The Classical Harnack Inequality Fails for Non-Local Operators

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Abstract

In the last years, several variants of the Harnack inequality have been studied for integro-differential operators of the form

$$\mathscr{L}u(x) = \int \big\{u(y) - u(x) - \mathbb{1}_{\{|y-x|\leq 1\}}(y-x)\nabla u(x)\big\}\nu(x,dy)\,,$$

where $\nu(x,dy)$ is a measure with singularity at $y=x$ and $\mathscr{L}$ generates a Markov jump process. It has remained open so far whether the classical Harnack inequality holds or not. In this note, we show that the classical Harnack inequality fails even in the most simple case. We construct a counter-example for $\alpha \in (0,2)$ and

$$\mathscr{L}u(x) = -(-\Delta)^{\frac{\alpha}{2}}u(x) = p.v.\int\limits_{\mathbb{R}^d} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}}\,dy\,.$$

1. *Introduction*

In 1887 Carl-Gustav Axel von Harnack published his book [**13**] on potential theory in two spatial dimensions. In Section 19 of Chapter 1, he proves the following assertion: If $u : B_R(x_0) \subset \mathbb{R}^2 \to \mathbb{R}$ is harmonic and does not change its sign, then the values of $u(x)$ for $x \in B_r(x_0)$, $r < R$, are bounded between

$$u(x_0)\frac{R-r}{R+r} \qquad \text{and} \qquad u(x_0)\frac{R+r}{R-r}\,.$$

One of its consequences, the so-called Harnack convergence result, became immediately well-known and important, partly due to the influential article [**22**]. The estimate itself had its glorious renaissance in the second half of the 20th century. It was extended to more general elliptic and parabolic operators and various underlying spaces. Given an operator $A : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$, one can phrase it abstractly as follows:

$$\begin{aligned}&\text{If } Au(x) = 0 \text{ and } u(x) \geq 0 \text{ for almost any } x \in B_R(x_0), \text{ then}\\ &\operatorname*{ess\text{-}sup}_{x\in B_{R/2}} u(x) \leq c \operatorname*{ess\text{-}inf}_{x\in B_{R/2}} u(x) \quad \text{ with } c>0 \text{ independent of } u, x_0, R\,.\end{aligned} \tag{1.1}$$

Here, in dependence of A the equation $Au(x) = 0$ needs to be interpreted in the right way. The main features of this inequality are its scale invariance and its pure locality, i.e., under assumptions on the behavior of u in $B_R(x_0)$ a property of u is established in $B_{R/2}(x_0)$. The parabolic version of (1.1) looks quite different, see [**15**].

The validity of the Harnack inequality is of great importance for several reasons. Hölder regularity *a priori* estimates follow, and together with an assumption on the underlying measure, the parabolic version of the inequality is equivalent to a L^2-Poincaré inequality. For various aspects of the inequality and (nonlinear) partial differential equations we refer to [**21**], [**18**], [**12**]. We refer to [**15**] for an introduction to the field of Harnack inequalities.

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In the last years, several variants of the Harnack inequality have been studied for integro-differential operators of the form

$$\mathscr{L}u(x) = \int_{\mathbb{R}^d} \left[u(x+h) - u(x) - \mathbb{1}_{\{|h|<1\}}\langle h, \nabla u(x)\rangle\right] \nu(x, dh)\,, \tag{1.2}$$

where

$$\operatorname*{ess-sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \min\left(|h|^2, 1\right) \nu(x, dh) < \infty\,.$$

For fixed x the measure $\nu(x,\cdot)$ is called Lévy-measure. Processes generated by $\mathscr{L}$ are called Lévy processes if $\nu(x, dh)$ does not depend on x. Basically, $x \mapsto \nu(x, dh)$ plays the same role as the coefficient functions $x \mapsto a_{ij}(x)$ for diffusion operators. In our form, the operator $\mathscr{L}$ generates quite general Markov jump processes and regularity of harmonic functions is closely linked to properties of the underlying stochastic process like the Feller property. Let us look at a simple case. For $\alpha \in (0, 2)$ set

$$\nu(x, dh) = n(x, h)\, dh = \mathscr{A}(d, -\alpha)|h|^{-d-\alpha}\, dh\,, \tag{1.3}$$

where $\mathscr{A}(d, -\alpha) = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)}$ is a normalization constant. With this choice of ν one obtains

$$\widehat{\left(\mathscr{L}u\right)}(\xi) = |\xi|^\alpha = \widehat{\left(\Delta^{\frac{\alpha}{2}}\right)}(\xi)\,.$$

