Poisson Cluster Measures: Quasi-Invariance, Integration by Parts and Equilibrium Stochastic Dynamics

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Abstract

The distribution $\mu_{cl}$ of a Poisson cluster process in $X = \mathbb{R}^d$ (with i.i.d. clusters) is studied via an auxiliary Poisson measure on the space of configurations in $\mathcal{X} = \sqcup_n X^n$, with intensity defined as a convolution of the background intensity of cluster centres and the probability distribution of a generic cluster. We show that the measure $\mu_{cl}$ is quasi-invariant with respect to the group of compactly supported diffeomorphisms of $X$ and prove an integration-by-parts formula for $\mu_{cl}$. The corresponding equilibrium stochastic dynamics is then constructed using the method of Dirichlet forms.

Key words: cluster point process; Poisson measure; configuration space; quasi-invariance; integration by parts; Dirichlet form; stochastic dynamics

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1 Introduction

In the mathematical modelling of multi-component stochastic systems, it is conventional to describe their behaviour in terms of random configurations of “particles” whose spatio-temporal dynamics is driven by interaction of particles with each other and the environment. Examples are ubiquitous and include various models in statistical mechanics, quantum physics, astrophysics, chemical physics, biology, computer science, economics, finance, etc. (see [16] and the extensive bibliography therein).
Initiated in statistical physics and theory of point processes, the development of a general mathematical framework for suitable classes of configurations was over decades a recurrent research theme fostered by widespread applications. More recently, there has been a boost of more specific interest in the analysis and geometry of configuration spaces. In the seminal papers [5,6], an approach was proposed to configuration spaces as infinite-dimensional manifolds. This is far from straightforward, since configuration spaces are not vector spaces and do not possess any natural structure of Hilbert or Banach manifolds. However, many “manifold-like” structures can be introduced, which appear to be nontrivial even in the Euclidean case. We refer the reader to papers [2,6,7,24,28] and references therein for further discussion of various aspects of analysis on configuration spaces and applications.

Historically, the approach in [5,6] was motivated by the theory of representations of diffeomorphism groups (see [18,20,30]). To introduce some notation, let $\Gamma_X$ denote the space of all countable locally finite subsets (configurations) in a topological space $X$ (e.g., a Euclidean space $\mathbb{R}^d$). Any probability measure $\mu$ on $\Gamma_X$, quasi-invariant with respect to the action of the group $\text{Diff}_0(X)$ of compactly supported diffeomorphisms of $X$ (lifted pointwise to transformations of $\Gamma_X$), generates a canonical unitary representation of $\text{Diff}_0(X)$ in $L^2(\Gamma_X, \mu)$. It has been proved in [30] that this representation is irreducible if and only if $\mu$ is $\text{Diff}_0(X)$-ergodic. Representations of such type are instrumental in the general theory of representations of diffeomorphism groups [30] and in quantum field theory [18,19].

According to a general paradigm described in [5,6], configuration space analysis is determined by the choice of a suitable probability measure $\mu$ on $\Gamma_X$ (quasi-invariant with respect to $\text{Diff}_0(X)$). It can be shown that such a measure $\mu$ satisfies a certain integration-by-parts formula, which enables one to construct, via the theory of Dirichlet forms, the associated equilibrium dynamics (stochastic process) on $\Gamma_X$ such that $\mu$ is its invariant measure [5,6,26]. In turn, the equilibrium process plays an important role in the asymptotic analysis of statistical-mechanical systems whose spatial distribution is controlled by the measure $\mu$; for instance, this process is a natural candidate for being an asymptotic “attractor” for motions started from a perturbed (non-equilibrium) configuration.

This programme has been successfully implemented in [5] for the Poisson measure, which is the simplest and most well-studied example of a $\text{Diff}_0(X)$-quasi-invariant measure on $\Gamma_X$, and in [6] for a wider class of Gibbs measures, which appear in statistical mechanics of classical continuous gases. In particular, it has been shown that in the Poisson case, the equilibrium dynamics amounts to the well-known independent particle process, that is, an infinite family of independent (distorted) Brownian motions started at the points of a random Poisson configuration. In the Gibbsian case, the dynamics is much more complex owing to interaction between the particles.

The Gibbsian class (containing the Poisson measure as a simple “interaction-free” case) is essentially the sole example so far that has been fully amenable to such analysis. In the present paper, our aim is to develop a similar framework for
a different class of random spatial structures, namely the well-known *cluster point processes* (see, e.g., [14,16,27]). Cluster process is a simple model to describe effects of grouping (“clustering”) in a sample configuration. The intuitive idea is to assume that the random configuration has a hierarchical structure, whereby independent clusters of points are distributed around a certain (random) configuration of invisible “centres”. The simplest model of such a kind is the Poisson cluster process, obtained by choosing a Poisson point process as the background configuration of the cluster centres.

Cluster models have been very popular in numerous practical applications ranging from neurophysiology (nerve impulses) and ecology (spatial distribution of offspring around the parents) to seismology (statistics of earthquakes) and cosmology (formation of constellations and galaxies). More recent examples include applications to trapping models of diffusion-limited reactions in chemical kinetics [1,9,12], where clusterization may arise due to binding of traps to a substrate (e.g., a polymer chain) or trap generation (e.g., by radiation damage). An exciting range of new applications in physics and biology is related to the dynamics of clusters consisting of a few to hundreds of atoms or molecules. Investigation of such “mesoscopic” structures, intermediate between bulk matter and individual atoms or molecules, is of paramount importance in the modern nanoscience and nanotechnology (for an authoritative account of the state of the art in this area, see a recent review [15] and further references therein).

In the present work, we consider Poisson cluster processes in \( X = \mathbb{R}^d \). We prove the Diff\(_0(X)\)-quasi-invariance of the Poisson cluster measure \( \mu_{cl} \), and establish the integration-by-parts formula. We then construct an associated Dirichlet form, which implies in a standard way the existence of equilibrium stochastic dynamics on the configuration space \( \Gamma_X \). Our technique is based on the representation of \( \mu_{cl} \) as a natural “projection” image of a certain Poisson measure on an auxiliary configuration space \( \Gamma_X \) over a disjoint union \( X = \sqcup_n X^n \), comprising configurations of “droplets” representing individual clusters of variable (finite) size. A suitable intensity measure in \( X \) is obtained as a convolution of the background intensity \( \sigma(dx) \) (of cluster centres) with the probability distribution \( \eta(dy) \) of a generic cluster. This approach enables one to apply the well-developed apparatus of Poisson measures to the study of the Poisson cluster measure \( \mu_{cl} \).

Let us point out that the “projection” construction of the Poisson cluster measure is very general, and in particular it works even in the case when “generalized” configurations (with possible accumulation or multiple points) are allowed. However, to be able to construct a well-defined differentiability structure on cluster configurations, we need to restrict ourselves to the space \( \Gamma_X \) of “proper” (i.e., locally finite and simple) configurations. Using the technique of Laplace transforms, we obtain necessary and sufficient conditions of almost sure (a.s.) properness for Poisson cluster configurations, set out in terms of the background intensity \( \sigma(dx) \) of cluster centres and the in-cluster distribution \( \eta(dy) \). To the best of our knowledge, these conditions appear to be new (cf., e.g., [16, Section 6.3]) and may be of interest for the general theory of cluster point processes.
Some of the results of this paper have been sketched in [11] (in the case of clusters of fixed size). We anticipate that the projection approach developed in the present paper can be applied to the study of more general cluster measures on configurations spaces, especially Gibbs cluster measure (see [10] for the case of fixed-size clusters). Such models, and related functional-analytic issues, will be addressed in our future work.

The paper is organized as follows. In Sections 2.1 and 2.2, we recall the definition and some basic properties of the Poisson point process and Poisson cluster point process, respectively, as measures on the space of generalized configurations $\Gamma^X_N$. In Section 2.3, we discuss criteria for Poisson cluster configurations to be a.s. locally finite and simple (Theorem 2.4, the proof of which is deferred to the Appendix). An auxiliary intensity measure $\sigma^*$ on the space $X = \sqcup_n X^n$ is introduced in Section 3.1, followed by Theorem 3.5 of Section 3.2 showing that the Poisson cluster measure $\mu_{cl}$ can be obtained as a push-forward of the Poisson measure $\pi_{\sigma^*}$ on $\Gamma^X_N$ under the “unpacking” map $X^n \ni \bar{x} \mapsto p(\bar{x}) = \{x_1, \ldots, x_n\}$ ($n \in \mathbb{N}$). In Section 3.3, we describe a more general construction of $\mu_{cl}$ using another Poisson measure defined in the space $\Gamma^X_N \times X$ of configurations of pairs $(x, \bar{y})$ ($x = \text{cluster centre}$, $\bar{y} = \text{in-cluster configuration}$), with the product intensity measure $\sigma(dx) \otimes \eta(dy)$. Further on, Section 4.1 deals with the property of quasi-invariance of the measure $\mu_{cl}$ with respect to the diffeomorphism group $\text{Diff}_0(X)$ (Theorem 4.3), and an integration-by-parts formula for $\mu_{cl}$ is established in Section 4.2 (Theorem 4.5). The Dirichlet form $E_{\mu_{cl}}$ associated with $\mu_{cl}$ is defined in Section 5.1, which enables us to construct in Section 5.2 the canonical equilibrium dynamics (i.e., diffusion on the space $\Gamma_X$ with invariant measure $\mu_{cl}$). In addition, we show that the form $E_{\mu_{cl}}$ is irreducible (Theorem 5.4, Section 5.3). Finally, the Appendix includes the proof of Theorem 2.4 (Section 6.1), a brief compendium on differentiable functions in configuration spaces (Section 6.2), and a proof of a general result on quasi-invariance of Poisson measures, adapted to our purposes (Section 6.3).

2 Point processes and measures in configuration space

2.1 Poisson measure

Let us recall some basic facts about Poisson measures in configuration spaces. As compared to a standard exposition (see, e.g., [14,16,17,27]), we adopt a more general standpoint by allowing configurations with multiple points and/or accumulation points.

Let $X$ be an arbitrary locally compact Hausdorff topological space with countable base (i.e., second-countable), equipped with the Borel sigma-algebra $\mathcal{B}(X)$ generated by open sets. Denote by $\mathcal{N}(X)$ the space of all non-negative integer-valued measures $N = N(\cdot)$ on $\mathcal{B}(X)$ with countable support $\text{supp} \ N := \{x \in X : \}$. 

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\[ N\{x\} > 0 \}, \]

\[ N = \sum_{x \in \text{supp } N} N\{x\} \delta_x, \]

where \( N\{x\} \in \mathbb{Z}_+: = \{0, 1, 2, \ldots\} \) and \( \delta_x \) is the Dirac measure at point \( x \) (i.e., \( \delta_x(B) = 1 \) if \( x \in B \) and \( \delta_x(B) = 0 \) otherwise). Each measure \( N \in \mathcal{N}(X) \) can be uniquely associated with a generalized configuration of points,

\[ N \leftrightarrow \gamma := \bigcup_{x \in \text{supp } N} (x \sqcup \cdots \sqcup x)_{N\{x\}}, \]

where the disjoint union \( x \sqcup \cdots \sqcup x \) signifies the inclusion of several distinct “copies” of point \( x \in \text{supp } N \). Hence, \( N(B) \) can be interpreted as the total number of points (counted with their multiplicities) in the restriction \( \gamma_B := \gamma \cap B \) of the configuration \( \gamma \leftrightarrow N \) to the set \( B \). In what follows, we shall identify configurations \( \gamma \) with the corresponding counting measures \( N \in \mathcal{N}(X) \), and we shall take the liberty of interpreting the notation \( \gamma \) either as a set of (multiple) points in \( X \) or as a counting measure or both, depending on the context.

Let us denote by \( \Gamma^d_X \) the set of all generalized configurations \( \gamma \) in \( X \), and let \( \mathcal{B}(\Gamma^d_X) \) be the smallest sigma-algebra containing all cylinder sets \( C_B^\gamma = \{ \gamma \in \Gamma^d_X : \gamma(B) = n \} \) \((B \in \mathcal{B}(X), n \in \mathbb{Z}_+)\). The Poisson measure on the configuration space \( \Gamma^d_X \) is defined as follows (cf. [16, Section 2.4]).

**Definition 2.1.** Let \( \sigma \) be a sigma-finite measure on \((X, \mathcal{B}(X))\). The Poisson measure \( \pi_\sigma \) with intensity \( \sigma \) is a probability measure on \( \mathcal{B}(\Gamma^d_X) \) such that for any finite collection of pairwise disjoint sets \( B_1, \ldots, B_k \in \mathcal{B}(X) \) and arbitrary non-negative integers \( n_1, \ldots, n_k \), the value of \( \pi_\sigma \) on the cylinder set

\[ C^{n_1, \ldots, n_k}_{B_1, \ldots, B_k} := \{ \gamma \in \Gamma^d_X : \gamma(B_i) = n_i, i = 1, \ldots, k \} \]

is given by the expression

\[ \pi_\sigma(C^{n_1, \ldots, n_k}_{B_1, \ldots, B_k}) = \prod_{i=1}^k \frac{\sigma(B_i)^{n_i} e^{-\sigma(B_i)}}{n_i!}, \quad B_i \cap B_j = \emptyset \quad (i \neq j). \quad (2.1) \]

That is to say, \( \gamma(B_i) \) are mutually independent Poisson random variables with parameters \( \sigma(B_i) \), respectively.

The Poisson measure is completely characterized by its Laplace transform (see [5,16])

\[ L_{\pi_\sigma}[f] := \int_{\Gamma^d_X} e^{\langle f, \gamma \rangle} \pi_\sigma(d\gamma) = \exp \left( \int_X (e^{f(x)} - 1) \sigma(dx) \right), \quad f \in \mathcal{D}, \quad (2.2) \]

where

\[ \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x) = \int_X f(x) \gamma(dx) \]

and the domain \( \mathcal{D} \) consists of measurable real-valued functions on \( X \) for which the integral on the right-hand side of (2.2) is well defined. It is sufficient to use a more
A well-known explicit construction of the Poisson measure $\pi_\sigma$ is as follows. Fix a set $\Lambda \in \mathcal{B}(X)$ such that $\sigma(\Lambda) < \infty$. For any $A \in \mathcal{B}(\Gamma_X^\sharp)$, set

$$A_{\Lambda,n} := \{\gamma_A : \gamma \in A \text{ and } \gamma(A) = n\}, \quad n \in \mathbb{Z}_+.$$ 

In particular, $(\Gamma_X^\sharp)_{X,n} =: \Gamma_{X,n}$ is the space of all $n$-point configurations. Let us define the measure $\pi^A_\sigma$ on $\mathcal{B}(\Gamma_X^\sharp)$ by

$$\pi^A_\sigma(A) := e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^\otimes n \left( A_{\Lambda,n} \right), \quad A \in \mathcal{B}(\Gamma_X^\sharp), \quad (2.3)$$

where $\sigma^\otimes n = \underbrace{\sigma \otimes \cdots \otimes \sigma}_{n}$ is the product measure on $(X^n, \mathcal{B}(X^n))$ (we formally set $X^0 := \{\emptyset\}$, $\sigma^\otimes 0 := \delta_{\emptyset}$) and $p$ is the operator of natural “projection” from the vector spaces $X^n$ to the spaces of $n$-point configurations $\Gamma_{X,n}$, respectively, whereby each vector is “unpacked” into distinct components (with multiplicities):

$$X^n \ni (x_1, \ldots, x_n) \mapsto p(x_1, \ldots, x_n) := \sum_{i=1}^{n} \delta_{x_i} \in \Gamma_{X,n}, \quad (2.4)$$

It is easy to check that the measure $\pi^A_\sigma$ satisfies equation (2.1) for any disjoint sets $B_i \subset \Lambda$. It is also clear that the family $\{\pi^A_\sigma, A \subset X\}$ is self-consistent: if $A_1 \subset A_2$ then $\pi^A_\sigma|_{A_1} = \pi^{A_2}_{\sigma}$. Taking a weak limit of $\pi^A_\sigma$ as $\Lambda \searrow X$, we obtain the Poisson measure $\pi_\sigma$ on the entire space $\Gamma_X^\sharp$.

