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Abstract

We consider in this article a class of systems of second order partial differential equations with non-linearity in the first order derivative and zero order term which can be super-quadratic. These problems are motivated by differential geometry and stochastic differential games. Up to now, in the case of systems, only quadratic growth had been considered.

Key Words. Differential Geometry, Stochastic Differential Games, Hamiltonians.

*This article is dedicated to Professor Philippe G. Ciarlet for his 70th anniversary

1 Introduction

In differential geometry and in the theory of stochastic differential games one finds many examples of systems of the type

$$(u_t^\nu) - \sum_{i,k=1}^{n} D_i\big(a_{ik}(x) D_k u^\nu\big) + \lambda_\nu(x,u,\nabla u) = H^\nu(x,u,\nabla u), \nu = 1,\dots,N \tag{1}$$

or equations which can be transformed to this type of system. The right hand side H^ν frequently is called Hamiltonian due to the application in stochastic control theory. The coefficients $a_{ik} \in L^\infty(\Omega)$ are assumed to satisfy a condition of uniform ellipticity. For applications to differential geometry see Hildebrandts survey [Hil82], books on geometric analysis, say [Jos02], for applications to stochastic differential games see Bensoussan-Frehse [BF02a] and [BF84]. In differential geometry, due to scaling invariance, the functions $H^\nu(x,u,\nabla u)$ are usually *quadratic* in ∇u. This is not the case in the theory of stochastic differential games where sub-quadratic, quadratic and even super-quadratic growth of $H_\nu(\nu,u,\nabla u)$ with respect to ∇u may occur in natural settings.
From the point of view of regularity theory the case of sub-quadratic growth is simple. The quadratic growth case is already difficult and creates a lot of interesting problems ([Fre01],[BF02b], [BF84], [BF95], [BF02a]).
Up to now, there is no result at all for the case of *systems* where the Hamiltonian grows super-quadratically in ∇u. In the *scalar* case ($N = 1$), L^∞-bounds for ∇u can be archived via barrier methods, thus cases of super quadratic Hamiltonians can be treated. See e.g. [Lio82].

In this note we present a special system of type (1) where the Hamiltonian $H^\nu = H^\nu(\nabla u)$ may have *any* polynomial growth and, nevertheless, there exist regular solutions. We have to confine to a periodic setting. We consider the method of proof as very simple and believe that there should be a lot of possibilities to generalize our conditions.

2 Formulation of the Theorem

Let $Q = [0,L]^n \subset \mathbb{R}^n$ be a cube and $H^{k,q}_\sharp = H^{k,q}_\sharp(Q;\mathbb{R}^N)$ be the usual Sobolev space of L-periodic functions $u = (u^1,\dots,u^N) : [0,L]^n \to \mathbb{R}^N$ with generalized derivatives up to order $k \in \mathbb{N}$ in the Lebesgue space L^q, $q \in [1,\infty]$. We look for solutions

$u \in H^{2,q}_\sharp$ of the system

$$-\Delta u^\nu + \lambda_\nu u^\nu = G^\nu(|\nabla u|^2) + G^o(|\nabla u|^2) \cdot \nabla u^\nu + f_\nu(x)\,, \qquad \nu = 1, \dots, N \tag{2}$$

where

$$\lambda_\nu = \text{const} > 0\,, \tag{3}$$
$$f_\nu \in H^{1,\infty}_\sharp(Q)\,, \tag{4}$$
$$G^\nu \in C^1(\mathbb{R})\,, \nu = 1, \dots, N\,, \tag{5}$$
$$G^o \in C^1(\mathbb{R}; \mathbb{R}^n)\,, \tag{6}$$

and we assume the growth conditions

$$\Big|\frac{\partial}{\partial\eta} G^\nu(\eta)\Big| + \Big|\frac{\partial}{\partial\eta} G^o(\eta)\Big| \le K|\eta|^{q-1} + K\,, \qquad \eta \in \mathbb{R}\,, \tag{7}$$

with some exponent $q \ge 1$ and a constant K. Note that $q >> 1$ is admitted. From (7) obviously we have

$$|G^\nu(\eta)| + |G^o(\eta)| \le K|\eta|^q + K \tag{8}$$

with some constants K.

