Structure of Derivations on Various Algebras of Measurable Operators for Type I von Neumann Algebras

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Structure of derivations on various algebras of measurable operators for type I von Neumann algebras

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Abstract

Given a von Neumann algebra M denote by $S(M)$ and $LS(M)$ respectively the algebras of all measurable and locally measurable operators affiliated with M. For a faithful normal semi-finite trace τ on M let $S(M,\tau)$ (resp. $S_0(M,\tau)$) be the algebra of all τ-measurable (resp. τ-compact) operators from $S(M)$. We give a complete description of all derivations on the above algebras of operators in the case of type I von Neumann algebra M. In particular, we prove that if M is of type I_∞ then every derivation on $LS(M)$ (resp. $S(M)$ and $S(M,\tau)$) is inner, and each derivation on $S_0(M,\tau)$ is spatial and implemented by an element from $S(M,\tau)$.

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Introduction

Derivations on unbounded operator algebras, in particular on various algebras of measurable operators affiliated with von Neumann algebras, appear to be a very attractive special case of the general theory of unbounded derivations on operator algebras. The present paper continues the series of papers of the authors [1]-[3] devoted to the study and a description of derivations on the algebra $LS(M)$ of locally measurable operators with respect to a von Neumann algebra M and on various subalgebras of $LS(M)$.

Let A be an algebra over the complex number. A linear operator $D : A \to A$ is called a *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations*. If the element a implementing the derivation D_a on A, belongs to a larger algebra B, containing A (as a proper ideal as usual) then D_a is called a *spatial derivation*.

In the particular case where A is commutative, inner derivations are identically zero, i.e. trivial. One of the main problems in the theory of derivations is automatic innerness or spatialness of derivations and the existence of non inner derivations (in particular, non trivial derivations on commutative algebras).

On this way A. F. Ber, F. A. Sukochev, V. I. Chilin [5] obtained necessary and sufficient conditions for the existence of non trivial derivations on commutative regular algebras. In particular they have proved that the algebra $L^0(0,1)$ of all (classes of equivalence of) complex measurable functions on the interval $(0,1)$ admits non trivial derivations. Independently A. G. Kusraev [14] by means of Boolean-valued analysis has established necessary and sufficient conditions for the existence of non trivial derivations and automorphisms on universally complete complex f-algebras. In particular he has also proved the existence of non trivial derivations and automorphisms on $L^0(0,1)$. It is clear that these derivations are discontinuous in the measure topology, and therefore they are neither inner nor spatial. It seems that the existence of such pathological example of derivations deeply depends on the commutativity of the underlying von Neumann algebra M. In this connection the present authors have initiated the study of the above problems in the non commutative case [1]-[4], by considering derivations on the algebra $LS(M)$ of all locally measurable operators with respect to a semi-finite von Neumann algebra M and on various subalgebras of $LS(M)$. Recently another approach to similar problems in the framework of type I AW^*-algebras has been outlined in [9].

The main result of the paper [1] states that if M is a type I von Neumann algebra, then every derivation D on $LS(M)$ which is identically zero on the center Z of the von Neumann algebra M (i.e. which is Z-linear) is automatically inner, i.e. $D(x) = ax - xa$ for an appropriate $a \in LS(M)$. In [1, Example 3.8] we also gave a construction of non inner derivations D_δ on the algebra $LS(M)$ for type I_{fin} von Neumann algebra M with non atomic center Z, where δ is a non trivial derivation on the algebra $LS(Z)$ (i.e. on the center of $LS(M)$) which is isomorphic with the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable functions on a non atomic measure space (Ω, Σ, μ).

The main idea of the mentioned construction is the following.

Let A be a commutative algebra and let $M_n(A)$ be the algebra of $n \times n$ matrices over A. If $e_{i,j}$, $i, j = \overline{1, n}$, are the matrix units in $M_n(A)$, then each element $x \in M_n(A)$ has the form

$$x = \sum_{i,j=1}^{n} \lambda_{i,j} e_{i,j},\ \lambda_{i,j} \in A,\ i, j = \overline{1, n}.$$

Let $\delta : A \to A$ be a derivation. Setting

$$D_\delta\Big(\sum_{i,j=1}^{n} \lambda_{i,j} e_{i,j}\Big) = \sum_{i,j=1}^{n} \delta(\lambda_{i,j}) e_{i,j} \tag{1}$$

we obtain a well-defined linear operator D_δ on the algebra $M_n(A)$. Moreover D_δ is a derivation on the algebra $M_n(A)$ and its restriction onto the center of the algebra $M_n(A)$ coincides with the given δ.

In papers [2], [4] we considered similar problems for derivations on the algebra $S_0(M, \tau)$ of τ-compact operators with respect to a type I von Neumann algebra M with a faithful normal semi-finite trace τ, and obtained necessary and sufficient conditions for derivations to be spatial. In [3] we have proved spatialness of all derivations on the non commutative Arens algebra $L^\omega(M, \tau)$ associated with an arbitrary von Neumann algebra M and a faithful normal semi-finite trace τ. Moreover if the trace τ is finite then every derivation on $L^\omega(M, \tau)$ is inner.

In the present paper we give a complete description of all derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M, and also on its subalgebras $S(M)$ – of measurable operators, $S(M,\tau)$ of τ-measurable operators and on $S_0(M,\tau)$ of all τ-compact operators with respect to M, where τ is a faithful normal semi-finite trace on M. We prove that the above mentioned construction of derivations D_δ from [1] gives the general form of pathological derivations on these algebras and these exist only in the type I_{fin} case, while for type I_∞ von Neumann algebras M all derivations on $LS(M)$, $S(M)$ and $S(M,\tau)$ are inner and for $S_0(M,\tau)$ they are spatial. Moreover we prove that an arbitrary derivation D on each of these algebras can be uniquely decomposed into the sum $D = D_a + D_\delta$ where the derivation D_a is inner (for $LS(M)$, $S(M)$ and $S(M,\tau)$) or spatial (for $S_0(M,\tau)$) while the derivation D_δ is constructed in the above mentioned manner from a non trivial derivation δ on the center of the corresponding algebra.

In section 1 we give necessary definition and preliminaries from the theory of measurable operators and Hilbert – Kaplansky modules.

In section 2 we describe derivations on the algebra $LS(M)$ of all locally measurable operators for a type I von Neumann algebra M.

Sections 3 and 4 are devoted to derivation respectively on the algebra $S(M)$ of all measurable operators and on the algebra $S(M,\tau)$ of all τ-measurable operators with respect to M, where M is a type I von Neumann algebra and τ is a faithful normal semi-finite trace on M.

In Section 5 we give the solution of the problem for derivations on the algebra $S_0(M,\tau)$ of all τ-compact operators affiliated with a type I von Neumann algebra M and a faithful normal semi-finite trace τ.

Finally, section 6 contains an application of the above results to the description of the first cohomology group for the considered algebras.

1. Preliminaries

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Consider a von Neumann algebra M in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M.

A linear subspace $\mathcal{D}$ in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra M, if

1) $\mathcal{D}\eta M$;

2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in M.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. Denote by $S(M)$ the set of all measurable operators with respect to M.

A closed linear operator x in H is said to be *locally measurable* with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$.

It is well-known [15] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator.

Let τ be a faithful normal semi-finite trace on M. We recall that a closed linear operator x is said to be *τ-measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is τ-dense in H, i.e. $\mathcal{D}(x)\eta M$ and given $\varepsilon > 0$ there exists a projection $p \in M$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^{\perp}) < \varepsilon$. The set $S(M, \tau)$ of all τ-measurable operators with respect to M is a solid *-subalgebra in $S(M)$ (see [16]).

Consider the topology t_τ of convergence in measure or *measure topology* on $S(M, \tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(M, \tau) : \exists e \in P(M), \tau(e^{\perp}) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where ε, δ are positive numbers.

It is well-known [16] that $S(M, \tau)$ equipped with the measure topology is a complete metrizable topological *-algebra.

An element x of the algebra $S(M, \tau)$ is said to be *τ-compact*, if given any $\varepsilon > 0$ there exists a projection $p \in P(M)$ such that $\tau(p^{\perp}) < \infty$, $xp \in M$ and $\|xp\|_M < \varepsilon$. The set $S_0(M, \tau)$ of all τ-compact operators is an *-ideal in the algebra $S(M, \tau)$ (see [15]).

It should be noted that the algebra of τ-compact operators were considered by Yeadon [21] and Fack and Kosaki [7] and one of the original definitions was the following: an operator $x \in S(M, \tau)$ is said to be τ-compact if

$$\lim_{t\to\infty} \mu_t(x) = 0,$$

where $\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda^\perp) \leq t\}$ and $\{e_\lambda\}_{\lambda>0}$ is the spectral resolution of $|x|$. The equivalence of this definition to the one given above was proved in [19].

Note that if the trace τ is a finite then

$$S_0(M, \tau) = S(M, \tau) = S(M) = LS(M).$$

The following result describes one of the most important properties of the algebra $LS(M)$ (see [15], [17]).

Proposition 1.1. *Suppose that the von Neumann algebra M is the C^*-product of the von Neumann algebras M_i, $i \in I$, where I is an arbitrary set of indices, i.e.*

$$M = \bigoplus_{i\in I} M_i = \{\{x_i\}_{i\in I} : x_i \in M_i, i \in I, \sup_{i\in I} \|x_i\|_{M_i} < \infty\}$$

*with coordinate-wise algebraic operations and involution and with the C^*-norm $\|\{x_i\}_{i\in I}\|_M = \sup\limits_{i\in I} \|x_i\|_{M_i}$. Then the algebra $LS(M)$ is *-isomorphic to the algebra $\prod\limits_{i\in I} LS(M_i)$ (with the coordinate-wise operations and involution), i.e.*

$$LS(M) \cong \prod_{i\in I} LS(M_i)$$

($\cong$ denoting *-isomorphism of algebras).

It should be noted that similar isomorphisms are not valid in general for the algebras $S(M)$, $S(M, \tau)$ and $S_0(M, \tau)$ (see [15]).

Proposition 1.1 implies that given any family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in M with $\bigvee\limits_{i\in I} z_i = \mathbf{1}$ and a family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum\limits_{i\in I} z_i x_i$.

It is well-known [18] that every commutative von Neumann algebra M is *-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu)$ of all (classes of equivalence of) complex essentially bounded measurable functions on a measure space (Ω, Σ, μ) and in this case $LS(M) = S(M) \cong L^0(\Omega)$, where $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ the algebra of all (classes of equivalence of) complex measurable functions on (Ω, Σ, μ).

Further we shall need the following remarkable description of centers of the algebras $S(M)$, $S(M, \tau)$ and $S_0(M, \tau)$ for type I_∞ von Neumann algebras.

