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A General Asset-Liability Management Model for the Efficient Simulation of Portfolios of Life Insurance Policies

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Abstract

New regulations and a stronger competition have increased the importance of stochastic asset-liability management (ALM) models for insurance companies in recent years. In this paper, we propose a discrete time ALM model for the simulation of simplified balance sheets of life insurance products. The model incorporates the most important life insurance product characteristics, the surrender of contracts, a reserve-dependent surplus declaration, a dynamic asset allocation and a two-factor stochastic capital market. All terms arising in the model can be calculated recursively which allows an easy implementation and efficient simulation. Furthermore, the model is designed to have a modular organisation which permits straightforward modifications and extensions to handle specific requirements. In a sensitivity analysis for sample portfolios and parameters, we investigate the impact of the most important product and management parameters on the risk exposure of the insurance company and show that the model captures the main behaviour patterns of the balance sheet development of life insurance products.

Keywords: asset-liability management, participating policies, numerical simulation

JEL classification: C15; G22; G32
MSC: IB10; IM22

1 Introduction

The scope of asset-liability management (ALM) is the responsible administration of the assets and liabilities of insurance contracts. Here, the insurance company has to attain two goals simultaneously. On the one hand, the available capital has to be invested profitably, usually in bonds but also, up to a certain percentage, in stocks (asset management). On the other hand, the obligations against policyholders, which depend on the specific insurance policies, have to be met (liability management). In this paper, we focus on portfolios of participating (with-profit) policies which make up a significant part of the life insurance market. The holder of such a policy gets a fixed guaranteed interest and, in addition, a variable reversionary bonus which is annually added to the policyholder's account and allows the policyholder to participate in the investment returns of the company. Thereby, the insurance company has to declare in each year which part of the investment returns is given to the policyholders as reversionary bonus, which part is saved in a

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reserve account for future bonus payments and which part is kept by the shareholders of the company. These management decisions depend on the financial situation of the company as well as on strategic considerations and legal requirements. A maximisation of the shareholders' benefits has to be balanced with a competitive bonus declaration for the policyholders. Moreover, the exposure of the company to financial, mortality and surrender risks has to be taken into account. These problems, which easily become quite complex due to the wide range of guarantees and option-like features of insurance products and management rules, are investigated with the help of ALM analyses. In this context, it is necessary to estimate the medium- and long-term development of all assets and liabilities as well as the interactions between them and to determine their sensitivity to the different types of risks. This can either be achieved by the computation of particular scenarios (stress tests) which are based on historical data, subjective expectations, and guidelines of regulatory authorities or by a stochastic modelling and simulation of the market development, the policyholder behaviour and all involved accounts.

In recent years, the latter approach has attracted more and more attention as it takes financial uncertainties more realistically into account than an analysis of a small number of deterministically given scenarios. Additional importance arises from the current need of insurance companies to move from an accounting based on book values to a market-based, fair value accountancy standard as required by Solvency II and the International Financial Reporting Standard (IFRS), see, e.g., [22]. This task can be achieved by performing stochastic simulations of ALM models in a risk-neutral environment. Much effort has been spent on the development of such models in the last years, see, e.g., [1, 2, 3, 8, 11, 15, 16, 17, 23, 25, 26, 29] and the references therein. Here, most authors focus on the fair valuation and contract design of unit-linked and participating life insurance policies. Exceptions are [15, 23] where the financial risks and returns of participating policies are analysed under the real world probability measure. Most of the articles in the existing literature (exceptions are [2, 3, 11, 26]) restrict themselves to single-premium contracts and neglect mortality to simplify the presentation or to obtain analytical solutions. However, in the presence of surrender, generalisations which include periodic premiums and mortality risk are not always straightforward, see, e.g., [4].

In this paper, we develop a general model framework for the ALM of life insurance products. The complexity of the model is chosen such that, on the one hand, most of the models previously proposed in the literature and the most important features of life insurance product management are included. As a consequence, closed-form solutions will only be available in special cases. On the other hand, the model is supposed to remain transparent and modular, and it should be possible to simulate the model efficiently. Therefore, we use a discrete time framework in which all terms can be derived easily and can be computed recursively. We use a stochastic two-factor model to simulate the behaviour of the capital markets, while the development of the biometric parameters is assumed to be deterministic. The asset allocation is dynamic with the goal of keeping a certain percentage of stocks. The bonus declaration mechanism is based on the reserve situation of the company as proposed in [16]. Surrender is modelled and analysed using experience-based surrender tables. Different life insurance product specifics are incorporated via premium, benefit and surrender characteristics in a fairly general framework. In contrast to most of the existing literature, where only the valuation or the development of a single policy is considered, we model the development of a portfolio of policies using model points. Each model point corresponds to an individual policyholder account or to a pool of similar policyholder accounts which can be used to reduce the computational complexity, in particular in the case of very large insurance portfolios. Thus we can also investigate effects which arise from the pooling of non-homogeneous contracts, as in [18], where the pooling of two contracts is considered.

The outline of this paper is as follows. In Section 2, we start with the main layout of the balance sheet. Then, in Section 3, the capital market model is described. In Section 4, management rules regarding the capital allocation, the bonus declaration and the shareholder participation are defined. The specification of the insurance products and the individual policyholder accounts is subject of Section 5. In Section 6, the future development of the balance sheet items introduced in Section 2 is derived. Numerical results for example portfolios and model parameters are shown in Section 7. Here, we illustrate the influence and the interaction of the parameters of the model by sensitivity analyses and iso-default probability curves. We particularly investigate the impact of mortality and surrender on the default probabilities of the insurance company. The paper closes in Section 8 with concluding remarks.

2 Balance Sheet Model

The main focus of our model is to simulate the temporal development of the most important balance sheet items for a portfolio of insurance policies. In this section, we indicate the overall structure of the balance sheet. The determination of the single balance sheet items and the modelling of their future development is the subject of the following sections.

We model all terms in discrete time. Here, we denote the start of the simulation by time $t = 0$ and the end of the simulation by $t = T$ (in years). The interval $[0, T]$ is decomposed into K periods $[t_{k-1}, t_k]$, $k = 1, \ldots, K$ with $t_k = k\,\Delta$. Throughout this paper, the period length $\Delta t = T/K$ is equal to one month, which is in line with conventions for insurance contract sales.[1] The balance sheet items at time t_k, $k = 0, \ldots, K$, used in our model are shown in Table 1.

assets		liabilities	
capital	C_k	actuarial reserve	D_k
		allocated bonus	B_k
		free reserve	F_k
		equity	Q_k

Table 1: Simplified balance sheet for a portfolio of life insurance policies.

The asset side consists of the market value C_k of the company's assets at time t_k. On the liability side, the first item is the book value of the actuarial reserve D_k, i.e., the guaranteed savings part of the policyholders after deduction of risk premiums and administrative costs. The second item is the book value of the allocated bonuses B_k which constitute the part of the surpluses that have been credited to the policyholders via the profit participation. The free reserve F_k is a buffer account for future bonus payments. It consists of surpluses which have not yet been credited to the individual policyholder accounts, and is used to smooth capital market oscillations and to achieve a stable and low-volatile return participation of the policyholders. The last item, the equity or company account Q_k, consists of the part of the surpluses which is kept by the shareholders of the company and is defined by

$$Q_k = C_k - D_k - B_k - F_k,$$

so that the sum of the assets equals the sum of the liabilities. Similar to the bonus reserve in [16], Q_k is a hybrid determined as the difference between a market value C_k and the three book values D_k, B_k and F_k. It may be interpreted as hidden reserve of the company as discussed in [23].

Similar balance sheets have already been considered in the literature. The sum $M_k = D_k + B_k$ corresponds to the policy reserve in [16], the policyholders' account in [23] or to the customer account in [25]. We prefer to separate the two accounts in order to thoroughly distinguish the effects of the bonus distribution from the guaranteed benefits. The free reserve F_k and the company account Q_k in our model correspond to the bonus account (also termed undistributed reserve) and to the insurer's account in [10]. These two accounts are sometimes merged into one single account. This, however, is only appropriate if the policyholders are also the owners of the company, see [16].