Theorem 2.1 *Under the above regularity and growth assumption (3)–(8) for the data there exists a periodic solution $u : Q \to \mathbb{R}^N$ of the system (2) which is contained in $H^{2,2}_\sharp \cap C^{2+\alpha}$ for $\alpha \in [0, 1)$.*

There is also a parabolic analogue of Theorem 2.1. We treat initial value problems

$$u^\nu_t - \Delta u^\nu = G^\nu(|\nabla u|^2) + G^0(|\nabla u|^2) \cdot \nabla u^\nu + f_\nu(t, x)\,, \qquad \nu = 1, \dots, N \tag{9}$$

with G^ν, G^o as before and

$$f_\nu \in L^\infty\big(0, T; H^{1,\infty}_\sharp\big) \tag{10}$$

in the space-time cylinder $[0, T] \times Q$ and look for smooth solutions $u \in C(0, T; L^2_\sharp) \cap L^2(0, T, H^1_\sharp)$ which are periodic in the space variables and satisfy the initial condition

$$u|_{t=0} = u_0 \in H^{2,\infty}_\sharp(Q), . \tag{11}$$

Theorem 2.2 *Let G^ν; G^o, f_ν, u_0 satisfy the regularity (5), (6), (10), (11) and the growth condition (7), (8). Then there exists a solution*

$$u \in L^r\big(0, T, H^{2,r}_\sharp(Q)\big)$$

of (9), (11) with

$$u_t \in L^r\big(0, T; L^r_\sharp(Q)\big)$$

for all $r < \infty$ and $T > 0$.

Remark: Even in the case where $G^\nu(|\nabla u|^2)$ and $G^o(|\nabla u|^2)\nabla u$ grow only quadratically in $|\nabla u|$ Theorem 2.1 and Theorem 2.2 seem to be "new" and are not contained in [BF02b] and [Fre01]

Let us sketch how to get estimates in the case of Neumann boundary conditions. We pose additional assumptions

$$G^1(\eta) \geq c_0|\eta|^{q'} - K,\, G^2(\eta) \geq c_0|\eta|^{q'} - K.$$

For simplicity, $q' > n/2$.

$$G^1(\eta) \geq (1-\delta_0)|G^0(\eta)||\eta|^{1/2} - K\,, \qquad \delta_0 > 0$$

$$G^2(\eta) \geq (1-\delta_0)|G^0(\eta)||\eta|^{1/2} - K\,, \qquad \delta_0 > 0$$

By maximum principle arguments one can achieve bounds from below

$$u^1 \geq -K,\, u^2 \geq -K.$$

Then one can use the function $\frac{1}{u^\nu + K + 1}$ as a test function, and if the G^ν have a stronger growth and coerciveness behaviour than G^0 we obtain an estimate

$$\int\limits_\Omega \frac{|\nabla u^\nu|^2}{(u^\nu + K + 1)^2} dx + \int\limits_\Omega \frac{|\nabla u^\nu|^{2q'}}{u^\nu + K + 1} dx \leq K. \tag{12}$$

Unfortunately, no bound for u itself is available. (We may treat a variational inequality with convex set $-c \leq u_\nu \leq c$; in this case the main estimates of our proof would work.) Approximating the system such that structure is maintained is also a delicate task. One can approximate the problem adding ϵ times the $\hat{p}$-Laplacian of u, $\hat{p} > 2p$. Then one has an approximate solution. Unfortunately, we were not able to localize the estimates in the proof of theorem 2.1 and 2.2, yet.

3 Proof of the Theorems

We approximate (8) by replacing the functions $G^\nu(\eta)$ and $G^o(\eta)$ with

$$G^\nu_\delta(\eta) := (1+\delta\eta)^{-q} G^\nu(\eta)$$
$$G^o_\delta(\eta) := (1+\delta\eta)^{-q-1} G^o(\eta)\,,$$

where $\eta \geq 0$, $\delta > 0$, $\delta \to 0$.
Since G^ν_δ and $G^o_\delta(|\nabla u|^2)\nabla u^\nu$ are bounded for any, say, H^1-function u, if δ is fixed, there is a solution $u_\delta \in H^{2,p}_\sharp$ of the system

$$-\Delta u^\nu_\delta + \lambda_\nu u^\nu_\delta = G^\nu_\delta(|\nabla u_\delta|^2) + G^o_\delta(|\nabla u_\delta|^2)\cdot \nabla u_\delta + f_\nu(x)\,, \tag{13}$$

where $p \in [1,\infty)$.
We will establish a uniform bound for $|\nabla u_\delta|^2 \exp(|\nabla u_\delta|^{2r})$ in L^1, r large enough, as $\delta \to 0$, and we will estimate related quantities.