The decomposition (2.3) implies that if $F(\gamma) \equiv F(\gamma \cap \Lambda)$ for some set $\Lambda \subset X$ such that $\sigma(\Lambda) < \infty$, then

$$\int_{\Gamma_X^\sharp} F(\gamma) \pi_\sigma(d\gamma) = e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} F(\{x_1, \ldots, x_n\}) \sigma(dx_1) \cdots \sigma(dx_n). \quad (2.5)$$

In particular, conditioned on the event $\gamma(\Lambda) = n$, the points of configuration $\gamma_A$ are distributed over $\Lambda$ independently of each other, with probability distribution $\sigma(dx)/\sigma(\Lambda)$ each.

We conclude this section by recalling the well-known (although not always stated explicitly in the literature, cf. [14,16,22,27]) necessary and sufficient conditions in order that $\pi_\sigma$-almost all (a.a.) configurations $\gamma \in \Gamma_X^\sharp$ have no accumulation points or multiple points.

**Definition 2.2.** Configuration $\gamma \in \Gamma_X^\sharp$ is said to be **locally finite** if $\gamma(K) < \infty$ for any compact set $K \subset X$. Configuration $\gamma \in \Gamma_X^\sharp$ is called **simple** if $\gamma\{x\} \leq 1$ for each $x \in X$. Configuration $\gamma \in \Gamma_X^\sharp$ is called **proper** if it is both locally finite and simple. The set of all proper configurations will be denoted by $\Gamma_X^\ast$ and called the **proper configuration space over $X$**.
Proposition 2.1. (a) In order that \( \pi_\sigma \)-a.a. configurations \( \gamma \in \Gamma_X \) be locally finite, it is necessary and sufficient that \( \sigma(K) < \infty \) for any compact set \( K \in \mathcal{B}(X) \).

(b) In order that \( \pi_\sigma \)-a.a. configurations \( \gamma \in \Gamma_X \) be simple, it is necessary and sufficient that \( \sigma \{ x \} = 0 \) for each \( x \in X \).

2.2 Poisson cluster measure

Let us first recall the notion of a general cluster point process (CPP). The intuitive idea is to construct its realizations in two steps: (i) take a background random configuration of (invisible) “centres” obtained as a realization of some point process \( \Gamma_c \) governed by a probability measure \( \mu_c \) on \( \Gamma_X \), and (ii) relative to each centre \( x \in \Gamma_c \), generate a set of observable secondary points (referred to as a cluster centred at \( x \)) according to a point process \( \Gamma'_x \) with probability measure \( \mu_x \) on \( \Gamma_X \) (\( x \in X \)).

The resulting (countable) assembly of random points, called cluster point process, can be symbolically expressed as

\[
\gamma = \bigsqcup_{x \in \Gamma_c} \gamma'_x \in \Gamma_X,
\]

where the disjoint union signifies that multiplicities of points should be taken into account. More precisely, the integer-valued measure corresponding to a CPP realization \( \gamma \) is given by

\[
\gamma(B) = \int_X \gamma'_x(B) \gamma_c(dx) = \sum_{x \in \Gamma_c} \gamma'_x(B) = \sum_{x \in \gamma_c} \sum_{y \in \gamma'_x} \delta_B(y), \quad B \in \mathcal{B}(X). \tag{2.6}
\]

A tractable model of such a kind is obtained when (a) \( X \) is a linear space and (b) random clusters are independent and identically distributed (i.i.d.), that is, mutually independent and governed by the same law translated to the cluster centres,

\[
\mu_x(A) = \mu_0(A - x), \quad A \in \mathcal{B}(\Gamma_X). \tag{2.7}
\]

Remark 2.1. From the description of the CPP given above, it only follows that its sample configurations are countable sets in \( X \), but possibly with multiple and/or accumulation points, even if the background point process \( \Gamma_c \) is proper. Therefore, the distribution \( \mu \) of the CPP (2.6) is a probability measure defined on the space \( \Gamma_X \) of generalized configurations. It is a matter of interest to obtain conditions in order that \( \mu \) be actually supported on the space of proper configurations \( \Gamma_X \), and we will address this issue in Section 2.3 below in the case of Poisson CPPs.

Let \( \nu_x := \gamma'_x(X) \) be the total (random) number of points in a cluster \( \gamma'_x \) centred at point \( x \in X \) (referred to as the cluster size). According to our assumptions, the random variables \( \nu_x \) are i.i.d. for different \( x \), with common distribution

\[
p_n := \mu_0 \{ \nu_0 = n \}, \quad n = 0, 1, 2, \ldots, \infty \tag{2.8}
\]
(so in principle the case $\nu_0 = \infty$ may have a positive probability).

**Remark 2.2.** One might argue that allowing for vacuous clusters (i.e., with $\nu_x = 0$) is superfluous since these are not visible in a sample configuration, and in particular the probability $p_0$ cannot be estimated statistically [16, Corollary 6.3.VI]. In fact, the possibility of vacuous cluster may be ruled out without loss of generality, at the expense of rescaling the background intensity measure, $\sigma \mapsto (1 - p_0) \sigma$. However, we keep this possibility in our model in order to provide a suitable framework for evolutionary cluster point processes with annihilation and creation of particles, which we intend to study elsewhere.

The following fact is well known (see, e.g., [16, Section 6.3]).

**Proposition 2.2.** The Laplace functional of the probability measure $\mu$ in $\Gamma^c_X$ corresponding to the CPP (2.6) is given by

$$L_\mu[f] := \int_{\Gamma^c_X} e^{(f, \gamma)} \mu(d\gamma) = L_{\mu_c}[\ln L_{\mu_x}[f]] = L_{\mu_c}[\ln L_{\mu_0}(f(\cdot + x))], \quad (2.9)$$

where $L_{\mu_c}$ acts in variable $x$.

**Proof.** The representation (2.6) of cluster configurations $\gamma$ implies

$$\langle f, \gamma \rangle = \sum_{z \in \gamma} f(z) = \sum_{x \in \gamma_c} \sum_{y \in \gamma'_x} f(y).$$

Conditioning on the background configuration $\gamma_c$ and using the independence of the clusters $\gamma'_x$ for different $x$, we obtain

$$\int_{\Gamma^c_X} e^{(f, \gamma)} \mu(d\gamma) = \int_{\Gamma^c_X} \prod_{x \in \gamma_c} \left( \int_{\Gamma^c_X} e^{\sum_{y \in \gamma'_x} f(y)} \mu_x(d\gamma'_x) \right) \mu_c(d\gamma_c)$$

$$= \int_{\Gamma^c_X} \exp \left\{ \sum_{x \in \gamma_c} \ln \left( L_{\mu_x}[f] \right) \right\} \mu_c(d\gamma_c) = L_{\mu_c}[\ln L_{\mu_x}[f]],$$

which proves the first formula in (2.9). The second one easily follows by shifting the measure $\mu_x$ to the origin using (2.7). \qed

In this paper, we are mostly concerned with the Poisson CPPs, which are specified by assuming that $\mu_c$ is a Poisson measure on configurations, with some intensity measure $\sigma$. The corresponding probability measure on the configuration space $\Gamma^c_X$ will be denoted by $\mu_{c\lambda}$ and called the Poisson cluster measure.

The combination of formulas (2.2) and (2.9) gives a formula for the Laplace functional of $\mu_{c\lambda}$.

**Proposition 2.3.** The Laplace functional of the Poisson cluster measure $\mu_{c\lambda}$ on $\Gamma^c_X$
is given by
\[
L_{\mu_{\text{cl}}}[f] = \exp \left\{ \int_X \left( \int_{\Gamma_X} \left( e^{\sum_{y \in \gamma'_0} f(y + x)} - 1 \right) \mu_0(d\gamma'_0) \right) \sigma(dx) \right\}
\]

(2.10)

(Here and below we use the convention that if \( \gamma'_0 = \emptyset \) then \( \sum_{y \in \gamma'_0} f(y + x) := 0 \).

2.3 Criteria of local finiteness and simplicity

In this section, we give criteria for the Poisson CPP to be locally finite and simple. For a given set \( B \in \mathcal{B}(X) \) and each in-cluster configuration \( \gamma'_0 \) centred at the origin, consider the set (referred to as a droplet cluster)

\[
D_B(\gamma'_0) := \bigcup_{y \in \gamma'_0} (B - y),
\]

which is a set-theoretic union of “droplets” of shape \( B \) shifted to the centrally reflected points of \( \gamma'_0 \).

**Theorem 2.4.** Let \( \mu_{\text{cl}} \) be a Poisson cluster measure on the generalized configuration space \( \Gamma_X^\mathbb{Z} \).

(a) In order that \( \mu_{\text{cl}} \)-a.a. configurations \( \gamma \in \Gamma_X^\mathbb{Z} \) be locally finite, it is necessary and sufficient that the following two conditions hold:

(a-i) in-cluster configurations \( \gamma'_0 \) are a.s. locally finite, that is, for any compact set \( K \in \mathcal{B}(X) \),

\[
\gamma'_0(K) < \infty \quad \mu_0\text{-a.s.}
\]

(2.12)

(a-ii) for any compact set \( K \in \mathcal{B}(X) \), the mean volume of the droplet cluster \( D_K(\gamma'_0) \) is finite,

\[
\int_{\Gamma_X^\mathbb{Z}} \sigma(D_K(\gamma'_0)) \mu_0(d\gamma'_0) < \infty.
\]

(2.13)

(b) In order that \( \mu_{\text{cl}} \)-a.a. configurations \( \gamma \in \Gamma_X^\mathbb{Z} \) be simple, it is necessary and sufficient that the following two conditions hold:

(b-i) in-cluster configurations \( \gamma'_0 \) are a.s. simple,

\[
\sup_{x \in X} \gamma'_0(x) \leq 1 \quad \mu_0\text{-a.s.}
\]

(2.14)

(b-ii) for each \( x \in X \), the “point” droplet cluster \( D_{\{x\}}(\gamma'_0) \) has a.s. zero volume,

\[
\sigma(D_{\{x\}}(\gamma'_0)) = 0 \quad \mu_0\text{-a.s.}
\]

(2.15)

The proof of Theorem 2.4 is given in the Appendix (Section 6.1).
Let us briefly discuss the conditions of properness. First of all, note that conditions (a-i) and (a-ii) taken in conjunction are incompatible with the possibility for the number of points in a generic cluster, \( \nu_0 = \gamma'_0(X) \), to be infinite, since due to (a-i) configuration \( \gamma'_0 \) must be a.s. locally finite, in which case the volume of the droplet cluster \( D_K(\gamma'_0) \) is necessarily infinite and hence (a-ii) is not satisfied.

Assuming that \( \nu_0 < \infty (\mu_0\text{-a.s.}) \), it is easy to give conditions sufficient for (a-ii). The first set of conditions is expressed in terms of the background intensity measure \( \sigma \) and the mean number of points in an individual cluster:

(a-ii') for any compact set \( K \in \mathcal{B}(X) \), the volume of its translates is uniformly bounded,
\[
C_K := \sup_{x \in X} \sigma(K - x) < \infty, \quad (2.16)
\]
and, moreover, the mean number of in-cluster points is finite,
\[
\int_{\mathbb{R}^d_X} \gamma'_0(X) \mu_0(d\gamma'_0) = \sum_{n=0}^{\infty} n p_n < \infty.
\]

Indeed, from (2.11) and (2.16) we obtain
\[
\sigma(D_K(\gamma'_0)) \leq \sum_{y \in \gamma'_0} \sigma(K - y) \leq C_K \gamma'_0(X) = C_K \nu_0,
\]
hence
\[
\int_{\mathbb{R}^d_X} \sigma(D_K(\gamma'_0)) \mu_0(d\gamma'_0) \leq C_K \int_{\mathbb{R}^d_X} \gamma'_0(X) \mu_0(d\gamma'_0) < \infty.
\]

Another sufficient condition is set in terms of the location of the in-cluster points:

(a-ii'') a generic cluster \( \gamma'_0 \), as a set in \( X \), is a.s. bounded, that is, there exists a compact \( K_0 \in \mathcal{B}(X) \) such that \( \gamma'_0 \subset K_0 \) (\( \mu_0\text{-a.s.})

Indeed, here we have
\[
D_K(\gamma'_0) \subset \bigcup_{y \in K_0} (K - y) =: K - K_0,
\]
where the set \( K - K_0 \) is compact. Therefore,
\[
\int_{\mathbb{R}^d_X} \sigma(D_K(\gamma'_0)) \mu_0(d\gamma'_0) \leq \sigma(K - K_0) \int_{\mathbb{R}^d_X} \mu_0(d\gamma'_0) = \sigma(K - K_0) < \infty.
\]

The impact of conditions (a-ii') and (a-ii'') on local finiteness of the Poisson CPP is clear: (a-ii') imposes a bound on the number of points which can be contributed from remote clusters, while (a-ii'') restricts the range of such contribution.

Similarly, one can work out simple conditions either of which is sufficient for (b-ii). The first condition below is set in terms of the background intensity measure \( \sigma \), whereas the second one exploits the in-cluster distribution:

(b-ii') the measure \( \sigma \) is continuous, that is, \( \sigma\{x\} = 0 \) for each \( x \in X \);
(b-ii’’) the in-cluster measure $\mu_0$ is continuous, that is, $\mu_0\{\gamma'_0 \in \Gamma^d_X : x \in \gamma'_0\} = 0$ for each $x \in X$.

Indeed, condition (b-ii’) readily implies (b-ii):

$$0 \leq \sigma\left(D_{\{x\}}(\gamma'_0)\right) \leq \sum_{y \in \gamma'_0} \sigma\{x - y\} = 0.$$

Further, if condition (b-ii’”) holds then we have

$$\int_{\Gamma^d_X} \sigma\left(D_{\{x\}}(\gamma'_0)\right) \mu_0(d\gamma'_0) = \int_X \left(\int_{\Gamma^d_X} \chi_{\gamma'_0 \subseteq \gamma'_0}(x-y)(z) \mu_0(d\gamma'_0)\right) \sigma(dz)$$

$$= \int_X \left(\int_{\Gamma^d_X} \chi_{\gamma'_0}(z-x) \mu_0(d\gamma'_0)\right) \sigma(dz)$$

$$= \int_X \mu_0\{\gamma'_0 \in \Gamma^d_X : z - x \in \gamma'_0\} \sigma(dz) = 0,$$  \hspace{1cm} (2.17)

and (b-ii) follows.

3 Poisson cluster processes via Poisson measures

From now on, we restrict ourselves to the case where $X = \mathbb{R}^d$. We shall also assume throughout that conditions (a-i) and (a-ii) of Proposition 2.1 are fulfilled, so that $\mu_{\text{cl}}$-a.a. configurations $\gamma \in \Gamma^d_X$ are locally finite. In this section, we give a description of the Poisson cluster measure $\mu_{\text{cl}}$ in terms of Poisson measures on suitable “vector” configuration spaces. This will enable us to apply the well-developed calculus on Poisson configuration spaces to the Poisson cluster measure.