3 Asset Model

We assume that the insurance company invests its capital either in fixed interest assets, i.e., bonds, or in a variable return asset, i.e., a stock or a basket of stocks. The future development of the capital market is specified by a coupled system of two continuous stochastic differential equations,

[1] Shorter or longer period lengths can be realised in a straightforward way, though.

one for the short interest rate and one for the stock price. This system is then discretized with a period length of Δt. The simulation of the model can either be performed under the objective probability measure which is used for risk analyses, see, e.g., [15, 23], and which is the main focus of this paper, or under the risk-neutral probability measure, which is appropriate for the fair valuation of embedded options or the identification of fair contract designs, see, e.g., [2, 16, 17, 25].

3.1 Continuous stochastic capital market model

For the modelling of the interest rate environment we use the Cox-Ingersoll-Ross (CIR) model [9]. The CIR model is an one-factor mean-reversion model which specifies the dynamics of the short interest rate $r(t)$ at time t by the stochastic differential equation

$$dr(t) = \kappa(\theta - r(t))dt + \sqrt{r(t)}\sigma_r dW_r(t), \tag{1}$$

where $W_r(t)$ is a standard Brownian motion, $\theta > 0$ denotes the mean reversion level, $\kappa > 0$ denotes the reversion rate and $\sigma_r \geq 0$ denotes the volatility of the short rate dynamic. The CIR model has the following appealing properties: First, it produces short interest rates which are always positive if the parameters fulfil the condition $\kappa\theta > \sigma_r^2/2$. In addition, assuming the absence of arbitrage and a market price $\lambda(t, r)$ of interest rate risk of the special form $\lambda(t, r) = \lambda_0\sqrt{r(t)}$ with a parameter $\lambda_0 \in \mathbb{R}$, the short interest rate under the risk-neutral measure follows the same square-root process as in (1) but with the parameters $\hat{\kappa} = \kappa + \lambda_0\sigma_r$ and $\hat{\theta} = \kappa\theta/\hat{\kappa}$. Moreover, the price $b(t, \tau)$ at time t of a zero coupon bond with a duration of τ periods and with maturity at time $T = t + \tau\Delta t$ can be derived in closed form by

$$b(t, \tau) = A(\tau)\, e^{-B(\tau)\, r(t)} \tag{2}$$

as an exponential affine function of the prevailing short interest rate $r(t)$ with

$$A(\tau) = \left(\frac{2he^{(\hat{\kappa}+h)\tau\Delta t/2}}{2h + (\hat{\kappa} + h)(e^{h\tau\Delta t} - 1)}\right)^{2\kappa\theta/\sigma_r^2}, \quad B(\tau) = \frac{2(e^{h\tau\Delta t} - 1)}{2h + (\hat{\kappa} + h)(e^{h\tau\Delta t} - 1)},$$

and $h = \sqrt{\hat{\kappa}^2 + 2\sigma_r^2}$.

To model the stock price uncertainty, we assume that the stock price $s(t)$ at time t evolve according to a geometric Brownian motion

$$ds(t) = \mu s(t)dt + \sigma_s s(t)dW_s(t), \tag{3}$$

where $\mu \in \mathbb{R}$ denotes the drift rate and $\sigma_s \geq 0$ denotes the volatility of the stock return.[2] By Itô's lemma, the explicit solution of this stochastic differential equation is given by

$$s(t) = s(0)\, e^{(\mu - \sigma_s^2/2)t + \sigma_s W_s(t)}. \tag{4}$$

Usually, stock and bond returns are correlated. We thus assume that the two Brownian motions satisfy $dW_s(t)dW_r(t) = \rho dt$ with a constant correlation coefficient $\rho \in [-1, 1]$.

The same two-factor model is used in [11]. In [8, 21], the Vasicek model is employed instead of the CIR model. In [23], stocks and bonds are modelled via a coupled system of two geometric Brownian motions with different drift and volatility parameters. In [5, 28], more complex jump-diffusion processes and Markov-modulated geometric Brownian motions are employed. The simulation of the latter models is more involved, though.

The capital market parameters $\kappa, \theta, \sigma_r, \mu, \sigma_s$ and ρ can be estimated on the basis of historical data. This way, the objective dynamics of $r(t)$ and $s(t)$ are characterised. The market price of risk parameter λ_0 can then be identified by calibrating the theoretical bond prices (2) to observed market prices, see, e.g., [12]. Alternatively, the parameters $\hat{\kappa}, \hat{\theta}, \sigma_r$ and σ_s, which identify the risk neutral measure, can be obtained by calibrating the models (1) and (3) to observed market prices, see, e.g., [7, 11]. To derive the remaining parameters, estimates of μ, θ and ρ are required.

[2]For a simulation under the risk-neutral measure Q, the drift μ is replaced by $r(t)$.

In a more complex model, the constant coefficients in (1) could be replaced by time-dependent parameter functions. Then, the model can be fitted to the currently observed term structure of interest rates. However, the bond prices $b(t, \tau)$ can not be derived analytically anymore, but have to be computed by numerical integration, see [7].

3.2 Discrete stochastic capital market model

In the discrete time case, the short interest rate and the stock and bond prices are defined by $r_k = r(t_k)$, $s_k = s(t_k)$ and $b_k(\tau) = b(t_k, \tau)$. For the solution of equation (1), we use an Euler-Maruyama discretization[3] with step size Δt, which yields

$$r_k = r_{k-1} + \kappa(\theta - r_{k-1})\Delta t + \sigma_r \sqrt{|r_{k-1}|}\sqrt{\Delta t}\,\xi_{r,k},$$

where $\xi_{r,k}$ is a $N(0,1)$-distributed random variable.[4] For the stock prices one obtains

$$s_k = s_{k-1} e^{(\mu - \sigma_s^2/2)\Delta t + \sigma_s \sqrt{\Delta t}(\rho \xi_{r,k} + \sqrt{1-\rho^2}\xi_{s,k})}, \tag{5}$$

where $\xi_{s,k}$ is a $N(0,1)$-distributed random variable independent of $\xi_{r,k}$. Since

$$\mathrm{Cov}(\rho\xi_{r,k} + \sqrt{1-\rho^2}\xi_{s,k}, \xi_{r,k}) = \rho,$$

the correlation between the two Wiener processes $W_s(t)$ and $W_r(t)$ is respected. Further information on diffusion processes in finance and their numerical treatment can be found, e.g., in [14, 24].

4 Management Model

In this section, we first discuss the allocation of the available capital between stocks and bonds, which determines the portfolio return rate p_k in each period k. Then, the bonus declaration mechanism is illustrated, which defines the interest rate z_k which is given to the policyholders. Finally, the shareholder participation is discussed.

4.1 Capital allocation

We assume that the company rebalances its assets at the beginning of each period. Thereby, the company aims to have a fixed portion $\beta \in [0, 1]$ of its assets invested in stocks, while the remaining capital is invested in zero coupon bonds with a fixed duration of τ periods.[5] We assume that no bonds are sold before their maturity.

Let P_k be the premium income at the beginning of period k and let C_{k-1} be the total capital at the end of the previous period. The part N_k of $C_{k-1} + P_k$ which is available for a new investment at the beginning of period k is then given by

$$N_k = C_{k-1} + P_k - \sum_{i=1}^{\tau-1} n_{k-i}\, b_{k-1}(\tau - i),$$

where n_j denotes the number of zero coupon bonds which were bought at the beginning of period j. The capital A_k which is invested in stocks at the beginning of period k is then determined by

$$A_k = \max\{\min\{N_k,\, \beta(C_{k-1} + P_k)\},\, 0\}$$

so that the side conditions $0 \le A_k \le \beta(C_{k-1} + P_k)$ are satisfied.[6] The remaining money $N_k - A_k$ is used to buy $n_k = (N_k - A_k)/b_{k-1}(\tau)$ zero coupon bonds with duration $\tau\Delta t$. Note that due

[3] An alternative to the Euler-Maruyama scheme, which is more time consuming but avoids a discretization error, is to sample from a noncentral chi-squared distribution, see [14].

[4] For the fast generation of normally distributed random variables, we use Moro's method [27].