A simple estimate shows, that

$$\Big|\frac{\partial}{\partial\eta} G^\nu_\delta\Big| \leq K_\delta\,, \qquad \Big|\frac{\partial}{\partial\eta} G^o_\delta\Big| \leq K_\delta$$

and, uniformly as $\delta \to 0$,

$$\Big|\frac{\partial}{\partial\eta} G^\nu_\delta(\eta)\Big| + \Big|\frac{\partial}{\partial\eta} G^o_\delta(\eta)\Big| \leq K|\eta|^{q-1} + K\,.$$

We now differentiate equation (13), i. e. apply D, and use the function

$$Du^\nu_\delta \exp(|\nabla u_\delta|^{2r})$$

as a test function where $r \geq 1$ will be chosen later. Here D stands for the first partial derivatives D_i, $i = 1, \ldots, n$. Note that we have sufficient regularity for justifying these operations since $\nabla u_\delta \in L^\infty \cap H^{1,p}$ due to regularity and imbedding theorems.

We obtain, dropping the index δ of the function during the estimates

$$\begin{aligned}
&\int\limits_Q |\nabla Du^\nu|^2 \exp(|\nabla u|^{2r})\, dx + \frac{1}{2}\int\limits_Q \nabla|Du^\nu|^2 \nabla \exp(|\nabla u|^{2r})\, dx + \\
&\quad + \lambda_\nu \int\limits_Q |Du^\nu|^2 \exp(|\nabla u|^{2r})\, dx = \\
&= \int\limits_Q D\big(|\nabla u|^2\big)\big\{G_\delta^{\nu\prime}(|\nabla u|^2) + G_\delta^{o\prime}(|\nabla u|^2)\cdot \nabla u^\nu\big\} Du^\nu \exp(|\nabla u|^2)\, dx + \\
&\quad + \frac{1}{2}\int\limits_Q G_\delta^o(|\nabla u|^2)\nabla(Du^\nu)^2 \exp(|\nabla u|^2)\, dx + \int\limits_Q Df_\nu Du^\nu \exp(|\nabla u|^{2r})\, dx\,.
\end{aligned} \tag{14}$$

We sum with respect to $\nu = 1, \ldots, M$ and $i = 1, \ldots, n$, $D = D_i$, and use vector notation and perform simple estimates. This yields, with $\lambda_0 = \min \lambda_\nu$,

$$\begin{aligned}
&\int\limits_Q |\nabla^2 u|^2 \exp(|\nabla u|^{2r})\, dx + r\int\limits_Q \big|\nabla\big(|\nabla u|^2\big)\big|^2 |\nabla u|^{2r-2} \exp(|\nabla u|^{2r})\, dx + \\
&\quad + \lambda_0 \int\limits_Q |\nabla u|^2 \exp(|\nabla u|^{2r})\, dx \le \\
&\le \int\limits_Q \big|\nabla\big(|\nabla u|^2\big)\big| \big\{K(|\nabla u|^{2q-2}) + K\big\}(|\nabla u| + |\nabla u|^2) \exp(|\nabla u|^{2r})\, dx + \\
&\quad + \int\limits_Q \big\{K|\nabla u|^{2q} + K\big\}\big|\nabla\big(|\nabla u|^2\big)\big| \exp(|\nabla u|^{2r})\, dx + K\int\limits_Q |\nabla u| \exp(|\nabla u|^{2r})\, dx \le \\
&\le A + B + C\,,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
A &= K\int\limits_Q \big|\nabla\big(|\nabla u|^2\big)\big|^2 |\nabla u|^{2q} \exp(|\nabla u|^{2r})\, dx\,, \\
B &= K\int\limits_Q \big|\nabla\big(|\nabla u|^2\big)\big| \exp(|\nabla u|^{2r})\, dx\,, \\
C &= K\int\limits_Q |\nabla u| \exp(|\nabla u|^{2r})\, dx,
\end{aligned}$$

uniformly as $\delta \to 0$, and K does not depend on r.