3.1 Intensity measure of an auxiliary Poisson process

It is more convenient to work with representation of clusters as ordered sequences, rather than sets of unordered points (cf. [16, page 129]). Let us consider the space generated by Cartesian powers of $X$, that is, the disjoint union

$$\mathfrak{X} := \bigsqcup_{n=0}^{\infty} X^n.$$

The natural Borel sigma-algebra $\mathcal{B}(\mathfrak{X})$, generated by Borel sets in the constituent spaces $X^n$, consists of all sets of the form $\mathcal{B} = \sqcup_n B_n$, $B_n \in \mathcal{B}(X^n)$.

The probability distribution $\mu_0$ of a generic cluster centred at the origin (see Section 2.2) can be canonically extended to a probability measure $\eta$ in $\mathfrak{X}$ which is symmetric with respect to permutation of coordinates. Conversely, $\mu_0 = p^*\eta$, where $p : \mathfrak{X} \to \Gamma^d_X$ is the canonical mapping defined by (2.4). Projections of $\eta$ onto
the spaces $X^n$ will be denoted $\eta_n$, so that (recall (2.8))

$$\eta(\bar{B}) = \sum_n p_n \eta_n(B_n), \quad \bar{B} = \bigcup_n B_n \in \mathcal{B}(X). \quad (3.1)$$

The following definition is fundamental for our construction.

**Definition 3.1.** We introduce the measure $\sigma^\ast$ on $X$ as a special “convolution” of the measures $\eta$ and $\sigma$:

$$\sigma^\ast(\bar{B}) := \int_{X} \eta(\bar{B} - x) \sigma(dx), \quad \bar{B} \in \mathcal{B}(X), \quad (3.2)$$

or, equivalently, for any measurable function $f : X \to \mathbb{R}$,

$$\int_X f(\bar{y}) \sigma^\ast(d\bar{y}) = \int_X \left( \int_X f(\bar{y} + x) \eta(d\bar{y}) \right) \sigma(dx). \quad (3.3)$$

Here and below we use the “shift” notation

$$\bar{y} + x := (y_1 + x, \ldots, y_n + x), \quad \bar{y} = (y_1, \ldots, y_n) \in X^n \quad (n \in \mathbb{N}).$$

If $\sigma^\ast_n := \sigma^\ast|_{X^n}$ stands for the projection of $\sigma^\ast$ onto the space $X^n$, then the definition (3.2) implies

$$\sigma^\ast_n(B_n) := \int_{X} \eta_n(B_n - x) \sigma(dx), \quad B_n \in \mathcal{B}(X^n), \quad (3.4)$$

and in view of (3.1) equation (3.2) can be represented as

$$\sigma^\ast(\bar{B}) = \sum_n p_n \sigma^\ast_n(B_n), \quad \bar{B} = \bigcup_n B_n \in \mathcal{B}(X). \quad (3.5)$$

If the measure $\eta$ is absolutely continuous (a.c.) with respect to the Lebesgue measure $d\bar{x} = \oplus_n dx_1 \otimes \cdots \otimes dx_n$ on $X$, with density $h$,

$$\eta(d\bar{y}) = h(\bar{y}) \, d\bar{y}, \quad \bar{y} \in X, \quad (3.6)$$

then the measure $\sigma^\ast$ also has density, $\sigma^\ast(d\bar{y}) = s(\bar{y}) \, d\bar{y}$, where

$$s(\bar{y}) = \int_{X} h(\bar{y} - x) \sigma(dx), \quad \bar{y} \in X. \quad (3.7)$$

**Remark 3.1.** In the case $n = 1$, the definition (3.4) is reduced to

$$\sigma^\ast_1(B_1) = \int_{X} \eta_1(B_1 - x) \sigma(dx) = \int_{X} \sigma(B_1 - x) \eta_1(dx), \quad B_1 \in \mathcal{B}(X).$$

In particular, if $\sigma$ is translation invariant (so that $\sigma(B_1 - x) = \sigma(B_1)$), then $\sigma^\ast_1 \equiv \sigma$.

**Example 3.1.** Let $X = \mathbb{R}$, and for $n \geq 1$ set

$$h_n(\bar{y}) = \frac{1}{(2\pi)^{n/2}} e^{-\|\bar{y}\|^2/2}, \quad \bar{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$
Thus, $\eta_n$ is the standard Gaussian measure on $\mathbb{R}^n$. Assume that $\sigma$ is the Lebesgue measure, $\sigma(dx) = dx$. For $n = 1$, from equation (3.7) we obtain

$$s_1(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(y-x)^2/2} \, dx = 1,$$

hence $\sigma_1^*$ coincides with $\sigma$, in accord with Remark 3.1.

If $n = 2$ then equation (3.7) yields

$$s_2(y_1, y_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(y_1-x)^2+(y_2-x)^2/2} \, dx = \frac{1}{2\sqrt{\pi}} e^{-(y_1-y_2)^2/4}.$$

Via the orthogonal transformation

$$z_1 = \frac{y_1 + y_2}{\sqrt{2}}, \quad z_2 = \frac{y_1 - y_2}{\sqrt{2}},$$

the measure $\sigma^*_2$ is reduced to

$$\sigma^*_2(dz_1, dz_2) = \frac{1}{2\sqrt{\pi}} e^{-z_2^2/2} \, dz_1 \, dz_2,$$

which is a direct product of the standard Gaussian measure (along the coordinate axis $z_1$) and the scaled Lebesgue measure $dz_2/\sqrt{2}$. Note that $\sigma^*_2$ is not finite, however any vertical or horizontal strip of finite width (in coordinates $\bar{y}$) has a finite $\sigma^*_2$-measure.

In general ($n \geq 2$), after integration in (3.7) we get

$$s_n(\bar{y}) = \frac{1}{(\sqrt{2\pi})^{n-1} \sqrt{n}} \exp \left(-\frac{1}{2} \left( \|\bar{y}\|^2 - \frac{1}{n} |(\bar{y}, \bar{1})|^2 \right) \right), \quad \bar{y} \in \mathbb{R}^n,$$

where $\bar{1} := (1, \ldots, 1) \in \mathbb{R}^n$. It is easy to check that after an orthogonal transformation $\bar{z} = \bar{y} U$ such that

$$z_1 = \frac{y_1 + \cdots + y_n}{\sqrt{n}},$$

the measure $\sigma^*_n$ takes the form

$$\sigma^*_n(d\bar{z}) = \frac{dz_1}{\sqrt{n}} \cdot \frac{1}{(\sqrt{2\pi})^{n-1}} e^{-z_2^2/2} \, dz_2 \cdots dz_n, \quad \bar{z} = (z_1, \ldots, z_n).$$

That is, $\sigma^*_n(d\bar{z})$ is a direct product of the scaled Lebesgue measure $dz_1/\sqrt{n}$ and the standard Gaussian measure in coordinates $z_2, \ldots, z_n$. Note that $\sigma^*_n(\mathbb{R}^n) = \infty$, but for any coordinate strip $C_i = \{\bar{y} \in \mathbb{R}^n : |y_i| \leq c\}$ we have $\sigma^*_n(C_i) < \infty$.

Example 3.1 can be generalized as follows.

**Proposition 3.1.** For each $n \geq 1$, consider an orthogonal linear transformation $\bar{z} = \bar{y} U_n$ of the space $X^n$ such that

$$z_1 = \frac{y_1 + \cdots + y_n}{\sqrt{n}}, \quad \bar{z} = (z_1, \ldots, z_n), \quad \bar{y} = (y_1, \ldots, y_n). \quad (3.8)$$
Set $\tilde{z}' := (z_2, \ldots, z_n)$ and consider the measures

$$\eta_n(B') := \int_X \eta_n(dz_1, B') = \eta_n(X \times B'), \quad B' \in \mathcal{B}(X^{n-1}), \quad (3.9)$$

and

$$\tilde{\sigma}(B_1|\tilde{z}') := \int_X \sigma \left( \frac{B_1 - z_1}{\sqrt{n}} \right) \eta_n(dz_1|\tilde{z}'), \quad B_1 \in \mathcal{B}(X), \quad (3.10)$$

where $\eta_n(dz_1|\tilde{z}')$ is the measure in $X$ obtained from $\eta_n$ via conditioning on $\tilde{z}'$. Then the measure $\sigma^*$ can be decomposed as

$$\sigma^*(d\tilde{z}) = p_0 \delta_{\{\emptyset\}}(d\tilde{z}) + \sum_{n=1}^\infty p_n \tilde{\sigma}(d\tilde{z}_1|\tilde{z}') \eta_n'(d\tilde{z}'). \quad (3.11)$$

In particular, if the measure $\sigma$ on $X = \mathbb{R}^d$ is translation invariant then

$$\sigma^*(d\tilde{z}) = p_0 \delta_{\{\emptyset\}}(d\tilde{z}) + \sum_{n=1}^\infty p_n \frac{\sigma(dz_1)}{p^{d/2}} \eta_n'(d\tilde{z}'). \quad (3.12)$$

Proof. For a fixed $n \geq 1$, let us pass to the coordinates $\tilde{z} = \tilde{y} U_n$ and consider an arbitrary Borel set in $X^n$ of the form $B_n = B_1 \times B'_n = \{\tilde{z} \in X^n : z_1 \in B_1, \tilde{z}' \in B'_n\}$. By equation (3.8) and orthogonality of $U_n$, we have

$$B_n - x = (B_1 - x\sqrt{n}) \times B'_n.$$ 

Therefore, from (3.4) we obtain

$$\begin{align*}
\sigma_n^*(B_n) &= \int_X \eta_n(B_n - x) \sigma(dx) \\
&= \int_X \left( \int_{X^n} 1_{(B_1 - x\sqrt{n}) \times B'_n}(\tilde{z}) \eta_n(d\tilde{z}) \right) \sigma(dx) \\
&= \int_{X^n} \left( \int_X 1_{B_1 - x\sqrt{n}}(z_1) \sigma(dx) \right) 1_{B'_n}(\tilde{z}') \eta_n(d\tilde{z}) \\
&= \int_{X \times X^{n-1}} \left( \int_X 1_{(B_1 - z_1)/\sqrt{n}}(x) \sigma(dx) \right) 1_{B'_n}(\tilde{z}') \eta_n(dz_1|\tilde{z}') \eta_n'(d\tilde{z}') \\
&= \int_{B'_n} \tilde{\sigma}(B_1|\tilde{z}') \eta_n'(d\tilde{z}'),
\end{align*}$$

and by inserting this into equation (3.5) we get (3.11). Finally, the translation invariance of $\sigma$ implies that $\sigma((B_1 - z_1)/\sqrt{n}) = n^{-d/2} \sigma(B_1)$. Formula (3.10) then gives $\tilde{\sigma}(B_1|\tilde{z}') = n^{-d/2} \sigma(B_1)$, and (3.12) readily follows from (3.11).

Using decomposition (3.11), it is easy to obtain the following criterion of absolute continuity of the measure $\sigma^*$. 

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Corollary 3.2. The measure $\sigma^*$ is a.c. with respect to the Lebesgue measure $d\bar{x} = \delta_0 \oplus \bigoplus_{n=1}^\infty dx_1 \otimes \cdots \otimes dx_n$ on $\mathcal{X}$ if and only if the following two conditions hold:

(i) for each $n \geq 1$, the measure $\eta'_n(d\bar{z}')$ is a.c. with respect to the Lebesgue measure $d\bar{z}'$ on $\mathcal{X}^{n-1}$;
(ii) for a.a. $\bar{z}'$, the measure $\tilde{\sigma}(dz_1|\bar{z}')$ is a.c. with respect to the Lebesgue measure $dz_1$ on $\mathcal{X}$.

In particular, if $\sigma$ is translation invariant then condition (ii) is automatically fulfilled and hence condition (i) alone is necessary and sufficient for absolute continuity of $\sigma^*$.

Remark 3.2. The absolute continuity of $\eta$ is sufficient (cf. (3.6), (3.7)), but not necessary, for condition (i). This is illustrated by the following example:

$$\eta(dy_1, dy_2) = \frac{1}{2} \delta_{(1)}(dy_1) f(y_2) dy_2 + \frac{1}{2} \delta_{(1)}(dy_2) f(y_1) dy_1, \quad (y_1, y_2) \in \mathbb{R}^2,$$

where $f(y) (y \in \mathbb{R})$ is some probability density function. Then the projection measure $\eta'$ on $\mathbb{R}$ (see (3.9)) is given by

$$\eta'(dz') = \frac{\sqrt{2}}{2} \left( f(1 - \sqrt{2} z') + f(1 + \sqrt{2} z') \right) dz', \quad z' = \frac{y_1 - y_2}{\sqrt{2}},$$

and so is absolutely continuous.

The next result shows that the absolute continuity of $\sigma^*$ implies that the Poisson cluster process with probability one has no multiple points (see Definition 2.2).

Proposition 3.3. Suppose that the measure $\sigma^*(d\bar{x})$ on $\mathcal{X}$ is a.c. with respect to the Lebesgue measure $d\bar{x}$. Then $\mu_{\mathrm{cl}}$-a.a. configurations $\gamma$ are simple.

Proof. By Theorem 2.4, it suffices to check conditions (b-i) and (b-ii). First, note that if condition (b-i) is not satisfied (i.e., if the set of points $\bar{y} \in \mathcal{X}$ with two or more coinciding coordinates has positive $\eta$-measure), then the projected measure $\eta'(d\bar{z}')$ charges a hyperplane (of codimension 1) in the space $\mathcal{X}'$ spanned over the coordinates $\bar{z}'$. But this contradicts the absolute continuity of $\sigma^*$, since such hyperplanes have zero Lebesgue measure.

Furthermore, similarly to (2.17) and using the definition (3.2), for each $x \in \mathcal{X}$ we obtain

$$\int_{\mathcal{X}} \sigma \left( \bigcup_{y \in \bar{y}} \{x - y\} \right) \eta(dy) = \int_{\mathcal{X}} \eta \{\bar{y} \in \mathcal{X} : z - x \in \bar{y}\} \sigma(dz) = \sigma^* \{\bar{y} \in \mathcal{X} : -x \in \text{p}(\bar{y})\} = 0,$$

by the absolute continuity of $\sigma^*$. Hence, $\sigma \left( \bigcup_{y \in \bar{y}} \{x - y\} \right) = 0 \ (\eta\text{-a.s.})$ and condition (b-ii) follows.
3.2 Poisson cluster measure via an auxiliary Poisson measure \( \pi_\sigma \)

Let us consider the cluster configuration space \( \Gamma_X^{\#} \), with generic elements \( \bar{\gamma} \), and let \( \pi_\sigma \) be the Poisson measure on \( \Gamma_X^{\#} \) with intensity \( \sigma^{\#} \) defined in the previous section. Recall (see (2.4)) that the “unpacking” map \( p \) from the space \( X = \bigsqcup_{n=0}^\infty X^n \) into the space \( \mathcal{N}(X) \) of non-negative integer-valued measures in \( X \) is defined on each component \( X^n \) by

\[
p(\bar{x}) := \sum_{x_i \in \bar{x}} \delta_{x_i}, \quad \bar{x} = (x_1, \ldots, x_n) \in X^n \quad (n \in \mathbb{N}). \tag{3.13}
\]

The image \( p(\bar{x}) (\bar{x} \in X) \) can be identified in the usual way with a (finite) generalized configuration of points in \( X \).

For any subset \( K \subset X \), denote

\[
X_K := \{ \bar{y} \in X : p(\bar{y}) \cap K \neq \emptyset \}. \tag{3.14}
\]

The following result is crucial for our purposes (cf. Example 3.1).