[5] A slightly more general trading strategy is discussed in [11].

[6] Alternatively, $\beta(C_{k-1} + P_k)$ can be replaced by $\beta\, F_{k-1}$ with $\beta \in \mathbb{R}^+$, so that the proportion of funds invested in stocks is linked to the current amount of reserves. This implements a CPPI capital allocation strategy, see [26].

to long-term investments in bonds it may happen that $N_k < 0$. This case of insufficient liquidity leads to $n_k < 0$ and thus to a short selling of bonds.

The portfolio return rate p_k in period k resulting from the above allocation procedure is then determined by

$$p_k = \left(\Delta A_k + \sum_{i=0}^{\tau-1} n_{k-i} \, \Delta b_{k,i} \right) / (C_{k-1} + P_k), \tag{6}$$

where $\Delta A_k = A_k(s_k/s_{k-1} - 1)$ and $\Delta b_{k,i} = b(t_k, \tau - i - 1) - b(t_{k-1}, \tau - i)$ denote the changes of the market values of the stock and of the bond investments from the beginning to the end of period k, respectively.

4.2 Bonus declaration

Due to regulatory requirements, companies can only guarantee a relatively low interest rate to their policyholders. As a compensation, policyholders are usually entitled to additional variable bonus payments, which are periodically credited to the policyholders' accounts and allow the policyholders to participate in the investment returns of the company (contribution principle). The exact specification of this reversionary bonus varies from one insurance product to another and often depends on the financial situation of the company as well as on strategical considerations and legal requirements. The bonus is declared annually by the company, with the goal to provide a low-volatile, stable and competitive return participation (average interest principle). The company builds up reserves in years of good returns, which are used to keep the bonus stable in years of low returns. Thereby, a high and thus competitive declaration has to be balanced with a solid financial strength of the company. Various mathematical models for the declaration mechanism are discussed in the literature, see, e.g., [2, 6, 10, 15, 16, 23, 25]. In [15, 16, 23], the bonus interest rate for the next year is already declared at the beginning of this year (principle of advance declaration) as it is required in some countries, e.g., Germany, by regulation. The declaration is based on the current reserve rate γ_{k-1} of the company, which is defined in our framework by the ratio of the free reserve to the allocated liabilities, i.e.,

$$\gamma_{k-1} = \frac{F_{k-1}}{D_{k-1} + B_{k-1}}.$$

In this paper, we follow the approach proposed in [16] where the annual interest rate is defined by

$$\hat{z}_k = \max\{\hat{z}, \omega(\gamma_{k-1} - \gamma)\}.$$

Here, $\hat{z}$ denotes the annual guaranteed interest rate, $\gamma \geq 0$ the target reserve rate of the company and $\omega \in [0, 1]$ the distribution ratio or participation coefficient which determines how fast excessive reserves are reduced. Typical values are $\omega \in [0.2, 0.3]$ and $\gamma \in [0.1, 0.4]$. This way, a fixed fraction of the excessive reserve is, in accordance with the average interest rate principle, distributed to the policyholders in case of a satisfactorily large reserve. If the reserve rate γ_{k-1} is below the target reserve rate γ, only the guaranteed interest is paid, see [16] for details.

In our model, the bonus is declared annually, always at the beginning of the first period of each year. In case of a monthly discretization, this bonus has to be converted into a monthly interest

$$z_k = \begin{cases} (1 + \hat{z}_k)^{1/12} - 1 & \text{if } k \bmod 12 = 1 \\ z_{k-1} & \text{otherwise} \end{cases}$$

which is given to the policyholders in each period k of this year.

4.3 Shareholder participation

Excess returns $p_k - z_k$, conservative biometry and cost assumptions as well as surrender fees lead to a surplus G_k in each period k which has to be divided among the two accounts free reserve F_k and equity Q_k. In case of a positive surplus, we assume that a fixed percentage $\alpha \in [0, 1]$ is saved in the free reserve while the remaining part is added to the equity account. Here, a typical assumption is a distribution according to the 90/10-rule which corresponds to the case $\alpha = 0.9$. If the surplus is negative, we assume that the required capital is taken from the free reserve. If the free reserves do not suffice, the company account has to cover the remaining deficit. The free reserve is then defined by

$$F_k = \max\{F_{k-1} + \min\{G_k, \alpha\, G_k\},\, 0\}. \tag{7}$$

The exact specification of the surplus G_k and the development of the equity Q_k is derived in Section 6.

5 Liability Model

In this section, we first discuss the modelling of the decrement of policies due to mortality and surrender. Then, the fixed guaranteed part and the variable bonus part of the insurance policies are specified. Finally, the development of the two policyholder accounts, the actuarial reserve for the guaranteed part and the bonus account for the bonus part is derived.

5.1 Decrement model

For efficiency, the portfolio of all insurance contracts is often represented by a reduced number m of model points. To this end, the individual policies are pooled into model points using criteria like the entry and maturity time, the age and the gender of the policyholders. This is performed in such a way that the cash flows and technical reserves of the representative and of the whole portfolio differ only within tolerable margins, see, e.g., [20]. A model point then contains averaged values for these criteria.

Let q_k^i denote the probability that a policyholder of model point i dies in the k-th period. In the following, we assume that this probability is given deterministically. This is motivated by the fact that the systematic development of mortality can be predicted much more accurately than, e.g., the development of the capital markets. Moreover, unsystematic mortality risk can be controlled by means of portfolio diversification. The probabilities q_k^i typically depend on the age, the year of birth and the gender of the policyholder. They are collected and regularly updated by the insurance companies. In the following, we always assume that mortality occurs deterministically in accordance with the actuarial assumptions.

Most insurance policies include surrender options for the policyholder. Usually, two different approaches for the valuation of these rights are distinguished, see [4]. Either surrender is considered as an exogenously determined event and modelled like death using experience-based decrement tables, or it is assumed that surrender options are rationally exercised by policyholders. While the second approach is extensively investigated in the literature, see, e.g., [1, 3, 16], only very few publications (see [19] and the references therein) investigate the effects of exogenously given surrender decisions. In this paper we assume that the probabilities u_k^i that a policyholder of model point i surrenders in the k-th period are exogenously given. This is the appropriate approach if surrender decisions are mainly driven by the personal consumption plans of the policyholders, see, e.g., [11]. Here, a typical assumption is that the probabilities u_k^i only depend on the elapsed contract time. As in [19], we assume that the time t until surrender follows an exponential distribution with exponent $\lambda = 0.03$. The probabilities of surrender are then given by

$$u_k^i = 1 - e^{-\lambda \Delta t}. \tag{8}$$

Since the probabilities of surrender are, in contrast to the probabilities of death, not included into the actuarial premium calculation, the effects of surrender on the success of the company significantly differ from the effects due to mortality.

Let δ_k^i denote the expected number of contracts in model point i at the end of period k. This number evolves over time according to

$$\delta_k^i = \left(1 - q_k^i - u_k^i\right) \delta_{k-1}^i. \tag{9}$$

By pooling, all contracts of a model point expire at the same time. In the simulation the model point is then simply dissolved. In this paper, we do not consider the evolution of new contracts during the simulation.

5.2 Insurance products

In this section, the guaranteed part and the bonus part of the insurance products are specified. We always assume that premiums are paid at the beginning of a period while benefits are paid at the end of the period. Furthermore, we assume that all administrative costs are already included in the premium.

For each model point $i = 1, \ldots, m$, the guaranteed part of the insurance product is defined by the specification of the following four characteristics:

- premium characteristic: $(P_1^i, \ldots, P_K^i)$ where P_k^i denotes the premium of an insurance holder in model point i at the beginning of period k if he is still alive at that time.
- survival benefit characteristic: $(E_1^{i,G}, \ldots, E_K^{i,G})$ where $E_k^{i,G}$ denotes the guaranteed payments to an insurance holder in model point i at the end of period k if he survives period k.
- death benefit characteristic: $(T_1^{i,G}, \ldots, T_K^{i,G})$ where $T_k^{i,G}$ denotes the guaranteed payment to an insurance holder in model point i at the end of period k if he dies in period k.
- surrender characteristic: $(S_1^{i,G}, \ldots, S_K^{i,G})$ where $S_k^{i,G}$ denotes the guaranteed payment to an insurance holder in model point i at the end of period k if he surrenders in period k.