Proposition 1.2. *Let M be a type I_∞ von Neumann algebra with the center Z. Then*

a) the centers of the algebras $S(M)$ and $S(M, \tau)$ coincide with Z;

b) the center of the algebra $S_0(M, \tau)$ is trivial, i.e. $Z(S_0(M, \tau)) = \{0\}$.

Proof. a) Suppose that $z \in S(M)$, $z \geq 0$, is a central element and let $z = \int\limits_0^\infty \lambda \, de_\lambda$ be its spectral resolution. Then $e_\lambda \in Z$ for all $\lambda > 0$. Assume that $e_n^\perp \neq 0$ for all $n \in \mathbb{N}$. Since M is of type I_∞, Z does not contain non-zero finite projections. Thus $e_n^\perp$ is infinite for all $n \in \mathbb{N}$, which contradicts the condition $z \in S(M)$. Therefore there exists $n_0 \in \mathbb{N}$ such that $e_n^\perp = 0$ for all $n \geq n_0$, i.e. $z \leq n_0 \mathbf{1}$. This means that $z \in Z$, i.e. $Z(S(M)) = Z$. Similarly $Z(S(M, \tau)) = Z$.

b) Let $z \in Z(S_0(M, \tau))$, $z \geq 0$. Take a projection $p \in M$ with $\tau(p) < \infty$. Then $p \in S_0(M, \tau)$ and therefore $zp = pz$. Since M is semi-finite this implies that $zp = pz$ for all $p \in P(M)$. Since the linear span of $P(M)$ is dense in $S(M, \tau)$ in the measure topology, we have that $zx = xz$ for all $x \in S(M, \tau)$, i.e. $z \in Z(S(M, \tau)) = Z$.

Suppose that $z = \int\limits_0^\infty \lambda \, de_\lambda$ is the spectral resolution of z. Then $e_\lambda \in Z$ for all $\lambda > 0$. Since $z \in S_0(M, \tau)$ we have that $e_\lambda^\perp$ is a finite projection for all $\lambda > 0$. But M does not contain any non zero central finite projection, because it is of type I_∞. Therefore $e_\lambda^\perp = 0$ for all $\lambda > 0$, i.e. $z = 0$. Thus $Z(S_0(M, \tau)) = \{0\}$. The proof is complete. ■

Now let us recall some notions and results from the theory of Hilbert – Kaplansky modules (for details we refer to [11], [12]).

Let (Ω, Σ, μ) be a measure space and let H be a Hilbert space. A map $s : \Omega \to H$ is said to be simple, if $s(\omega) = \sum\limits_{k=1}^n \chi_{A_k}(\omega) c_k$, where $A_k \in \Sigma, A_i \cap A_j = \emptyset$, $i \neq j$, $c_k \in H$, $k = \overline{1, n}$, $n \in \mathbb{N}$. A map $u : \Omega \to H$ is said to be measurable, if there is a sequence (s_n) of simple maps such that $\|s_n(\omega) - u(\omega)\| \to 0$ almost everywhere on any $A \in \sum$ with $\mu(A) < \infty$.

Let $\mathcal{L}(\Omega, H)$ be the set of all measurable maps from Ω into H, and let $L^0(\Omega, H)$ denote the space of all equivalence classes with respect to the equality almost everywhere.

Denote by $\hat{u}$ the equivalence class from $L^0(\Omega, H)$ which contains the measurable map $u \in \mathcal{L}(\Omega, H)$. Further we shall identify the element $u \in \mathcal{L}(\Omega, H)$ and the class $\hat{u}$. Note that the function $\omega \to \|u(\omega)\|$ is measurable for any $u \in \mathcal{L}(\Omega, H)$. The equivalence class containing the function $\|u(\omega)\|$ is denoted by $\|\hat{u}\|$. For $\hat{u}, \hat{v} \in L^0(\Omega, H), \lambda \in L^0(\Omega)$ put $\hat{u} + \hat{v} = \widehat{u(\omega) + v(\omega)}, \lambda\hat{u} = \widehat{\lambda(\omega)u(\omega)}$. Equipped with the $L^0(\Omega)$-valued inner product

$$\langle x, y\rangle = \langle x(\omega), y(\omega)\rangle_H,$$

where $\langle \cdot, \cdot\rangle_H$ in the inner product in H, $L^0(\Omega, H)$ becomes a Hilbert – Kaplansky module over $L^0(\Omega)$. The space

$$L^\infty(\Omega, H) = \{x \in L^0(\Omega, H) : \langle x, x\rangle \in L^\infty(\Omega)\}$$

is a Hilbert – Kaplansky module over $L^\infty(\Omega)$. Denote by $B(L^0(\Omega, H))$ the algebra of all $L^0(\Omega)$-bounded $L^0(\Omega)$-linear operators on $L^0(\Omega, H)$ and denote by $B(L^\infty(\Omega, H))$ the algebra of all $L^\infty(\Omega)$-bounded $L^\infty(\Omega)$-linear operators on $L^\infty(\Omega, H)$.

Now consider a von Neumann algebra M which is homogeneous of type I_α with the center $L^\infty(\Omega)$, where α is a cardinal number. Then M is *-isomorphic to the algebra $B(L^\infty(\Omega, H))$, where $\dim H = \alpha$, while the algebra $LS(M)$ is *-isomorphic to $B(L^0(\Omega, H))$ (see for details [1]).

It is known [20] that given a type I von Neumann algebra M there exists a unique (cardinal-indexed) family of central orthogonal projections $(q_\alpha)_{\alpha\in J}$ in $P(M)$ with $\sum\limits_{\alpha\in J} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is a homogeneous type I_α von Neumann algebra, i.e. $q_\alpha M \cong B(L^\infty(\Omega_\alpha, H_\alpha))$ with $\dim H_\alpha = \alpha$ and

$$M \cong \bigoplus_{\alpha\in J} B(L^\infty(\Omega_\alpha, H_\alpha)).$$

The direct product

$$\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha)$$

equipped with the coordinate-wise algebraic operations and inner product forms a Hilbert — Kaplansky module over $L^0(\Omega) \cong \prod\limits_{\alpha\in J} L^0(\Omega_\alpha)$.

In [1] we have proved that if the von Neumann algebra M is *-isomorphic with $\bigoplus\limits_{\alpha\in J} B(L^\infty(\Omega_\alpha, H_\alpha))$ then the algebra $LS(M)$ is *-isomorphic with $B(\prod\limits_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$.

Therefore there exists a map $||\cdot|| : LS(M) \to L^0(\Omega)$ such that for all $x, y \in LS(M), \lambda \in L^0(\Omega)$ one has

$$||x|| \geq 0, ||x|| = 0 \Leftrightarrow x = 0;$$

$$||\lambda x|| = |\lambda| ||x||;$$

$$||x + y|| \leq ||x|| + ||y||;$$

$$||xy|| \leq ||x|| ||y||;$$

$$||xx^*|| = ||x||^2.$$

This map $||\cdot|| : LS(M) \to L^0(\Omega)$ is called the *center-valued* norm on $LS(M)$.

2. Derivations on the algebra $LS(M)$

In this section we shall give a complete description of derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M. It is clear that if a derivation D on $LS(M)$ is inner then it is Z-linear, i.e. $D(\lambda x) = \lambda D(x)$ for all $\lambda \in Z$, $x \in LS(M)$, where Z is the center of the von Neumann algebra M. The following main result of [1] asserts that the converse is also true.

Theorem 2.1. *Let M be a type I von Neumann algebra with the center Z. Then every Z-linear derivation D on the algebra $LS(M)$ is inner.*

Proof. (see [1, Theorem 3.2]). ■

We are now in position to consider arbitrary (non Z-linear, in general) derivations on $LS(M)$. The following simple but important remark is crucial in our further considerations.

Remark 1. Let A be an algebra with the center Z and let $D : A \to A$ be a derivation. Given any $x \in A$ and a central element $\lambda \in Z$ we have

$$D(\lambda x) = D(\lambda)x + \lambda D(x)$$

and

$$D(x\lambda) = D(x)\lambda + xD(\lambda).$$

Since $\lambda x = x\lambda$ and $\lambda D(x) = D(x)\lambda$, it follows that $D(\lambda)x = xD(\lambda)$ for any $\lambda \in A$. This means that $D(\lambda) \in Z$, i.e. $D(Z) \subseteq Z$. Therefore given any derivation D on the algebra A we can consider its restriction $\delta : Z \to Z$.

Now let M be a homogeneous von Neumann algebra of type $I_n, n \in \mathbb{N}$, with the center Z. Then the algebra M is *-isomorphic with the algebra $M_n(Z)$ of all $n \times n$-matrices over Z, and the algebra $LS(M) = S(M)$ is *-isomorphic with the algebra $M_n(S(Z))$ of all $n \times n$ matrices over $S(Z)$, where $S(Z)$ is the algebra of measurable operators for the commutative von Neumann algebra Z.

The algebra $LS(Z) = S(Z)$ is isomorphic to the algebra $L^0(\Omega) = L(\Omega, \Sigma, \mu)$ of all measurable functions on a measure space (see section 2) and therefore it admits (in non atomic cases) non zero derivations (see [5], [14]).

Let $\delta : S(Z) \to S(Z)$ be a derivation and D_δ be a derivation on the algebra $M_n(S(Z))$ defined by (1) in Introduction.

The following lemma describes the structure of an arbitrary derivation on the algebra of locally measurable operators for homogeneous type I_n, $n \in \mathbb{N}$, von Neumann algebras.

Lemma 2.2. *Let M be a homogenous von Neumann algebra of type $I_n, n \in \mathbb{N}$. Every derivation D on the algebra $LS(M)$ can be uniquely represented as a sum*

$$D = D_a + D_{\delta},$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$ while D_δ is the derivation of the form (1) generated by a derivation δ on the center of $LS(M)$ identified with $S(Z)$.

Proof. Let D be an arbitrary derivation on the algebra $LS(M) \cong M_n(S(Z))$. Consider its restriction δ onto the center $S(Z)$ of this algebra, and let D_δ be the derivation on the algebra $M_n(S(Z))$ constructed as in (1). Put $D_1 = D - D_\delta$. Given any $\lambda \in S(Z)$ we have

$$D_1(\lambda) = D(\lambda) - D_\delta(\lambda) = D(\lambda) - D(\lambda) = 0,$$

i.e. D_1 is identically zero on $S(Z)$. Therefore D_1 is Z-linear and by Theorem 2.1 we obtain that D_1 is inner derivation and thus $D_1 = D_a$ for an appropriate $a \in M_n(S(Z))$. Therefore $D = D_a + D_\delta$.