**Proposition 3.4.** Let \( K \subset X \) be a compact set. Then condition (a-ii) of Theorem 2.4 (see (2.13)) is necessary and sufficient in order that

\[
\sigma^{\#}(X_K) < \infty, \tag{3.15}
\]

or equivalently,

\[
\bar{\gamma}(X_K) < \infty \quad \text{for } \pi_\sigma \text{-a.a. } \bar{\gamma} \in \Gamma_X. \tag{3.16}
\]

**Proof.** Using the definition (3.2), we obtain

\[
\sigma^{\#}(X_K) = \int_X \eta(X_K - x) \sigma(dx) = \int_X \left( \int_X 1_{X_K}(\bar{y} + x) \sigma(dx) \right) \eta(d\bar{y}). \tag{3.17}
\]

From (3.14), it follows that \( \bar{y} + x \in X_K \) if and only if \( x \in \bigcup_{y \in \bar{y}} (K - y) = D_K(\bar{y}) \) (see (2.11)). Hence, (3.17) is reduced to

\[
\int_X \left( \int_X 1_{D_K(\bar{y})}(x) \sigma(dx) \right) \eta(d\bar{y}) = \int_X \sigma(D_K(\bar{y})) \eta(d\bar{y}) < \infty,
\]

according to condition (2.13), and the bound (3.15) follows.

The second part (see (3.16)) follows by observing that the probability distribution of the random variable \( \bar{\gamma}(X_K) \) under the measure \( \pi_\sigma \) is given by the Poisson law with parameter \( \sigma^{\#}(X_K) \) (cf. (2.1)), and so \( \bar{\gamma}(X_K) \) is finite a.s. if and only if \( \sigma^{\#}(X_K) < \infty. \)

We can lift the mapping (3.13) to the configuration space \( \Gamma_X^{\#} \) by setting

\[
p(\bar{\gamma}) := \bigsqcup_{\bar{x} \in \bar{\gamma}} p(\bar{x}) \subset X, \quad \bar{\gamma} \in \Gamma_X^{\#}. \tag{3.18}
\]
Disjoint union in (3.18) highlights the fact that the set $p(\bar{\gamma})$ may have multiple points, even if the cluster configuration $\bar{\gamma}$ is proper. Thus, formula (3.18) defines a mapping $p : \Gamma^d_X \to \Gamma^d_X$ into the space of generalized configurations in $X$.

Finally, we introduce the measure $\mu_{cl}$ on $\Gamma^d_X$ as the push-forward of the Poisson measure $\pi_{\sigma^*}$ under the mapping $p$,

$$\mu_{cl}(A) := (p^*\pi_{\sigma^*})(A) = \pi_{\sigma^*}(p^{-1}(A)), \quad A \in B(\Gamma^d_X). \quad (3.19)$$

Equivalently, for any measurable function $f : \Gamma^d_X \to \mathbb{R}$,

$$\int_{\Gamma^d_X} f(\gamma) \mu_{cl}(d\gamma) = \int_{\Gamma^d_X} f(p(\bar{\gamma})) \pi_{\sigma^*}(d\bar{\gamma}). \quad (3.20)$$

The next theorem is the main result of this section.

**Theorem 3.5.** The measure $\mu_{cl}$ on $\Gamma^d_X$ defined by (3.19) coincides with the Poisson cluster measure.

**Proof.** Let us compute the Laplace transform of $\mu_{cl}$. For any measurable function $f : \Gamma^d_X \to \mathbb{R}$, by the change of measure (3.19) we have

$$\int_{\Gamma^d_X} e^{\langle f, \gamma \rangle} \mu_{cl}(d\gamma) = \int_{\Gamma^d_X} e^{\langle f, p(\bar{\gamma}) \rangle} \pi_{\sigma^*}(d\bar{\gamma}) = \int_{\Gamma^d_X} e^{\langle \tilde{f}, \bar{\gamma} \rangle} \pi_{\sigma^*}(d\bar{\gamma}),$$

where $\tilde{f}(\bar{y}) := \sum_{y \in \bar{y}} f(y)$. According to (2.2) and (3.3), the right-hand side of (3.20) takes the form

$$\exp \left( \int_X \left( e^{\tilde{f}(\bar{y})} - 1 \right) \sigma^*(d\bar{y}) \right) = \exp \left( \int_X \int_X \left( e^{\tilde{f}(\bar{y}+x)} - 1 \right) \eta(d\bar{y}) \sigma(dx) \right)$$

$$= \exp \left( \int_X \left( \int_X \left( e^{\sum_{y \in \bar{y}} f(y+x)} - 1 \right) \eta(d\bar{y}) \right) \sigma(dx) \right),$$

which coincides (up to a change of notation) with the expression (2.10) for the Laplace transform of the Poisson cluster measure. \qed

**Remark 3.3.** As an elegant application of the technique developed here, let us give a transparent proof of Theorem 2.4(a) (cf. the Appendix, Section 6.1). Indeed, in order that a given compact set $K \subset X$ contain finitely many points of configuration $\gamma = p(\tilde{\gamma})$, it is necessary and sufficient that (i) each cluster “point” $\bar{x} \in \bar{\gamma}$ is locally finite, which is equivalent to the condition (a-i), and (ii) there are finitely many points $\bar{x} \in \bar{\gamma}$ which contribute to the set $K$ under the mapping $p$, the latter being equivalent to condition (a-ii) by Proposition 3.4.

**3.3 An alternative construction of the measures $\pi_{\sigma^*}$ and $\mu_{cl}$**

The measure $\pi_{\sigma^*}$ was introduced in the previous section as a Poisson measure on the configuration space $\Gamma_X$ with a certain intensity measure $\sigma^*$ prescribed *ad hoc* by equation (3.2). In this section, we show that $\pi_{\sigma^*}$ can be obtained in a more natural
way as a suitable skew projection of a “canonical” Poisson measure \( \tilde{\pi} \) defined on a bigger configuration space \( \Gamma_{X}^{d} \), with the product intensity measure \( \sigma \otimes \eta \).

More specifically, given a Poisson measure \( \pi_{\sigma} \) in \( \Gamma_{X}^{d} \), let us construct a new measure \( \tilde{\mu} \) in \( \Gamma_{X \times \mathcal{X}}^{d} \) as the probability distribution of random configurations \( \hat{\gamma} \in \Gamma_{X \times \mathcal{X}}^{d} \) obtained from Poisson configurations \( \gamma \in \Gamma_{X}^{d} \) by the rule

\[
\gamma \mapsto \hat{\gamma} := \{(x, \bar{y}_x) : x \in \gamma, \bar{y}_x \in \mathcal{X}\}, \tag{3.21}
\]

where the random vectors \( \{\bar{y}_x\} \) are i.i.d., with common distribution \( \eta(d\bar{y}) \). Geometrically, such a construction may be viewed as pointwise i.i.d. translations of the Poisson configuration \( \gamma \in \mathcal{X} \) into the space \( X \times \mathcal{X} \),

\[ X \ni x \mapsto (x, 0) \mapsto (x, \bar{y}_x) \in X \times \mathcal{X}. \]

**Remark 3.4.** Vector \( \bar{y}_x \) in each pair \( (x, \bar{y}_x) \in X \times \mathcal{X} \) can be interpreted as a mark attached to the point \( x \in X \), so that \( \hat{\gamma} \) becomes a marked configuration, with the mark space \( \mathcal{X} \) (see [16,23]).

**Theorem 3.6.** The probability distribution \( \tilde{\mu} \) of random configurations \( \hat{\gamma} \in \Gamma_{X \times \mathcal{X}}^{d} \) constructed in (3.21) is given by the Poisson measure \( \pi_{\sigma} \) on the configuration space \( \Gamma_{X \times \mathcal{X}}^{d} \) with the product intensity measure \( \tilde{\sigma} := \sigma \otimes \eta \).

**Proof.** Let us check that, for any measurable function \( f(x, \bar{y}) \) on \( X \times \mathcal{X} \), the Laplace transform of the measure \( \tilde{\mu} \) is given by formula (2.2). Using independence of the vectors \( \bar{y}_x \) corresponding to different \( x \), we obtain

\[
\int_{\Gamma_{X \times \mathcal{X}}^{d}} e^{\langle f, \hat{\gamma} \rangle} \tilde{\mu}(d\hat{\gamma}) = \int_{\Gamma_{X \times \mathcal{X}}^{d}} \prod_{x \in \gamma} \left( \int_{X} e^{\langle f(x, \bar{y}) \rangle} \eta(d\bar{y}) \right) \pi_{\sigma}(d\gamma) \\
= \exp \left\{ \int_{X} \left( \int_{\mathcal{X}} e^{\langle f(x, \bar{y}) \rangle} \eta(d\bar{y}) - 1 \right) \sigma(dx) \right\} \\
= \exp \left\{ \int_{X} \int_{\mathcal{X}} \left( e^{\langle f(x, \bar{y}) \rangle} - 1 \right) \eta(d\bar{y}) \sigma(dx) \right\} \\
= \exp \left\{ \int_{X \times \mathcal{X}} \left( e^{\langle f(x, \bar{y}) \rangle} - 1 \right) \tilde{\sigma}(dx, d\bar{y}) \right\} = \int_{\Gamma_{X \times \mathcal{X}}^{d}} e^{\langle f, \hat{\gamma} \rangle} \pi_{\sigma}(d\hat{\gamma}),
\]

where we have applied formula (2.2) for the Laplace transform of the Poisson measure \( \pi_{\sigma} \) with the function \( f(x) = \ln \left( \int_{X} e^{\langle f(x, \bar{y}) \rangle} \eta(d\bar{y}) \right) \).

**Remark 3.5.** The measure \( \tilde{\mu} \), originally defined on configurations \( \hat{\gamma} \) of the form (3.21), naturally extends to a probability measure on the entire space \( \Gamma_{X \times \mathcal{X}}^{d} \).

**Remark 3.6.** Theorem 3.6 can be regarded as a generalization of the well-known invariance property of Poisson measures under random i.i.d. translations (see, e.g., [14,22]). A novel element here is that starting from a Poisson point field in \( X \), random translations create a new (Poisson) point field in a bigger space, \( X \times \mathcal{X} \), with the product intensity measure. On the other hand, note that the pointwise coordinate projection \( X \times \mathcal{X} \ni (x, \bar{y}_x) \mapsto x \in X \) recovers the original Poisson measure.
πσ, in accord with the Kingman mapping theorem [22]. Therefore, Theorem 3.6 provides a converse counterpart to the Kingman mapping theorem. To the best of our knowledge, these interesting properties of Poisson measures have not so far been pointed out in the literature.

Theorem 3.6 can be easily extended to more general (skew) translations. We will need the following result.

**Theorem 3.7.** Suppose that random configurations \( \hat{\gamma} \in \Gamma_{X \times X}^d \) are obtained from Poisson configurations \( \gamma \in \Gamma_{X}^d \) by pointwise translations of the form \( x \mapsto (x, \bar{y}_x + x) \), where \( \bar{y}_x \in X (x \in X) \) are i.i.d. with the common distribution \( \eta(d\bar{y}) \). Then the corresponding probability measure \( \hat{\mu} \) on \( \Gamma_{X \times X}^d \) coincides with the Poisson measure of intensity

\[
\sigma(d\bar{y}) := \eta(d\bar{y} - x).
\]

**Corollary 3.8.** Under the pointwise coordinate projection of the form \( (x, \bar{y}) \mapsto \bar{y} \) applied to configurations \( \hat{\gamma} \in \Gamma_{X \times X}^d \), the Poisson measure \( \hat{\mu} \) of Theorem 3.7 is pushed forward to the Poisson measure \( \pi_{\sigma^*} \) on \( \Gamma_{X}^d \) with intensity measure \( \sigma^* \) defined in (3.2).

**Proof.** By the Kingman mapping theorem (see [22, Section 2.3]), the image of the measure \( \hat{\mu} \) under the projection \( (x, \bar{y} + x) \mapsto \bar{y} + x \) is a Poisson measure with intensity given by the push-forward of the measure (3.22), that is,

\[
\int_X \hat{\sigma}(dx, B) = \int_X \eta(B - x) \sigma(dx) = \sigma^*(B), \quad B \in \mathcal{B}(X),
\]

according to the definition (3.2).

Finally, combining the results of Theorem 3.7 and Corollary 3.8 with Theorem 3.5, we obtain that under the composition mapping

\[
x \mapsto (x, \bar{y}_x + x) \mapsto \bar{y}_x + x \mapsto p(\bar{y}_x + x),
\]

the measure \( \hat{\mu} \) of Theorem 3.7 is pushed forward from the space \( \Gamma_{X \times X}^d \) directly to the space \( \Gamma_{X}^d \) where it coincides with the prescribed Poisson cluster measure \( \mu_{\text{cl}} \). The latter construction may prove instrumental for more complex (e.g., Gibbs) cluster processes, as it enables one to avoid the intermediate space \( \Gamma_{X \times X}^2 \) where the pushed-forward measure (analogous to \( \pi_{\sigma^*} \)) may have no explicit description available.

### 4 Quasi-invariance and integration by parts

#### 4.1 Diff\(0\)-quasi-invariance of the measure \( \mu_{\text{cl}} \)

In this section we discuss the property of quasi-invariance of the measure \( \mu_{\text{cl}} \) with respect to diffeomorphisms of \( X \). To this end, we need to require absolute continuity of the intensity measure \( \sigma^* \) (corresponding necessary and sufficient conditions
are described in Corollary 3.2). By Proposition 3.3, this automatically ensures that \( \mu_{cl} \)-a.a. configurations \( \gamma \) are simple (i.e., have no multiple points), which will enable us to work in the proper configuration space \( \Gamma_X \).

Let us start by describing how diffeomorphisms of \( X \) act on configuration spaces. For a mapping \( \varphi : X \to \bar{X} \), let \( \text{supp} \, \varphi \) be the smallest closed set containing all \( x \in X \) such that \( \varphi(x) \neq x \). Let \( \text{Diff}_0(X) \) be the group of diffeomorphisms of \( X \) with compact support. For any \( \varphi \in \text{Diff}_0(X) \), we define the “diagonal” diffeomorphism \( \bar{\varphi} : \bar{X} \to \bar{X} \) acting on each space \( X^n \) \((n = 1, 2, \ldots)\) as follows:

\[
X^n \ni \bar{x} = (x_1, \ldots, x_n) \mapsto \bar{\varphi}(\bar{x}) := (\varphi(x_1), \ldots, \varphi(x_n)) \in X^n.
\]

Obviously, \( \bar{\varphi} \) is a continuous transformation of \( \bar{X} \).

**Remark 4.1.** Note that \( \bar{\varphi} \not\in \text{Diff}_0(\bar{X}) \). Indeed, \( \text{supp} \, \bar{\varphi} = \bar{X}_K \), where \( K := \text{supp} \, \varphi \).

The set \( \bar{X}_K \) is not compact, however \( \sigma^*(\bar{X}_K) < \infty \) (by Proposition 3.4), which is sufficient for our purposes.

The transformations \( \varphi \) and \( \bar{\varphi} \) can be lifted to the “diagonal” transformations (denoted by the same letters) of the configuration spaces \( \Gamma_X \) and \( \bar{\Gamma}_X \), respectively:

\[
\varphi(\gamma) := \{ \varphi(x), \ x \in \gamma \}, \quad \gamma \in \Gamma_X,
\]

\[
\bar{\varphi}(\bar{\gamma}) := \{ \bar{\varphi}(\bar{x}), \ \bar{x} \in \bar{\gamma} \}, \quad \bar{\gamma} \in \bar{\Gamma}_X.
\]

Let \( I : L^2(\Gamma_X, \mu_{cl}) \to L^2(\Gamma_X, \pi_{\sigma^*}) \) be the isometry defined by the projection \( p \), that is,

\[
(IF)(\bar{\gamma}) := F(p(\bar{\gamma})), \quad \bar{\gamma} \in \bar{\Gamma}_X,
\]

and let \( I^* : L^2(\Gamma_X, \pi_{\sigma^*}) \to L^2(\Gamma_X, \mu_{cl}) \) be the adjoint operator. The next assertion shows that the action of \( \text{Diff}_0(X) \) commutes with the operators \( p \) and \( I \).

