An Irregular Complex Valued Solution to a Scalar Linear Parabolic Equation

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An Irregular Complex Valued Solution to a Scalar Linear Parabolic Equation

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Abstract: We show that Nash’s celebrated regularity result [Nas58] does not hold for partial differential operators with complex valued coefficients. In detail, we present a linear scalar parabolic equation on $[0, T] \times \mathbb{R}^n$, $n \geq 3$, with complex valued measurable coefficients whose solution $\in L^\infty(L^2) \cap L^2(H^1)$ is discontinuous or may be unbounded.

Keywords: parabolic equations, counterexample, Hölder continuity

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1 Introduction

A famous result of John Nash [Nas58] states that a weak solution
\[ u \in L^\infty(0, T; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega)) , \]
\( \Omega = \) open subset of \( \mathbb{R}^n \), to a scalar parabolic equation
\[ u_t - \sum_{j,k=1}^{n} D_j (a_{jk}(t, x) D_j u) = f , \] (1)
is Hölder continuous in \( \Omega \) provided that the coefficients \( a_{jk} \) are only bounded and measurable and satisfy the condition of uniform ellipticity and, say, \( f \in L^r(L^r) \) with \( r > 1 + n/2 \), cf. [LSU67] for further results. In the present paper we present an example of a scalar parabolic equation of type (1) with \( f = 0 \) and complex valued coefficients
\[ a_{jk} : (0, T) \times \mathbb{R}^n \rightarrow \mathbb{C} \]
having an irregular complex valued solution
\[ u \in L^\infty(0, T; L^2_{loc}(\mathbb{R}^n; \mathbb{C})) \cap L^2(0, T; H^1_{loc}(\mathbb{R}^n; \mathbb{C})) , \]
provided \( n > 2 \). Here the \( a_{jk} \) satisfy a condition of uniform ellipticity adapted to the complex case
\[ \text{Re} \sum_{j,k=1}^{n} a_{jk}(t, x) \zeta_j \bar{\zeta}_k \geq c_0 |\zeta|^2 , \quad a_{ik} \in L^\infty , \] (2)
with some positive constant \( c_0 \in \mathbb{R} \). For the definition of the usual spaces \( L^p(0, T; V) \) cf. [LSU67], [Lio69]. So, the complex valued analog to the Nash-Theorem does not hold in more than two space dimensions. Note that the scalar problem with complex valued coefficients can also be considered as a real valued parabolic system in two unknown functions.

Other examples, together with further references, can be found in John and Stará [JoS95]. They present a system with coefficients depending analytically on the unknown function \( u \), which can be interpreted as a linear system with measurable coefficients.
This paper is motivated also by the corresponding elliptic case: In their book [MNP91] Mazja, Plamenevsky, and Nasarov present a general approach of how to construct uniformly elliptic equations with complex valued measurable coefficients having an irregular complex valued $H^1$-solution. This approach has so far been successfully applied for dimensions $n \geq 5$. Furthermore, in a recent paper [Fre07], the first author presents an example of a linear uniformly elliptic equation with an irregular $H^1$-solution for dimension $n \geq 3$, in the case with complex valued coefficients.

The parabolic counterexample presented here is significantly different from the analogous elliptic counterexample. Due to the additional dependence on the variable $t$ we have to introduce certain so called decontamination terms, see (6) and (7), which would vanish in the purely elliptic case.

## 2 Construction of the Example

By elementary calculations which may be found in the appendix we obtain for

$$u = x_1 \exp \left( \lambda \ln(|x|^2 + T - t) \right)$$

with $\lambda \in \mathbb{C}$:

$$u_t = -\lambda \frac{u}{|x|^2 + T - t},$$

$$\Delta u = \left[ (4\lambda + 2n\lambda) + (4\lambda^2 - 4\lambda) \frac{|x|^2}{|x|^2 + T - t} \right] \frac{u}{|x|^2 + T - t},$$

$$L_1 u := \sum_{j=1}^n D_j \left( \frac{|x|^2}{|x|^2 + T - t} D_j u \right) - \sum_{j,k=1}^n D_j \left( \frac{x_j x_k}{|x|^2 + T - t} D_k u \right) = (1 - n) \frac{u}{|x|^2 + T - t},$$