Note that we used

$$\Big|\sum_{i=1}^{n}\sum_{\nu=1}^{N} G_\delta^o \nabla (D_i u^\nu)^2 \exp(|\nabla u|^2)\Big| = \big|G_\delta^o \nabla\big(|\nabla u|^2\big)\big| \exp(|\nabla u|^2) \le$$
$$\le (K|\nabla u|^{2q} + K)\big|\nabla\big(|\nabla u|^2\big)\big| \exp(|\nabla u|^2)\,.$$

We estimate $\big(\exp = \exp(|\nabla u|^{2r})\big)$:

$$\begin{aligned} A \le{}& \frac{r}{2} \int\limits_{(|\nabla u| \ge 1)} \big|\nabla\big(|\nabla u|^2\big)\big|^2 |\nabla u|^{2r-2} \exp\, dx + \\ &+ \frac{K}{2r} \int\limits_{(|\nabla u| \ge 1)} |\nabla u|^{4q+4-2r+2} \exp\, dx + \\ &+ \varepsilon_0 \int\limits_{|\nabla u| \le 1} |\nabla^2 u|^2 \exp\, dx + K_{\varepsilon_0} \int\limits_{|\nabla u| \le 1} \exp\, dx = \\ ={}& A_1 + A_2 + A_3 + A_4\,. \end{aligned}$$

If we choose $r \ge 2q+3$ we have that

$$A_2 \le \frac{K}{2r} \int\limits_{|\nabla u| \ge 1} \exp(|\nabla u|^{2r})\, dx$$

The term A_1 is absorbed by a corresponding one on the left hand side of (15). The term A_3 (the one with ε_0) is absorbed by the left hand side, too. The term A_4 is obviously bounded. Thus we arrive at

$$\begin{aligned} (1-\varepsilon_0) \int\limits_Q |\nabla^2 u|^2 \exp\, dx + \frac{r}{2} \int\limits_Q \big|\nabla\big(|\nabla u|^2\big)\big|^2 |\nabla u|^{2r-2} \exp\, dx + \\ + \lambda_0 \int\limits_Q |\nabla u|^2 \exp\, dx \le \frac{K}{2r} \int\limits_{|\nabla u| \ge 1} \exp\, dx + C + K\,. \end{aligned} \tag{16}$$

The first summand on the left hand side of (16) can be absorbed by the term with factor λ_0 if r is chosen larger than $2\lambda_0/K$ since K does not depend on r here. The term C is estimated by

$$C = K\Big(\int\limits_{|\nabla u| \le \Lambda} + \int\limits_{|\nabla u| \ge \Lambda} \Big) |\nabla u| \exp\, dx \le$$

$$\le K_\Lambda + K \int\limits_{|\nabla u| \ge \Lambda} |\nabla u| \exp\, dx \le K_\Lambda + \frac{1}{2}\lambda_0 \int\limits_Q |\nabla u|^2 \exp\, dx$$

if we choose $\Lambda = 2K/\lambda_0$.

Again, the term $\frac{1}{2}\lambda_0 \int |\nabla u|^2 \exp\, dx$ is absorbed by a corresponding term at the left hand side and we arrive at the inequality

$$\frac{r}{2} \int\limits_Q \big|\nabla\big(|\nabla u|^2\big)\big|^2 |\nabla u|^{2r-2} \exp(|\nabla u|^{2r})\, dx +$$

$$+ \frac{1}{2} \int\limits_Q |\nabla^2 u|^2 \exp(|\nabla u|^{2r})\, dx + \frac{1}{4}\lambda_0 \int |\nabla u|^2 \exp(|\nabla u|^{2r})\, dx \le$$

$$\le K\,.$$

This holds uniformly for $u = u_\delta$, $\delta \to 0$. We need only the summand with factor λ_0 and fix

Proposition 3.1 *The approximate solutions*

$$u_\delta \in H^{2,p}_\sharp(Q, \mathbb{R}^N)$$

of equation (13) obey the uniform bound

$$\int\limits_Q |\nabla u_\delta|^2 \exp(|\nabla u_\delta|^{2r})\, dx \le K$$

uniformly for $\delta \to 0$.