**Lemma 4.1.** For any \( \varphi \in \text{Diff}_0(X) \), we have

\[
\varphi(p(\bar{\gamma})) = p(\bar{\varphi}(\bar{\gamma})), \quad \bar{\gamma} \in \bar{\Gamma}_X,
\]

and moreover, for any \( F \in L^2(\Gamma_X, \mu_{cl}) \),

\[
I(F \circ \varphi) = (IF) \circ \bar{\varphi}.
\]

**Proof.** The first statement follows from the definition (3.18) of the mapping \( p \) and the diagonal form of \( \bar{\varphi} \) (see (4.1)). The second statement readily follows from (4.3) and the definition (4.2) of the operator \( I \).

Let us now consider the configuration space \( \Gamma_X \) equipped with the Poisson measure \( \pi_{\sigma^*} \), where the intensity measure \( \sigma^* \) is defined in Section 3.1. We shall assume that \( \sigma^* \) is a.c. with respect to the Lebesgue measure on \( \bar{X} \) and, moreover,

\[
s(\bar{x}) := \frac{\sigma^*(d\bar{x})}{d\bar{x}} > 0 \quad \text{for a.a. } \bar{x} \in \bar{X}.
\]
This implies that the measure $\sigma^*$ is quasi-invariant with respect to the action of the diagonal transformation $\bar{\varphi} : \mathcal{X} \to \mathcal{X}$ ($\varphi \in \text{Diff}_0(X)$), and the corresponding Radon–Nikodym derivative is given by

$$\rho_{\sigma^*}^\bar{\varphi}(x) = \frac{s(\bar{\varphi}^{-1}(x))}{s(x)} J_{\bar{\varphi}}(x)^{-1} \quad \text{for \ } \sigma^*\text{-a.a. } x, \quad (4.5)$$

where $J_{\bar{\varphi}}$ is the Jacobian determinant of $\bar{\varphi}$. We set $\rho_{\sigma^*}^\bar{\varphi}(x) = 1$ if $s(x) = 0$ or $s(\bar{\varphi}^{-1}(x)) = 0$.

**Proposition 4.2.** The Poisson measure $\pi_{\sigma^*}$ is quasi-invariant with respect to the action of diagonal diffeomorphisms $\bar{\varphi}$ of $\varGamma_X$ ($\varphi \in \text{Diff}_0(X)$). The Radon–Nikodym density $R_{\pi_{\sigma^*}}^\bar{\varphi}$ is given by

$$R_{\pi_{\sigma^*}}^\bar{\varphi}(\bar{\gamma}) = \exp \left( \int_X (1 - \rho_{\sigma^*}^\bar{\varphi}(x)) \sigma^*(d\bar{x}) \right) \prod_{x \in \bar{\gamma}} \rho_{\sigma^*}^\bar{\varphi}(x), \quad \bar{\gamma} \in \varGamma_X, \quad (4.6)$$

where $\rho_{\sigma^*}^\bar{\varphi}$ is defined in (4.5).

**Proof.** The result follows from Remark 4.1 and Proposition 6.1 in the Appendix below (with $X = \mathcal{X}$, $\nu = \sigma^*$ and $\vartheta = \varphi$).

**Remark 4.2.** The function $R_{\pi_{\sigma^*}}^\bar{\varphi}$ is local in the sense that, for $\pi_{\sigma^*}$-a.a. $\bar{\gamma} \in \varGamma_X$, we have $R_{\pi_{\sigma^*}}^\bar{\varphi}(\bar{\gamma}) = R_{\pi_{\sigma^*}}^\bar{\varphi}(\bar{\gamma} \cap \mathcal{X}_K)$, where $K := \text{supp} \varphi$.

**Remark 4.3** (Explicit form of $R_{\pi_{\sigma^*}}^\bar{\varphi}$). According to (4.5), we have

$$\rho_{\sigma^*}^\bar{\varphi}(\bar{y}) = \frac{\int_X h(\varphi^{-1}(y_1) - x, \ldots, \varphi^{-1}(y_n) - x) \sigma(dx)}{\int_X h(y_1 - x, \ldots, y_n - x) \sigma(dx)} \prod_{i=1}^n J_{\varphi}(y_i)^{-1}, \quad \bar{y} \in X^n,$$

where $J_{\varphi}(\bar{y}) = \det(\partial \varphi_i/\partial y_j)$ is the Jacobian determinant of $\varphi$ (note that $J_{\bar{\varphi}}(\bar{y}) = \prod_{i=1}^n J_{\varphi}(y_i)$ for $\bar{y} = (y_1, \ldots, y_n) \in X^n$). Then $R_{\pi_{\sigma^*}}^\bar{\varphi}(\bar{\gamma})$ can be calculated using formula (4.6). In particular, if the components of the random vector $\bar{y}$ are i.i.d.,

$$h(\bar{y}) = \prod_{i=1}^n h_0(y_i), \quad \bar{y} = (y_1, \ldots, y_n) \in X^n,$$

we have

$$\rho_{\sigma^*}^\bar{\varphi}(\bar{y}) = \frac{\int_X \prod_{i=1}^n J_{\varphi}(y_i)^{-1} h_0(\varphi^{-1}(y_i) - x) \sigma(dx)}{\int_X \prod_{i=1}^n h_0(y_i - x) \sigma(dx)}, \quad \bar{y} = (y_1, \ldots, y_n) \in X^n,$$

and

$$R_{\pi_{\sigma^*}}^\bar{\varphi}(\bar{\gamma}) = C \prod_{\bar{y} \in \bar{\gamma}} \frac{\int_X \prod_{\bar{y} \in \bar{y}} J_{\varphi}(y)^{-1} h_0(\varphi^{-1}(y) - x) \sigma(dx)}{\int_X \prod_{y \in \bar{y}} h_0(y - x) \sigma(dx)}, \quad \bar{\gamma} \in \varGamma_X,$$

where $C := \exp \left( \int_X (1 - \rho_{\sigma^*}^\bar{\varphi}(\bar{y})) \sigma^*(d\bar{y}) \right)$ is a normalizing constant.

Now we can prove the main result of this section.
Theorem 4.3. Under condition (4.4), the Poisson cluster measure $\mu_{cl}$ on $\Gamma_X$ is quasi-invariant with respect to the action of $\text{Diff}_0(X)$ on $\Gamma_X$. The corresponding Radon–Nikodym density is given by $R_{\mu_{cl}} = \mathcal{I}^* R_{\bar{\phi}^* \pi_{\sigma^*}}$.

Proof. By Proposition 4.2, the Poisson measure $\pi_{\sigma^*}$ is quasi-invariant with respect to $\bar{\phi}$, with the density $R_{\bar{\phi}^* \pi_{\sigma^*}}$. According to (4.3), the measure $\varphi^* \mu_{cl}$ is the image of the measure $\bar{\phi}^* \pi_{\sigma^*}$ under the projection $p$. Thus, the absolute continuity of $\bar{\phi}^* \pi_{\sigma^*}$ with respect to $\pi_{\sigma^*}$ implies that $\varphi^* \mu_{cl}$ is absolutely continuous with respect to $\mu_{cl}$. Moreover,

$$\int_{\Gamma_X} F(\gamma) \varphi^* \mu_{cl}(d\gamma) = \int_{\Gamma_X} \mathcal{I} F(\gamma) \bar{\phi}^* \pi_{\sigma^*}(d\bar{\gamma})$$

$$= \int_{\Gamma_X} \mathcal{I} F(\gamma) R_{\pi_{\sigma^*}}(\bar{\phi}; \gamma) \pi_{\sigma^*}(d\bar{\gamma})$$

$$= \int_{\Gamma_X} F(\gamma) (\mathcal{I}^* R_{\pi_{\sigma^*}}(\bar{\phi})) (\gamma) \mu_{cl}(d\gamma),$$

which implies that $R_{\mu_{cl}} = \mathcal{I}^* R_{\bar{\phi}^* \pi_{\sigma^*}}$.

Remark 4.4. The Poisson cluster measure $\mu_{cl}$ on the configuration space $\Gamma_X$ can be used to construct the canonical unitary representation $U$ of the diffeomorphism group $\text{Diff}_0(X)$ by operators in $L^2(\Gamma_X, \mu_{cl})$, given by the formula

$$U_\varphi F(\gamma) = \sqrt{R_{\mu_{cl}}(\gamma)} F(\varphi^{-1}(\gamma)), \quad F \in L^2(\Gamma_X, \mu_{cl}).$$

Such representations, which can be defined for arbitrary quasi-invariant measures on $\Gamma_X$, play a significant role in the representation theory of the diffeomorphism group $\text{Diff}_0(X)$ [20,30] and quantum field theory [18,19]. An important question is whether the representation $U$ is irreducible. According to [30], this is equivalent to the $\text{Diff}_0(X)$-ergodicity of the measure $\mu_{cl}$, which in our case is equivalent to the ergodicity of the measure $\pi_{\sigma^*}$ with respect to the group of transformations $\bar{\phi}$, where $\varphi \in \text{Diff}_0(X)$. The latter is an open question.

4.2 Integration-by-parts formula

The main objective of this section is to establish an integration-by-parts (IBP) formula for the Poisson cluster measure $\mu_{cl}$, in the spirit of the IBP formula for Poisson measures proved in [5]. To this end, we will use the projection operator $p$ and the properties of the auxiliary Poisson measure $\pi_{\sigma^*}$. Since our framework is somewhat different from that in [5], we give a proof of the corresponding IBP formula for $\pi_{\sigma^*}$.

Let us recall the classical IBP formula for a Borel measure $\varpi$ on a Euclidean space $Y = \mathbb{R}^m$ (see, e.g., [13, Chapter 5]). We say that $\varpi$ satisfies an IBP formula if the following identity holds for any $v \in \text{Vect}_0(Y)$ and all $f, g \in C_0^2(Y)$:

$$\int_Y \nabla^v f(y) \, g(y) \varpi(dy) = - \int_Y f(y) \nabla^v g(y) \varpi(dy) - \int_Y \beta^v_{\varpi}(y) f(y) \varpi(dy),$$
where $\nabla^v h(y)$ is the derivative of $h$ along $v$ at point $y \in Y$ and $\beta^v_\varpi \in L^1_{\text{loc}}(Y, \varpi)$ is a measurable function called the logarithmic derivative of $\varpi$ along the vector field $v$. It is easy to see that $\beta^v_\varpi$ can be represented in the form

$$
\beta^v_\varpi(y) = \beta_\varpi(y) \cdot v(y) + \text{div } v(y)
$$

for some mapping $\beta_\varpi : Y \to Y$ called the vector logarithmic derivative of $\varpi$. If the measure $\varpi$ is a.c. with respect to the Lebesgue measure $dy$, with density $w$ such that $w^{1/2} \in H^{1,2}_{\text{loc}}(Y)$ ( := the local $(1, 2)$-Sobolev space), then $\beta_\varpi$ is given by the formula $\beta_\varpi(y) = w(y)^{-1} \nabla w(y)$ (note that $w(y) \neq 0$ for $\varpi$-a.a. $y \in Y$).

In what follows, we always assume that the density $s(x) = \sigma^*(d\bar{x})/d\bar{x}$ satisfies the condition $s^{1/2} \in H^{1,2}_{\text{loc}}(\mathcal{X})$; equivalently, $s^{1/2} \in H^{1,2}_{\text{loc}}(X^n)$ for each $n \in \mathbb{N}$. This condition ensures that for each measure $\sigma^*_n$, the IBP formula holds with the vector logarithmic derivative $\beta_{\sigma^*_n}(\bar{y}) = (\beta_i(\bar{y}))_{i=1}^n$, where

$$
\beta_i(\bar{y}) := \frac{\nabla_i s_n(\bar{y})}{s_n(\bar{y})} = \frac{\int_{Y} \nabla_i h_n(y_1 - x, \ldots, y_n - x) \sigma(d\bar{x})}{\int_{Y} h_n(y_1 - x, \ldots, y_n - x) \sigma(d\bar{x})} \quad (4.7)
$$

if $s_n(\bar{y}) \neq 0$ and $\beta_i(\bar{y}) := 0$ if $s_n(\bar{y}) = 0$.

For any $v \in \text{Vect}_0(X)$, let us define the vector field $\bar{v} : \mathcal{X} \to \mathcal{X}$ by setting

$$
\bar{v}(\bar{x}) := (v(x_i))_{i=1}^n, \quad \bar{x} = (x_1, \ldots, x_n) \in X^n \quad (n \in \mathbb{N}).
$$

Then the logarithmic derivative $\beta_{\sigma^*_n}$ of the measure $\sigma^*_n$ along the vector field $\bar{v}$ is given by

$$
\beta_{\sigma^*_n}(\bar{y}) = \sum_{i=1}^n [\beta_i(\bar{y}) \cdot v(y_i) + \text{div } v(y_i)]. \quad (4.8)
$$

**Proposition 4.4.** The measure $\sigma^*$ satisfies the following IBP formula:

$$
\int_{X} \nabla^\bar{v} f(\bar{x}) g(\bar{x}) \sigma^*(d\bar{x}) = -\int_{X} f(\bar{x}) \nabla^\bar{v} g(\bar{x}) \sigma^*(d\bar{x}) - \int_{X} \beta^\bar{v}_{\sigma^*} (\bar{x}) f(\bar{x}) \sigma^*(d\bar{x}),
$$

where $f, g \in C^2_0(\mathcal{X})$ and $\beta^\bar{v}_{\sigma^*} (\bar{x}) = \beta^\bar{v}_{\sigma^*_n} (\bar{x})$ if $\bar{x} \in X^n$.

**Proof.** The result easily follows from the definition of the measure $\sigma^*$ and the IBP formula for each $\sigma^*_n (n \in \mathbb{N})$. \qed

**Remark 4.5.** Formula (4.9) can be rewritten in the form

$$
\int_{X} \sum_{x \in \mathcal{P}(\bar{x})} \nabla_x f(\bar{x}) g(\bar{x}) \cdot v(x) \sigma^*(d\bar{x}) = -\int_{X} f(\bar{x}) \sum_{x \in \mathcal{P}(\bar{x})} \nabla_x g(\bar{x}) \cdot v(x) \sigma^*(d\bar{x}) - \int_{X} \beta^\bar{v}_{\sigma^*} (\bar{x}) f(\bar{x}) \sigma^*(d\bar{x}).
$$

In what follows, we will use some convenient “manifold-like” notations introduced in [5] (see the Appendix, Section 6.2).
Theorem 4.5. For each \( v \in \text{Vect}_0(X) \) and any \( F, G \in \mathcal{F}C(\Gamma_X) \), the following IBP formula holds:

\[
\int_{\Gamma_X} \nabla_v^F F(\gamma) G(\gamma) \mu_{cl}(d\gamma) = - \int_{\Gamma_X} F(\gamma) \nabla_v^F G(\gamma) \mu_{cl}(d\gamma) - \int_{\Gamma_X} F(\gamma) G(\gamma) B_{\mu_{cl}}^v(\gamma) \mu_{cl}(d\gamma),
\]

where \( B_{\mu_{cl}}^v(\gamma) := \mathcal{I}^* (\beta_{\mu_{cl}}^v, \bar{\gamma}) \) and \( \beta_{\mu_{cl}}^v \) is the logarithmic derivative of \( \sigma^* \) along the vector field \( \bar{v} \).