Suppose that

$$D = D_{a_1} + D_{\delta_1} = D_{a_2} + D_{\delta_2}.$$

Then $D_{a_1} - D_{a_2} = D_{\delta_2} - D_{\delta_1}$. Since $D_{a_1} - D_{a_2}$ is identically zero on the center of the algebra $M_n(S(Z))$ this implies that $D_{\delta_2} - D_{\delta_1}$ is also identically zero on the center of

$M_n(S(Z))$. This means that $\delta_1 = \delta_2$, and therefore $D_{a_1} = D_{a_2}$, i.e. the decomposition of D is unique. The proof is complete. $\blacksquare$

Now let M be an arbitrary finite von Neumann algebra of type I with the center Z. There exists a family $\{z_n\}_{n\in F}$, $F \subseteq \mathbb{N}$, of central projections from M with $\sup\limits_{n\in F} z_n = \mathbf{1}$ such that the algebra M is *-isomorphic with the C^*-product of von Neumann algebras z_nM of type I_n respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} z_nM.$$

By Proposition 1.1 we have that

$$LS(M) \cong \prod_{n\in F} LS(z_nM).$$

Suppose that D is a derivation on $LS(M)$, and δ is its restriction onto its center $S(Z)$. Since δ maps each $z_nS(Z) \cong Z(LS(z_nM))$ into itself, δ generates a derivation δ_n on $z_nS(Z)$ for each $n \in F$.

Let D_{δ_n} be the derivation on the matrix algebra $M_n(z_nZ(LS(M))) \cong LS(z_nM)$ defined as in (1). Put

$$D_\delta(\{x_n\}_{n\in F}) = \{D_{\delta_n}(x_n)\},\ \{x_n\}_{n\in F} \in LS(M). \tag{2}$$

Then the map D is a derivation on $LS(M)$.

Now Lemma 2.2 implies the following result:

Lemma 2.3. *Let M be a finite von Neumann algebra of type I. Each derivation D on the algebra $LS(M)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$, and D_δ is a derivation given as (2).

In order to consider the case of type I_∞ von Neumann algebra we need some auxiliary results concerning derivations on the algebra $L^0(\Omega) = L(\Omega, \Sigma, \mu)$.

Recall that a net $\{\lambda_\alpha\}$ in $L^0(\Omega)$ (o)-converges to $\lambda \in L^0(\Omega)$ if there exists a net $\{\xi_\alpha\}$ monotone decreasing to zero such that $|\lambda_\alpha - \lambda| \leq \xi_\alpha$ for all α.

Denote by ∇ the complete Boolean algebra of all idempotents from $L^0(\Omega)$, i. e. $\nabla = \{\tilde{\chi}_A : A \in \Sigma\}$, where $\tilde{\chi}_A$ is the element from $L^0(\Omega)$ which contains the characteristic function of the set A. A *partition of the unit* in ∇ is a family (π_α) of orthogonal idempotents from ∇ such that $\bigvee\limits_\alpha \pi_\alpha = \mathbf{1}$.

Lemma 2.4. *Any derivation δ on the algebra $L^0(\Omega)$ commutes with the mixing operation on $L^0(\Omega)$, i.e.*

$$\delta(\sum_\alpha \pi_\alpha \lambda_\alpha) = \pi_\alpha \delta(\sum_\alpha \lambda_\alpha)$$

for an arbitrary family $(\lambda_\alpha) \subset L^0(\Omega)$ and any partition $\{\pi_\alpha\}$ of the unit in ∇.

Proof. Consider a family $\{\lambda_\alpha\}$ in $L^0(\Omega)$ and a partition of the unit $\{\pi_\alpha\}$ in $\nabla \subset L^0(\Omega)$. Since $\delta(\pi) = 0$ for any idempotent $\pi \in \nabla$, we have $\delta(\pi_\alpha) = 0$ for all α and thus $\delta(\pi_\alpha \lambda) = \pi_\alpha \delta(\lambda)$ for any $\lambda \in L^0(\Omega)$. Therefore for each π_{α_0} from the given partition of the unit we have

$$\pi_{\alpha_0} \delta(\sum_\alpha \pi_\alpha \lambda_\alpha) = \delta(\pi_{\alpha_0} \sum_\alpha \pi_\alpha \lambda_\alpha) = \delta(\pi_{\alpha_0} \lambda_{\alpha_0}) = \pi_{\alpha_0} \delta(\lambda_{\alpha_0}).$$

By taking the sum over all α_0 we obtain

$$\delta(\sum_\alpha \pi_\alpha \lambda_\alpha) = \sum_\alpha \pi_\alpha \delta(\lambda_\alpha).$$

The proof is complete. ■

Recall [12] that a subset $K \subset L^0(\Omega)$ is called *cyclic*, if $\sum\limits_{\alpha \in J} \pi_\alpha u_\alpha \in K$ for each family $(u_\alpha)_{\alpha \in J} \subset K$ and for any partition of the unit $(\pi_\alpha)_{\alpha \in J}$ in ∇.

Lemma 2.5. *Given any non trivial derivation $\delta : L^0(\Omega) \to L^0(\Omega)$ there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ in $L^\infty(\Omega)$ with $|\lambda_n| \leq \mathbf{1}$, $n \in \mathbb{N}$, and an idempotent $\pi \in \nabla$, $\pi \neq 0$ such that*

$$|\delta(\lambda_n)| \geq n\pi$$

for all $n \in \mathbb{N}$.

Proof. Suppose that the set $\{\delta(\lambda) : \lambda \in L^0(\Omega), |\lambda| \leq \mathbf{1}\}$ is order bounded in $L^0(\Omega)$. Then δ maps any uniformly convergent sequence in $L^\infty(\Omega)$ to an (o)-convergent sequence in $L^0(\Omega)$. The algebra $L^\infty(\Omega)$ coincides with the uniform closure of the linear span of idempotents from ∇. Since δ is identically zero on ∇ it follows that $\delta \equiv 0$ on $L^\infty(\Omega)$. Since δ commutes with the mixing operation and every element $\lambda \in L^0(\Omega)$ can be represented as $\lambda = \sum\limits_\alpha \pi_\alpha \lambda_\alpha$, where $\{\lambda_\alpha\} \subset L^\infty(\Omega)$, and $\{\pi_\alpha\}$ is a partition of unit in ∇, we have $\delta(\lambda) = \delta(\sum\limits_\alpha \pi_\alpha \lambda_\alpha) = \sum\limits_\alpha \pi_\alpha \delta(\lambda_\alpha) = 0$, i.e. $\delta \equiv 0$ on $L^0(\Omega)$. This contradiction shows that the set $\{\delta(\lambda) : \lambda \in L^0(\Omega), |\lambda| \leq \mathbf{1}\}$ is not order bounded in $L^0(\Omega)$. Further, since δ commutes with the mixing operations and the set $\{\lambda : \lambda \in L^0, |\lambda| \leq \mathbf{1}\}$ is cyclic, the set $\{\delta(\lambda) : \lambda \in L^0(\Omega), |\lambda| \leq \mathbf{1}\}$ is also cyclic. By [8, Proposition 3] there exist a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $L^{\infty}(\Omega)$ with $|\lambda_n| \leq \mathbf{1}$ and an idempotent $\pi \in \nabla$, $\pi \neq 0$, such that $|\delta(\lambda_n)| \geq n\pi$, $n \in \mathbb{N}$. The proof is complete. ■

Now we are in position to consider derivations on the algebra of locally measurable operators for type I_∞ von Neumann algebras.

Theorem 2.6. *If M is a type I_∞ von Neumann algebra, then any derivation on the algebra $LS(M)$ is inner.*

Proof. Since M is of type I_∞ there exists a sequence of mutually orthogonal and mutually equivalent abelian projections $\{p_n\}_{n=1}^{\infty}$ in M with the central cover $\mathbf{1}$ (i.e. faithful projections).

For any bounded sequence $\Lambda = \{\lambda_k\}$ in Z define an operator x_Λ by

$$x_\Lambda = \sum_{k=1}^{\infty} \lambda_k p_k.$$

Then

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n. \tag{3}$$

Let D be a derivation on $LS(M)$, and let δ be its restriction onto the center of $LS(M)$, identified with $L^0(\Omega)$.

Take any $\lambda \in L^0(\Omega)$ and $n \in \mathbb{N}$. From the identity

$$D(\lambda p_n) = D(\lambda)p_n + \lambda D(p_n)$$

multiplying it by p_n from both sides we obtain

$$p_n D(\lambda p_n) p_n = p_n D(\lambda) p_n + \lambda p_n D(p_n) p_n.$$

Since p_n is a projection, one has that $p_n D(p_n) p_n = 0$, and since $D(\lambda) = \delta(\lambda) \in L^0(\Omega)$, we have

$$p_n D(\lambda p_n) p_n = \delta(\lambda) p_n. \tag{4}$$

Now from the identity

$$D(x_\Lambda p_n) = D(x_\Lambda) p_n + x_\Lambda D(p_n),$$

in view of (3) one has similarly

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n + \lambda p_n D(p_n) p_n,$$

i.e.

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n. \tag{5}$$

(4) and (5) imply

$$p_n D(x_\Lambda) p_n = \delta(\lambda_n) p_n.$$

Further for the center-valued norm $\|\cdot\|$ on $LS(M)$ (see Section 1) we have :

$$\|p_n D(x_\Lambda) p_n\| \leq \|p_n\| \|D(x_\Lambda)\| \|p_n\| = \|D(x_\Lambda)\|$$

and

$$\|\delta(\lambda_n) p_n\| = |\delta(\lambda_n)|.$$

Therefore

$$\|D(x_\Lambda)\| \geq |\delta(\lambda_n)|$$

for any bounded sequence $\Lambda = \{\lambda_n\}$ in Z.

If we suppose that $\delta \neq 0$ then by Lemma 2.5 there exist a bounded sequence $\Lambda = \{\lambda_n\}$ in Z and an idempotent $\pi \in \nabla$, $\pi \neq 0$, such that

$$|\delta(\lambda_n)| \geq n\pi$$

for any $n \in \mathbb{N}$. Thus

$$\|D(x_\Lambda)\| \geq n\pi \tag{6}$$

for all $n \in \mathbb{N}$, i.e. $\pi = 0$ – that is a contradiction. Therefore $\delta \equiv 0$, i.e. D is identically zero on the center of $LS(M)$, and therefore it is Z-linear. By Theorem 2.1 D is inner. The proof is complete. ■

We shall now consider derivations on the algebra $LS(M)$ of locally measurable operators with respect to an arbitrary type I von Neumann algebra M.

Let M be a type I von Neumann algebra. There exists a central projection $z_0 \in M$ such that

a) $z_0 M$ is a finite von Neumann algebra;

b) $z_0^{\perp} M$ is a von Neumann algebra of type I_∞.

Consider a derivation D on $LS(M)$ and let δ be its restriction onto its center $Z(S)$. By Theorem 2.6 $z_0^{\perp} D$ is inner and thus we have $z_0^{\perp}\delta \equiv 0$, i.e. $\delta = z_0\delta$.