Here, K denotes the last period of the simulation. The characteristics, which can be different for each model point, are usually derived from an insurance tariff which contains the functional interrelations between premium, benefit, death and surrender characteristics. Given the benefit characteristics of a product, the premiums are determined by the equivalence principle which states that the present value of the death and survival benefits must equal the present value of the premium at the start of the insurance, see, e.g., [2, 30]. Here, the present values are computed according to a given technical interest rate z using the traditional actuarial approach. Typically, z is fixed at the start of the contract and constitutes an interest rate guarantee. We assume here that z is equal for all contracts.

The actuarial reserve D_k^i of an insurance contract at each point in time is defined as the difference of the present value of the expected future death and survival benefits and the present value of the expected future premiums which are calculated according to the actuarial assumptions. An efficient computation of the actuarial reserve D_k^i of model point i at the end of period k is possible by using the recursion

$$D_k^i = \frac{1+z}{1-q_k^i}(D_{k-1}^i + P_k^i) - E_k^{i,G} - \frac{q_k^i}{1-q_k^i} T_k^{i,G}. \tag{10}$$

Multiplication by $(1-q_k^i)$ shows that this equation results from the fact that the expected actuarial reserve $(1-q_k^i)D_k^i$ at the end of period k is given by the sum of the actuarial reserve of the previous period and the premium after guaranteed interest $(1+z)(D_{k-1}^i + P_k^i)$ minus the expected survival and death benefits $(1-q_k^i)E_k^{i,G} + q_k^i T_k^{i,G}$, see, e.g., [2, 30].

In addition to the guaranteed benefits which depend on the technical interest rate z, policyholders are also entitled to a bonus interest $z_k - z$ defined in Section 4.2, which depends on the development of the financial markets. Depending on the specific insurance product, the allocated bonuses are distributed to each policyholder either at maturity of his contract or earlier in case of death or surrender. Let $E_k^{i,B}$, $T_k^{i,B}$ and $S_k^{i,B}$ denote the bonus payments to an insurance holder in model point i at the end of period k in case of survival, death and surrender, respectively. The sum of all bonuses allocated to a policyholder of model point i at the end of period k is collected in an individual bonus account B_k^i. Its value is recursively defined by

$$B_k^i = \frac{1+z_k}{1-q_k^i} B_{k-1}^i + \frac{z_k - z}{1-q_k^i}(D_{k-1}^i + P_k^i) - E_k^{i,B} - \frac{q_k^i}{1-q_k^i} T_k^{i,B}. \tag{11}$$

Similar to above, this equation results from the fact that the expected value[7] $(1-q_k^i)B_k^i$ of the bonus account at the end of period k is given by the sum of allocated bonuses in the past after interest $(1+z_k)B_{k-1}^i$ and the bonus payment $(z_k - z)(D_{k-1}^i + P_k^i)$ of period k minus the expected bonus payments $(1-q_k^i)E_k^{i,G} + q_k^i T_k^{i,G}$ in case of survival and death, respectively.

The total payments E_k^i, T_k^i and S_k^i to a policyholder of model point i at the end of period k in case of survival, death and surrender are then given by

$$E_k^i = E_k^{i,G} + E_k^{i,B}, \quad T_k^i = T_k^{i,G} + T_k^{i,B} \ \text{ and } \ S_k^i = S_k^{i,G} + S_k^{i,B}. \tag{12}$$

By adding (10) and (11), we see that the sum $M_k^i = D_k^i + B_k^i$ of the two policyholder accounts satisfies

$$M_k^i = \frac{1+z_k}{1-q_k^i}(M_{k-1}^i + P_k^i) - E_k^i - \frac{q_k^i}{1-q_k^i} T_k^i \tag{13}$$

which has a similar structure as the recursion for the actuarial reserve (10).

Example 5.1 *For illustration, an endowment insurance with death benefit and constant premium payments is considered. Let P^i denote the constant premium which is paid by each of the policyholders in model point i in every period. If they are still alive, the policyholders receive at maturity d^i a guaranteed benefit $E^{i,G}$ and the value of the bonus account. In case of death prior to maturity, the sum of all premium payments and the value of the bonus account is returned. In case of surrender, the policyholder capital and the bonus is reduced by a surrender factor $\vartheta \in [0,1]$. The guaranteed components of the four characteristics are then defined by*

$$P_k^i = P^i, \quad E_k^{i,G} = \chi_k(d^i)\, E^{i,G}, \quad T_k^{i,G} = k\, P^i \quad \text{and} \quad S_k^{i,G} = \vartheta D_k^i,$$

where $\chi_k(d^i)$ denotes the indicator function which is one if $k = d^i$ and zero otherwise. The bonus payments at the end of period k are given by

$$E_k^{i,B} = \chi_k(d^i) B_k^i, \quad T_k^{i,B} = B_k^i \quad \text{and} \quad S_k^{i,B} = \vartheta\, B_k^i.$$

We will return to this example in Section 7.

6 Projection of the Balance Sheet

In this section, we derive the recursive development of all items in the simplified balance sheet introduced in Section 2. We then discuss relations to other mathematical models previously proposed in the literature and study various performance figures, which can be used to illustrate the results of the balance sheet projection.

6.1 Projection of the assets

In order to define the capital C_k at the end of period k, we first determine the cash flows which are occurring to and from the policyholders in our model framework. The premium P_k, which is obtained by the company at the beginning of period k, and the survival payments E_k, the death payments T_k, and the surrender payments S_k to policyholders, which take place at the end of period k, are obtained by summation of the individual cash flows (12), i.e.,

$$P_k = \sum_{i=1}^m \delta_{k-1}^i\, P_k^i, \quad E_k = \sum_{i=1}^m \delta_k^i\, E_k^i, \quad T_k = \sum_{i=1}^m q_k^i \delta_{k-1}^i\, T_k^i \ \text{ and } \ S_k = \sum_{i=1}^m u_k^i \delta_{k-1}^i\, S_k^i, \tag{14}$$

[7] Note that B_k^i still depends on financial uncertainty. The expected value is only taken with respect to the death probabilities.

where the numbers δ_k^i are given by (9). Note that the cashflows are expected values with respect to our deterministic mortality and surrender assumptions from Section 5.1. They are random numbers with respect to our stochastic capital market model from Section 3.1. Taking into account the portfolio return rate p_k from Section 4.1, one obtains that the capital C_k is recursively given by

$$C_k = (C_{k-1} + P_k)\,(1 + p_k) - E_k - T_k - S_k. \tag{15}$$

6.2 Projection of the liabilities

The actuarial reserve D_k and the allocated bonus B_k are derived by summation of the individual policyholder accounts (10) and (11), i.e.,

$$D_k = \sum_{i=1}^{m} \delta_k^i\, D_k^i \quad \text{and} \quad B_k = \sum_{i=1}^{m} \delta_k^i\, B_k^i.$$

From the equations (9), (13) and (14), we derive that the sum $M_k = \sum_i \delta_k^i\, M_k^i = D_k + B_k$ is recursively given by

$$M_k = (1 + z_k)(M_{k-1} + P_k) - E_k - T_k - S_k/\vartheta, \tag{16}$$

where we assumed $S_k^{i,G} = \vartheta\, D_k^i$ as in Example 5.1 and $\vartheta > 0$.

In order to define the free reserve F_k, we next determine the gross surplus G_k in period k. By the so-called contribution formula, the surplus is usually divided into the components interest surplus, risk surplus, cost surplus and surrender surplus. In our model only the interest surplus and the surrender surplus show up. The interest surplus is given by the difference between the total capital market return $p_k\,(F_{k-1} + M_{k-1} + P_k)$ on policyholder capital and the interest payments $z_k\,(M_{k-1} + P_k)$ to policyholders. The surrender surplus is given by the difference $S_k/\vartheta - S_k$. The gross surplus in period k is thus given by

$$G_k = p_k\, F_{k-1} + (p_k - z_k)\,(M_{k-1} + P_k) + (1/\vartheta - 1)S_k.$$

The development of the free reserve F_k is then derived using equation (7). Altogether, the company account Q_k is determined by

$$Q_k = C_k - M_k - F_k.$$

For convenience, the complete model is summarised in Figure 1.