Proof. Denote

\[
\Phi(\bar{\gamma}) := \nabla_v^F F(\gamma) G(\gamma) = \left( \sum_{x \in \gamma} \nabla_x F(\gamma) \cdot v(x) \right) G(\gamma).
\]

Then

\[
\mathcal{I} \Phi(\bar{\gamma}) = \left( \sum_{x \in p(\bar{\gamma})} \nabla_x F(p(\bar{\gamma})) \cdot v(x) \right) \mathcal{I} G(\bar{\gamma}).
\]

Note that \( \mathcal{I} \Phi \in \mathcal{F}C_{\sigma^*}(\Gamma_X) \), so we can use (2.5) in order to integrate \( \mathcal{I} \Phi \) with respect to \( \pi_{\sigma^*} \). Applying formula (4.9), we have

\[
\int_{\Gamma_X} \nabla_v^F F(\gamma) G(\gamma) \mu_{cl}(d\gamma) = \int_{\Gamma_X} \sum_{x \in p(\bar{\gamma})} \left( \nabla_x F(p(\bar{\gamma})) \cdot v(x) \right) \mathcal{I} G(\bar{\gamma}) \pi_{\sigma^*}(d\bar{\gamma})
\]

\[
eq e^{-\sigma^*(x_K)} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(x_K)^m} \sum_{i=1}^{m} \nabla_x F(\{p(x_1), \ldots, p(x_m)\}) \cdot v(x) \times G(\{p(x_1), \ldots, p(x_m)\}) \bigotimes_{i=1}^{m} \sigma^*(d\bar{x}_i)
\]

\[
= e^{-\sigma^*(x_K)} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i=1}^{m} \left( \int_{x_K} \sum_{x \in p(\bar{x}_i)} \nabla_x F(\{p(x_1), \ldots, p(x_m)\}) \cdot v(x) \times G(\{p(x_1), \ldots, p(x_m)\}) \sigma^*(d\bar{x}_i) \bigotimes_{j \neq i} \sigma^*(d\bar{x}_j) \right)
\]

Using the IBP formula for \( \sigma^* \), the inner integral in (4.11) can be rewritten as

\[
-\int_{x_K} F(\{p(x_1), \ldots, p(x_m)\}) \left( \sum_{x \in p(\bar{x}_i)} \nabla_x G(\{p(x_1), \ldots, p(x_m)\}) \cdot v(x) \right.
\]

\[
+ G(\{p(x_1), \ldots, p(x_m)\}) \beta_{\mu_{cl}}(\bar{x}_i) \bigotimes_{j \neq i} \sigma^*(d\bar{x}_j) \right).
\]

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Remark 4.6. The logarithmic derivative \( B_{\pi^*, \bar{x}}^\sigma(\bar{\gamma}) \) can be rewritten in the form (cf. (4.7))

\[
B_{\pi^*, \bar{x}}^\sigma(\bar{\gamma}) = \sum_{\bar{y} \in \bar{\gamma}} \left[ \sum_k \alpha_k(\bar{y}) \cdot v(y_k) + \text{div} v(y_k) \right]
\]

Hence, the right-hand side of (4.11) is reduced to

\[
- e^{-\sigma^*(x_K)} \sum_{m=0}^\infty \frac{1}{m!} \int_{(x_K)^m} F(\{p(\bar{x}_1), \ldots, p(\bar{x}_m)\}) \times \left( \sum_{i=1}^m \nabla_x G(\{p(\bar{x}_1), \ldots, p(\bar{x}_m)\}) \cdot v(x) \right)
\]

\[
+ G(\{p(\bar{x}_1), \ldots, p(\bar{x}_m)\}) \sum_{i=1}^m \beta_{\alpha^*}(\bar{x}_i) \bigotimes_{i=1}^m \sigma^*(d\bar{x}_i)
\]

\[
= - \int_{\Gamma_X} F(p(\bar{\gamma})) \left( \sum_{x \in p(\bar{\gamma})} \nabla_x G(p(\bar{\gamma})) \cdot v(x) + G(p(\bar{\gamma})) \right) B_{\pi^*, \bar{x}}^\sigma(\bar{\gamma}) \pi(\text{d}\bar{\gamma})
\]

\[
= - \int_{\Gamma_X} F(\gamma) \nabla_{\bar{y}} G(\gamma) \mu(\text{d}\gamma) - \int_{\Gamma_X} F(\gamma) G(\gamma) B_{\mu(\gamma), \bar{x}}^\sigma(\gamma) \mu(\text{d}\gamma),
\]

where

\[
B_{\pi^*, \bar{x}}^\sigma(\bar{\gamma}) := \sum_{\bar{x} \in \bar{\gamma}} \beta_{\alpha^*}(\bar{x})
\]

and

\[
B_{\mu(\gamma), \bar{x}}^\sigma(\gamma) := \mathcal{T}^\sigma B_{\pi^*, \bar{x}}^\sigma.
\]

Note that \( B_{\pi^*, \bar{x}}^\sigma \) is well defined since \( \sigma^*(\text{supp } \bar{v}) < \infty \), and thus there are only finitely many non-zero terms in the sum (4.12). Moreover, the finiteness of the first and second moments of \( \pi_{\alpha^*} \) implies that \( B_{\pi^*, \bar{x}}^\sigma \in L^2(\Gamma_X, \pi_{\alpha^*}) \). \( \square \)

**Remark 4.6.** The logarithmic derivative \( B_{\pi^*, \bar{x}}^\sigma(\bar{\gamma}) \) can be rewritten in the form (cf. (4.7))

\[
B_{\pi^*, \bar{x}}^\sigma(\bar{\gamma}) = \sum_{\bar{y} \in \bar{\gamma}} \left[ \sum_k \alpha_k(\bar{y}) \cdot v(y_k) + \text{div} v(y_k) \right]
\]

\[
= \sum_{\bar{y} \in \bar{\gamma}} \left[ \alpha(\bar{y}) \cdot \bar{v}(\bar{y}) + \text{div} \bar{v}(\bar{y}) \right].
\]

Formula (4.10) can be extended to more general vector fields on \( \Gamma_X \). For any \( V \in \mathcal{FV}(\Gamma_X) \) of the form (6.4) (see the Appendix, Section 6.2), we set

\[
B_{\mu(\gamma), \bar{x}}^V(\gamma) := \sum_{j=1}^N \left( G_j(\gamma)B_{\mu(\gamma), \bar{x}}^\sigma(\gamma) + \sum_{x \in \gamma} \nabla_x G_j(\gamma) \cdot v_j(x) \right).
\]

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Theorem 4.6. For arbitrary $F, G \in \mathcal{FC}(\Gamma_X)$ and $V$ as above, we have

$$\int_{\Gamma_X} \nabla^V_F F(\gamma) G(\gamma) \mu_{cl}(d\gamma)$$

$$= - \int_{\Gamma_X} F(\gamma) \nabla^V G(\gamma) \mu_{cl}(d\gamma) - \int_{\Gamma_X} F(\gamma) G(\gamma) B^V_{\mu_{cl}}(\gamma) \mu_{cl}(d\gamma).$$

(4.13)

Proof. The result readily follows from Theorem 4.5 and linearity of the right-hand side of (4.8) with respect to $v$. 

Remark 4.7. The logarithmic derivative $B^V_{\mu_{cl}}$ can be interpreted as

$$B^V_{\mu_{cl}} = I^* B^IV_{\pi^*},$$

where $B^IV_{\pi^*}$ is the logarithmic derivative of $\pi_{\sigma^*}$ along the vector field $IV(\bar{\gamma}) := V(p(\bar{\gamma})).$ Note that the equality

$$T_{\bar{\gamma}} \Gamma_X = \bigoplus_{x \in \bar{\gamma}} T_x X = \bigoplus_{x \in \bar{\gamma}} \bigoplus_{x \in \bar{x}} T_x X = \bigoplus_{x \in p(\bar{\gamma})} T_x X = T_{p(\bar{\gamma})} \Gamma_X$$

implies that $V(p(\bar{\gamma})) \in T_{\bar{\gamma}} \Gamma_X,$ and thus $IV(\bar{\gamma})$ is a vector field on $\Gamma_X.$

5 Dirichlet forms and equilibrium stochastic dynamics

Throughout this section, we assume that the measure $\sigma^*$ satisfies the conditions of Section 4.2.

5.1 The Dirichlet form associated with $\mu_{cl}$

Let us introduce the pre-Dirichlet form $\mathcal{E}_{\mu_{cl}}$ on $\mathcal{FC}(\Gamma_X)$ associated with the Poisson cluster measure $\mu_{cl},$

$$\mathcal{E}_{\mu_{cl}}(F, G) := \int_{\Gamma_X} \langle \nabla^F F(\gamma), \nabla^G G(\gamma) \rangle \gamma \mu_{cl}(d\gamma).$$

The next proposition shows that the form $\mathcal{E}_{\mu_{cl}}$ is well defined.

Proposition 5.1. We have $\mathcal{E}_{\mu_{cl}}(F, G) < \infty$ for any $F, G \in \mathcal{FC}(\Gamma_X).$

Proof. The statement follows from the existence of the first moments of $\mu_{cl}.$ Indeed, a direct calculation shows that

$$\langle \nabla^F F(\gamma), \nabla^G G(\gamma) \rangle \gamma = \sum_{x \in \gamma} \nabla_x F(\gamma) \cdot \nabla_x G(\gamma) = \sum_{i,j} \Phi_{ij}(\gamma) \langle \varphi_{ij}, \gamma \rangle,$$

where

$$\Phi_{ij}(\gamma) := \nabla_i g_F(\langle f^F_{i1}, \gamma \rangle, \ldots, \langle f^F_{in}, \gamma \rangle) \nabla_j g_G(\langle f^G_{j1}, \gamma \rangle, \ldots, \langle f^G_{jm}, \gamma \rangle).$$
and \( \varphi_{ij}(x) := \nabla f_i^F(x) \cdot \nabla f_j^G(x) \). Obviously, \( \Phi_{ij} \in \mathcal{FC}(\Gamma_X) \) (hence, \( \Phi_{ij} \) is a bounded measurable function on \( \Gamma_X \)) and \( \varphi_{ij} \in C_0(\mathcal{X}) \). Denoting for brevity \( \varphi = \varphi_{ij} \) and setting \( \bar{\varphi}(x) := \sum_{x \in \mathcal{X}} \varphi(x) \), we have

\[
\int_{\Gamma_X} \langle \varphi, \gamma \rangle \mu_{\mathcal{X},\mu}(d\gamma) = \int_{\Gamma_X} \langle \varphi, p(\gamma) \rangle \mu_{\sigma^*}(d\gamma) = \int_{\Gamma_X} \langle \tilde{\varphi}, \tilde{\gamma} \rangle \mu_{\sigma^*}(d\tilde{\gamma}) = \int_{\mathcal{X}} \tilde{\varphi}(\tilde{y}) \sigma^*(d\tilde{y}) < \infty,
\]

because \( \text{supp} \bar{\varphi} = \mathcal{X}_K \), where \( K = \text{supp} \varphi \), and, by Proposition 3.4, \( \sigma^*(\mathcal{X}_K) < \infty \). Therefore, \( \langle \varphi, \gamma \rangle \in L^1(\Gamma_X, \mu_{\mathcal{X},\mu}) \) and the required result follows.

Let us also consider the pre-Dirichlet form \( \mathcal{E}_{\pi_{\sigma^*}} \) associated with \( \pi_{\sigma^*} \), defined on the space \( \mathcal{FC}(\Gamma_X) \subset L^2(\Gamma_X, \pi_{\sigma^*}) \) by

\[
\mathcal{E}_{\pi_{\sigma^*}}(f, g) := \int_{\Gamma_X} \langle \nabla^\Gamma f(\gamma), \nabla^\Gamma g(\gamma) \rangle \pi_{\sigma^*}(d\gamma).
\]

Pre-Dirichlet forms of such type associated with general Poisson measures were introduced and studied in [5]. The finiteness of the first moments of the Poisson measure \( \pi_{\sigma^*} \) implies that \( \mathcal{E}_{\pi_{\sigma^*}} \) is well defined. It follows from the IBP formula for \( \pi_{\sigma^*} \) that, for any \( f, g \in \mathcal{FC}(\Gamma_X) \),

\[
\mathcal{E}_{\pi_{\sigma^*}}(f, g) = \int_{\Gamma_X} H_{\pi_{\sigma^*}} f(\gamma) g(\gamma) \pi_{\sigma^*}(d\gamma). \tag{5.1}
\]

Here \( H_{\pi_{\sigma^*}} \) is the Dirichlet operator of the Poisson measure \( \pi_{\sigma^*} \) (see [5]),

\[
H_{\pi_{\sigma^*}} f(\gamma) := \sum_{x \in \gamma} \left[ \Delta_x f(\gamma) + (\beta_{\sigma^*}(x), \nabla_x f(\gamma))_x \right], \tag{5.2}
\]

where we use a short-hand notation \( (\cdot, \cdot)_x := (\cdot, \cdot)_{\mathcal{X}^n} \) when \( x \in \mathcal{X}^n \).

\textbf{Remark 5.1.} Note that the operator \( H_{\pi_{\sigma^*}} \) is well defined on a bigger set \( \mathcal{FC}_{\sigma^*}(\Gamma_X) \). Indeed, for any \( f \in \mathcal{FC}_{\sigma^*}(\Gamma_X) \) and \( \sigma^* \)-a.a configurations \( \gamma \), we have \( \gamma(\text{supp} f) < \infty \) because \( \sigma^*(\text{supp} f) < \infty \), which implies that there are only finitely many non-zero terms on the right-hand side of (5.2). Similar arguments show that the pre-Dirichlet form \( \mathcal{E}_{\pi_{\sigma^*}}(f, g) \) is well defined on \( \mathcal{FC}_{\sigma^*}(\Gamma_X) \), and formula (5.1) holds for any \( f, g \in \mathcal{FC}_{\sigma^*}(\Gamma_X) \).

\textbf{Theorem 5.2.} For \( F, G \in \mathcal{FC}(\Gamma_X) \),

\[
\mathcal{E}_{\mu_{\mathcal{X}}} (F, G) = \int_{\Gamma_X} H_{\mu_{\mathcal{X}}} F(\gamma) G(\gamma) \mu_{\mathcal{X},\mu}(d\gamma), \tag{5.3}
\]

where \( H_{\mu_{\mathcal{X}}} := I^* H_{\pi_{\sigma^*}} I \).

