
\sup_{0<T} \int_{\mathbb{R}^N} |x|^2 \left| \nabla u(x, t) \right|^2 < \infty. \quad (3.10)
\]
On the other hand, (2.8) entails that
\[
\sup_{0<T} \int_{\mathbb{R}^N} u^2(x, t) \, dx < \infty.
\]
Then, the result follows by application of the following lemma. \qed

Lemma 3.4. Assume that a function $v \in H^1(\mathbb{R}^N)$ satisfies
\[
\int_{\mathbb{R}^N} x^2 |\nabla v|^2 \, dx + \int_{\mathbb{R}^N} v^2 \, dx < \infty. \quad (3.11)
\]
Then there is a positive constant $C_3$ such that
\[
\int_{\partial B_{R}(0)} v^2 \, d\mathcal{H}^{N-1} \leq C_2 \cdot R^{-1} \quad (3.12)
\]
for arbitrary $R > 0$.

Proof. Let us denote the transformation of $v$ to polar coordinates by $\hat{v}$. Consider the quantity
\[
r \cdot \|\hat{v}\|_{L^2_{\partial B_{R}(0)}}^2 := r \cdot \int_{S^{N-1}} \hat{v}^2(r, \theta) \cdot r^{N-1} \, dS^{N-1}. \quad (3.13)
\]
Differentiation with respect to $r$ gives

$$\frac{\partial}{\partial r} (r \| \hat{v} \|^2_{L^2(\partial B_r(0))}) = 2 r \int_{S^{N-1}} \hat{v} \hat{vv}_r r^{N-1} dS^{N-1} + N \int_{S^{N-1}} \hat{v}^2 r^{N-1} dS^{N-1}.$$ 

Integrating this identity over the interval $(0, R)$ with respect to $r$ implies that

$$R \int_{S^{N-1}} \hat{v}^2 R^{N-1} dS^{N-1} = 2 \int_0^R \int_{S^{N-1}} r \hat{v} \hat{vv}_r r^{N-1} dS^{N-1} dr + N \int_0^R \int_{S^{N-1}} \hat{v}^2 r^{N-1} dS^{N-1} dr$$

$$\leq C \left( \int_0^R \int_{S^{N-1}} r^2 \hat{v}^2 r^{N-1} dS^{N-1} dr + \int_0^R \int_{S^{N-1}} \hat{v}^2 r^{N-1} dS^{N-1} dr \right)$$

$$\leq C \left( \int_{B_R} |x|^2 |\nabla v|^2 dx + \int_{B_R} v^2 dx \right).$$

This proves the assertion of Lemma 3.4. \qed

4. The Main Result

This section is devoted to the proof of

**Theorem 4.1.** Let $n \in \left( 2 - \sqrt{1 - \frac{N}{8+N}}, 3 \right)$, $N < 4$, assume initial data $u_0 \in H^1(\mathbb{R}^N)$ to be non-negative with compact support in the sense that $u_0(x) = 0$ almost everywhere on $\mathbb{R}^N \setminus B_{R_0}(0)$ for a positive number $R_0$. Let $u$ be the strong solution to the Cauchy problem constructed in Theorem 2.1. Then $u$ has finite speed of propagation in the sense of Definition 1.1. More precisely, supp$(u(\cdot, t)) \subset B_{R(t)}(0)$ where

$$R(t) = R_0 + C \cdot t^\frac{1}{\alpha} \left( \int_0^t \left( \int_{\mathbb{R}^N \setminus B_{R_0}} u^2 \right)^\frac{n+2}{4} dt \right)^\frac{2}{n+2}$$

(4.1)

with a positive constant $C = C(n, N)$ and $\alpha = \frac{(8+N)n(n+2)}{4}$. 

Remark: 1. The notion of finite speed of propagation formulated in Definition 1.1 is still a rather weak one. However, it is possible to replace balls by general convex sets having sufficiently smooth boundary. With – sometimes rather tedious – technical changes, the results to be proved in this section continue to hold. For further improvements, e.g., the treatment of initial data with non-convex support, refined versions of Hardy’s inequality will be necessary. In the forthcoming paper [17], we will prove a Hardy-type inequality valid on infinite cones. On the basis of the finite speed of propagation result established in the present paper, new weighted energy estimates will be formulated for which the spatial support will be given by an infinite cone. These estimates will be the key ingredient to prove local results both on finite speed of propagation and on the occurrence of a waiting time phenomenon wherever the support of initial data locally satisfies an exterior cone condition.
2. Note that (4.1) can be combined with the global energy estimate (2.9) to yield the estimate

\[ R(t) \leq R_0 + \hat{C} \cdot t^{\frac{\gamma}{8 + \gamma N}} \]

with a constant \( \hat{C} \) depending on \( n, N, \| \nabla u_0 \|_2 \) and the initial mass. However, the exponent \( \gamma = \frac{2}{8 + \gamma N} \) is not optimal as a comparison with self-similar solutions reveals (see Ferreira, Bernis [13]). Nevertheless, the merely qualitative result presented here is the starting point to provide optimal quantitative estimates on the diameter of \( \text{supp}(u(\cdot, t)) \). This is the subject of the forthcoming paper [22].