6.3 Relation to other mathematical models

Our model framework includes many of the models previously proposed in the literature as special cases. If we consider only one policy with a single initial payment P^i at time $t = 0$, a term of K years, a yearly discretization and neglect mortality and surrender, then the non-zero entries in the characteristics of this (pure savings) product are defined by $P_1^i = P^i$ and $E_K^{i,G} = (1 + z)^K P^i$. Moreover, by (16) it holds that

$$M_k^i = (1 + z_k)M_{k-1}^i, \quad M_0^i = P^i \;\text{ for } k = 1, \ldots, K.$$

If we further replace (7) by $F_k = F_{k-1} + G_k$, we exactly recover the model for European-type participation contracts proposed in [16]. If we include mortality, define the guaranteed death benefit by $T_k^{i,G} = (1+z)^k P^i$ and declare the policyholder interest according to $z_k = \max\{\omega\, p_k, z\}$, we further recover the model for single premium contracts proposed in [2]. The definition $P_k^i = P^i$ for all $k = 1, \ldots, K$ leads to the constant premium case in [3] but without the linear approximation for the benefit adjustment.

Data given at the start of the simulation:

Insurance portfolio	$D_0^i,\, B_0^i,\, \delta_0^i$ for $i = 1, \ldots, m$
Product parameters	$z \in \mathbb{R}^+,\, \vartheta \in [0,1];\; P_k^i,\, E_k^{i,G},\, T_k^{i,G},\, S_k^{i,G}$ for all i, k
Biometric parameters	$q_k^i,\, u_k^i$ for $i = 1, \ldots, m$ and $k = 1, \ldots, K$
Capital market parameters	$\kappa,\, \theta,\, \sigma_r,\, r_0,\, \sigma_s \in \mathbb{R}^+;\; \mu,\, \lambda_0 \in \mathbb{R};\; \rho \in [-1,1]$
Management parameters	$\tau \in \mathbb{N};\; \beta,\, \omega,\, \gamma,\, \alpha \in [0,1]$
Initial values	$F_0,\, Q_0,\, n_{1-\tau}, \ldots, n_0 \in \mathbb{R}$

Asset model (see Section 3):

Short interest rates	r_k	$=$	$r_{k-1} + \kappa(\theta - r_{k-1})\Delta t + \sigma_r \sqrt{\lvert r_{k-1}\rvert}\sqrt{\Delta t}\, \xi_{r,k}$
Stock prices	s_k	$=$	$s_{k-1} \exp\{(\mu - \sigma_s^2/2)\Delta t + \sigma_s \sqrt{\Delta t}(\rho \xi_{r,k} + \sqrt{1-\rho^2}\xi_{s,k})\}$
Bond prices	$b_k(\tau)$	$=$	$A(\tau) \exp\{-B(\tau)\, r_k\}$

Management model (see Section 4):

New investment	N_k	$=$	$C_{k-1} + P_k - \sum_{i=1}^{\tau-1} n_{k-i}\, b_{k-1}(\tau - i)$
Investment in stocks	A_k	$=$	$\max\{\min\{N_k,\, \beta(C_{k-1} + P_k)\},\, 0\}$
Number of new bonds	n_k	$=$	$(N_k - A_k)/b_{k-1}(\tau)$
Portfolio return rate	p_k	$=$	$(\Delta A_k + \sum_{i=0}^{\tau-1} n_{k-i}\, \Delta b_{k,i})/(C_{k-1} + P_k)$
Annual policyholder interest	$\hat{z}_k$	$=$	$\max\{\hat{z},\, \omega(F_{k-1}/M_{k-1} - \gamma)\}$
Policyholder interest	z_k	$=$	$(1 + \hat{z}_k)^{\Delta t} - 1$ if $k \bmod (1/\Delta t) = 1$, else z_{k-1}

Liability model (see Section 5):

Number of contracts	δ_k^i	$=$	$\left(1 - q_k^i - u_k^i\right) \delta_{k-1}^i$
Actuarial reserve	D_k^i	$=$	$((1+z)(D_{k-1}^i + P_k^i) - q_k^i T_k^{i,G})/(1 - q_k^i) - E_k^{i,G}$
Allocated bonus	B_k^i	$=$	$(1 + z_k)\, B_{k-1}^i + (z_k - z)\,(D_{k-1}^i + P_k^i)$

Balance sheet projection (see Section 6):

Surplus	G_k	$=$	$p_k\, F_{k-1} + (p_k - z_k)\,(M_{k-1} + P_k) + (1/\vartheta - 1) S_k$
Capital	C_k	$=$	$(1 + p_k)(C_{k-1} + P_k) - E_k - T_k - S_k$
Policyholder accounts	M_k	$=$	$(1 + z_k)(M_{k-1} + P_k) - E_k - T_k - S_k/\vartheta$
Free reserve	F_k	$=$	$\max\{F_{k-1} + \min\{G_k, \alpha\, G_k\},\, 0\}$
Equity	Q_k	$=$	$C_k - M_k - F_k$

Figure 1: Summary of the input parameters and of the important model equations.

6.4 Performance figures

A single simulation run can be analysed by looking at the balance sheet positions or at cross sections of the portfolio at certain times. In a stochastic simulation, however, a large number of scenarios is generated, such that an analysis of all individual scenarios is not possible anymore. Instead, statistical measures are considered which result from an averaging over all scenarios. As a risk measure we consider the path-dependent cumulative probability of default

$$\mathrm{PD}_k = \mathbb{P}\left(\min_{j=1,\ldots,k} Q_j < 0\right).$$

In the next section, we also take a look at the expected values of the balance sheet items $\mathbb{E}[C_k]$, $\mathbb{E}[B_k]$, $\mathbb{E}[F_k]$ and $\mathbb{E}[Q_k]$ for $k = 1, \ldots, K$. These profit and risk figures can easily be computed during the simulation. Similarly, it is straightforward to include the computation of further performance measures like the variance, the value-at-risk, the expected shortfall or the return on risk capital.

To determine the sensitivity of a given performance figure f to one of the model parameters v, we compute the partial derivative $f'(v) = \partial f(v)/\partial v$ by a finite difference approximation. For a better comparison, often also the relative change in f to a small change in v is considered, which is given by $f'(v)/f(v)$. For $v = r_0$, this measure of sensitivities is also called effective duration or interest-rate elasticity. For a discussion and further collection of useful risk measures we refer to [8, 11, 20].

7 Numerical Results

In this section, numerical results are presented for various sample products and model parameters. Due to the complex path-dependence of the stochastic terms in our model, closed-form solutions are only available in very special cases. The modular and recursive design of our model permits efficient Monte Carlo simulations[8] of all performance figures, however. The Monte Carlo samples $\mathbf{x} \in (0,1)^{2K}$ are transformed to $N(0,1)$-distributed random variables $\xi_{r,1}, \ldots, \xi_{r,K}$ and $\xi_{s,1}, \ldots, \xi_{s,K}$ by Moro's method [27]. From these numbers, the development of all involved accounts can then be derived by evaluating the model equations summarised in Figure 1. The number of operations for the simulation of a single scenario is of order $O(m \cdot K)$. For a representative portfolio with 500 model points and a time horizon of $K = 120$ periods, the simulation of a single scenario takes about 0.04 seconds on a dual Intel(R) Xeon(TM) CPU 3.06GH workstation.

We first describe the basic setting for our numerical experiments. Then, the expected developments of the balance sheet items and of the default probabilities are shown for four different sample products in order to investigate the influence of mortality and surrender on the success of the company. Next, we investigate the sensitivities of the performance figures from Section 6.4 to the input parameters of the model. Finally, iso-default probability curves are discussed to illustrate the effects which result from the management parameters. In particular, optimal bond durations τ and stock ratios β are determined.