The potential theory for $\mathscr{L} = \Delta^{\frac{\alpha}{2}}$ has been worked out in [**23**], [**3**], [**19**], see also [**24**]. M. Riesz obtained precise Poisson formulae, thereby proving the following generalized Harnack inequality:

THEOREM 1.1. *(generalized Harnack inequality) Assume $\alpha \in (0, 2)$ and $R > 0$. Let $\nu(x, dh)$ and $\mathscr{L}$ be defined by (1.3) and (1.2). There exists $c > 0$ such that for any $u \in L^\infty(\mathbb{R}^d) \cap C^2(B_R(0))$ with*

$$\begin{aligned}\left(\mathscr{L}u\right)(x) &= 0 \text{ for } x \in B_R(x_0)\,,\\ u(x) &\geq 0 \text{ for } x \in \mathbb{R}^d\,,\end{aligned}$$

the following estimate holds true:

$$\sup_{B_{R/2}(x_0)} u(x) \leq c \inf_{B_{R/2}(x_0)} u(x)\,.$$

The constant $c > 0$ can be computed explicitely; it is independent of u, x_0, and R.

One could also prove this result using [**20**] or [**10**] together with results from degenerate partial differential operators. The aim of this note is to show that the assumption of u being non-negative in all of $\mathbb{R}^d$ cannot be relaxed to $u \geq 0$ in $B_R(x_0)$. This question has recently attracted some attention, see the discussion below. Here is our result.

THEOREM 1.2. *Assume $\alpha \in (0, 2)$ and $R > 0$. Let $\nu(x, dh)$ and $\mathscr{L}$ be defined by (1.3) and (1.2). Then there exists a function $u \in L^\infty(\mathbb{R}^d) \cap C^2(B_R(0))$ satisfying*

$$\begin{aligned}\left(\mathscr{L}u\right)(x) &= 0 \text{ for } x \in B_R(0)\,,\\ u(x) &> 0 \text{ for } x \in B_R(0) \setminus \{0\}\,,\\ |u(x)| &\leq 1 \text{ for } x \in \mathbb{R}^d\,,\end{aligned}$$

but at the same time $u(0) = 0$. Therefore, the classical Harnack inequality as well as the local maximum principle fail for the operator $\mathscr{L}$.

Note that, for any s with $|s| < 1$, we could construct an example with $u(0) = s$.

Let us briefly discuss the significance of Theorem 1.2. The functions u considered above need to be defined on $\mathbb{R}^d$ for $\mathscr{L}u(x)$ to make sense in $B_R(x_0)$ but this is not our main point here. Theorem 1.1 is still good enough for a uniform convergence theorem and a Liouville theorem. But it is weaker than (1.1) since one requires u to be non-negative in all of $\mathbb{R}^d$. This is a strong restriction and it rules out the possibility to derive Hölder regularity estimates, at least directly. These observations are one of the driving forces behind the research of local regularity for nonlocal operators. Under the assumption

$$\nu(x, dh) = n(x, h)\, dh, \qquad \kappa_0 |h|^{-d-\alpha} \le n(x, h) \le \kappa_1 |h|^{-d-\alpha} \tag{1.4}$$

where κ_0 and κ_1 are positive constants the assertion of Theorem 1.1 is proved in [**7**]. Extensions of this are worked out in [**26**]. In [**5**] examples of measures $\nu(x, dh)$ are given where a result like Theorem 1.1 fails because the constant c would need to depend on R. The relation of parabolic versions of Theorem 1.1, heat kernel bounds and certain geometric conditions for random walks of unbounded range is currently of interest, see [**4**], [**2**]. In [**9**] the authors show for Lévy-stable measures $\nu(x, dh) = \nu(dh)$ which are scale invariant, but otherwise pretty general, that a generalized Harnack inequality is equivalent to the so-called relative Kato condition.