**Proof of Theorem 4.1.** Starting point is the estimate (3.9) which can be simplified for arbitrary \( R \geq R_0 \) and \( T > 0 \) in the following way.

\[
\sup_{t \in (0, T)} \int_{\mathbb{R}^N \setminus B_R} (|x| - R)^4 u(\cdot, t)^2 + C_2^{-1} \int_0^T \int_{\mathbb{R}^N \setminus B_R} (|x| - R)^6 \| \nabla u \|_{\frac{n+2}{n}}^6 \leq C_2 \int_0^T \int_{\mathbb{R}^N \setminus B_R} u^{n+2}. \tag{4.2}
\]

Consider positive numbers \( \varrho > R \). Obviously, \( |x| - R > \varrho - R \) on \( \mathbb{R}^N \setminus B_{\varrho} \). This implies that

\[
\sup_{t \in (0, T)} \int_{\mathbb{R}^N \setminus B_{\varrho}} u(\cdot, t)^2 + (\varrho - R)^2 \int_0^T \int_{\mathbb{R}^N \setminus B_{\varrho}} \| \nabla u \|_{\frac{n+2}{n}}^6 \leq \frac{C}{(\varrho - R)^4} \int_0^T \int_{\mathbb{R}^N \setminus B_R} u^{n+2} \tag{4.3}
\]

for \( \varrho > R \geq R_0 \).

By virtue of Lemma 2.2, the term on the right-hand side can be estimated as follows.

\[
\int_0^T \int_{\mathbb{R}^N \setminus B_R} u^{n+2} \leq K_1 \left( \int_0^T \int_{\mathbb{R}^N \setminus B_R} \| \nabla u \|_{\frac{n+2}{n}}^6 \right)^{\frac{nN}{n+2(N+1)}} \left( \int_0^T \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \right)^{\frac{1}{n+2(N+1)}} \tag{4.4}
\]
Young’s inequality yields

\[
\frac{1}{(\rho - R)^4} \int_0^T \int_{\mathbb{R}^N \setminus B_R} u^{n+2} \leq (\rho - R)^{\frac{2nN}{nN+1}} K_1 \left( \int_0^T \int_{\mathbb{R}^N \setminus B_R} |\nabla u^{\frac{n+2}{6}}|^6 \right)^{\frac{\frac{2nN}{nN+1}}{6}} \cdot (\rho - R)^{-4 \cdot \frac{2nN}{nN+1}} \left( \int_0^T \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \right)^{\frac{12}{nN+1}}
\]

\[
\leq \varepsilon (\rho - R)^2 \int_0^T \int_{\mathbb{R}^N \setminus B_R} |\nabla u^{\frac{n+2}{6}}|^6 + C_\varepsilon (\rho - R)^{-\frac{4+\frac{2nN}{2}}{2}} \left( \int_0^T \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \right)
\]

(4.5)

Putting everything together gives

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^N \setminus B_\rho} u^2(\cdot,t) + (\rho - R)^2 \int_0^T \int_{\mathbb{R}^N \setminus B_R} |\nabla u^{\frac{n+2}{6}}|^6 
\]

\[
\leq \varepsilon (\rho - R)^2 \int_0^T \int_{\mathbb{R}^N \setminus B_R} |\nabla u^{\frac{n+2}{6}}|^6 + \frac{C_\varepsilon}{(\rho - R)^{4+\frac{2nN}{2}}} \int_0^T \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}}
\]

(4.6)

for all \( \rho > R > R_0 \).

Introducing the quantities

\[
V(\rho) := \sup_{t \in [0,T]} \int_{\mathbb{R}^N \setminus B_\rho} u^2, \quad U(\rho) := \int_0^T \int_{\mathbb{R}^N \setminus B_\rho} |\nabla u^{\frac{n+2}{6}}|^6, \\
F_\varepsilon(\rho, R) := \frac{C_\varepsilon}{(\rho - R)^{4+\frac{2nN}{2}}} \int_0^T \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}}
\]

(4.6) can be written as follows.

\[
V(\rho) + (\rho - R)^2 U(\rho) \leq \varepsilon (\rho - R)^2 U(R) + F_\varepsilon(\rho, R)
\]

for all \( \varepsilon > 0 \) and all \( \rho > R \geq R_0 \). An application of the subsequent iteration result Lemma 4.2 shows that

\[
V(\rho) + \frac{(\rho - R)^2}{4} U(\rho) \leq K_\varepsilon F_\varepsilon(\rho, R)
\]
for sufficiently small, but fixed $\varepsilon > 0$. Therefore,

$$
\sup_{t \in (0, T)} \int_{\mathbb{R}^N \setminus B_R} T \int_{\mathbb{R}^N \setminus B_R} |\nabla u|^{\frac{n+2}{2}} \leq \frac{K_x}{(\rho - R)^{4 + \frac{nN+2}{2}}} \int_{\mathbb{R}^N \setminus B_R} \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \cdot (\rho - R)^{4 + \frac{nN+2}{2}}.
$$