7.1 Setting

We consider a representative model portfolio with 50,000 contracts in 500 equal-sized model points. The data of each model point i is generated according to the following distribution assumptions: entry age $\underline{x}^i \sim N(36, 10)$, exit age $\overline{x}^i \sim N(62, 4)$, current age $x_0^i \sim U(\underline{x}^i, \overline{x}^i)$ and monthly premium $P^i \sim U(50, 500)$ where $N(\mu, \sigma)$ denotes the normal distribution with mean μ and variance σ, and $U(a, b)$ denotes a uniform distribution in the interval $[a, b]$. In addition, the side conditions

[8]For more information on Monte Carlo methods and possible approaches, as variance reduction techniques and quasi-Monte Carlo methods to reduce the number of required scenarios to obtain a prescribed accuracy, see, e.g., [14]. Also, sparse grid methods [13] can be used for the generation of the sample points $\mathbf{x} \in (0,1)^{2K}$. The comparison of the different numerical methods will be the focus of a future paper.

$15 \leq \underline{x}^i \leq 55$ and $55 \leq \overline{x}^i \leq 70$ are respected. The probability that the contracts of a model point belong to female policyholders is assumed to be 55%. From the difference of exit age and current age the maturity time $d^i = \overline{x}^i - \underline{x}^i$ of the contracts is computed. For simplicity, we assume that the policies have not received any bonus payments before the start of the simulation, i.e., $B_0^i = 0$ for all $i = 1, \ldots, m$. We consider the following four sample products:

- $p^{(1)}$: pure savings account ($q_k^i \equiv 0$, $u_k^i \equiv 0$),
- $p^{(2)}$: endowment insurance with death benefit ($u_k^i \equiv 0$),
- $p^{(3)}$: endowment insurance with death benefit and surrender option (no surrender fee),
- $p^{(4)}$: endowment insurance with death benefit and surrender option (10% surrender fee),

which are all equipped with the contract characteristics of Example 5.1 but differ in their dependence on mortality and surrender. We take the probabilities q_k^i of death from the DAV 2004R mortality table and choose the probabilities u_k^i of surrender as in (8) with $\lambda = 0.03$. In our numerical tests we use the capital market, product and management parameters as displayed in the second rows of Table 2 and 3 unless stated otherwise. The parameters for the short rate and the stock prices are taken from [12] and [23], respectively, where they have been estimated based on historical data for the German market. For simplicity, the values are rounded off to two decimal places. The parameters for the bonus declaration correspond to the neutral scenario in [16]. At time t_0, we assume a uniform bond allocation, i.e., $n_j = (1-\beta)C_0 / \sum_{i=0}^{\tau-1} b_0(i)$ for $j = 1-\tau, \ldots, 0$. We assume $Q_0 = 0$ which means that the shareholders will not make additional payments to the company to avoid a ruin. This way, $\mathbb{E}[Q_k]$ serves as a direct measure for the investment returns of the shareholders in the time interval $[0, t_k]$. The total initial reserves of the company are then given by $F_0 = \gamma_0 \, D_0$.

We simulated $n = 10,000$ Monte Carlo scenarios. For the set of parameters considered, this number is sufficient to obtain approximations to all balance sheet items with relative standard errors smaller than 10^{-4}. In the following sensitivity analyses, the simulations for varying parameters are based on the same sequence of normally distributed random numbers $\xi_{s,k}$ and $\xi_{r,k}$ in order to allow for a direct comparison of the results.

7.2 Impact of mortality, surrender and surrender fees

For the setting described in Section 7.1, we now analyse the expected development of all balance sheet items and of the default probability for the example products $p^{(1)}$, $p^{(2)}$, $p^{(3)}$ and $p^{(4)}$, respectively. We choose a simulation horizon of $T = 30$ years and a period length of $\Delta t = 1/12$ years, i.e., $K = 360$. The time horizon is chosen exactly in a way such that all policies are expired at the end of the simulation. The results are given in Figure 2 where the expected values are displayed for each period $k = 1, \ldots, 360$.

In Figure 2 (top left and top right), we see that the expected value $\mathbb{E}[C_k]$ of the capital and the actuarial reserve[9] D_k increases in the first two years due to the premium income and the capital market returns. Then, the expected values of the accounts decrease with time due to the decrement of policies. Here, the presence of surrender has a much higher impact than the presence of mortality as can be seen from the comparison of the products $p^{(1)}$, $p^{(2)}$ and $p^{(3)}$. The expected capital is higher in case $p^{(4)}$ compared to $p^{(3)}$ because of the additional surpluses generated by the surrender fees.

In Figure 2 (middle left), one can see that the expected bonus account B_k does not start to increase significantly until the fifth year of the simulation. This is explained by the choice of the initial reserve rate $\gamma_0 = 10\%$ which is below the target reserve rate $\gamma = 15\%$. In the first years of the simulation, the company is therefore only paying the guaranteed interest in order to build up sufficient reserves.

[9]Note that the development of the actuarial reserve is, in contrary to all other balance sheet items, independent of the development of the financial markets and not affected by the choice of the surrender fee.