Let D_δ be the derivation on $z_0 LS(M)$ defined as in (2) and consider its extension D_δ on $LS(M) = z_0 LS(M) \oplus z_0^\perp LS(M)$ which is defined as

$$D_\delta(x_1 + x_2) := D_\delta(x_1),\ x_1 \in z_0 LS(M), x_2 \in z_0^\perp LS(M). \tag{7}$$

The following theorem is the main result of this section, and gives the general form of derivations on the algebra $LS(M)$.

Theorem 2.7. *Let M be a type I von Neumann algebra. Each derivation D on $LS(M)$ can be uniquely represented in the form*

$$D = D_a + D_\delta$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$, and D_δ is a derivation of the form (7), generated by a derivation δ on the center of $LS(M)$.

Proof. It immediately follows from Lemma 2.3 and Theorem 2.6. ■

3. Derivations on the algebra $S(M)$

In this section we describe derivations on the algebra $S(M)$ of measurable operators affiliated with a type I von Neumann algebra M.

Let M be a type I von Neumann algebra and let $\mathcal{A}$ be an arbitrary subalgebra of $LS(M)$ containing M. Consider a derivation $D : \mathcal{A} \to LS(M)$ and let us show that D can be extended to a derivation $\tilde{D}$ on the whole $LS(M)$.

Since M is a type I, for an arbitrary element $x \in LS(M)$ there exists a sequence $\{z_n\}$ of mutually orthogonal central projections with $\bigvee\limits_{n\in\mathbb{N}} z_n = \mathbf{1}$ and $z_n x \in M$ for all $n \in \mathbb{N}$. Set

$$\tilde{D}(x) = \sum_{n\geq 1} z_n D(z_n x). \tag{8}$$

Since every derivation $D : \mathcal{A} \to LS(M)$ is identically zero on central projections of M, the equality (8) gives a well-defined derivation $\tilde{D} : LS(M) \to LS(M)$ which coincides with D on $\mathcal{A}$.

In particular, if D is Z-linear on $\mathcal{A}$, then $\tilde{D}$ is also Z-linear and by Theorem 2.1 the derivation $\tilde{D}$ is inner on $LS(M)$ and therefore D is a spatial derivation on $\mathcal{A}$, i. e. there exists an element $a \in LS(M)$ such that

$$D(x) = ax - xa$$

for all $x \in \mathcal{A}$.

Therefore we obtain the following

Theorem 3.1. *Let M be a type I von Neumann algebra with the center Z, and let $\mathcal{A}$ be an arbitrary subalgebra in $LS(M)$ containing M. Then any Z-linear derivation $D : \mathcal{A} \to LS(M)$ is spatial and implemented by an element of $LS(M)$.*

Corollary 3.2. *Let M be a type I von Neumann algebra with the center Z and let D be a Z-linear derivation on $S(M)$ or $S(M, \tau)$. Then D is spatial and implemented by an element of $LS(M)$.*

We are now in position to improve the last result by showing that in fact such derivations on $S(M)$ and $S(M, \tau)$ are inner.

Let us start by the consideration of the type I_∞ case.

Let M be a type I_∞ von Neumann algebra with the center Z identified with the algebra $L^\infty(\Omega)$ and let ∇ be the Boolean algebra of projection from $L^\infty(\Omega)$.

Denote by $St(\nabla)$ the set of all elements $\lambda \in L^\infty(\Omega)$ of the form $\lambda = \sum\limits_\alpha \pi_\alpha t_\alpha$, where $\{\pi_\alpha\}$ is a partition of the unit in ∇, and $\{t_\alpha\} \subset \mathbb{R}$ (so called step-functions).

Suppose that $a \in LS(M)$, $a = a^*$ and consider the spectral family $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ of the operator a. For $\lambda \in St(\nabla)$, $\lambda = \sum\limits_\alpha \pi_\alpha t_\alpha$ put $e_\lambda = \sum\limits_\alpha \pi_\alpha e_{t_\alpha}$.

Denote by $P_\infty(M)$ the family of all faithful projections p from M such that pMp is of type I_∞.

Set

$$\Lambda_- = \{\lambda \in St(\nabla) : e_\lambda \in P_\infty(M)\}$$

and

$$\Lambda_+ = \{\lambda \in St(\nabla) : e_\lambda^\perp \in P_\infty(M)\}.$$

Lemma 3.3. *a) $\Lambda_- \neq \emptyset$ and $\Lambda_+ \neq \emptyset$;*

b) the set Λ_+ (resp. Λ_-) is bounded from above (resp. from below);

c) if $\lambda_+ = \sup \Lambda_+$ (resp. $\lambda_- = \inf \Lambda_-$) then $\lambda \in \Lambda_+$ (resp. $\lambda \in \Lambda_-$) for all $\lambda \in St(\nabla)$ with $\lambda + \varepsilon \mathbf{1} \leq \lambda_+$ (resp. $\lambda - \varepsilon \mathbf{1} \geq \lambda_-$) for some $\varepsilon > 0$.

d) if $\lambda_+ \in L^\infty(\Omega)$, then $a \in S(M)$.

Proof. a) Take a sequence of projections $\{z_n\}$ from ∇ such that $z_n a \in M$ for all $n \in \mathbb{N}$. Then for $t_n < - \| z_n a \|_M$ we have $e_{t_n} = 0$. Therefore for $\lambda = \sum z_n t_n$ one has $e_\lambda^\perp = \mathbf{1}$, i.e. $\lambda \in \Lambda_+$ and hence $\Lambda_+ \neq \emptyset$. Similarly $\Lambda_- \neq \emptyset$.

b) Suppose that the element $\lambda = \sum \pi_\alpha \lambda_\alpha \in St(\nabla)$, satisfies the condition $\pi_0 \lambda \geq \pi_0||a|| + \varepsilon\pi_0$ for an appropriate non zero $\pi_0 \in \nabla$, where $||\cdot||$ is the center-valued norm on $LS(M)$. Without loss of generality we may assume that $\pi_0 = \pi_\alpha$ for some α, i.e. $\pi_\alpha t_\alpha \geq \pi_\alpha||a|| + \varepsilon\pi_\alpha$. Then $t_\alpha \geq ||\pi_\alpha a||_M + \varepsilon$ and therefore $\pi_\alpha e_{t_\alpha} = \pi_\alpha \mathbf{1}$. Thus $\pi_\alpha e^\perp_{t_\alpha} = 0$, i.e. $\lambda \notin \Lambda_+$. Therefore Λ_+ is bounded from above by the element $||a||$. Similarly the set Λ_- is bounded from below by the element $-||a||$.

c) Put

$$\lambda_+ = \sup \Lambda_+$$

and

$$\lambda_- = \inf \Lambda_-.$$

Take an element $\lambda \in St(\nabla)$ such that $\lambda + \varepsilon\mathbf{1} \leq \lambda_+$, where $\varepsilon > 0$. Suppose that $e^\perp_\lambda \notin P_\infty(M)$. Then $\pi_0 e^\perp_\lambda M e^\perp_\lambda$ is a finite von Neumann algebra for some non zero $\pi_0 \in \nabla$. Without loss of generality we may assume that $\pi_0 = \pi_\alpha$ for some α, i.e. $\pi_\alpha e^\perp_{t_\alpha}$ is a finite projection. Then $\pi_\alpha e^\perp_t$ is finite for all $t > t_\alpha$. This means that $\pi_\alpha \lambda_+ \leq \pi_\alpha t_\alpha$.

On the other hand multiplying by π_α the unequality $\lambda + \varepsilon\mathbf{1} \leq \lambda_+$ we obtain that $\pi_\alpha t_\alpha + \pi_\alpha \varepsilon \leq \pi_\alpha \lambda_+$. Therefore $\pi_\alpha \varepsilon \leq 0$. This contradiction implies that $\lambda \in \Lambda_+$ for all $\lambda \in St(\nabla)$ with $\lambda + \varepsilon\mathbf{1} \leq \lambda_+$.

d) Let $\lambda_+ \in L^\infty(\Omega)$. Take a number $n \in \mathbb{N}$ such that $\lambda_+ \leq n\mathbf{1}$. Then by the definition of λ_+ it follows that $e^\perp_{n+1}$ is a finite projection, i.e. $a \in S(M)$. The proof is complete. $\blacksquare$

Lemma 3.4. *If M is a type I_∞ von Neumann algebra then every derivation $D : M \to S(M)$ has the form*

$$D(x) = ax - xa, \quad x \in M$$

for an appropriate $a \in S(M)$.

Proof. By the Remark 1 D maps the center Z of M into the center of $S(M)$ which coincides with Z by Proposition 1.2, i.e. we obtain a derivation D on commutative von Neumann algebra Z. Therefore $D|_Z = 0$. Thus $D(\lambda x) = D(\lambda)x + \lambda D(x) = \lambda D(x)$ for all $\lambda \in Z$, i.e. D is Z-linear.

By Theorem 3.1 there exists an element $a \in LS(M)$ such that $D(x) = ax - xa$ for all $x \in M$.

Let us prove that one can choose the element a from $S(M)$.

For $x \in M$ we have

$$(a + a^*)x - x(a + a^*) = (ax - xa) - (ax^* - x^*a)^* = D(x) - D(x^*)^* \in S(M)$$

and

$$(a - a^*)x - x(a - a^*) = D(x) + D(x^*)^* \in S(M).$$

This means that the elements $a + a^*$ and $a - a^*$ implement derivations from M into $S(M)$. Since $a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i}$, it is sufficient to consider the case where a is a self-adjoint element.

Consider the elements $\lambda_+, \lambda_- \in L^0$ defined in Lemma 3.3 c) and let us prove that $\lambda_+ - \lambda_- \in L^\infty(\Omega)$. Lemma 3.3 c) implies that there exists an element $\lambda_1 \in \Lambda_-$ such that $\lambda_- + \frac{1}{16} \leq \lambda_1 \leq \lambda_- - \frac{1}{2}$. Since $D(x) = (a - \lambda_1)x - x(a - \lambda_1)$, replacing a by $a - \lambda_1$, we may assume that $\lambda_- \leq \frac{1}{8}$. Then $e_\varepsilon \in P_\infty(M)$ for all $\varepsilon > \frac{1}{8}$.

Suppose that $\lambda_+ \notin L^\infty(\Omega)$. Passing if necessary to the subalgebra zM, where z is a non zero central projection in M with $z\lambda_+ \geq z$, we may assume without loss of generality that $\lambda_+ \geq \mathbf{1}$.