Firstly, integrating the equation over the periodicity cube, one sees that proposition 3.1 yields that the mean values of the u^ν are bounded. Then one obtains bounds for $\|u_\delta\|_p$.

From linear elliptic regularity theory we then obtain

$$\|u_\delta\|_p + \|\nabla u_\delta\|_p + \|\nabla^2 u_\delta\|_p \le K_p$$

uniformly for $\delta \to 0$, for any fixed $p < \infty$. This allows us to subtract a subsequence $(u_\delta)_{\delta\in\Gamma}$ where Γ is a sequence of positive numbers tending to zero such that

$$\nabla^2 u_\delta \to \nabla^2 u\,, \qquad \text{weakly}$$
$$u_\delta \to u\,, \nabla u_\delta \to \nabla u \qquad \text{strongly}$$

in all L^p, $p < \infty$.
Thus we may pass to the limit $\delta \to 0$ in (13) and obtain that the weak limit

$$u \in H^{2,p}_\sharp(Q; \mathbb{R}^N)$$

satisfies the primal equation (2). Further regularity $-u \in C^{2+\alpha}$ follows from (4)–(6) and elliptic regularity theory. This proves Theorem 2.1. ✠

Let us indicate the proof of Theorem 2.2 concerning the parabolic system.. With the new unknown variable

$$v = e^{-t} u$$

the function v satisfies the system

$$v^\nu_t - \Delta v^\nu + v^\nu = e^{-t} G^\nu\big(e^{2t}|\nabla v|^2\big) + G^o\big(e^{2t}|\nabla v|^2\big) \cdot \nabla v^\nu + e^{-t} f_\nu(t,x)\,.$$

We replace G^ν, G^o by G^ν_δ, G^o_δ as in the proof of Theorem 2.1 and obtain a regular solution $v = v_\delta$. Then one applies the operation $D = D_j$, $j = 1, \ldots, n$, i. e. the derivatives in space direction, and uses the test function

$$Dv^\nu \exp\big(|\nabla v|^p\big)\,.$$

Then the calculations run as in the proof of Theorem 2.1; we only have to handle with the factors e^{-t}, e^{2t}, which is simple since only space derivatives are involved in the calculation.
This finally gives an estimate

$$\int_0^T \int_Q |\nabla v|^2 \exp\big(|\nabla v|^{2p}\big)\, dx\, dt \le K_T$$

uniformly as $\delta \to 0$ and the L^r-estimate for ∇u follows. Integrating the equation with respect to x, we derive a differential equation for the mean value of u and hence a bound. Thereafter, from the theory of linear parabolic equations we obtain the

Theorem. ✠

4 Further Generalization

It is a challenging task to generalize Theorem 2.1 and 2.2 for a richer class of Hamiltonians.
In this section we present one of the lot of possibilities to do so. The technique of proof reminds somewhat to our papers [BF02a], [BF84], [Fre01], using iterated exponential functions of u as test functions, although the situation here is completely different.
For describing the new situation let

$$F(\eta) = \int_1^\eta \exp(s^p)\, ds\,, \quad p \geq 1\,, \quad \eta \geq 0\,,$$

and we assume that the unknown functions consist of two groups of variables

$$u = (u^1, u^2, \dots, u^N)\,, \quad v = (v^1, v^2, \dots, v^M)$$

and we consider the system

$$\begin{aligned}
&-\Delta u^\nu + \lambda_\nu u^\nu = G^{u\nu}\big(|\nabla u|^2\big) + G^{uo}\big(|\nabla u|^2\big)\cdot \nabla u^\nu + L\Big(F(|\nabla u|^2) + F(|\nabla v|^2)\Big)\cdot \nabla u^\nu + f^u_\nu \\
&\nu = 1, \dots, N \\
&-\Delta v^\nu + \lambda^v_\nu v^\nu = G^{v\nu}\big(|\nabla v|^2\big) + G^{vo}\big(|\nabla v|^2\big)\cdot \nabla v^\nu + L\Big(F(|\nabla u|^2) + F(|\nabla v|^2)\Big)\cdot \nabla v^\nu + f^v_\nu \\
&\nu = 1 \dots, M\,,
\end{aligned} \tag{17}$$