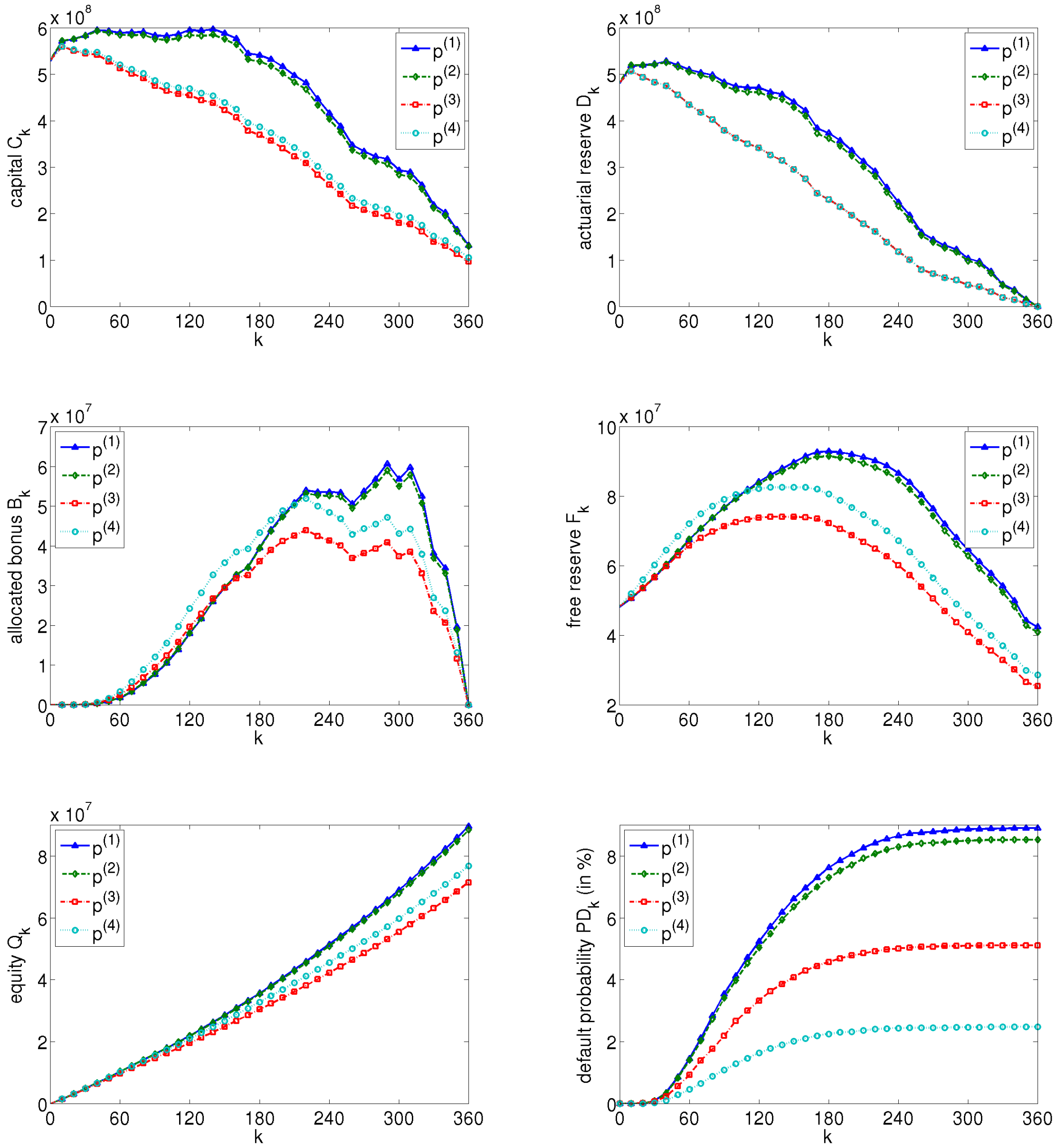

Figure 2: For each period $k = 0, \ldots, 360$, the expected values $\mathbb{E}[C_k]$ (top left), $\mathbb{E}[B_k]$ (middle left), $\mathbb{E}[F_k]$ (middle right), $\mathbb{E}[Q_k]$ (bottom left) and the default probability PD_k (bottom right) are displayed for the portfolio, parameters and sample products from Section 7.1. The actuarial reserve D_k is shown in the top right place. In contrast to the other balance sheet items, it is independent of the random numbers $\xi_{s,k}$ and $\xi_{r,k}$ and equal for the products $p^{(3)}$ and $p^{(4)}$.

In Figure 2 (middle right), the expected values $\mathbb{E}[F_k]$ of the free reserves are shown. They increase in the first 10 years for the products $p^{(3)}$ and $p^{(4)}$ with surrender option and in the first 15 years for the products $p^{(1)}$ and $p^{(2)}$ without surrender option, and decrease afterwards. The expected values of the reserve rates $\gamma_k = F_k/(D_k+B_k)$ at $k = 120$ are given by 17.2%, 17.4%, 20.4% and 22.4% for the products $p^{(1)}$, $p^{(2)}$, $p^{(3)}$ and $p^{(4)}$. The different values can be explained by the fact that the decrement of contracts leads to a reduction of the actuarial reserve and of the bonus accounts while it does not directly affect the free reserve. As a consequence, the expected reserve rate increases with a higher decrement of contracts. In the case $p^{(4)}$, the company profits additionally from the surrender fees. In all cases, γ_k tends to infinity in the last periods of the simulation[10] as the policyholder capital $D_k + B_k$ converges to zero for $k \to K$.

In Figure 2 (bottom left), we see that the expected value of the equity account Q_k increases almost linearly with time. Depending on the product, smaller values are attained in the order $p^{(1)}, p^{(2)}, p^{(4)}$ and $p^{(3)}$, which is explained by the differences of the expected asset bases $\mathbb{E}[C_k]$ as these lead to differently large expected surpluses.

Of particular interest is the development of the default probabilities PD_k (Figure 2, bottom right). We observe that the default risk of the company is almost zero in the first years of the simulation which is due to the 10% initial buffer in the free reserve. Then, the default probabilities start to increase significantly with time up to a certain limit which is reached after 15 years in case of a surrender option and which is reached after 20 years in the other case. At these points in time a major part of the insurance portfolio is expired or has surrendered, so that the company faces almost no default risk anymore. The default risk significantly depends on the insurance product under consideration: If we compare $p^{(1)}$ and $p^{(2)}$, we can see that the presence of mortality reduces the default risk slightly from 5.2% to 5.0% for a 10 year time horizon and from 8.9% to 8.5% for a 30 year time horizon. A significant reduction results from the surrender option. In the case of product $p^{(3)}$, one obtains $\mathrm{PD}_{120} = 3.3\%$ and $\mathrm{PD}_{360} = 5.1\%$, and in case of product $p^{(4)}$ the default probabilities are $\mathrm{PD}_{120} = 1.6\%$ and $\mathrm{PD}_{360} = 2.5\%$.

7.3 Sensitivities

For the product $p^{(4)}$, we next investigate the sensitivity of the model to the various input parameters to show that the model captures the most important behaviour patterns of the balance sheet development of life insurance products. Note that all of the following results are only true for the chosen set of parameters and can change if one or more of the parameters are varied. In the following, we choose a simulation horizon of $T = 10$ years and a period length of $\Delta t = 1/12$ years, i.e., $K = 120$. The sensitivities are described by the partial derivatives $f'(v)/f(v)$, see Section 6.4, which are displayed in Table 2 and 3. Here, each row corresponds to a different function $f(v)$ and each column to a different model input parameter v, e.g., $\partial\mathrm{PD}_K/(\partial\mu\,\mathrm{PD}_K) = -0.431$. For the chosen set of parameters, we observe that the default probability decreases in $\mu, \kappa, \theta, r_0, \tau, \gamma_0$ and increases in $\sigma_s, \sigma_r, \lambda_0, \rho, \beta, \vartheta, z$ as expected. The highest sensitivity arises with respect to the guaranteed interest z and the stock ratio β. Moreover, the parameters r_0, θ and σ_r of the interest rate model are of high importance, which indicates that these parameters should be estimated by the insurance company with particular care. The parameters λ_0 and ρ are only of moderate importance. For our setting, the default probability is almost insensitive to ω, γ and α. This, however, is no longer true if other parameters, like the initial solvency rate, are changed as shown in Section 7.4 where the interaction of several parameters is investigated.

The expected values of the equity account and of the free reserves decrease in $\sigma_r, \rho, \lambda_0, \omega, \vartheta$ and increase in the remaining parameters. The most important input parameter is the initial interest rate r_0 due to its relevance in the specification of the surplus in the first years of the simulation. For longer time horizons the importance of θ increases. While $\mathbb{E}[F_K]$ is also rather sensitive to z and not much affected by small changes of α, the equity account is more affected by changes in α than by changes in z. It is striking that $\mathbb{E}[Q_K]$ profits significantly from higher values of β and higher values of σ_s, while $\mathbb{E}[F_K]$ increases only slightly in case of higher values of β and even

[10] To avoid unrealistic high interest rates z_k in these periods, we capped the bonus declaration at 10%.

	stock price model		interest rate model					correlation
	$\mu = 8\%$	$\sigma_s = 20\%$	$\kappa = 0.1$	$\theta = 4\%$	$\sigma_r = 5\%$	$r_0 = 3\%$	$\lambda_0 = -5\%$	$\rho = -0.1$
$\mathbb{E}[Q_K]$	0.028	0.035	0.007	0.085	-0.001	0.156	-0.001	-0.0008
$\mathbb{E}[F_K]$	0.039	-0.008	0.009	0.136	-0.0014	0.212	-0.0014	-0.0002
PD_K	-0.431	0.219	-0.172	-0.884	0.729	-2.122	0.005	0.04

Table 2: Capital market parameters p used in the simulation and their partial derivatives $f'(p)/f(p)$ for $f \in \{\mathrm{PD}_K, \mathbb{E}[Q_K], \mathbb{E}[F_K]\}$.

	asset allocation		bonus declaration		shareholder	product parameters		solv. rate
	$\beta = 10\%$	$\tau = 3$	$\omega = 25\%$	$\gamma = 15\%$	$\alpha = 90\%$	$\vartheta = 90\%$	$z = 3\%$	$\gamma_0 = 10\%$
$\mathbb{E}[Q_K]$	0.083	0.004	-0.002	0.009	-0.101	-0.006	-0.086	0.011
$\mathbb{E}[F_K]$	0.002	0.002	-0.009	0.03	0.013	-0.01	-0.22	0.034
PD_K	0.265	-0.054	0	-0.002	0.001	0.08	2.706	-0.504

Table 3: Solvency rate and management and product parameters p used in the simulation and their partial derivatives $f'(p)/f(p)$ for $f \in \{\mathrm{PD}_K, \mathbb{E}[Q_K], \mathbb{E}[F_K]\}$.

decreases slightly for higher values of σ_s. These effects are a consequence of equation (7) which states that the shareholder account usually does not suffer under negative surpluses while the free reserve is affected by negative as well as by positive surpluses.