First let us consider the particular case where M is of type $\mathrm{I}_{\aleph_0}$, where $\aleph_0$ is the countable cardinal number. Take an element $\lambda_0 \in St(\nabla)$ such that $\lambda_+ - \frac{1}{2} \leq \lambda_0 \leq \lambda_+ - \frac{1}{4}$. By Lemma 3.3 c) we have $e_{\lambda_0}^\perp \in P_\infty(M)$. Since $e_{\lambda_0}^\perp M e_{\lambda_0}^\perp$ and $e_{\frac{1}{4}} M e_{\frac{1}{4}}$ are algebras of type $\mathrm{I}_{\aleph_0}$, the projections $p_1 = e_{\lambda_0}^\perp$ and $p_2 = e_{\frac{1}{4}}$ are equivalent. From $\lambda_0 e_{\lambda_0}^\perp \leq a e_{\lambda_0}^\perp$ it follows that $\lambda_0 p_1 \leq p_1 a p_1$. Since $p_1 M p_1$ is of type $\mathrm{I}_{\aleph_0}$, the center of the algebra $S(p_1 M p_1)$ coincides with the center of the algebra $p_1 M p_1$ (Proposition 1.2) and therefore $\lambda_0 p_1 \notin S(p_1 M p_1)$, because $\lambda_0 p_1$ is a central unbounded element in $LS(p_1 M p_1)$. Therefore $a p_1 = p_1 a p_1 \notin S(p_1 M p_1)$.

Let u be a partial isometry in M such that $uu^* = p_1$, $u^*u = p_2$. Put $p = p_1 + p_2$. Consider the derivation D_1 from pMp into $pS(M)p = S(pMp)$ defined as

$$D_1(x) = pD(x)p,\ x \in pMp.$$

This derivation is implemented by the element $ap = pap$, i.e.

$$D_1(x) = apx - xap,\ x \in pMp.$$

Since $ap_2 \in pMp$, the element $b = ap_1 = ap - ap_2$ implements a derivation D_2 from pMp into $S(pMp)$.

Since $D_2(u+u^*) = b(u+u^*) - (u+u^*)b$, it follows that $b(u+u^*) - (u+u^*)b \in S(M)$. From $up_1 = p_1u^* = 0$ it follows that $bu - u^*b \in S(M)$. Multiplying this by u from the left side we obtain $ubu - uu^*b \in S(M)$. From $ub = 0$, $uu^* = p_1$, it follows that $p_1b \in S(M)$, i.e. $ap_1 \in S(M)$. This contradicts the above relation $ap_1 \notin S(M)$. The contradiction shows that $\lambda_+ \in L^\infty(\Omega)$. Now Lemma 3.3 d) implies that $a \in S(M)$.

Let us consider the case of general type I_∞ von Neumann algebra M. Take an element $\lambda_0 \in St(\nabla)$ such that $\lambda_+ - \frac{1}{2} \leq \lambda_0 \leq \lambda_+ - \frac{1}{4}$. Lemma 3.3 c) implies that $e^\perp_{\lambda_0} \in P_\infty(M)$. Consider projections p_1 and p_2 with the central cover $\mathbf{1}$ such that $p_1 \leq e^\perp_{\lambda_0}$, $p_2 \leq e_{\frac{1}{4}}$ and such that p_iMp_i are of type $\mathrm{I}_{\aleph_0}$, $i = 1, 2$. Put $p = p_1 + p_2$. Consider the derivation D_p from pMp into $pS(M)p$ defined as

$$D_p(x) = pD(x)p,\ x \in pMp.$$

Since pMp is of type $\mathrm{I}_{\aleph_0}$ the above case implies that $pap \in S(M)$ and therefore $p_1ap_1 \in S(M)$. On the other hand $\lambda_0p_1 \leq p_1ap_1$ and $\lambda_0p_1 \notin S(M)$. From this contradiction it follows that $\lambda_+ \in L^\infty(\Omega)$. By Lemma 3.3 d) we obtain that $a \in S(M)$. The proof is complete. ■

From the above results we obtain

Lemma 3.5. *Let M be a type I von Neumann algebra with the center Z. Then every Z-linear derivation D on the algebra $S(M)$ is inner. In particular, if M is a type I_∞ then every derivation on $S(M)$ is inner.*

Now let M be an arbitrary type I von Neumann algebra and let z_0 be the central projection in M such that z_0M is a finite von Neumann algebra and $z_0^\perp M$ is a von Neumann algebra of type I_∞. Consider a derivation D on $S(M)$ and let δ be its restriction onto its center $Z(S)$. By Lemma 3.5 the derivation $z_0^\perp D$ is inner and thus we have $z_0^\perp\delta \equiv 0$, i.e. $\delta = z_0\delta$.

Since z_0M is a finite type I von Neumann algebra, we have that $z_0LS(M) = z_0S(M)$. Let D_δ be the derivation on $z_0S(M) = z_0LS(M)$ defined as in (2).

Finally Lemma 2.3 and Lemma 3.5 imply the following main result the present section.

Theorem 3.6. *Let M be a type I von Neumann algebra. Then every dereivation D on the algebra $S(M)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is inner and implemented by an element $a \in S(M)$ and D_δ is the derivation of the form (7) generated by a derivation δ on the center of $S(M)$.

4. Derivations on the algebra $S(M, \tau)$

In this section we present a general form of derivations on the algebra $S(M, \tau)$ of τ-measurable operators affiliated with a type I von Neumann algebra M and a faithful normal semi-finite trace τ.

Theorem 4.1. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Then every Z-linear derivation D on the algebra $S(M, \tau)$ is inner. In particular, if M is a type I_∞ then every derivation on $S(M, \tau)$ is inner.*

Proof. By Theorem 3.1 $D(x) = ax - xa$ for some $a \in LS(M)$ and all $x \in S(M, \tau)$. Let us show that the element a can be chosen from the algebra $S(M, \tau)$.

Case 1. M is a homogeneous type I_n, $n \in \mathbb{N}$ von Neumann algebra. Then $LS(M) = S(M) \cong M_n(L^0(\Omega))$. As in Lemma 3.3 we may assume that $a = a^*$. By [13, Theorem 3.5] *-isomorphism between $S(M)$ and $M_n(L^0(\Omega))$ can be a chosen such that the element a can be represented as $a = \sum\limits_{i=1}^{n} \lambda_i e_{i,i}$, where $\lambda_i = \overline{\lambda_i} \in L^0(\Omega)$, $i = \overline{1, n}$, $\lambda_1 \geq \cdots \geq \lambda_n$.

Put $u = \sum\limits_{j=1}^{n} e_{j,n-j+1}$. Then

$$D_a(u) = au - ua = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1}) e_{i,n-i+1}$$

and

$$D_a(u)^* = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1}) e_{n-i+1,i}.$$

Therefore $D_a(u)^* D_a(u) = \sum\limits_{i=1}^{n} (\lambda_i - \lambda_{n-i+1})^2 e_{i,i}$, and thus $|D_a(u)| = \sum\limits_{i=1}^{n} |\lambda_i - \lambda_{n-i+1}| e_{i,i}$, Since $\lambda_1 \geq \cdots \geq \lambda_n$, we have

$$|\lambda_i - \lambda_{n-i+1}| \geq |\lambda_i - \lambda_{[\frac{n+1}{2}]}| \tag{9}$$

for all $i \in \overline{1, n}$.

Set $b = \sum_{i=1}^{n} |\lambda_i - \lambda_{[\frac{n+1}{2}]}| e_{i,i}$. From (9) we obtain that $|D_a(u)| \geq b$, and thus $b \in S(M, \tau)$.

Set $v = \sum_{i=1}^{[\frac{n+1}{2}]} e_{i,i} - \sum_{j=[\frac{n+1}{2}]}^{n} e_{j,j}$. Then $vb = a - \lambda_{[\frac{n+1}{2}]}\mathbf{1}$ and $vb \in S(M, \tau)$. Therefore $a - \lambda_{[\frac{n+1}{2}]}\mathbf{1} \in S(M, \tau)$ and this element also implements the derivation D_a.

Case 2. Let M be a finite type I von Neumann algebra. Then

$$LS(M) = S(M) \cong \prod_{n \in F} M_n(L^0(\Omega_n),$$

where $F \subseteq \mathbb{N}$. Therefore $a = \{a_n\}$, where $a_n = \sum_{i=1}^{n} \lambda_i^{(n)} e_{i,i}^{(n)}$, $\lambda_1^{(n)} \geq \cdots \geq \lambda_n^{(n)}$, $\lambda_i^{(n)} \in L^0(\Omega_n)$ and $e_{i,j}^{(n)}$ are the matrix units in $M_n(L^0(\Omega_n))$, $i, j = \overline{1, n}$, $n \in F$.

For each $n \in F$ consider the following elements in $M_n(L^0(\Omega_n)$

$$b_n = \sum_{i=1}^{n} |\lambda_i^{(n)} - \lambda_{[\frac{n+1}{2}]}^{(n)}| e_{i,i}^{(n)}$$

and

$$v_n = \sum_{i=1}^{[\frac{n+1}{2}]} e_{i,i}^{(n)} - \sum_{j=[\frac{n+1}{2}]}^{n} e_{j,j}^{(n)}.$$

Set $b = \{b_n\}_{n \in F}$ and $v = \{v_n\}_{n \in F}$. Consider the element

$$\lambda = \{\lambda_{[\frac{n+1}{2}]}\}_{n \in F} \in L^0(\Omega) \cong \prod_{n \in F} L^0(\Omega_n).$$

Similar to the case 1 we obtain that $a - \lambda\mathbf{1} = vb \in S(M, \tau)$.

Case 3. M is a type I_∞ von Neumann algebra. Since $S(M, \tau) \subseteq S(M)$ by Lemma 3.4 there exists an element $a \in S(M)$ such that $D(x) = ax - xa$ for all $x \in M$. Let us show that a can be picked from the algebra $S(M, \tau)$. Since $a \in S(M)$, there exists $\lambda \in \mathbb{R}$, $\lambda > 0$ such that $e_\lambda^\perp$ is a finite projection. Then $e_\lambda M e_\lambda$ is a type I_∞ von Neumann algebra and thus there exists a projection $q \leq e_\lambda$ such that $q \sim p$. Let u be a partial isometry in M such that $uu^* = p$, $u^*u = q$. Similar to Lemma 3.4 we obtain that $uapu - uu^*ap \in S(M, \tau)$ and $ap \in S(M, \tau)$. Therefore $a \in S(M, \tau)$. The proof is complete. ■

Let N be a commutative von Neumann algebra, then $N \cong L^\infty(\Omega)$ for an appropriate measure space (Ω, Σ, μ). It has been proved in [5], [14] that the algebra $LS(N) =$

$S(N) \cong L^0(\Omega)$ admits non trivial derivations if and only if the measure space (Ω, Σ, μ) is not atomic.

