On Output Functionals of Boundary Value Problems on Stochastic Domains

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no. 423
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Abstract. We consider the computation of output functionals of random solutions to elliptic boundary value problems in domains with random boundary perturbations. We use a second order shape calculus to linearize the problem around a fixed nominal domain. For known mean and two-point correlation function of the boundary perturbation, we derive, with leading order, deterministic expressions for the mean and the variance of the random output functional. These expressions include the solution of the boundary value problem on the nominal domain and a further, deterministic solution of the so-called adjoint equation. The theoretical findings are supported and quantified by numerical experiments.

Keywords: Elliptic boundary value problem, stochastic domain, output functionals.

1. Introduction

Many problems in physics and engineering sciences lead to boundary value problems for an unknown function. In general, the numerical simulation is well understood provided that the input parameters are given exactly. Often, however, the input parameters are not known exactly. Thus, the modeling of stochastic input parameters becomes recently a topic of growing interest in research.

A principal approach to solve boundary value problems with stochastic input parameters is the Monte Carlo Approach, see e.g. [27] and the references therein. However, it is hard and extremely expensive to generate a large number of suitable samples and to solve a deterministic boundary value problem on each sample. Thus, we aim here at a direct, deterministic computation of the stochastic solution.

Deterministic approaches to solve stochastic partial differential equations have been proposed in e.g. [1, 2, 3, 7, 8, 14, 15, 20, 28, 29, 30]. Therein, loadings and coefficients have been considered as stochastic input parameter. Recently, in [6, 17, 34], also the underlying domain has been modeled as stochastic input parameter \( D(\omega) \). For example, one may think of tolerances in the shape of products fabricated by line

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This work was supported by the SFB 611 “Singular Phenomena and Scaling in Mathematical Models”, funded by the Deutsche Forschungsgemeinschaft.
production, or shapes which stem from inverse problems, like e.g. tomography. Other applications arise from unsharp interfaces like cell membranes or molecular surfaces.

In the present paper we will consider boundary value problems on stochastic domains. Often, however, the quantity of interest is not the random solution \( u(\omega) \) itself but an output functional of domain or boundary integral type that invokes the random solution or its gradient. This induces to investigate stochastic shape functionals of domain integral type

\[
J(D(\omega); u(\omega)) = \int_{D(\omega)} j(x, u(x, \omega), \nabla u(x, \omega)) \, dx
\]

or of boundary integral type

\[
J(D(\omega); u(\omega)) = \int_{\partial D(\omega)} j(x, u(x, \omega), \nabla u(x, \omega)) \, d\sigma_x.
\]

The random solution \( u(\omega) \) is supposed to be determined by a boundary value problem, that is

\[
A u(\omega) = f \text{ on } D(\omega), \quad B u(\omega) = g \text{ on } \partial D(\omega),
\]

where \( A \) corresponds to a well-posed elliptic second order differential operator in the domain \( D(\omega) \) and \( B \) operates on the functions supported at the boundary \( \partial D(\omega) \).

In Section 2 we will specify particular examples of such output functionals.

If a statistical description of the stochastic domain is available, one can mathematically describe data and solutions as random fields and aim at the computation of corresponding deterministic statistics of the unknown random solution. By identifying the stochastic random domain with its boundary, the problem under consideration can thus be formulated as follows: given the mean and the two-point correlation of the random boundary perturbation field, compute, to leading order, the mean

\[
E_J = \int_\Omega J(D(\omega); u(\omega)) \, dP(\omega)
\]

and the variance

\[
\text{Var}_J = \int_\Omega \left( J(D(\omega); u(\omega)) - E_J \right)^2 \, dP(\omega)
\]

of the random output functionals.

The functional’s nonlinear dependence on the shape of the domain is Fréchet differentiable \([11, 12, 13, 26]\). Thus, we achieve our goal by linearizing around an fixed, nominal domain \( D \). For shape calculus we refer to e.g. \([9, 21, 31, 33]\) and the references therein. We use the second order shape calculus developed in \([11, 12, 13]\) to derive deterministic expressions for the random functional’s second order statistics with respect to random perturbations of the domain.
The second moment equation originates from the explicit Hadamard representation of the shape gradient. It involves only the deterministic solution of the boundary value problem on the nominal domain. In addition the so-called adjoint equation with respect to the nominal domain needs to be solved. This is in contrast to the problem of computing directly the solution to a boundary value problem on a stochastic domain. There, one always arrives at a high dimensional boundary value problem, cf. [6, 17, 34].

The outline of the paper is as follows. In Section 2 we motivate the present setup by several examples. Then, in Section 3 we briefly introduce to shape calculus. We consider exemplarily the Poisson equation with Dirichlet boundary conditions and functionals of domain integral type. In Section 4 we model stochastic domains. Stochastic shape functionals are introduced in Section 5. Their first and second moments are computed with leading order in Section 6. Numerical experiments are carried out in Section 7. Finally, in Section 8 we state concluding remarks.