and we pose the following conditions on G^u, G^v, G^{ov}, L

$$
\begin{aligned}
& G^u, G^v : \mathbb{R} \to \mathbb{R} \text{ are locally Lipschitz} && (18)\\
& G^{ou}, G^{ov}, L : \mathbb{R} \to \mathbb{R}^n \text{ are locally Lipschitz} && (19)\\
& |L(\eta)| \le K\eta^{1/2} + K\,, \qquad \eta \ge 0 && (20)\\
& |L'(\eta)| \le K\eta^{-1/2}\big(\log(2+|\eta|)\big)^{-r_0} && (21)\\
& \text{with some } r_0 > \frac{1}{2p} + \frac{1}{2}\,, && \\
& \big|G^u(\eta)\big| + \big|G^v(\eta)\big| \le K|\eta|^q + K\,, && \\
& \big|G^{u\prime}(\eta)\big| + \big|G^{v\prime}(\eta)\big| \le K|\eta|^{q-1} + K\,, && (22)\\
& \big|G^{ou}(\eta)\big| + \big|G^{ov}(\eta)\big| \le K|\eta|^{q-1/2} + K && (23)\\
& \big|G^{ou\prime}(\eta)\big| + \big|G^{ov\prime}(\eta)\big| \le K|\eta|^{q-3/2} + K\,, && (24)\\
& \text{where } q < \frac{1+p}{2}\,. &&
\end{aligned}
$$

The sign $'$ on $G^u, G^{ov}, ..., L$ denotes the derivative.
Since $q >> 2$ is admissible we see that a strong super-quadratic growth of the Hamiltonian is possible.
Furthermore, we see that the term $L\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)$ has even exponential growth in $|\nabla u|$ and $|\nabla v|$. This follows from the following simple calculation:

$$F(\eta) = \int_1^\eta \frac{1}{p}\big(\exp(s^p)\big)' s^{1-p}\, ds = \frac{1}{p}\eta^{1-p}\exp(\eta^p) - \frac{e}{p} + \frac{p-1}{p}\int_1^\eta s^{-p}\exp(s^p)\, ds\,.$$

Since $0 \le \int_1^\eta s^{-p}\exp(s^p)\, ds \le \int_1^\eta \exp(s^p)\, dx$ for $\eta \ge 1$, we conclude

$$F(\eta) \le \eta^{1-p}\exp(\eta^p) - e\,, \qquad \eta \ge 1 \tag{25}$$

and

$$F(\eta) \ge \frac{1}{p}\eta^{1-p}\exp(\eta^p) - \frac{e}{p}\,, \qquad \eta \ge 1\,. \tag{26}$$

Clearly, one has to admit that the term L does not look very natural, however one has a lot of possibilities modifying the test functions to obtain other structure conditions.

Theorem 4.1 *Assume the growth and Lipschitz conditions (4), and (19) up to (24) for the functions occurring at the right hand side of equation (17) and let* λ_ν^u, $\nu = 1, \dots, N$, λ_ν^v, $\nu = 1, \dots, M$ *be non-negative. Then there is a solution* $(u,v) \in H_\sharp^{2,2}(Q) \cap C^{2+\alpha}$, $\alpha \in [0,1)$.

Proof of Theorem 4.1: We approximate the problem as in the proof of Theorem 2.1 as far as G^{uv}, $G^{u\nu}$, G^{uo}, G^{vo} concerns. The factor $L\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)$ is approximated by

$$L_\delta = \big(1 + \delta L\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)\big)^{-2} L\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)\,.$$

Again, it is clear that the approximate problem where G^u, G^v, G^{uo}, G^{vo}, L are approximated by corresponding terms with index δ, has a solution $(u,v) = (u_\delta, v_\delta)$

$$u_\delta \in H_\sharp^{2,\infty}(Q;\mathbb{R}^N)\,,\ v_\delta \in H_\sharp^{2,\infty}(Q,\mathbb{R}^n)\,.$$