Let τ be a faithful normal semi-finite trace on the commutative von Neumann algebra N and suppose that the Boolean algebra $P(N)$ of projections is not atomic. This means that there exists a projection $z \in N$ with $\tau(z) < \infty$ such that the Boolean algebra of projection in zN is continuous (i.e. has no atom). Since $zS(N, \tau) = zS_0(N, \tau) = zS(N) = S(zN)$, the algebra $zS(N, \tau)$ (resp. $zS_0(N, \tau)$) admits a non trivial derivation δ. Putting

$$\delta_0(x) = \delta(zx),\ x \in S(N, \tau)$$

we obtain a non trivial derivation δ_0 on the algebra $S(N, \tau)$. Therefore, we have that if a commutative von Neumann algebra N has a non atomic Boolean algebra of projections then the algebra $S(N, \tau)$ admits a non zero derivation.

Given an arbitrary derivation δ on $S(N, \tau)$ or $S_0(N, \tau)$ the element

$$z_\delta = \inf\{z \in P(N) : z\delta = \delta\}$$

is called the support of the derivation δ.

Lemma 4.2. *If N is a commutative von Neumann algebra with a faithful normal semi-finite trace τ and δ is a derivation on $S(N, \tau)$ or $S_0(N, \tau)$, then $\tau(z_\delta) < \infty$.*

Proof. Let us give proof for the algebra $S_0(N, \tau)$, since the case of $S(N, \tau)$ is similar and simpler. Suppose the opposite, i.e. $\tau(z_\delta) = \infty$. Then there exists a sequence of mutually orthogonal projections $z_n \in N$, $n = 1, 2...$, with $z_n \leq z_\delta,\ 1 \leq \tau(z_n) < \infty$. For $z = \sup\limits_n z_n$ we have $\tau(z) = \infty$. Since $\tau(z_n) < \infty$ for all $n = 1, 2...$, it follows that $z_n S_0(N, \tau) = z_n S(N) = S(z_n N)$. Define a derivation $\delta_n : S(z_n N) \to S(z_n N)$ by

$$\delta_n(x) = z_n \delta(x),\ x \in S(z_n N).$$

Since $z_{\delta_n} = z_n$, Lemma 3.5 implies that for each $n \in \mathbb{N}$ there exists an element $\lambda_n \in z_n N$ such that $|\lambda_n| \leq n^{-1} z_n$ and $|\delta_n(\lambda_n)| \geq z_n$.

Put $\lambda = \sum\limits_{n\geq 1} \lambda_n$. Then $|\lambda| \leq \sum\limits_{n\geq 1} n^{-1} z_n$ and therefore $\lambda \in S_0(N, \tau)$. On other hand

$$|\delta(\lambda)| = |\delta(\sum_{n\geq 1} \lambda_n)| = |\delta(\sum_{n\geq 1} z_n \lambda_n)| = |\sum_{n\geq 1} z_n \delta(\lambda_n)| = \sum_{n\geq 1} |\delta_n(\lambda_n)| \geq \sum_{n\geq 1} z_n = z,$$

i.e. $|\delta(\lambda)| \geq z$. But $\tau(z) = \infty$, i.e. $z \notin S_0(N, \tau)$. Therefore $\delta(\lambda) \notin S_0(N, \tau)$. The contradiction shows that $\tau(z_\delta) < \infty$. The proof is complete. ■

Let M be a homogeneous von Neumann algebra of type $\mathrm{I}_n, n \in \mathbb{N}$, with the center Z and a faithful normal semi-finite trace τ. Then the algebra M is *-isomorphic with the algebra $M_n(Z)$ of all $n \times n$- matrices over Z, and the algebra $S(M, \tau)$ is *-isomorphic with the algebra $M_n(S(Z, \tau_Z))$ of all $n \times n$ matrices over $S(Z, \tau_Z)$, where τ_Z is the restriction of the trace τ onto the center Z.

Now let M be an arbitrary finite von Neumann algebra of type I with the center Z and let $\{z_n\}_{n\in F}$, $F \subseteq \mathbb{N}$, be a family of central projections from M with $\sup\limits_{n\in F} z_n = \mathbf{1}$ such that the algebra M is *-isomorphic with the C^*-product of von Neumann algebras $z_n M$ of type I_n respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} z_n M.$$

In this case we have that

$$S(M, \tau) \subseteq \prod_{n\in F} S(z_n M, \tau_n),$$

where τ_n is the restriction of the trace τ onto $z_n M$, $n \in F$.

Suppose that D is a derivation on $S(M, \tau)$, and let δ be its restriction onto the center $S(Z, \tau_Z)$. Since δ maps each $z_n S(Z, \tau_Z) \cong Z(S(z_n M, \tau_n))$ into itself, δ generates a derivation δ_n on $z_n S(Z, \tau_Z)$ for each $n \in F$.

Let D_{δ_n} be the derivation on the matrix algebra $M_n(z_n Z(S(M, \tau))) \cong S(z_n M, \tau_n)$ defined as in (1). Put

$$D_\delta(\{x_n\}_{n\in F}) = \{D_{\delta_n}(x_n)\},\ \{x_n\}_{n\in F} \in S(M, \tau). \tag{10}$$

By Lemma 4.2 $\tau(z_\delta) < \infty$, thus

$$z_\delta S(M, \tau) = z_\delta S(M) \cong z_\delta \prod_{n\in F} S(z_n M) = z_\delta \prod_{n\in F} S(z_n M, \tau_n),$$

and therefore $\{D_{\delta_n}(x_n)\} \in z_\delta S(M, \tau)$ for all $\{x_n\}_{n\in F} \in S(M, \tau)$. Hence we obtain that the map D is a derivation on $S(M, \tau)$.

Similar to Lemma 2.3 one can prove the following.

Lemma 4.3. *Let M be a finite von Neumann algebra of type I with a faithful normal semi-finite trace τ. Each derivation D on the algebra $S(M, \tau)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in S(M, \tau)$, and D_δ is a derivation given as (10).

Finally Theorem 4.1 and Lemma 4.3 imply the following main result the present section.

Theorem 4.4. *Let M be a type I von Neumann algebra with a faithful normal semi-finite trace τ. Then every derivation D on the algebra $S(M, \tau)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is inner and implemented by an element $a \in S(M, \tau)$ and D_δ is the derivation of the form (10) generated by a derivation δ on the center of $S(M, \tau)$.

If we consider the measure topology t_τ on the algebra $S(M, \tau)$ (see Section 1) then it is clear that every non-zero derivation of the form D_δ is discontinuous in t_τ. Therefore the above Theorem 4.4 implies

Corollary 4.5. *Let M be a type I von Neumann algebra with a faithful normal semi-finite trace τ. A derivation D on the algebra $S(M, \tau)$ is inner if and only if it is continuous in the measure topology.*

5. Derivations on the algebra $S_0(M, \tau)$

In this section we describe derivations on the algebra $S_0(M, \tau)$ of all τ-compact operators for type I von Neumann algebra M with a faithful normal semi-finite trace τ.

It should be noted that the centers of the algebras $LS(M)$, $S(M)$ and $S(M, \tau)$ for general von Neumann algebra M contain Z. This was an essential point in the proof of theorems concerning the description of derivations on these algebras. Proposition 1.2 shows that this is not the case for the algebra $S_0(M, \tau)$ because the center of this algebra may be trivial. Thus the methods of previous sections can not be directly applied for the description of derivations of the algebra $S_0(M, \tau)$.

First recall the following main result of the paper [2].

Theorem 5.1. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Then every Z-linear derivation D on the algebra $S_0(M, \tau)$ is spatial and implemented by an element from $S(M, \tau)$.*

The main result of this section will be proved step by step in several particular cases.

For a finite type I von Neumann algebras we have

Lemma 5.2. *Let M be a finite von Neumann algebra of type I with the center Z and let $D : S_0(M,\tau) \to S_0(M,\tau)$ be a derivation. If $D(\lambda) = 0$ for every λ from the center $Z(S_0(M,\tau))$ of $S_0(M,\tau)$, then D is Z-linear.*

Proof. Take $\lambda \in Z$ and choose a central projection z in M with $\tau(z) < \infty$. Since z, $z\lambda \in Z(S_0(M,\tau))$, we have that $D(z) = D(z\lambda) = 0$.

For $x \in S_0(M,\tau)$ one has

$$D(z\lambda x) = D(z\lambda)x + z\lambda D(x) = z\lambda D(x),$$

i.e.

$$D(z\lambda x) = z\lambda D(x).$$

On the other hand

$$D(z\lambda x) = D(z)\lambda x + zD(\lambda x) = zD(\lambda x),$$

i.e.

$$D(z\lambda x) = zD(\lambda x).$$

Therefore $zD(\lambda x) = z\lambda D(x)$. Since z is an arbitrary with $\tau(z) < \infty$ this implies (taking $z \uparrow \mathbf{1}$) that $D(\lambda x) = \lambda D(x)$ for all $\lambda \in Z$ and $x \in S_0(M,\tau)$, i.e. D is Z-linear. The proof is complete. ■

Now let M be a type I_n von Neumann algebra with a finite trace τ. Then $S_0(M,\tau) = S(M,\tau) = S(M)$. Consider a family $\{e_i\}_{i=1}^n$ of mutually orthogonal and mutually equivalent abelian projections in the von Neumann algebra M. Put $e = \sum\limits_{i=1}^{n-1} e_i$. Then eMe is a von Neumann algebra of type I_{n-1}, and

$$S_0(Z,\tau_Z) \cong Z(eS_0(M,\tau)e) \cong Z(S_0(M,\tau)).$$

Remark 2. From now on we shall identify these isomorphic abelian von Neumann algebras. In this case the element λ from $S_0(Z,\tau_Z)$ corresponds to λe from $Z(eS_0(M,\tau)e)$ and to $\lambda\mathbf{1}$ from $Z(S_0(M,\tau))$.

Consider a derivation D on the algebra $S_0(M,\tau)$. Since D maps $Z(S_0(M,\tau))$ into itself, its restriction $D|_{Z(S_0(M,\tau))}$ induces a derivation δ on $S_0(Z,\tau_Z) \cong Z(S_0(M,\tau))$, i.e.

$$D(\lambda\mathbf{1}) = \delta(\lambda)\mathbf{1},\ \lambda \in S_0(Z,\tau_Z).$$

Let D_e be the derivation on $eS_0(M,\tau)e$ defined as

$$D_e(x) = eD(x)e,\ x \in eS_0(M,\tau)e.$$

Since $Z(eS_0(M,\tau)e) \cong Z(S_0(M,\tau))$, the restriction of D_e onto $Z(eS_0(M,\tau)e)$ also generates a derivation, denoted by δ_e, on $S_0(Z,\tau_Z)$, i.e.

$$D_e(\lambda e) = \delta_e(\lambda)e,\ \lambda \in S_0(Z,\tau_Z).$$

Lemma 5.3. *The derivations δ and δ_e on $S_0(Z,\tau_Z)$ coincide.*

Proof. Since e is a projection it is clear that $eD(e)e = 0$ and therefore

$$\delta_e(\lambda)e = D_e(\lambda e) = eD(\lambda e)e = eD(\lambda \mathbf{1})e + e\lambda D(e)e = eD(\lambda \mathbf{1})e = \delta(\lambda)e,$$

i.e.

$$\delta_e(\lambda)e = \delta(\lambda)e$$

for any $\lambda \in S_0(M,\tau)$. Therefore (see Remark 2) $\delta_e(\lambda) = \delta(\lambda)$, i.e. $\delta \equiv \delta_e$. The proof is complete. ■

Now similar to the proof of Lemma 2.2 from Lemma 5.2 we obtain following results which describes derivations on the algebra of τ-compact operators for type I_n, $n \in \mathbb{N}$, von Neumann algebras.