$$= (1 - n) \frac{u}{|x|^2 + T - t},$$
\begin{equation}
L_2 u := \sum_{j=1}^{n} D_j \left( \frac{|x|^4}{(|x|^2 + T - t)^2} D_j u \right) - \sum_{j,k=1}^{n} D_j \left( \frac{x_j x_k |x|^2}{(|x|^2 + T - t)^2} D_k u \right) = \end{equation}

\begin{equation}
= (1 - n) \left( \frac{|x|^2}{(|x|^2 + T - t)^2} \right) u.
\end{equation}

The terms $L_1 u$, $L_2 u$ are used as decontamination terms.

From (3) up to (6) we obtain

\begin{equation}
u_t + \alpha \Delta u + \beta L_1 u + \gamma L_2 u = \end{equation}

\begin{equation}
= \left\{ \left[ - \lambda + \alpha(4\lambda + 2n\lambda) + \beta(1 - n) \right] + \left[ \alpha(4\lambda^2 - 4\lambda) + \gamma(1 - n) \right] \frac{|x|^2}{|x|^2 + T - t} \right\} \times
\end{equation}

\begin{equation}
\times \frac{u}{|x|^2 + T - t}.
\end{equation}

By an appropriate choice of $\beta$ and $\gamma$ the brackets $[\ldots]$ in (8) vanish; we have to choose

\begin{equation}
\beta = -\frac{\lambda}{n - 1} + \alpha \frac{4\lambda + 2n\lambda}{n - 1}, \quad \gamma = \alpha \frac{4\lambda^2 - 4\lambda}{n - 1},
\end{equation}

and the right hand side of (8) vanishes.

We define

\begin{equation}
-a_{jk}(t, x) = \alpha \delta_{jk} + \beta \frac{|x|^2}{|x|^2 + T - t} \delta_{jk} - \beta \frac{x_j x_k}{|x|^2 + T - t} + \gamma \frac{|x|^4}{(|x|^2 + T - t)^2} \delta_{jk} - \gamma \frac{|x|^2 x_j x_k}{(|x|^2 + T - t)^2}
\end{equation}

and obtain

**Theorem 2.1** The function

\begin{equation}
u = x_1 \exp \left( \lambda \ln(|x|^2 + T - t) \right)
\end{equation}

solves the equation

\begin{equation}
u_t - \sum_{j,k=1}^{n} D_j (a_{jk} D_k u) = 0 \quad \text{for } 0 \leq t \leq T, x \neq 0,
\end{equation}

where $a_{jk}$ is defined in (10) and $\alpha, \beta, \gamma, \lambda$ satisfy the equations in (9).
3 Discussion of the Example

We want an irregular solution \( u \); so we require either

\[
\text{Re} \lambda = -\frac{1}{2},
\]

which leads to a bounded discontinuous solution or

\[
\text{Re} \lambda < -\frac{1}{2},
\]

which gives an unbounded solution. We further want

\[
u \in L^\infty(L^2_{\text{loc}}(\mathbb{R}^n)).
\]

This is the case if

\[x_1 |x|^{2 \text{Re} \lambda} \in L^2_{\text{loc}}(\mathbb{R}^n),\]

which follows from

\[
\text{Re} \lambda > -\frac{1}{2} - \frac{n}{4}.
\]

For \( n = 2 \) this implies that the solution \( u \) above is Hölder continuous and (13) cannot be satisfied.

In order to have

\[
u \in L^2(H^{1,2}_{\text{loc}}(\mathbb{R}^n)),
\]

we need

\[(|x|^2 + T - t)^{2 \text{Re} \lambda} \in L^1(L^1_{\text{loc}}(\mathbb{R}^n))\]

which is satisfied, if (14) holds.