We apply to the operation $D = D_j$ and use the function

$$Du^\nu \exp\big(|\nabla u|^{2p}\big) \exp\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)$$

for the system in u and

$$Dv^\nu \exp\big(|\nabla v|^{2p}\big) \exp\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)$$

for the system in v as test functions.
We then sum with respect to $\nu = 1, \dots, N$, for the components of u, $\nu = 1, \dots, M$ for the components of v, and with respect to j (from $D = D_j$), then we obtain an equation like (14), but the integrals have the additional factor $\exp\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)$ and on the left hand side there occurs an additional summand

$$\frac{1}{2}\int\limits_Q \nabla\big(|\nabla u|\big)^2 e^{|\nabla u|^{2p}} \nabla \exp\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big) + \textit{a corresponding term with } v\,.$$

This leads to, an additional term

$$Z_0 := \int\limits_Q |\nabla\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)^2 \exp\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)\,dx$$

arises. Again, on the left hand side of the analogue of equation (14),there occurs the term

$$2p\int\limits_Q \big|\nabla|\nabla u|\big|^2|\nabla u|^{2p}\exp\left(|\nabla u|^{2p}\right)\exp\Big(F(|\nabla u|^2)+F(|\nabla u|^2)\Big)\,.$$

On the right hand side of the analogue of equation (14) there arise the terms

$$\begin{aligned}
T_1^u &= \int\limits_Q \sum_{\nu=1}^N \nabla\left(|\nabla u|^2\right)G_\delta^{u\nu\prime}\left(|\nabla u|^2\right)\nabla u^\nu \exp\left(|\nabla u|^{2p}\right)\exp\left(F(|\nabla u|^2)+F(|\nabla v|^2)\right)dx\\
T_0^u &= \int\limits_Q \sum_{\nu=1}^N \nabla\left(|\nabla u|^2\right)\left(G_\delta^{o\nu\prime}\left(|\nabla u|^2\right)\cdot\nabla u^\nu\right)\nabla u^\nu \exp(.)\exp\left(F(.)+F(.)\right)dx+\\
&\quad + G_\delta^{o\nu}\left(|\nabla u|^2\right)\cdot\nabla\left(|\nabla u|^2\right)\exp\left(|\nabla u|^{2p}\right)\exp\left(F(|\nabla u|^2)+F(|\nabla v|^2)\right)dx
\end{aligned} \tag{27}$$

and corresponding terms $T_1^{v\nu}$, T_0^v, and further

$$\begin{aligned}
T_L^u &= \int\limits_Q \sum_{v=1}^N \nabla\Big(F(|\nabla u|^2)+F(|\nabla v|^2)\Big)\Big(L'(F(|\nabla u|^2)+F(|\nabla v|^2))\cdot\nabla u^\nu\Big)\cdot\nabla u^\nu\times\\
&\quad\times\exp\left(|\nabla u|^2\right)\exp\left(F(|\nabla u|^2)+F(|\nabla v|^2)\right)dx+\\
&\quad+\frac{1}{2}\int\limits_Q L\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\cdot\nabla F\big(|\nabla u|^2\big)\exp\left(F(|\nabla u|^2)+F(|\nabla v|^2)\right)dx\,.
\end{aligned}$$

The terms T_1^u, T_1^v are treated as in the proof of Theorem 2.1. The difference is that p is not a free parameter anymore. We estimate

$$\begin{aligned}
&\left|\nabla(|\nabla u|^2)G_\delta^{u\nu\prime}(|\nabla u|^2)\nabla u^\nu\right|\le\\
&\varepsilon_0\big|\nabla|\nabla u|\big|^2|\nabla u|^{2p}+K_{\varepsilon_0}|\nabla u|^{4+4q-4-2p}\,,\qquad \big(|\nabla u|\ge 1\big)\,.
\end{aligned}$$

Thus we may dominate both terms by the right hand side if $4q-2p<2$ which is true due to the hypothesis (one uses also the terms with factor λ_0).