\textbf{Proof.} Let us fix \( F, G \in \mathcal{FC}(\Gamma_X) \) and set \( \Phi(\gamma) := \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \rangle_\gamma \). From the
definition of the operator $\mathcal{I}$, it readily follows that
\[
\mathcal{I} \Phi(\bar{\gamma}) = \sum_{\bar{x} \in \Phi(\bar{\gamma})} \nabla_{\bar{x}} \mathcal{I} F(\bar{\gamma}) \cdot \nabla_{\bar{x}} \mathcal{I} G(\bar{\gamma}),
\]
where $\nabla_{\bar{x}} = \nabla_{x_1} + \cdots + \nabla_{x_n}$ when $\bar{x} = (x_1, \ldots, x_n) \in X^n$. Thus,
\[
\mathcal{E}_{\mu_{cl}}(F, G) = \int_{\Gamma_X} \Phi(\gamma) \mu_{cl}(d\gamma) = \int_{\Gamma_X} \mathcal{I} \Phi(\bar{\gamma}) \pi_{\sigma^*}(d\bar{\gamma})
\]
\[
= \int_{\Gamma_X} \sum_{\bar{x} \in \bar{\gamma}} \nabla_{\bar{x}} \mathcal{I} F(\bar{\gamma}) \cdot \nabla_{\bar{x}} \mathcal{I} G(\bar{\gamma}) \pi_{\sigma^*}(d\bar{\gamma}) = \mathcal{E}_{\pi_{\sigma^*}}(\mathcal{I} F, \mathcal{I} G)
\]  
(5.4)

(note that $\mathcal{I} F, \mathcal{I} G \in \mathcal{FC}_{\pi^*}(\Gamma_X) \subset D(\mathcal{E}_{\pi_{\sigma^*}})$). Finally, combining (5.4) with formula (5.1) we get (5.3). □

**Remark 5.2.** For any $F \in \mathcal{FC}(\Gamma_X)$
\[
H_{\mu_{cl}} F(\gamma) = \mathcal{I}^* H_{\pi_{\sigma^*}} \mathcal{I} F(\gamma)
\]
\[
= \sum_{x \in \gamma} \Delta_x F(\gamma) + \mathcal{I}^* \sum_{\bar{x} \in \gamma} \beta_{\sigma^*}(\bar{x}) \cdot \nabla_{\bar{x}} \mathcal{I} F(\bar{\gamma}).
\]  
(5.5)

**Remark 5.3.** Formulas (5.3) and (5.5) can also be obtained directly from the IBP formula (4.13).

### 5.2 The associated equilibrium stochastic dynamics

Formula (5.3) implies that the form $\mathcal{E}_{\mu_{cl}}$ is closable on $L^2(\Gamma_X, \mu_{cl})$, and we preserve the same notation for its closure. Its domain $D(\mathcal{E}_{\mu_{cl}})$ is obtained as a completion of $\mathcal{FC}(\Gamma_X)$ with respect to the norm
\[
\|F\|_{D(\mathcal{E}_{\mu_{cl}})} := \left( \mathcal{E}_{\mu_{cl}}(F, F) + \int_{\Gamma_X} F^2 \, d\mu_{cl} \right)^{1/2}.
\]

According to a general result in [26, Section 4], $\mathcal{E}_{\mu_{cl}}$ is a quasi-regular local Dirichlet form on the bigger state space $\bar{\Gamma}_X$ consisting of all $\mathbb{Z}_+^n$-valued Radon measures on $X$. Then, by the general theory of Dirichlet forms (see, e.g., [25]), we obtain the following result.

**Theorem 5.3.** There exists a conservative diffusion process $X = (X_t, t \geq 0)$ on $\bar{\Gamma}_X$, properly associated with the Dirichlet form $\mathcal{E}_{\mu_{cl}}$; that is, for any function $F \in L^2(\bar{\Gamma}_X, \mu_{cl})$ and all $t \geq 0$, the mapping
\[
\bar{\Gamma}_X \ni \gamma \mapsto p_tF(\gamma) := \int_{\Omega} F(X_t) \, dP_\gamma
\]
is an $\mathcal{E}_{\mu_{cl}}$-quasi-continuous version of $\exp(t H_{\mu_{cl}}) F$. Here $\Omega$ is the canonical sample space (of all $\bar{\Gamma}_X$-valued continuous functions on $\mathbb{R}_+$) and $(P_\gamma, \gamma \in \bar{\Gamma}_X)$ is
the family of probability distributions of the process \(X\) conditioned on the initial value \(\gamma = X_0\). The process \(X\) is unique up to \(\mu_{\text{cl}}\)-equivalence. In particular, \(X\) is \(\mu_{\text{cl}}\)-symmetric (i.e., \(\int G \, p_t \, d\mu_{\text{cl}} = \int F \, p_t \, G \, d\mu_{\text{cl}}\) for all measurable functions \(F, G : \tilde{\Gamma}_X \to \mathbb{R}_+\)) and \(\mu_{\text{cl}}\) is its invariant measure.

**Remark 5.4.** Formula (5.1) implies that the “pre-projection” form \(E_{\pi_*}\) is closable. According to the general theory of Dirichlet forms \([25,26]\), its closure is a quasi-regular local Dirichlet form on \(\tilde{\Gamma}_X\) and as such generates a diffusion process \(\bar{X}\) on \(\tilde{\Gamma}_X\). This process coincides with the independent infinite particle process, which amounts to independent distorted Brownian motions in \(X\) with drift given by the vector logarithmic derivative of \(\sigma\) (see [5]). However, it is not clear in what sense the process \(X\) constructed in Theorem 5.3 can be obtained directly via the projection of \(\bar{X}\) from \(\tilde{\Gamma}_X\) onto \(\Gamma_X\).

### 5.3 Irreducibility of the Dirichlet form \(E_{\mu_{\text{cl}}}\)

Let us recall that a Dirichlet form \(E\) is called *irreducible* if the condition \(E(F, F) = 0\) implies that \(F = \text{const}\).

**Theorem 5.4.** The Dirichlet form \((E_{\mu_{\text{cl}}}, D(E_{\mu_{\text{cl}}}))\) is irreducible.

**Proof.** For any \(F \in FC(\Gamma_X)\) we have
\[
\|F\|^2_{D(E_{\mu_{\text{cl}}})} = E_{\mu_{\text{cl}}}(F, F) + \int_{\Gamma_X} F^2 \, d\mu_{\text{cl}}
= E_{\pi_*}((I_\Gamma F, I_\Gamma F) + \int_{\Gamma_X} (I_\Gamma F)^2 \, d\pi_* = \|I_\Gamma F\|^2_{D(E_{\pi_*})},
\]
which implies that \(I_\Gamma D(E_{\mu_{\text{cl}}}) \subset D(E_{\pi_*})\). It is obvious that if \(I_\Gamma F = \text{const (}\pi_*\text{-a.s.)} then F = \text{const (}\mu_{\text{cl}}\text{-a.s.)}. Therefore, according to formula (5.4), it suffices to prove that the Dirichlet form \((E_{\pi_*}, D(E_{\pi_*}))\) is irreducible, which is established in Lemma 5.6 below.

We first need the following general result (see \([3, \text{Lemma 3.3}]\)).

**Lemma 5.5.** Let \(A\) and \(B\) be self-adjoint, non-negative operators in separable Hilbert spaces \(\mathcal{H}\) and \(\mathcal{K}\), respectively. Then
\[
\text{Ker}(A \boxtimes B) = \text{Ker} A \otimes \text{Ker} B,
\]
where \(A \boxplus B\) is the closure of the operator \(A \otimes I + I \otimes B\) from the algebraic tensor product of the domains of \(A\) and \(B\).

**Proof.** \(\text{Ker} A\) and \(\text{Ker} B\) are closed subspaces of \(\mathcal{H}\) and \(\mathcal{K}\), respectively, and so their tensor product \(\text{Ker} A \otimes \text{Ker} B\) is a closed subspace of the space \(\mathcal{H} \otimes \mathcal{K}\). The inclusion \(\text{Ker} A \otimes \text{Ker} B \subset \text{Ker}(A \boxplus B)\) is trivial. Let \(f \in \text{Ker}(A \boxplus B)\). Using the theory of operators admitting separation of variables (see, e.g., [8, Chapter 6]), we

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have

\[ 0 = (A \boxplus B f, f) = \int_{\mathbb{R}_+^2} (x_1 + x_2) d(E(x_1, x_2)f, f) \]

\[ = \int_{\mathbb{R}_+^2} x_1 d(E(x_1, x_2)f, f) + \int_{\mathbb{R}_+^2} x_2 d(E(x_1, x_2)f, f) \]

\[ = (A \otimes I f, f) + (I \otimes B f, f), \quad (5.6) \]

where \( E \) is the joint resolution of the identity of the commuting operators \( A \otimes I \) and \( I \otimes B \). Since both operators \( A \otimes I \) and \( I \otimes B \) are non-negative, we conclude from (5.6) that

\[ f \in \text{Ker}(A \otimes I) \cap \text{Ker}(I \otimes B) = \text{Ker}A \otimes \text{Ker}B, \]

which completes the proof of the lemma. \( \square \)

**Lemma 5.6.** The Dirichlet form \((\mathcal{E}_{\pi_*}, D(\mathcal{E}_{\pi_*}))\) is irreducible.

**Proof.** The irreducibility of Dirichlet forms associated with Poisson measures on configuration spaces of connected Riemannian manifolds was shown in [5]. However, the space \( \mathcal{X} \) consists of countably many disjoint connected components \( X^n \), so we need to adapt the result of [5] to this situation.

Let us recall that, according to the general theory (see, e.g., [4]), irreducibility of a Dirichlet form is equivalent to the fact that the kernel of its generator consists of constants (uniqueness of the ground state). Thus, it suffices to prove that \( \text{Ker} H_{\pi_*} = \{\text{const}\} \).

Let us set

\[ \tilde{X}_n := \bigcup_{k=n+1}^{\infty} X^k, \quad n \in \mathbb{N}, \]

so that \( \mathcal{X} = \{\emptyset\} \sqcup X^1 \sqcup \cdots \sqcup X^n \sqcup \tilde{X}_n \). The space \( \tilde{X}_n \) is endowed with the measure

\[ \tilde{\sigma}_n^* := \sum_{k=n+1}^{\infty} p_k \sigma_k^*. \]

In terms of configuration spaces, we have

\[ \Gamma_{\mathcal{X}} = \Gamma_{\{\emptyset\}} \times \Gamma_X \times \cdots \times \Gamma_X^n \times \Gamma_{\tilde{X}_n} \]

and, because of infinite divisibility of Poisson measures,

\[ \pi_{\sigma^*} = \delta_{\emptyset} \otimes \pi_1 \otimes \cdots \otimes \pi_n \otimes \tilde{\pi}_n, \]

where we use a short-hand notation \( \pi_n := \pi_n \sigma_n^*, \tilde{\pi}_n := \pi_n \sigma_n^* \). Therefore, there is the isomorphism of Hilbert spaces

\[ L^2(\Gamma_{\mathcal{X}}, \pi_{\sigma^*}) \cong L^2(\Gamma_X, \pi_1) \otimes \cdots \otimes L^2(\Gamma_X^n, \pi_n) \otimes L^2(\Gamma_{\tilde{X}_n}, \tilde{\pi}_n). \]
Consequently, the Dirichlet operator $H_{\pi_{\sigma^*}}$ can be decomposed as

$$H_{\pi_{\sigma^*}} = H_{\pi_1} \boxplus \cdots \boxplus H_{\pi_n} \boxplus H_{\tilde{\pi}_n}. \quad (5.7)$$

Since all operators on the right-hand side of (5.7) are self-adjoint and non-negative, it follows by Lemma 5.5 that

$$\text{Ker } H_{\pi_{\sigma^*}} = \text{Ker } H_{\pi_1} \otimes \cdots \otimes \text{Ker } H_{\pi_n} \otimes \text{Ker } H_{\tilde{\pi}_n}. \quad (5.8)$$

The Dirichlet forms of all measures $\pi_k$ are irreducible (as Dirichlet forms of Poisson measures on connected manifolds), hence $\text{Ker } H_{\pi_k} = \mathbb{R}$ and (5.8) implies

$$\text{Ker } H_{\pi_{\sigma^*}} = \text{Ker } H_{\tilde{\pi}_n}.$$  

Since $n$ is arbitrary, it follows that every function $f \in \text{Ker } H_{\pi_{\sigma^*}}$ does not depend on any finite number of variables, and thus $f = \text{const } (\pi_{\sigma^*} \text{-a.s.})$. □

**Remark 5.5.** The result of Lemma 5.6 (and the idea of its proof) can be viewed as a functional-analytic analogue of Kolmogorov’s zero–one law (see, e.g., [21, Chapter 3]), stating that for a sequence of independent random variables $(X_n)$, the corresponding tail sigma-algebra $\mathcal{F}_\infty := \cap_n \sigma(X_m, m \geq n)$ is trivial (in particular, all $\mathcal{F}_\infty$-measurable random variables are a.s.-constants).

**Remark 5.6.** According to the general theory of Dirichlet forms (see, e.g., [4]), the irreducibility of $\mathcal{E}_{\mu_{\text{cl}}}$ implies the following results:

1. the semigroup $e^{-tH_{\mu_{\text{cl}}}}$ is $L^2$-ergodic, that is, as $t \to \infty$,
   $$\int_{\Gamma_X} \left( e^{-tH_{\mu_{\text{cl}}}} F(\gamma) - \int_{\Gamma_X} F(\gamma) \mu_{\text{cl}}(d\gamma) \right)^2 \mu_{\text{cl}}(d\gamma) \to 0;$$

2. if $F \in D(H_{\mu_{\text{cl}}})$ and $H_{\mu_{\text{cl}}}F = 0$ then $F = \text{const}$.

6 Appendix

6.1 Proof of Theorem 2.4

Note that the droplet cluster $D_B(\gamma_0') = \bigcup_{y \in \gamma_0}(B - y)$ (see (2.11)) can be decomposed into disjoint components according to the number of constituent “layers” (including infinitely many):

$$D_B(\gamma_0') = \bigcup_{1 \leq \ell \leq \infty} D^\ell_B(\gamma_0'),$$

where

$$D^\ell_B(\gamma_0') := \{ x \in X : \gamma_0'(B - x) = \ell \}, \quad \ell = 1, 2, \ldots, \infty.$$
(a) First of all, it is clear that condition (2.12) is necessary in order that the overall configuration $\gamma$ be a.s. locally finite. On the other hand, (2.12) implies that $D_K^\infty(\gamma_0') = \emptyset$ ($\mu_0$-a.s.) for any compact $K \in \mathcal{B}(X)$. Indeed, if $x \in D_K^\infty(\gamma_0')$ then infinitely many $y \in \gamma_0'$ belong to the compact set $K - x$, which is only possible with $\mu_0$-probability zero.

Furthermore, for the Laplace functional of the measure $\mu_{c1}$, applied to the function $f_q(x) := \ln q \cdot 1_K(x)$ ($0 < q < 1$), we obtain

$$L_{\mu_{c1}}[f_q] = \int_{F_X^+} q^{\ell(K)} \mu_{c1}(d\gamma) = \sum_{n=0}^\infty q^n \mu_{c1}\{\gamma : \gamma(K) = n\} \rightarrow \mu_{c1}\{\gamma : \gamma(K) < \infty\} \quad (q \uparrow 1).$$

Hence, the local finiteness of $\pi$-a.s. $\gamma$ is equivalent to $L_{\pi_a}[f_q] \rightarrow 1$ as $q \uparrow 1$. According to (2.10) and using that $D_K^\infty(\gamma_0') = \emptyset$ ($\mu_0$-a.s.), we have

$$-\ln L_{\mu_{c1}}[f_q] = \int_{F_X^+} \left( \int_{F_X^+} \left(1 - q^{\gamma_0'(K-x)}\right) \mu_0(d\gamma_0') \right) \sigma(dx)$$

$$= \int_{F_X^+} \left( \int_{F_X^+} \sum_{\ell=1}^\infty \left(1 - q^\ell\right) 1_{D_0^\ell(\gamma_0')(x)} \sigma(dx) \right) \mu_0(d\gamma_0')$$

$$= \int_{F_X^+} \sum_{\ell=1}^\infty \left(1 - q^\ell\right) \sigma(D_0^\ell(\gamma_0')). \quad (6.1)$$

Note that for $0 < q < 1$,

$$0 \leq \sum_{\ell=1}^\infty \left(1 - q^\ell\right) \sigma(D_0^\ell(\gamma_0')) \leq \sum_{\ell=1}^\infty \sigma(D_0^\ell(\gamma_0')) = \sigma(D_0(\gamma_0')),$$

so if the condition (2.13) is satisfied then we can apply Lebesgue’s dominated convergence theorem and pass termwise to the limit on the right-hand side of (6.1) as $q \uparrow 1$, which gives $\lim_{q \uparrow 1} \ln L_{\mu_{c1}}[f_q] = 0$, that is, $\lim_{q \uparrow 1} L_{\mu_{c1}}[f_q] = 1$, as required.