As mentioned above, Theorem 1.1 does not imply Hölder regularity by Moser's iteration procedure. Nevertheless such regularity estimates are established and discussed in [**17**], [**7**] [**26**],[**6**], [**25**], [**14**].

Knowing that continuity *a priori* estimates can be proved under much weaker assumptions than (1.4) and for all cases where a generalized Harnack inequality holds, it is a natural question whether the classical Harnack inequality (1.1) holds true. This question is raised by several authors, see for example [**17**]. The classical Harnack inequality would explain the above phenomenon nicely. The aim of this note is to give a negative answer to this question. More precisely, we show that, even for the most simple case $\mathscr{L} = \Delta^{\frac{\alpha}{2}}$, $\alpha \in (0, 2)$, local assumptions, i.e., assumptions on $u(x)$ for $x \in B_R(x_0)$ do not ensure finiteness of ess-sup$\{\frac{u(x)}{u(y)} : x, y \in B_{\frac{R}{2}}(0)\}$ in general.

2. *Proof of Theorem 1.2*

We prove Theorem 1.2. Although the proof is purely analytic, it has a probabilistic motivation.

Proof Theorem 1.2. We prove the result in dimension $d = 2$. As will become clear, the idea and method of proof carry over to any dimension $d \in \mathbb{N}$. The main idea is to construct a function $g : \mathbb{R}^2 \setminus B_R(0) \to \mathbb{R}$ and to define $u(y)$ for $y \in B_R(0)$ by the Poisson formula for α-harmonic functions. The function $u(y) = u(y)\mathbb{1}_{y \in B_R(0)} + g(y)\mathbb{1}_{y \notin B_R(0)}$ then becomes the desired function. Define $g : \mathbb{R}^2 \setminus B_R(0) \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} 1 & ; R \le |x| < S\,, \\ -1 & ; S \le |x| < T\,, \\ 0 & ; T \le |x|\,. \end{cases} \tag{2.1}$$

We choose $S > 0$ and $T > S$ further down, see the remarks following the proof. The Poisson formula for α-harmonic functions reads as follows [**3**, **19**, **23**]:

$$u(y) = \mathscr{C}_\alpha \left(R^2 - |y|^2\right)^{\frac{\alpha}{2}} \int\limits_{\mathbb{R}^2 \setminus B_R(0)} g(x) \left|R^2 - |x|^2\right|^{\frac{-\alpha}{2}} |x - y|^{-2}\, dx\,, \quad y \in B_R(0)\,,$$

where $\mathscr{C}_\alpha = \frac{\sin(\frac{\pi\alpha}{2})}{\pi^2}$.

The probabilistic motivation behind this proof becomes clear if one notes that $u(y) = \mathbb{E}^y\Big(g(X_{\tau_{B_r(0)}})\Big)$ where X_t is the rotational symmetric α-stable process and $\tau_{B_r(0)}$ the first time when X_t exits $B_r(0)$, $r < R$. Let us compute $u(y)$ for $y \in B_R(0)$ and g as above.

$$\begin{aligned} u(y) &= \mathscr{C}_\alpha \big(R^2 - |y|^2\big)^{\frac{\alpha}{2}} \int\limits_{B_S(0)\setminus B_R(0)} \big|R^2 - |x|^2\big|^{\frac{-\alpha}{2}} |x-y|^{-2}\, dx \\ &\quad - \mathscr{C}_\alpha \big(R^2 - |y|^2\big)^{\frac{\alpha}{2}} \int\limits_{B_T(0)\setminus B_S(0)} \big|R^2 - |x|^2\big|^{\frac{-\alpha}{2}} |x-y|^{-2}\, dx \\ &= I_{R,S}(y) - I_{S,T}(y)\,. \end{aligned}$$