Now we want that our equation is parabolic, i.e. the coefficients \( a_{jk} \) defined in (10) satisfy the ellipticity condition (2). Since the operators \( \Delta, L_1, L_2 \) are symmetric and have real valued coefficients, the imaginary parts of \( \alpha, \beta, \gamma \) do not influence the ellipticity condition (c.f. the paper [Fre07] concerning
the counter example in the elliptic case), so we have only to prove

\[ E_0 := (\text{Re} \alpha)|\xi|^2 + (\text{Re} \beta) \left( \frac{|x|^2}{|x|^2 + T - t} \right)|\xi|^2 - \\
- (\text{Re} \beta) \sum_{j,k=1}^{n} \frac{x_j x_k}{|x|^2 + T - t} \xi_j \xi_k + (\text{Re} \gamma) \left( \frac{|x|^4}{(|x|^2 + T - t)^2} \right)|\xi|^2 - \\
- (\text{Re} \gamma) \sum_{j,k=1}^{n} \frac{|x|^2 x_j x_k}{(|x|^2 + T - t)^2} \xi_j \xi_k \leq -c_0|\xi|^2 \]  \hspace{2cm} (15)

for all \( \xi \in \mathbb{R}^n \), with some \( c_0 > 0 \). Since \( x \) may vanish, we certainly need

\[ \text{Re} \alpha < 0. \]  \hspace{2cm} (16)

We rewrite (15) in the form

\[ E_0|\xi|^{-2} = \text{Re} \alpha + \frac{|x|^2}{|x|^2 + T - t} \left( 1 - \frac{(\xi \cdot x)^2}{|\xi|^2} \right) \left[ \text{Re} \beta + (\text{Re} \gamma) \frac{|x|^2}{|x|^2 + T - t} \right]. \]

So, uniform ellipticity holds if (16) and

\[ \text{Re} \beta + (\text{Re} \gamma) \frac{|x|^2}{|x|^2 + T - t} \leq 0. \]  \hspace{2cm} (17)

We set \( \alpha = \alpha_0 + i\alpha_1, \lambda = \lambda_0 + i\lambda_1, \alpha_0, \alpha_1, \lambda_0, \lambda_1 \in \mathbb{R} \) and use (9) to reformulate (17). This leads to the condition

\[ C_0 := \frac{1}{n-1} \left[ -\lambda_0 + \alpha_0(4\lambda_0 + 2n\lambda_0) + \alpha_1(-2n - 4)\lambda_1 + \\
+ \frac{|x|^2}{|x|^2 + T - t} \alpha_0(-4\lambda_0 + 4\lambda_0^2 + 4\lambda_1^2) + \frac{|x|^2}{|x|^2 + T - t} \alpha_1(4\lambda_1 - 8\lambda_1 \lambda_0) \right] \leq 0. \]

If \( C_0 \leq 0 \) and (16), then ellipticity holds.

Let us analyze the term \( T_0 \) with coefficient \( \alpha_1 \):

\[ T_0 = \alpha_1 \lambda_1 \left[ -2n - 4 + (4 - 8\lambda_0) \frac{|x|^2}{|x|^2 + T - t} \right]. \]

We want that the term in the brackets is strictly less then 0. Since \( \lambda_0 \) is negative (in order to have a singularity), we may estimate

\[ T_0(\alpha_1 \lambda_1)^{-1} = -2n - 4 + (4 - 8\lambda_0) \frac{|x|^2}{|x|^2 + T - t} \leq -2n - 4 + 4 - 8\lambda_0 = \\
= -2n - 8\lambda_0 < k_0 < 0 \]
if $\Re \lambda > -n/4$. Thus we arrive at
\[ C_0 \leq K(\alpha_0, \lambda_0, \lambda_1, n) + \alpha_1 \lambda_1 k_0 \]
and it follows that for given $\alpha_0, \lambda_0, \lambda_1, n$, with $\lambda_0 > -n/4$, $\lambda_1 \neq 0$, one may choose $\alpha_1 \lambda_1 > 0$ and $|\alpha_1|$ so large such that (17) holds and we have obtained

**Theorem 3.1** The equation (11) from Theorem 2.1 is parabolic in $[0, T] \times \mathbb{R}^n$ in the sense that the elliptic condition (2) holds provided that $\Re \alpha < 0$, $\Im (\lambda) \neq 0$, $\Re \lambda > -\frac{n}{4}$ and $\Im (\lambda) \geq K(\Re (\alpha), \lambda)$ with some constant $K(\Re (\alpha), \lambda)$ being large enough.