Conversely, since

$$\sum_{\ell=1}^\infty \left(1 - q^\ell\right) \sigma(D_0^\ell(\gamma_0')) \geq (1 - q) \sum_{\ell=1}^\infty \sigma(D_0^\ell(\gamma_0'))$$

$$= (1 - q) \sigma(D_0(\gamma_0')) \geq 0,$$

from (6.1) we must have

$$(1 - q) \int_{F_X^+} \sigma(D_0(\gamma_0')) \mu_0(d\gamma_0') \rightarrow 0 \quad (q \uparrow 1),$$

which implies (2.13).

(b) Let us prove the “only if” part. Clearly, the condition (2.14) is necessary in order to avoid any in-cluster ties. Furthermore, each fixed $x_0 \in X$ cannot belong to
more than one cluster; in particular, for any \(2 \leq \ell \leq \infty\),

\[
\sigma(D_{\{x_0\}}(\gamma')_0) = 0 \quad \mu_0\text{-a.s.} \tag{6.2}
\]

Let us now take \(K = \{x_0\}\) and consider \(f_q(x) := \ln q \cdot 1_{\{x_0\}}(x) (0 < q < 1)\). Then, as explained in the proof of Proposition 2.1(b), the Laplace transform \(L_{\mu} [f_q]\) must be a linear function of \(q\). But from (6.1) and (6.2) we have

\[
L_{\mu} [f_q] = \exp\left(-(1 - q) \int_X \sigma(D_{\{x_0\}}(\gamma')_0) \mu_0(d\gamma'_0)\right),
\]

and it follows that \(\sigma(D_{\{x_0\}}(\gamma')_0) = 0 (\mu_0\text{-a.s.})\). Together with (6.2), this gives

\[
\sigma(D_{\{x_0\}}(\gamma')_0) = \sum_{1 \leq \ell \leq \infty} \sigma(D_{\{x_0\}}(\gamma')_0) = 0 \quad \mu_0\text{-a.s.,}
\]

and condition (2.15) follows.

To prove the “if” part, it suffices to show that, under conditions (2.14) and (2.15), with probability one there are no cross-ties between the clusters whose centres belong to a set \(\Lambda \subset X, \sigma(\Lambda) < \infty\). Conditionally on the total number of cluster centres in \(\Lambda\) (which are then i.i.d. and have the distribution \(\sigma(\cdot)/\sigma(\Lambda)\)), the probability of the tie between a given pair of (independent) clusters is given by

\[
\frac{1}{\sigma(\Lambda)^2} \int_{f^2} \sigma^{\otimes 2}(B_A(\gamma_1, \gamma_2)) \mu_0(d\gamma_1) \mu_0(d\gamma_2),
\]

where

\[
B_A(\gamma_1, \gamma_2) := \{(x_1, x_2) \in A^2 : x_1 + y_1 = x_2 + y_2 \text{ for some } y_1 \in \gamma_1, y_2 \in \gamma_2\}.
\]

But

\[
\sigma^{\otimes 2}(B_A(\gamma_1, \gamma_2)) = \int_A \sigma\left(\bigcup_{y_1 \in \gamma_1} \bigcup_{y_2 \in \gamma_2} \{x_1 + y_1 - y_2\}\right) \sigma(dx_1)
\leq \sum_{y_1 \in \gamma_1} \int_A \sigma\left(\bigcup_{y_2 \in \gamma_2} \{x_1 + y_1 - y_2\}\right) \sigma(dx_1)
= \sum_{y_1 \in \gamma_1} \int_A \sigma(D_{\{x_1+y_1\}}(\gamma_2)) \sigma(dx_1) = 0 \quad (\mu_0\text{-a.s.}),
\]

since, by assumption (2.15), \(\sigma(D_{\{x_1+y_1\}}(\gamma_2)) = 0 \ (\mu_0\text{-a.s.})\) and \(\gamma_1\) is a countable set. Thus, the proof is complete.

6.2 Differentiable functions on configuration spaces

In this section, we recall some convenient “manifold-like” notations introduced in [5]. First, we define the tangent space to \(\Gamma_X\) at point \(\gamma\) as the Hilbert space

\[
T_\gamma \Gamma_X := L^2(X \to TX; \ d\gamma),
\]

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or equivalently
\[ T_\gamma \Gamma X = \bigoplus_{x\in \gamma} T_x X, \]
the direct sum of Euclidean spaces \( T_x X \). The scalar product in \( T_\gamma \Gamma X \) will be denoted by \( \langle \cdot, \cdot \rangle_\gamma \).

A vector field \( V \) over \( \Gamma X \) is given by a mapping

\[ \Gamma X \ni \gamma \mapsto V(\gamma) = (V(\gamma)_x)_{x \in \gamma} \in T_\gamma \Gamma X. \]

Thus for any vector fields \( V_1, V_2 \) over \( \Gamma X \) we have

\[ \langle V_1(\gamma), V_2(\gamma) \rangle_\gamma = \sum_{x \in \gamma} V_1(\gamma)_x \cdot V_2(\gamma)_x. \]

For \( \gamma \in \Gamma X \) and \( x \in \gamma \), we denote by \( \mathcal{O}_{\gamma,x} \) an arbitrary open neighborhood of \( x \) in \( X \) such that \( \mathcal{O}_{\gamma,x} \cap \gamma = \{x\} \). For any measurable function \( F : \Gamma X \to \mathbb{R} \), define the function \( F_x(\gamma, \cdot) : \mathcal{O}_{\gamma,x} \to \mathbb{R} \) by

\[ F_x(\gamma, y) := F((\gamma \setminus \{x\}) \cup \{y\}). \]

Also, set

\[ \nabla_x F(\gamma) := \nabla F_x(\gamma, y) \big|_{y=x}, \]

provided \( F_x(\gamma, \cdot) \) is differentiable at \( x \).

Following [5], consider the class \( \mathcal{F}C(\Gamma X) \) of functions on \( \Gamma X \) of the form

\[ F(\gamma) = g(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle), \quad \gamma \in \Gamma X, \quad (6.3) \]

where \( N \in \mathbb{N} \), \( g \in C_0^\infty(\mathbb{R}^N) \) (:= the set of all \( C^\infty \)-functions on \( \mathbb{R}^N \) bounded together with their derivatives) and \( \varphi_1, \ldots, \varphi_N \in C_0^\infty(X) \) (:= the set of all \( C^\infty \)-functions on \( X \) with compact support). Each \( F \in \mathcal{F}C(\Gamma X) \) is local in the sense that there exists a compact \( K_F \subset X \) such that \( F(\gamma) = F(\gamma \cap K_F) \) for all \( \gamma \in \Gamma X \). Thus, for a fixed \( \gamma \), there are only finitely many non-zero partial derivatives \( \nabla_x F(\gamma) \).

For a function \( F \in \mathcal{F}C(\Gamma X) \), let us introduce the \( \Gamma \)-gradient \( \nabla^\Gamma F \) by setting

\[ \nabla^\Gamma F(\gamma) := (\nabla_x F(\gamma))_{x \in \gamma} \in T_\gamma \Gamma X, \quad \gamma \in \Gamma X, \]

and define the directional derivative of \( F \) along a vector field \( V \) by

\[ \nabla^\Gamma_V F(\gamma) := \langle \nabla^\Gamma F(\gamma), V(\gamma) \rangle_\gamma = \sum_{x \in \gamma} \nabla_x F(\gamma)_x \cdot V(\gamma)_x. \]

Note that the sum on the right-hand side contains only finitely many non-zero terms.

Let us introduce the class \( \mathcal{F}V(\Gamma X) \) of cylinder vector fields \( V \) on \( \Gamma X \) of the form

\[ V(\gamma)_x := \sum_{j=1}^N G_j(\gamma) v_j(x) \in T_x X, \quad (6.4) \]

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where \( G_j \in \mathcal{F}C(\Gamma_X) \) and \( v_j \in \text{Vect}_0(X) \) (:= the space of compactly supported smooth vector fields on \( X \)), \( j = 1, \ldots, N \) (\( N \in \mathbb{N} \)).

Any vector filed \( v \in \text{Vect}_0(X) \) generates a constant vector field \( V \) on \( \Gamma_X \) defined by \( V(\gamma)_x := v(x) \). We will preserve the notation \( v \) for it. Thus

\[
\nabla^\Gamma_v F(\gamma) = \sum_{x \in \gamma} \nabla_x F(\gamma) \cdot v(x).
\]

Let us point out that for any set \( \Lambda \in \mathcal{B}(X) \) such that \( \sigma(\Lambda) < \infty \) and for \( \pi_\sigma \)-a.a. \( \gamma \in \Gamma_X \), we have \( \gamma(\Lambda) < \infty \). The latter fact motivates the definition of the class \( \mathcal{F}C_\sigma(\Gamma_X) \) of functions on \( \Gamma_X \) of the form (6.3), where \( \varphi_1, \ldots, \varphi_N \) are smooth functions with \( \sigma(\text{supp} \varphi_k) < \infty \), \( k = 1, \ldots, N \). Any function \( F \in \mathcal{F}C_\sigma(\Gamma_X) \) is local in the sense that there exists a set \( K_F \subset X \) such that \( \sigma(K_F) < \infty \) and \( F(\gamma) = F(\gamma \cap K_F) \) for all \( \gamma \in \Gamma_X \), which implies that \( F \) is measurable and thus \( \pi_\sigma \)-integrable. As in the case of functions from \( \mathcal{F}C(\Gamma_X) \), for a fixed \( \gamma \) there are only finitely many non-zero partial derivatives \( \nabla_x F(\gamma) \).

Remark 6.1. Notions introduced in this section can be extended to the case of more general \( X \) (e.g., a Riemannian manifold or more general linear spaces). In particular, we use the spaces \( \mathcal{F}C(\Gamma_X), \mathcal{F}C_\sigma(\Gamma_X) \) and \( \mathcal{F}V(\Gamma_X) \) of differentiable local functions and vector fields on the configuration space \( \Gamma_X \), respectively. In this situation, \( T_{\gamma} \Gamma_X = \bigoplus_{\bar{x} \in \bar{\gamma}} T_{\bar{x}} \chi \), where \( T_{\bar{x}} \chi = T_{\bar{x}} \chi^n \) if \( \bar{x} \in \chi^n \).

6.3 Quasi-invariance of Poisson measures

Let us return to the framework of Section 1 and consider a general Poisson measure \( \pi_\nu \) on the space \( \Gamma_X \) of proper configurations in a topological space \( X \), with intensity measure \( \nu \). Let \( \varphi : X \rightarrow X \) be a continuous mapping. It can be lifted to a transformation (denoted by the same letter) of the configuration space \( \Gamma_X \):

\[
\varphi(\gamma) := \{ \varphi(x), x \in \gamma \}, \quad \gamma \in \Gamma_X.
\]

It is clear that the lifted mapping \( \varphi : \Gamma_X \rightarrow \Gamma_X \) is continuous.

The next general result is essentially well known. Its first part follows from the definition of Poisson measures, and the second part is a direct consequence of Skorokhod’s theorem [29] on the absolute continuity of Poisson measures (see also [5]). We include its simple proof adapted to our setting.

Proposition 6.1. (1) Under the mapping (6.5), the push-forward measure \( \varphi^* \pi_\nu := \pi_\nu \circ \varphi^{-1} \) is a Poisson measure on \( \Gamma_X \) with intensity measure \( \varphi^* \nu \):

\[
\varphi^* \pi_\nu = \pi_{\varphi^* \nu}.
\]

(2) Let us assume that \( \varphi^* \nu \) is a.c. with respect to \( \nu \), with density

\[
\rho_\varphi(x) := \frac{\varphi^* \nu(dx)}{\nu(dx)}, \quad x \in X,
\]
and suppose that the set \( K_\varphi := \text{supp} \varphi = \{ x \in X : \varphi(x) \neq 0 \} \) is \( \nu \)-finite. Then the measure \( \pi_\varphi \) is quasi-invariant with respect to the action (6.5),

\[
\varphi^* \pi_\nu(d\gamma) = R^{\varphi}_\pi(\gamma) \pi_\nu(d\gamma), \quad \gamma \in \Gamma_X,
\]

where the density \( R^{\varphi}_\pi \) is given by

\[
R^{\varphi}_\pi(\gamma) = \exp \left( \int_X (1 - \rho^{\varphi}_\nu(x)) \nu(dx) \right) \prod_{x \in \gamma} \rho^{\varphi}_\nu(x) \quad (6.6)
\]

and, moreover, \( R^{\varphi}_\pi \in L^2(\Gamma_X, \pi_\nu) \).

Proof. (1) The statement follows by comparing the Laplace transforms of the measures \( \varphi^* \pi_\nu \) and \( \pi_{\varphi^* \nu} \). Indeed, for any \( f \in C_0(X) \) we have

\[
\int_{\Gamma_X} e^{(f, \gamma)} \varphi^* \pi_\nu(d\gamma) = \int_{\Gamma_X} e^{(f, \varphi(\gamma))} \pi_\nu(d\gamma) = \exp \left( \int_X (e^{f(x)} - 1) \nu(dx) \right) \int_{\Gamma_X} \pi_{\varphi^* \nu}(d\gamma),
\]

(2) Note that \( \rho^{\varphi}_\nu \equiv 1 \) outside \( K_\varphi \). The condition \( \nu(K_\varphi) < \infty \) implies that, for \( \pi_\nu \)-a.a. \( \gamma \), there are only finitely many terms in the product \( \prod_{x \in \gamma} \rho^{\varphi}_\nu(x) \) not equal to 1, thus the right-hand side of equation (6.6) is well defined. The statement now follows by comparing the Laplace transforms of the measures \( R^{\varphi}_\pi \pi_\nu \) and \( \pi_{\varphi^* \nu} \):

\[
\int_{\Gamma_X} e^{(f, \gamma)} R^{\varphi}_\pi \pi_\nu(d\gamma) = \exp \left( \int_X (1 - \rho^{\varphi}_\nu(x)) \nu(dx) \right) \int_{\Gamma_X} \prod_{x \in \gamma} \rho^{\varphi}_\nu(x) \pi_\nu(d\gamma)
\]

To check that \( R^{\varphi}_\pi \in L^2(\Gamma_X, \pi_\nu) \), let us compute its \( L^2 \)-norm:

\[
\int_{\Gamma_X} |R^{\varphi}_\pi(\gamma)|^2 \pi_\nu(d\gamma) = \exp \left( \int_X (1 - \rho^{\varphi}_\nu(x)) \nu(dx) \right) \int_{\Gamma_X} e^{2|\ln \rho^{\varphi}_\nu|, \gamma} \pi_\nu(d\gamma)
\]

\[
= \exp \left( \int_X (1 - \rho^{\varphi}_\nu(x)) \nu(dx) \right) \cdot \exp \left( \int_X \left( e^{2|\ln \rho^{\varphi}_\nu|} - 1 \right) \nu(dx) \right)
\]

\[
= \exp \left( \int_X \left( |\rho^{\varphi}_\nu(x)|^2 - \rho^{\varphi}_\nu(x) \right) \nu(dx) \right) < \infty,
\]

because \( |\rho^{\varphi}_\nu(x)|^2 - \rho^{\varphi}_\nu(x) = 0 \) outside \( K_\varphi \). \( \square \)
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References


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