The tricks used in the following computations seem to be well-known among specialists in potential theory. Since they are far from being trivial, we provide all details. Using polar coordinates we compute

$$\begin{aligned} A(y) &= \int\limits_{B_T(0)\setminus B_S(0)} \big|R^2 - |x|^2\big|^{\frac{-\alpha}{2}} |x-y|^{-2}\, dx \\ &= \int\limits_S^T \frac{r}{\big|R^2 - r^2\big|^{\frac{\alpha}{2}}} \Big(\int\limits_0^{2\pi} \frac{d\theta}{r^2 + |y|^2 - 2r|y|\big(\frac{y_1}{|y|}\cos(\theta) + \frac{y_2}{|y|}\sin(\theta)\big)} \Big)\, dr \end{aligned}$$

Assume now $y \neq 0$. Choose $\gamma \in [0, 2\pi)$ such that $\frac{y_1}{|y|} = \cos(\gamma)$ and $\frac{y_2}{|y|} = \sin(\gamma)$. Note that $\cos(\gamma)\cos(\theta) + \sin(\gamma)\sin(\theta) = \cos(\gamma - \theta)$. Set $\phi = \gamma - \theta$. We obtain

$$A(y) = \int\limits_S^T \frac{r}{\big|R^2 - r^2\big|^{\frac{\alpha}{2}}} \Big(\int\limits_{\gamma - 2\pi}^{\gamma} \frac{d\phi}{r^2 + |y|^2 - 2r|y|\cos(\phi)} \Big)\, dr$$

Setting $r = t|y|$ we obtain $t > 1$ and

$$A(y) = \int\limits_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t}{\big(t^2|y|^2 - R^2\big)^{\frac{\alpha}{2}}} \Big(\int\limits_0^{2\pi} \frac{d\phi}{t^2 + 1 - 2t\cos(\phi)} \Big)\, dt \tag{2.2}$$

The interior integral $B(t) = \int\limits_0^{2\pi} \frac{d\phi}{t^2+1-2t\cos(\phi)}$ can be computed explicitely as follows. Set $\delta = \frac{2t}{1+t^2} < 1$. Then

$$B(t) = \frac{1}{t^2+1} \int\limits_0^{2\pi} \frac{d\phi}{1 - \delta\cos(\phi)} = \frac{1}{t^2+1} \frac{2\pi}{\sqrt{1-\delta^2}} = \frac{2\pi}{t^2 - 1}\,,$$

where we used

$$\int \frac{d\phi}{1 - \delta\cos(\phi)} = \begin{cases} \frac{2}{\sqrt{1-\delta^2}} \arctan\Big(\sqrt{\frac{1+\delta}{1-\delta}}\tan(\frac{\phi}{2})\Big) + \frac{2\pi}{\sqrt{1-\delta^2}} \Big[\frac{x+\pi}{2\pi}\Big] \\ \qquad \text{for } x \notin \{\pi + 2k\pi\}, k \in \mathbb{Z}\,, \\ \frac{\pi}{\sqrt{1-\delta^2}}(2k+1) \\ \qquad \text{for } x = \pi + 2k\pi, k \in \mathbb{Z}\,. \end{cases}$$

which follows after the substitution $z = \tan(\frac{\phi}{2})$. Setting $\frac{R}{|y|} = \rho > 1$ we obtain from (2.2)

$$A(y) = \frac{2\pi}{|y|^\alpha} \int\limits_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t}{\big(t^2-\rho^2\big)^{\frac{\alpha}{2}}(t^2-1)}\, dt\,,$$

$$I_{S,T}(y) = 2\mathscr{C}_\alpha \pi\big(\rho^2-1\big)^{\frac{\alpha}{2}} \int\limits_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t}{\big(t^2-\rho^2\big)^{\frac{\alpha}{2}}(t^2-1)}\, dt\,.$$