Lemma 5.4. *Let M be a homogenous von Neumann algebra of type $I_n, n \in \mathbb{N}$, with a faithful normal semi-finite trace τ. Every derivation D on the algebra $S_0(M,\tau)$ can be uniquely represented as a sum*

$$D = D_a + D_\delta,$$

where D_a is a spatial derivation implemented by an element $a \in S(M,\tau)$ while D_δ is the derivation of the form (1) generated by a derivation δ on the center of $S_0(M,\tau)$ identified with $S_0(Z,\tau_Z)$.

We are now in position to prove one of the main results of this section.

Theorem 5.5. *If M is a type I_∞ von Neumann algebra with a faithful normal semi-finite trace τ, then every derivation on the algebra $S_0(M,\tau)$ is spatial and implemented by an element of the algebra $S(M,\tau)$.*

The proof of the theorem consists of several lemmata.

Lemma 5.6. *Let $z \in Z$ be a central projection from M and let $x \in S_0(M, \tau)$. Then*

$$D(zx) = zD(x).$$

Proof. Without loss of generality we may suppose that $x \geq 0$, i.e. $x = y^2$ for some $y \in S_0(M, \tau)$. From the Leibniz rule for derivations we obtain

$$D(zx) = D(zyzy) = D(zy)zy + zyD(zy) = z[D(zy)y + yD(zy)].$$

Therefore

$$z^{\perp}D(zx) = 0.$$

Similarly we have that

$$zD(z^{\perp}x) = 0.$$

Further

$$zD(x) = zD((z + z^{\perp})x) = zD(zx) + zD(z^{\perp}x) = zD(zx),$$

i.e.

$$zD(x) = zD(zx).$$

On the other hand

$$D(zx) = (z + z^{\perp})D(zx) = zD(zx) + z^{\perp}D(zx) = zD(zx),$$

i.e.

$$D(zx) = zD(zx).$$

Therefore

$$D(zx) = zD(x).$$

The proof is complete. ■

Lemma 5.7. *Suppose that $\lambda \in Z$, $p \in P(M)$, $\tau(p) < \infty$. Put $y = D(\lambda p) - \lambda D(p)$. Then*

$$p^{\perp}yp^{\perp} = 0.$$

Proof. From

$$D(p) = D(pp) = D(p)p + pD(p)$$

and

$$D(\lambda p) = D(\lambda pp) = D(\lambda p)p + \lambda pD(p)$$

we obtain

$$p^{\perp}D(\lambda p)p^{\perp} = p^{\perp}\lambda D(p)p^{\perp} = 0$$

and in particular $p^{\perp}yp^{\perp} = 0$. The proof is complete. ■

Lemma 5.8. *For each $\lambda \in Z$ and for every abelian projection $p \in P(M)$ with $\tau(p) < \infty$ we have*

$$D(\lambda p) = \lambda D(p).$$

Proof. Let z be the central cover of the projection p. Lemma 5.6 implies that the derivation D maps the algebra $zS_0(M,\tau)$ into itself. Therefore passing if necessary to the algebra zM and to the derivation zD we may assume without loss of generality that $z = \mathbf{1}$, i.e. that p is a faithful projection. Take an arbitrary faithful projection p_0 such that $p_0 \leq p^{\perp}$ and such that the von Neumann algebra p_0Mp_0 is of type $\mathrm{I}_{\aleph_0}$, where $\aleph_0$ is the countable cardinal number. Then there exists a sequence of mutually orthogonal and pairwise equivalent abelian projections $\{p_n\}_{n=2}^{\infty}$ in M with $\sum\limits_{n=2}^{\infty} p_n = p_0$. Putting $p_1 = p$ we obtain that the projections p_1 and p_n are equivalent ($p_1 \sim p_n$) and thus $\tau(p_n) = \tau(p_1) < \infty$ for all $n \geq 2$.

Set $e_n = \sum\limits_{k=1}^{n} p_k,\ n \geq 1$. Then e_nMe_n is a homogeneous von Neumann algebra of type I_n, and the restriction τ_n of the trace τ onto e_nMe_n is finite, and therefore $e_nS_0(M,\tau)e_n = S(e_nMe_n)$, $n = 1, 2....$

Define a derivation D_n on $e_nS_0(M,\tau)e_n$ as follows

$$D_n(x) = e_nD(x)e_n,\ x \in e_nS_0(M,\tau)e_n.$$

By Lemma 5.4 for each n there exists an element $a_n \in e_nS_0(M,\tau)e_n$ and a derivation δ_n on $e_1S_0(M,\tau)e_1$ identified with $Z(e_nS_0(M,\tau)e_n)$ (see Remark 2) such that

$$D_n = D_{a_n} + D_{\delta_n}. \tag{11}$$

Since $D_n = e_nD_{n+1}e_n$ Lemma 5.3 implies that $\delta_n = \delta_{n+1},\ n \geq 1$. Denote $\delta = \delta_n$.

Given a sequence $\Lambda = \{\lambda_n\}$ in Z with $|\lambda_n| \leq \frac{1}{n}\mathbf{1},\ n \in \mathbb{N}$, put

$$x_{\Lambda} = \sum_{n=1}^{\infty} \lambda_n p_n.$$

Let us show that $x_{\Lambda} \in S_0(M,\tau)$. For an arbitrary $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < \varepsilon$. Set

$$p_{\varepsilon} = \mathbf{1} - \sum_{n=1}^{n_0-1} p_n,$$

then $\tau(p_\varepsilon^\perp) = \tau(\sum\limits_{n=1}^{n_0-1} p_n) = (n_0-1)\tau(p_1) < \infty$. Moreover

$$||x_\Lambda p_\varepsilon||_M = ||\sum_{n=n_0}^{\infty} \lambda_n p_n||_M = \sup_{n\geq n_0} ||\lambda_n||_M \leq \frac{1}{n_0} < \varepsilon.$$

This means that $x_\Lambda \in S_0(M,\tau)$. For each $n \in \mathbb{N}$ we have

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n.$$

Similar to the proof of (5) in Theorem 2.5 we obtain

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n. \tag{12}$$

On the other hand

$$p_n D(\lambda_n p_n) p_n = p_n e_n D(\lambda_n p_n) e_n p_n = p_n D_n(\lambda_n p_n) p_n.$$

From (11) we obtain

$$p_n D(\lambda_n p_n) p_n = p_n D_{a_n}(\lambda_n p_n) p_n + p_n D_\delta(\lambda_n p_n) p_n.$$

Since D_{a_n} is a spatial derivation (and hence it is Z-linear), we have that

$$p_n D_{a_n}(\lambda_n p_n) p_n = \lambda_n p_n D_{a_n}(p_n) p_n = 0.$$

From

$$p_n D_\delta(\lambda_n p_n) p_n = \delta(\lambda_n) p_n,$$

we obtain

$$p_n D(\lambda_n p_n) p_n = \delta(\lambda_n) p_n. \tag{13}$$

Now (12) and (13) imply

$$p_n D(x_\Lambda) p_n = \delta(\lambda_n) p_n.$$

Suppose that $\delta \neq 0$. Then Lemma 2.5 implies the existence of a sequence $\Lambda = \{\lambda_n\}$ in Z with $|\lambda_n| \leq \frac{1}{n}\mathbf{1},\ n \in \mathbb{N}$, and a projection $\pi \in Z$, $\pi \neq 0$ such that

$$|\delta(\lambda_n)| \geq n\pi,\ n \in \mathbb{N}.$$

Similar to the proof of (6) in Theorem 2.5 we obtain

$$||D(x_\Lambda)|| \geq \pi n,\ n \geq 1.$$

The last inequality contradicts the choice of $\pi \neq 0$. Therefore $\delta \equiv 0$, i.e. from (11) we obtain that $D_n = D_{a_n}$. Since D_{a_n} is a spatial derivation and the center of the algebra $Z(e_n M e_n)$ coincides with $e_n Z$, it follows that D_n is $e_n Z$-linear. Thus

$$D_n(\lambda e_n p e_n) = \lambda e_n D_n(e_n p e_n) \tag{14}$$

for all $\lambda \in Z$. Since the projection e_n is in $S_0(M, \tau)$ and it commutes with p we have

$$D_n(e_n p e_n) = D_n(e_n p) = e_n D(e_n p) e_n = e_n D(e_n) p e_n + e_n D(p) e_n =$$

$$= e_n D(e_n) e_n p + e_n D(p) e_n = e_n D(p) e_n,$$

i.e.

$$\lambda D_n(e_n p e_n) = \lambda e_n D(p) e_n. \tag{15}$$

In a similar way we obtain

$$D_n(\lambda e_n p e_n) = e_n D(\lambda p) e_n. \tag{16}$$

Now (14), (15) and (16) imply

$$e_n D(\lambda p) e_n = e_n \lambda D(p) e_n$$

for all $n \in \mathbb{N}$.

Set $y = D(\lambda p) - \lambda D(p)$. Then $e_n y e_n = 0$. From $e_1 = p_1 = p$, we have $pyp = 0$. By Lemma 5.7 we have $p^\perp y p^\perp = 0$. Multiplying the equality $e_n y e_n = 0$ by p from the left side we obtain $p y e_n = 0$ for all $n \in \mathbb{N}$. Since $e_n \uparrow p_0 + p$, it follows that $py(p_0 + p) = 0$, i.e. $pyp_0 = 0$. Since p_0 is an arbitrary projection with the central cover $\mathbf{1}$ such that $p_0 \leq p^\perp$ and such that the von Neumann algebra $p_0 M p_0$ is of type $\mathrm{I}_{\aleph_0}$, we obtain that $pyp^\perp = 0$.