In order to have the correct function space for $u$, we have to satisfy condition (14), i.e.
\[ \Re \lambda > -\frac{1}{2} - \frac{n}{4}. \]
That is why the condition
\[ -\frac{1}{2} \geq \Re \lambda > -\frac{n}{4} \]
allows the solution $u$ to be contained in a better function space than $L^\infty(L^2_{loc} \cap L^2(H^1_{loc}))$ and still to be irregular.

For $n = 2$ the condition $\Re \lambda > -\frac{n}{4}$ implies that the solution $u$ of Theorem 2.1 is Hölder continuous, so there is no example of an irregular solution for $n = 2$. For $n \geq 3$, both bounded and unbounded irregular solutions can be constructed by our approach.

**Remark:** By the usual theory of parabolic equations the solution $u$ can be extended as a weak solution to any larger interval $[0, T_1]$, $T_1 > T$, if the coefficients are extended preserving ellipticity say $a_{jk} = \delta_{jk}$ (Kronecker Symbol) on $[T, T_1] \times \mathbb{R}^n$.

**References**

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4 Appendix

We present some elementary calculations. With \( u = x_1 \exp(\lambda \ln(|x|^2 + T - t)) \) we get

\[
 u_t = \frac{-\lambda x_1}{|x|^2 + T - t} \exp(\lambda \ln(|x|^2 + T - t))
\]

and

\[
 \Delta u = 2D_1 \exp(\lambda \ln(|x|^2 + T - t)) + x_1 \Delta \exp(\lambda \ln(|x|^2 + T - t))
\]

\[
 = \frac{4\lambda x_1}{|x|^2 + T - t} \exp(\lambda \ln(|x|^2 + T - t))
\]

\[
 + x_1 \left[ \frac{4\lambda^2 |x|^2}{(|x|^2 + T - t)^2} + \frac{2n\lambda}{|x|^2 + T - t} - \frac{4\lambda |x|^2}{(|x|^2 + T - t)^2} \right] \exp(\lambda \ln(|x|^2 + T - t))
\]

\[
 = (4\lambda + 2n\lambda) \frac{x_1 |x|^2}{|x|^2 + T - t} \exp(\lambda \ln(|x|^2 + T - t))
\]

\[
 + (4\lambda^2 - 4\lambda) \frac{x_1 |x|^2}{(|x|^2 + T - t)^2} \exp(\lambda \ln(|x|^2 + T - t))
\]

Now we introduce two semi-elliptic operators \( L_1, L_2 \) in order to have a sufficiently large degree of freedom.

\[
 L_1 u := \sum_{j=1}^{n} D_j \left( \frac{|x|^2}{|x|^2 + T - t} D_j u \right) - \sum_{j,k=1}^{n} D_j \left( \frac{x_j x_k}{|x|^2 + T - t} D_k u \right)
\]

\[
 = \left[ D_1 \left( \frac{|x|^2}{|x|^2 + T - t} \exp(\lambda \ln(|x|^2 + T - t)) \right) + \sum_{j=1}^{n} D_j \left( \frac{2\lambda x_1 x_j |x|^2}{(|x|^2 + T - t)^2} \exp(\lambda \ln(|x|^2 + T - t)) \right) \right]
\]

\[
 - \left[ \sum_{j=1}^{n} D_j \left( \frac{x_1 x_j}{|x|^2 + T - t} \exp(\lambda \ln(|x|^2 + T - t)) \right) \right]
\]

\[
 + \sum_{j=1}^{n} D_j \left( \frac{2\lambda x_1 x_j |x|^2}{(|x|^2 + T - t)^2} \exp(\lambda \ln(|x|^2 + T - t)) \right)
\]
Hence,

\[ L_1 u = \left[ \frac{2x_1}{|x|^2 + T - t} - \frac{2x_1|x|^2}{(|x|^2 + T - t)^2} + \frac{2\lambda x_1|x|^2}{(|x|^2 + T - t)^2} \right] \exp \left( \lambda \ln(|x|^2 + T - t) \right) \]

\[ - \left[ \frac{(n + 1)x_1}{|x|^2 + T - t} - \frac{2x_1|x|^2}{(|x|^2 + T - t)^2} + \frac{2\lambda x_1|x|^2}{(|x|^2 + T - t)^2} \right] \exp \left( \lambda \ln(|x|^2 + T - t) \right) \]

\[ = \frac{(1 - n)x_1}{|x|^2 + T - t} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \]