Setting $t^2-\rho^2 = \kappa$ and then $\frac{\kappa}{\rho^2-1} = \tau$ we obtain

$$I_{S,T}(y) = \mathscr{C}_\alpha \pi\big(\rho^2-1\big)^{\frac{\alpha}{2}} \int\limits_{\frac{S^2-R^2}{|y|^2}}^{\frac{T^2-R^2}{|y|^2}} \frac{d\kappa}{\kappa^{\frac{\alpha}{2}}(\kappa+\rho^2-1)} = \mathscr{C}_\alpha \pi \int\limits_{\frac{S^2-R^2}{R^2-|y|^2}}^{\frac{T^2-R^2}{R^2-|y|^2}} \frac{d\tau}{\tau^{\frac{\alpha}{2}}(\tau+1)}\,,$$

which implies $I_{R,\infty}(y) = \mathscr{C}_\alpha \pi \mathscr{B}(1-\frac{\alpha}{2},\frac{\alpha}{2}) = \mathscr{C}_\alpha \frac{\pi^2}{\sin(\frac{\pi\alpha}{2})} = 1$ for all $y \in B_R(0), y \neq 0$. The case $y = 0$ is similar:

$$\begin{aligned} I_{S,T}(0) &= \mathscr{C}_\alpha 2\pi R^\alpha \int\limits_S^T \frac{dr}{r(r^2-R^2)^{\frac{\alpha}{2}}} = \mathscr{C}_\alpha 2\pi \int\limits_S^T \frac{dr}{r(\frac{r^2}{R^2}-1)^{\frac{\alpha}{2}}} \\ &= \mathscr{C}_\alpha 2\pi \int\limits_{\frac{S}{R}}^{\frac{T}{R}} \frac{dv}{v(v^2-1)^{\frac{\alpha}{2}}} = \mathscr{C}_\alpha \pi \int\limits_{(\frac{S}{R})^2-1}^{(\frac{T}{R})^2-1} \frac{dw}{(w+1)w^{\frac{\alpha}{2}}} \end{aligned}$$

Altogether we obtain

$$u(y) = \pi\mathscr{C}_\alpha \int\limits_0^{\frac{S^2-R^2}{R^2-|y|^2}} \frac{d\tau}{\tau^{\frac{\alpha}{2}}(\tau+1)} - \int\limits_{\frac{S^2-R^2}{R^2-|y|^2}}^{\frac{T^2-R^2}{R^2-|y|^2}} \frac{d\tau}{\tau^{\frac{\alpha}{2}}(\tau+1)} \tag{2.3}$$

Define $J(a,b) = \int\limits_a^b \frac{d\tau}{(\tau+1)\tau^{\frac{\alpha}{2}}}$, $E = (\frac{S}{R})^2-1$, $F = (\frac{T}{R})^2-1$ and $q = \frac{R^2}{R^2-|y|^2}$. Then

$$u(y) = \pi\mathscr{C}_\alpha \begin{cases} J(0,qE) - J(qE,qF) & ; y \neq 0\,, \\ J(0,E) - J(E,F) & ; y = 0\,. \end{cases}$$

Now the task is to find $E^* > 0$, $F^* > E^*$ such that

$$J(0,qE^*) - J(qE^*,qF^*) \begin{cases} = 0 & , q = 1\,, \\ > 0 & , q > 1\,. \end{cases} \tag{2.4}$$

First, we study the function $q \mapsto J(0,qE^*) - J(qE^*,qF^*)$, $q \geq 1$. Note

$$\frac{d}{dq}\big[J(0,qE) - J(qE,qF)\big] = \frac{2E}{(qE+1)(qE)^{\frac{\alpha}{2}}} - \frac{F}{(qF+1)(qF)^{\frac{\alpha}{2}}}\,,$$

and

$$\frac{2E}{(qE+1)(qE)^{\frac{\alpha}{2}}} - \frac{F}{(qF+1)(qF)^{\frac{\alpha}{2}}} \geq 0 \quad \Leftrightarrow \quad \frac{F}{(qF+1)(qF)^{\alpha/2}} \leq \frac{2E}{(qE+1)(qE)^{\alpha/2}}$$
$$\Leftrightarrow \quad F^{\frac{2-\alpha}{2}}(qE+1) \leq 2E^{\frac{2-\alpha}{2}}(qF+1) \quad \Leftrightarrow \quad F^{\frac{2-\alpha}{2}} - 2E^{\frac{2-\alpha}{2}} \leq q\big(2E^{\frac{2-\alpha}{2}}F - EF^{\frac{2-\alpha}{2}}\big)$$
$$\Leftrightarrow \quad E^{\frac{\alpha-2}{2}} - 2F^{\frac{\alpha-2}{2}} \leq q\big(2F^{\alpha/2} - E^{\alpha/2}\big)\,.$$