7.4 Impact of the management parameters

Now, we investigate the interaction of the management parameters $\beta, \tau, \omega, \gamma$ and the initial solvency rate γ_0. Similar to [16, 23] we illustrate the results by iso-default probability curves, i.e., by providing pairs of parameters which result in the same default probability PD_K.

We first consider the parameter β which determines the target stock ratio of the asset allocation, see Section 4.1. In Figure 3 (left) the combinations of (β, γ_0) which give $\mathrm{PD}_{120} = 5\%$ are shown for each product. Note that higher values of β result in higher but more volatile expected capital market returns. We see that the slightly negative correlation $\rho = -0.1$ results in a diversification effect such that the lowest default risk is not attained at $\beta = 0$ but at about $\beta = 5\%$. As expected, companies with higher initial reserves can afford to have higher stock ratios. In the case of product $p^{(1)}$, the value of β which maximises the capital market returns under the condition $\mathrm{PD}_K \leq 5\%$ is given by 11% for a company with $\gamma_0 = 10\%$ and by 21% for a company with $\gamma_0 = 20\%$. In the case of product $p^{(4)}$, these optimal values shift to 16% and 25%, respectively.

Next, we consider the parameter τ which defines the duration of the bond investments, see Section 4.1. Higher values of τ result in higher bond yields, since we assumed a negative market price of risk parameter λ_0. We thus also expect a reduction of the default risk for higher values of τ. On the other hand, longer bond durations increase the probability that the company runs into growing liquidity problems during the simulation, which in turn increases PD_K. The interaction of these two effects are illustrated in Figure 3 (right) where the default probabilities are displayed as a function of τ for different solvency rates γ_0. For τ varying from one month to eight years, PD_{120} varies between $2 - 4\%$ for a company with $\gamma_0 = 10\%$. The bond duration which minimizes the default risk is given by $\tau^{\min} = 4$ years. Interestingly, it depends on the solvency rate of the company. For the case $\gamma_0 = 13\%$, we obtain $\tau^{\min} = 5$ years. The optimal bond duration increases for companies with larger reserves.[11]

[11] As an interesting extension one could try to further optimise the risk-return profile of the company by using a more complex capital allocation model which dynamically matches the bond duration with the duration of the liabilities.

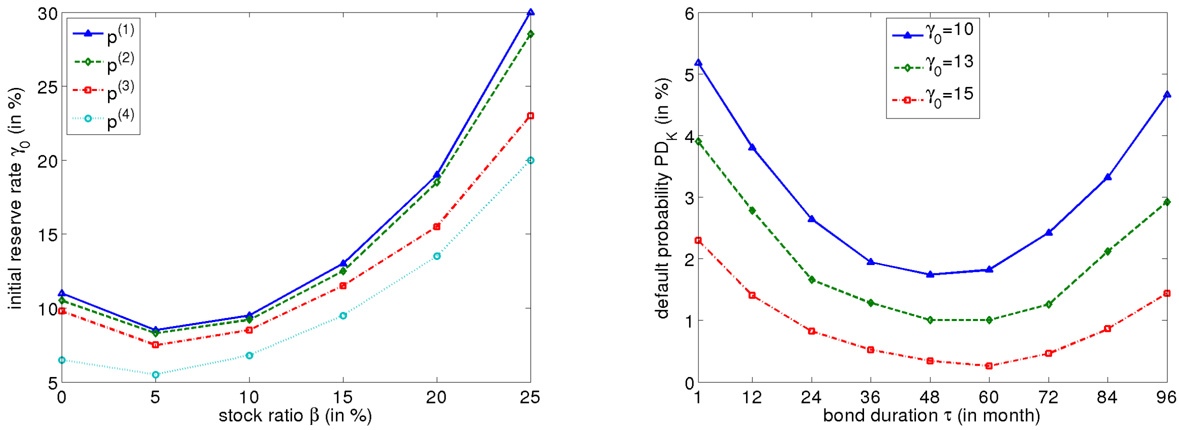

Figure 3: Left: Parameter pairs (β, γ_0) for the example products $p^{(1)} - p^{(4)}$ which lead to a default probability of 5%. Right: Default probability PD_{120} in the case $p^{(1)}$ in dependence of the bond duration τ for different initial solvency rates γ_0.

Finally, we consider the parameters ω and γ which specify the bonus declaration mechanism, see Section 4.2. In Figure 4, pairs of (ω, γ) are shown which lead to $\mathrm{PD}_{120} = 5\%$ for different solvency situations (left figure) and for different products (right figure). We see that higher values of the participation coefficient ω, which increases PD_k, have to be balanced with smaller values for the target reserve rate γ, which decreases PD_k. For the set of parameters considered, only very aggressive bonus declarations result in default probabilities larger than 5%. Furthermore, it is interesting that not only the solvency situation but also the product features have a significant influence on the iso-default probability curves. For participating insurance policies in a geometric Brownian motion framework, a similar investigation of the bonus declaration mechanism and additional details can be found in [16].

8 Concluding Remarks

In this paper, we proposed a discrete time model framework for the asset-liability management of life insurance products. The model incorporates fairly general product characteristics, a surrender option, a reserve-dependent bonus declaration, a dynamic capital allocation and a two-factor stochastic capital market model. The recursive formulation of the model allows for an efficient simulation. Furthermore, the model structure is modular and allows to be extended easily. In a series of examples, we showed that the model captures the most important behaviour patterns of the balance sheet development of life insurance products. In particular, we analysed the impact of mortality and surrender and the influence of the most important product, management and capital market parameters on the risk exposure of the insurance company and showed that different product features may have a significant impact on the default risk. Compared to the results presented in [16, 23], which are based on a geometric Brownian motion framework, the incorporation of a bond trading strategy and of a mean reverting process for the short rate results in significantly smaller and maybe more realistic default probabilities. Furthermore, in line with many other results presented in the literature, our results lead to the conclusion that static regulations, like the prescription of the maximum portion of stocks or the minimal participation rates for policyholders, are insufficient to control the company's default risk or to ensure an appropriate policyholder's participation. Instead, regulation as well as internal risk management guidelines should lay more emphasis on prescribing stress tests and stochastic simulations as these methods are much better suited to take into account the complex interaction of the assets and liabilities of a life insurance company.

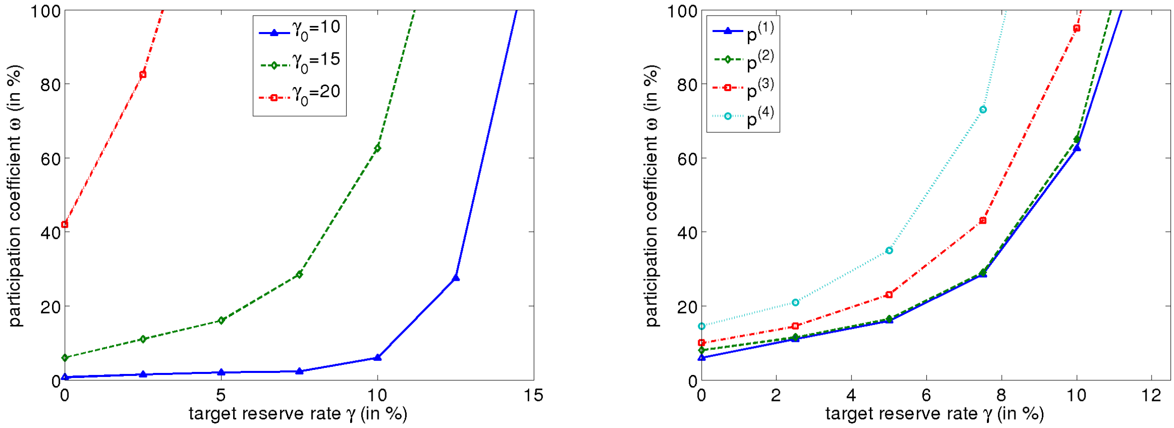

Figure 4: Parameter pairs (γ, ω) for different initial solvency rates γ_0 which lead to a default probability of 5% in case of product $p^{(1)}$ (left) and for the products $p^{(1)} - p^{(4)}$ in case of $\gamma_0 = 15\%$ (right).

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