(4.7)

Using the estimate

$$
\int_{0}^{T} \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \leq T \sup_{t \in (0, T)} \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}},
$$

we find that

$$
\int_{0}^{T} \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \leq \frac{CT}{(\rho - R)^{4 + \frac{nN+2}{2}}} \cdot \left( \int_{0}^{T} \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \right)^{\frac{n+2}{2}}.
$$

(4.8)

Introducing

$$
G(\rho) := \int_{0}^{T} \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}}
$$

$$
\alpha := (4 + \frac{nN}{2}) \frac{n+2}{2}, \quad \beta := \frac{n+2}{2} > 1,
$$

(4.8) can be rewritten in the form

$$
G(\rho) \leq \frac{CT}{(\rho - R)^{\alpha}} \cdot G(R)^{\beta}
$$

for all $\rho > R \geq R_0$. An application of Stampacchia’s iteration lemma (Lemma 4.3) shows that $G(\rho) = 0$ provided

$$(\rho - R_0)^\alpha \geq C \cdot T \left( \int_{0}^{T} \left( \int_{\mathbb{R}^N \setminus B_{R_0}} u^2 \right)^{\frac{n+2}{2}} \right)^{\frac{4}{\beta}}.$$

Hence, we obtain for $R(T)$ the estimate given in (4.1). Recalling in addition that $u \in L^\infty(\mathbb{R}^+; L^\beta(\mathbb{R}^N))$ for all $1 < \beta < \frac{2N}{N-2}$, finite speed of propagation is proven. □

We used the following iteration lemma which is a slight modification of an argument presented in the proof of Theorem 6.1 of [10] (see also [25]). For the reader’s convenience, we sketch the proof.
Lemma 4.2. Assume that
\[ V(\varrho') + (\varrho' - R')^2 U(\varrho') \leq \varepsilon (\varrho' - R')^2 U(R') + F_\varepsilon(\varrho', R') \]  
(4.9)
for \( \varepsilon > 0 \) sufficiently small and \( 0 \leq R_0 \leq R < R' < \varrho' < \varrho \). Then there exists a positive constant \( K_\varepsilon \) such that
\[ V(\varrho) + \frac{(\varrho - R)^2}{4} U(\varrho) \leq K_\varepsilon F_\varepsilon(\varrho, R). \]  
(4.10)

Proof. Introduce for \( k \in \mathbb{N} \) points \( \varrho_k, R_k \) such that
\[ \varrho_k = R + \frac{(\varrho - R)}{2^{k-1}}, \quad R_k := R + \frac{(\varrho - R)}{2^k}, \]  
(4.11)
i.e.
\[ R_k = \varrho_{k+1} \quad \text{and} \quad \varrho_k - R_k = \frac{(\varrho - R)}{2^k}. \]  
(4.12)
Along the lines of the corresponding result in [10], we may prove that
\[ V(\varrho) + \frac{(\varrho - R)^2}{4} U(\varrho) \leq \frac{\varepsilon^M}{4} (\varrho - R)^2 U(R_M) + \sum_{k=1}^{M} (4\varepsilon)^{k-1} F_\varepsilon(\varrho_k, R_k). \]  
(4.13)
Now estimating
\[ F_\varepsilon(\varrho_k, R_k) \leq 2^{k(4+\frac{n\alpha}{2})} \frac{C_\varepsilon}{(\varrho - R)^{4+\frac{n\alpha}{2}}} \int_0^T \left( \int_{\mathbb{R}^N \setminus B_R} u^2 \right)^{\frac{n+2}{2}} \],

it becomes evident that for \( \varepsilon \) sufficiently small the right-hand side of (4.13) is convergent which proves the lemma.

For the sake for completeness, we state

Lemma 4.3 (Stampacchia’s iteration lemma). Assume that a given non-negative, non-increasing function \( G : (0, \varrho_0) \to \mathbb{R} \) satisfies
\[ G(\xi) \leq \frac{c_0}{(\xi - \eta)^{\alpha}} G(\eta)^{\beta} \]
for \( 0 \leq \eta < \xi \leq \varrho_0 \) and positive numbers \( c_0, \alpha, \beta \) with \( \beta > 1 \). Assume further that
\[ \varrho_0^{\alpha} \geq 2^{\frac{\alpha\beta}{4\beta - 1}} \cdot c_0 \cdot G(0)^{\beta-1}. \]
Then, \( G \) has a root in \( \varrho_0 \).
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