In particular we obtain that the function $q \mapsto J(0,qE) - J(qE,qF)$, $q \geq 1$, is non-decreasing for $F \geq E$ if

$$\begin{aligned} & E^{\frac{\alpha-2}{2}} - 2F^{\frac{\alpha-2}{2}} \leq \big(2F^{\alpha/2} - E^{\alpha/2}\big) \\ \Leftrightarrow \quad & E^{\alpha/2}(1+E^{-1}) \leq 2F^{\alpha/2}(1+F^{-1})\,. \end{aligned} \tag{2.5}$$

Next, let us prove the following two observations:

$$\alpha \leq 1: \quad J(1,\infty) \geq J(0,1) \quad \Leftrightarrow J(0,\infty) \geq 2J(0,1)\,. \tag{2.6}$$
$$\alpha > 1: \quad J(1,\infty) < J(0,1) \quad \Leftrightarrow J(0,\infty) < 2J(0,1)\,. \tag{2.7}$$

Inequality (2.6) follows from

$$J(1,\infty) = \int_1^\infty \frac{d\tau}{\tau^{\alpha/2}(1+\tau)} \geq \int_1^\infty \frac{d\tau}{\sqrt{\tau}(1+\tau)} \stackrel{\theta=1/\tau}{=} \int_0^1 \frac{d\theta}{\sqrt{\theta}(1+\theta)} \geq \int_0^1 \frac{d\theta}{\theta^{\alpha/2}(1+\theta)} \geq J(0,1)\,.$$

The proof of inequality (2.7) is analogous. Now we are in the position to prove (2.4). First, consider $\alpha > 1$. Choose $F > 1$. Next, choose $E < 1$ such that $J(0,F) = 2J(0,E)$ or equivalently $J(E,F) = J(0,E)$. Such choice is possible because of the intermediate value theorem, $J(0,0) = 0$ and (2.7). Choose $F^* > 1$ large enough such that (2.5) holds which is possible because the right-hand side tends to infinity. Note that (2.5) still holds if E is replaced by any $E' \in [E,1]$ because the left-hand side is non-increasing for $E \in (0,1]$. Now, choose $E^* \in [E,1]$ such that $J(0,F^*) = 2J(0,E^*)$ or equivalently $J(E^*,F^*) = J(0,E^*)$. Again, such choice is possible because of (2.7). The tuple (E^*,F^*) now has the desired properties. The case $\alpha \leq 1$ is much simpler. Note that (2.5) holds true if $E = 1$ and $F > 1$. Now, choose $E^* = 1$ and F^* so large such that (2.6) holds. Then we are done.

Setting $S = R\sqrt{E^*+1}$ and $T = R\sqrt{F^*+1}$, we have constructed g defined in (2.1). Finally, set $u(x) = u(x)\mathbb{1}_{x\in B_R(0)} + g(x)\mathbb{1}_{x\notin B_R(0)}$. Then $\mathscr{L}u(x) = 0$ for all $x \in B_R(x_0)$ following from standard results, see Theorem 3.9 of [**1**] or [**19**]. The function $u : \mathbb{R}^d \to \mathbb{R}$ is bounded by the values of g which follows from the maximum principle. The proof of Theorem 1.2 is complete. □

Let us make some remarks.

(1) In the proof of Theorem 1.2, the closer α is to 2, the closer one needs to choose S to R. Already for $\alpha = 1.5$ and $R = 1$ one would need to choose S so close to R such that the difference would hardly been visible in a figure. This reflects the fact that smaller jumps of the corresponding stochastic process outweigh large jumps for values of α close to 2.
(2) One could choose a function $g : \mathbb{R}^2 \setminus B_R(0) \to \mathbb{R}$ which is continuous. This would lead to a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$. The function u shows a sharp increase for $|y| \nearrow R$ and does not seem to be regular at the boundary of $B_R(0)$.
(3) The counter-example can be constructed without using very far jumps of the underlying process but just jumps across the boundary $\partial\Omega$. Hence the classical Harnack inequality presumably also fails for truncated kernels resp. truncated stable processes.

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