Similarly $p^\perp y p = 0$. Therefore

$$pyp = pyp^\perp = p^\perp y p = p^\perp y p^\perp = 0$$

and hence

$$y = pyp + pyp^\perp + p^\perp y p + p^\perp y p^\perp,$$

i.e. $D(\lambda p) = \lambda D(p)$. The proof is complete. ■

Lemma 5.9. *Suppose that $\lambda \in Z$ and $x \in S_0(M, \tau)$. Then*

$$D(\lambda x) = \lambda D(x).$$

Proof. Case (i). $x = p$ is a projection and

$$p = \sum_{i=1}^{k} p_i, \tag{17}$$

where p_i, $i = \overline{1, k}$ are mutually orthogonal abelian projections with $\tau(p_i) < \infty$. By Lemma 5.8 we have $D(\lambda p_i) = \lambda D(p_i)$. Therefore

$$D(\lambda p) = D(\lambda \sum_{i=1}^{k} p_i) = \sum_{i=1}^{k} D(\lambda p_i) = \sum_{i=1}^{k} \lambda D(p_i) = \lambda D(\sum_{i=1}^{k} p_i) = \lambda D(p),$$

i.e.

$$D(\lambda p) = \lambda D(p).$$

Case (ii). $x = p$ is a projection with $\tau(p) < \infty$. Then pMp is a finite von Neumann algebra of type I, and therefore there exists a sequence of mutually orthogonal central projections $\{z_n\}$ such that each $p_n = z_n p$ is a projection of the form (17). From the above case we have $D(\lambda p_n) = \lambda D(p_n)$. This and Lemma 5.6 imply that

$$z_n D(\lambda p) = D(\lambda z_n p) = D(\lambda p_n) = \lambda D(p_n) = \lambda D(z_n p) = \lambda z_n D(p).$$

i.e.

$$z_n D(\lambda p) = z_n \lambda D(p)$$

for all n. Therefore

$$D(\lambda p) = \lambda D(p)$$

Case (iii). Let $x \in S_0(M, \tau)$ be an element such that $xp = x$ for some projection p with $\tau(p) < \infty$. Then

$$D(\lambda x) = D(\lambda xp) = D(x\lambda p) = D(x)\lambda p + xD(\lambda p) =$$

$$= D(x)\lambda p + x\lambda D(p) = \lambda(D(x)p + xD(p)) = \lambda D(xp) = \lambda D(x),$$

i.e. $D(\lambda x) = \lambda D(x)$.

Case (iv). x is an arbitrary element from $S_0(M, \tau)$. Take a projection p with finite trace $\tau(p)$. Put $x_0 = xp$. From the case (iii) we have $D(\lambda x_0) = \lambda D(x_0)$. Now one has

$$D(\lambda x_0) = D(\lambda xp) = D(\lambda x)p + \lambda xD(p),$$

i.e.

$$D(\lambda x)p = D(\lambda x_0) - \lambda x D(p).$$

On the other hand

$$D(\lambda x_0) = \lambda D(x_0) = \lambda D(xp) = \lambda D(x)p + \lambda x D(p),$$

i.e.

$$\lambda D(x)p = D(\lambda x_0) - \lambda x D(p).$$

Therefore $\lambda D(x)p = D(\lambda x)p$. Since p is an arbitrary with $\tau(p) < \infty$, this implies

$$D(\lambda x) = \lambda D(x).$$

The proof is complete. ■

Proof of Theorem 5.5.

By Lemma 5.9 the derivation $D : S_0(M, \tau) \to S_0(M, \tau)$ is Z-linear. By Theorem 5.1 D is spatial and moreover

$$D(x) = ax - xa,\ x \in S_0(M, \tau)$$

for an appropriate $a \in S(M, \tau)$. The proof is complete. ■

Now we can describe the structure of derivations on the algebra $S_0(M, \tau)$ of τ-compact operators with respect to a type I von Neumann algebra M with a faithful normal semi-finite trace τ.

Let M be a type I von Neumann algebra and let z_0 be the central projection in M such that $z_0 M$ is a finite von Neumann algebra and $z_0^{\perp} M$ is a von Neumann algebra of type I_∞. Consider a derivation D on $S_0(M, \tau)$ and let δ be its restriction onto the center $Z(S_0(M, \tau))$. By Proposition 1.2 we have $z_0^{\perp} Z(S_0(M, \tau)) = \{0\}$, and therefore $z_0^{\perp}\delta \equiv 0$, i.e. $\delta = z_0\delta$.

By Lemma 4.2 $\tau(z_\delta) < \infty$ and therefore the derivation D_δ defined in (2) maps $z_0 S_0(M, \tau)$ into itself. Consider its extension D_δ on $S_0(M, \tau) = z_0 S_0(M, \tau) \oplus z_0^{\perp} S_0(M, \tau)$ which is defined as

$$D_\delta(x_1 + x_2) := D_\delta(x_1),\ x_1 \in z_0 S_0(M, \tau), x_2 \in z_0^{\perp} S_0(M, \tau). \tag{18}$$

Similar to the cases of the algebras $LS(M)$, $S(M)$ and $S(M, \tau)$ for a finite von Neumann algebra M of type I, every derivation on the algebra $S_0(M, \tau)$ admits the decomposition $D = D_a + D_\delta$.

The following is the main result of this section, which gives the general form of derivations on the algebra $S_0(M,\tau)$ (cf. [4]).

Theorem 5.10. *Let M be a type I von Neumann algebra with a faithful normal semi-finite trace τ. Each derivation D on $S_0(M,\tau)$ can be uniquely represented in the form*

$$D = D_a + D_{\delta,} \tag{19}$$

where D_a is a spatial derivation implemented by an element $a \in S(M,\tau)$, and D_δ is a derivation of the form (18), generated by a derivation δ on the center of $S_0(M,\tau)$.

Similar to Corollary 4.5 we obtain the following result.

Corollary 5.11. *Under the conditions of Theorem 5.10 every t_τ- continuous derivation on the algebra $S_0(M,\tau)$ is spatial and implemented by an element of $S(M,\tau)$.*

Finally from Theorems 2.7, 3.6, 4.4, 5.10 and from [5, Theorem 3.4] we obtain the following corollary

Corollary 5.12. *Let M be a type I von Neumann algebra. The following conditions are equivalent:*

(i) Every derivation on the algebra $LS(M)$ (resp. $S(M)$, $S(M,\tau)$) is inner.

(ii) Every derivation on the algebra $S_0(M,\tau)$ is spatial and implemented by an element of $S(M,\tau)$.

(iii) The center of the type I_{fin} part of M is atomic.

6. An application to the description of the first cohomology group

Let A be an algebra. Denote by $Der(A)$ the space of all derivations (in fact it is a Lie algebra with respect to the commutator), and denote by $InDer(A)$ the subspace of all inner derivations on A (it is a Lie ideal in $Der(A)$).

The factor-space $H^1(A) = Der(A)/InDer(A)$ is called the first (Hochschild) cohomology group of the algebra A (see [6]). It is clear that $H^1(A)$ measures how much the space of all derivations on A differs from the space on inner derivations.

The following result shows that the first cohomology groups of the algebras $LS(M)$, $S(M)$ and $S(M,\tau)$ are completely determined by the corresponding cohomology groups of their centers (cf. [5, Corollary 3.1]).

Theorem 6.1. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Suppose that z_0 is a central projection such that z_0M is a finite von Neumann algebra, and $z_0^{\perp}M$ is of type I_∞. Then*

a) $H^1(LS(M)) = H^1(S(M)) \cong H^1(S(z_0Z))$;

b) $H^1(S(M,\tau)) \cong H^1(S(z_0Z,\tau_0))$, where τ_0 is the restriction of τ onto z_0Z.

Proof. It immediately follows from Theorems 2.7, 3.6 and 4.4. ∎

Further we need the following property of the algebra of τ-compact operators from [19]:

$$S(M,\tau) = M + S_0(M,\tau). \tag{20}$$

Set $C(M,\tau) = M \cap S_0(M,\tau)$ and consider $M/(C(M,\tau)+Z)$ – the factor space of M with respect to the space $C(M,\tau)+Z$.

For $D_1, D_2 \in Der(S_0(M,\tau))$ put

$$D_1 \sim D_2 \Leftrightarrow D_1 - D_2 \in InDer(S_0(M,\tau)).$$

Suppose that $D_1 \sim D_2$. From Theorem 5.10 these derivation can be represented in the form (19):

$$D_1 = D_a + D_\delta,\ D_2 = D_b + D_\sigma.$$

Since $D_1 - D_2 = D_c$, where $c \in S_0(M,\tau) \subset S(M,\tau)$, from the uniqueness of a the representation in the form (19) it follows that $D_a - D_b \in InDer(S_0(M,\tau))$ and $D_\delta = D_\sigma$. Therefore $\delta \equiv \sigma$ and

$$a - b \in S_0(M,\tau) + Z(S(M,\tau)). \tag{21}$$

According to (20) we have

$$a = a_1 + a_2, a_1 \in M,\ a_2 \in S_0(M,\tau),$$

$$b = b_1 + b_2, b_1 \in M,\ b_2 \in S_0(M,\tau).$$

From (21) it follows that

$$a_1 - b_1 \in (b_2 - a_2) + S_0(M,\tau) + Z(S(M,\tau)) \subset S_0(M,\tau) + Z(S(M,\tau)).$$

Since $a_1, b_1 \in M$, we have that

$$a_1 - b_1 \in (S_0(M,\tau) + Z(S(M,\tau))) \cap M \subset C(M,\tau) + Z$$

because $Z(S(M, \tau)) \cap M = Z$ (cf. Proposition 1.2). Therefore

$$D_1 \sim D_2 \Leftrightarrow a_1 - b_1 \in C(M, \tau) + Z, \, \delta \equiv \sigma.$$

Thus we have the following result.

Theorem 6.2. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Suppose that z_0 is a central projection such that z_0M is a finite von Neumann algebra, and $z_0^{\perp}M$ is of type I_∞. Then the group $H^1(S_0(M, \tau))$ is isomorphic with the group $M/(C(M, \tau)+Z) \oplus H^1(S_0(z_0Z, \tau_0))$, where τ_0 is the restriction of τ onto z_0Z. In particular, if M is of type I_∞, then $H^1(S_0(M, \tau)) \cong M/(C(M, \tau)+Z)$.*

Remark 3. In the algebras $S(M, \tau)$ and $S_0(M, \tau)$ equipped with the measure topology t_τ one can consider another possible cohomology theories. Similar to [10] consider the space $Der_c(A)$ of all continuous derivation on a topological algebra A and define the first cohomology group $H^1_c(A) = Der_c(A)/InDer(A)$.

Under these notations the above results and Corollaries 4.6 and 5.11 imply the following result (cf. [10, Theorem 4.4]).

Corollary 6.3. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Consider the topological algebras $S(M, \tau)$ and $S_0(M, \tau)$ equipped with the measure topology. Then $H^1_c(S(M, \tau)) = \{0\}$ and $H^1_c(S_0(M, \tau)) \cong M/(C(M, \tau) + Z)$.*

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