We remark that two ugly terms of the form \( \pm \sum_{j=1}^{n} D_j[2\lambda \ldots] \) cancel.

\[ L_2 u := \sum_{j=1}^{n} D_j \left( \frac{|x|^4}{(|x|^2 + T - t)^2} D_j u \right) - \sum_{j,k=1}^{n} D_j \left( \frac{x_j x_k|x|^2}{(|x|^2 + T - t)^2} D_k u \right) \]

\[ = \left[ D_1 \left( \frac{|x|^4}{(|x|^2 + T - t)^2} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right) \right. \]

\[ + \sum_{j=1}^{n} D_j \left( \frac{2\lambda x_1 x_j|x|^4}{(|x|^2 + T - t)^3} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right) \]

\[ - \left[ \sum_{j=1}^{n} D_j \left( \frac{x_1 x_j|x|^2}{|x|^2 + T - t} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right) \right) \]

\[ + \sum_{j=1}^{n} D_j \left( \frac{2\lambda x_1 x_j|x|^4}{(|x|^2 + T - t)^3} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right) \]

We see again that the two terms \( \pm \sum_{j=1}^{n} D_j[2\lambda \ldots] \) cancel:

\[ L_2 u = \left[ \frac{4x_1|x|^2}{(|x|^2 + T - t)^2} - \frac{4x_1|x|^4}{(|x|^2 + T - t)^3} + \frac{2\lambda x_1|x|^4}{(|x|^2 + T - t)^3} \right] \exp \left( \lambda \ln(|x|^2 + T - t) \right) \]

\[ - \left[ \frac{x_1|x|^2}{(|x|^2 + T - t)^2} - \frac{2x_1|x|^2}{(|x|^2 + T - t)^2} + \frac{nx_1|x|^2}{(|x|^2 + T - t)^2} - \frac{4x_1|x|^4}{(|x|^2 + T - t)^3} \right] \]
\[ + \frac{2\lambda x_1 |x|^4}{(|x|^2 + T - t)^2} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \]
\[ = \frac{(1 - n) x_1 |x|^2}{(|x|^2 + T - t)^2} \exp \left( \lambda \ln(|x|^2 + T - t) \right). \]

Considering now
\[ u_t + \alpha \Delta u + \beta L_1 u + \gamma L_2 u \]
\[ = - \frac{\lambda x_1}{|x|^2 + T - t} \exp \left( \lambda \ln(|x|^2 + T - t) \right) + \alpha \left[ \frac{(4\lambda + 2n\lambda)x_1}{|x|^2 + T - t} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right. \]
\[ + \left. \frac{(4\lambda^2 - 4\lambda)x_1 |x|^2}{(|x|^2 + T - t)^2} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right] \]
\[ + \beta \left[ \frac{(1 - n)x_1}{|x|^2 + T - t} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right] \]
\[ + \gamma \left[ \frac{(1 - n)x_1 |x|^2}{(|x|^2 + T - t)^2} \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right] \]
\[ = \frac{x_1}{|x|^2 + T - t} \left[ - \lambda + \alpha (4\lambda + 2n\lambda + \beta (1 - n)) \exp \left( \lambda \ln(|x|^2 + T - t) \right) \right] \]
\[ + \frac{x_1 |x|^2}{(|x|^2 + T - t)^2} \left[ \alpha (4\lambda^2 - 4\lambda) + \gamma (1 - n) \right] \exp \left( \lambda \ln(|x|^2 + T - t) \right) = 0, \]

we obtain
\[ \beta = \frac{-\lambda}{n - 1} + \alpha \frac{4\lambda + 2n\lambda}{n - 1} \]
\[ \gamma = \alpha \frac{4\lambda^2 - 4\lambda}{n - 1} \]
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