For estimating T_0^u we observe that the term

$$\sum_{\nu=1}^N \nabla\left(|\nabla u|^2\right)\left(G^{o\nu\prime}\left(|\nabla u|^2\right)\nabla u^\nu\right)\nabla u^\nu$$

which occurs in the integrand of the first summand in 27 is estimated for $|\nabla u| \geq 1$ by

$$\begin{aligned} &K\big|\nabla|\nabla u|\big||\nabla u|^3\big|G^{o\nu\prime}\big(|\nabla u|^2\big) \leq K\big|\nabla|\nabla u|\big||\nabla u|^3|\nabla u|^{2q-3} = \\ &= K\big|\nabla|\nabla u|\big||\nabla u|^{2q} \leq \varepsilon\big|\nabla|\nabla u|\big|^2|\nabla u|^{2p} + K|\nabla u|^{4q-2p}\,, \end{aligned}$$

so, this part of T_0^u (and T_0^v) is estimated by the right hand side if ε is small and $4q - 2p < 2$. (Again, for dominating the second summand of the last inequality for u, the term with λ_0 is used.)

The second summand in the expression defining T_0^u (and T_0^v) is estimated analogously. Thus the terms T_1^u, T_0^u and T_1^v, T_0^v can be dominated by the left hand side. This is also true for the terms containing f^u, f^v (see Section 3.1).

There remains the term T_L^u which is estimated in the following way: The factor

$$\begin{aligned} \mathrm{Fakt}_L^u := \sum_{\nu=1}^{N} \nabla\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)\big(L'(F(|\nabla u|^2) + F(|\nabla v|^2))\big) \cdot (\nabla u^\nu)\times \\ \times \nabla u^\nu \exp(|\nabla u|^{2p}))\big) \end{aligned}$$

is estimated by

$$\begin{aligned} |\mathrm{Fakt}_L^u| \leq \varepsilon\big|(F(|\nabla u|^2) + F(|\nabla v|^2)\big|^2 + \\ + K_\varepsilon\big|L'\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big|^2\big)|\nabla u|^4 \exp\big(2|\nabla u|^{2p}\big) = \\ = F_{21}^u + F_{22}^u\,. \end{aligned}$$

The term with F_{21}^u then is absorbed by the left hand side. For $|\nabla u| \geq K_0$ the term with F_{22}^u is estimated by

$$\epsilon|\nabla u|^2 \exp\big(|\nabla u|^{2p}\big)$$

since, by assumption on L' and inequality (25), (26)

$$\begin{aligned} &\big|L'\big(F(|\nabla u|^2) + F(|\nabla v|^2)\big)\big|^2|\nabla u|^4 \leq K\big(F(|\nabla u|^2) + 1\big)^{-1}|\nabla u|^4\big(\log(1 + F((|\nabla u|^2)))\big)^{-2r_0} \leq \\ &\leq K\exp\big(-|\nabla u|^{2p}\big)|\nabla u|^{2p+2}|\nabla u|^{-4pr_0} \leq \\ &\leq K\exp\big(-|\nabla u|^{2p}\big)|\nabla u|^{2-\delta_0} \end{aligned}$$

since $r_0 > \frac{1}{2p} + \frac{1}{2}$. Thus, the term is absorbed by the term with factor λ_0. The terms F_{21}^v, F_{22}^v are estimated analogously.
The second term in $T_L^u + T_L^v$ is estimated by

$$\Big|\frac{1}{2}\int\limits_Q L\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\nabla\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\exp\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\,dx\Big| \leq$$

$$\leq \frac{1}{2}\int\limits_Q \big|\nabla\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\big|^2 \exp\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\,dx +$$

$$+ \frac{1}{8}\int\limits_Q \big|L\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\big|^2 \exp\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\,dx\,.$$

Due to the hypothesis on L and inequality (25), we may estimate

$$\big|L\big(F(|\nabla u|^2)+F(|\nabla v|^2)\big)\big|^2 \leq K|\nabla u|^{1-p}\exp\big(|\nabla u|^{2p}\big)+$$
$$+ K|\nabla v|^{1-p}\exp\big(|\nabla v|^{2p}\big)+K\,,\qquad |\nabla u|\geq 1\,,\ |\nabla v|\geq 1\,,$$

and also this term can be absorbed. Thus, finally, we habe established an L^p-bound for $\nabla u, \nabla v$. We then procees in a similar way as we did in the proof of 2.1 and after proposition 3.1
The proof of Theorem 4.1 is now complete. ✠

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