2. Background and Motivation

In many applications engineers are not directly interested in the solution $u$ of a boundary value problem. Often the quantities of interest are given as integrals over the domain $D$ or its boundary $\partial D$, involving the solution $u$ or its gradient $\nabla u$, that is

$$J(D) = \int_D j(x, u(x), \nabla u(x)) dx$$

or

$$J(D) = \int_{\partial D} j(x, u(x), \nabla u(x)) d\sigma_x.$$

For illustrational reasons we will present some examples in more detail.

2.1. Torsional rigidity. We consider a cylindric circular bar which is homogeneous and isotropic with a planar, simply connected cross section $D \subset \mathbb{R}^2$. Define the stress function $u \in H^1_0(D)$ by

$$-\Delta u = 2 \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

Then, the torsional rigidity $T(D)$ of the bar is expressed by

$$T(D) = 2G \int_D u dx$$

with $G$ being the shear modulus. Notice that the computation of the torsion will be considered in the numerical results.
2.2. **Heat conduction.** We consider a long pipe carrying turbulent flow of a coolant, e.g. water. The cross section \( D \subset \mathbb{R}^2 \) of the pipe is fixed and has the interior boundary \( \Gamma_i \) and exterior boundary \( \Gamma_e \), see Fig. 1.

![Figure 1. The cross section of the pipe.](image)

The heat conductivity of the pipe material (assumed to be homogeneous isotropic) is \( A \in \mathbb{R} \). Moreover, we suppose constant temperatures \( u_e, u_i \in \mathbb{R} \) at the boundaries \( \Gamma_e \) and \( \Gamma_i \). Thus, the temperature distribution \( u \) is governed by the equation

\[
\Delta u = 0 \quad \text{in} \quad D, \quad u = u_i \quad \text{on} \quad \Gamma_i, \quad u = u_e \quad \text{on} \quad \Gamma_e.
\]

The mean temperature of the pipe material is

\[
M(D) = \frac{1}{|D|} \int_D u \, dx
\]

while the heat flux between the coolant and the pipe is computed by

\[
F(D) = A \int_{\Gamma_i} \frac{\partial u}{\partial n} d\sigma.
\]

Provided that \( u_i \neq u_e \), we conclude

\[
F(D) = \int_{\partial D} \frac{A}{u_i - u_e} \frac{\partial u}{\partial n} (u - u_e) d\sigma = \int_D \frac{A}{u_i - u_e} \|\nabla u\|^2 dx.
\]

2.3. **Potential flow.** The potential flow around an obstacle \( D \subset \mathbb{R}^2 \) is defined as the gradient \( \nabla u \) of the harmonic function \( u \), satisfying

\[
\Delta u = 0 \quad \text{in} \quad D^c, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial D,
\]

with an appropriate given far field \( u_\infty \), cf. Fig. 2.

The circulation around the obstacle is defined according to

\[
C(D) = \int_{\partial D} \langle \nabla u, t \rangle d\sigma
\]
where \( t \) is the tangent to the boundary curve. By the Kutta-Joukowski theorem, the lift is computed from (2.1) by

\[ L(D) = -\rho u_\infty C(D) \]

with \( \rho \) being the density of the fluid.

By introducing the stream function \( \nu(y, x) = u(x, -y) \) we can rewrite the functional (2.1) as

\[ C(D) = \int_{\partial D} \frac{\partial \nu}{\partial n} \, d\sigma, \]

where \( \nu \) satisfies the Dirichlet problem

\[ \Delta \nu = 0 \quad \text{in} \quad D, \quad \nu = 0 \quad \text{on} \quad \partial D. \]

The pressure at an arbitrary point \( x \in \mathbb{R}^2 \) can be related to the pressure \( p_\infty \), taken at the same hydrostatic level at a point far upstream, by Bernoulli’s equation,

\[ p_\infty + \frac{\rho}{2} \| \nabla u_\infty \|^2 =: c = p(x) + \frac{\rho}{2} \| \nabla u(x) \|^2. \]

Thus, due to the homogeneous Neumann boundary condition, the total pressure acting on the obstacle is

\[ P(D) = c|\partial D| - \int_{\partial D} \frac{\rho}{2} |(\nabla u, t)|^2 d\sigma = c|\partial D| - \int_{\partial D} \frac{\rho}{2} \left( \frac{\partial \nu}{\partial n} \right)^2 d\sigma. \]

3. Shape Calculus

In order to assess the impact of random boundary perturbations of the domain \( D \subset \mathbb{R}^n, n = 2, 3 \), on the shape functional \( J(D) \), we use shape calculus via boundary variations. For a general overview on shape calculus, mainly based on the perturbation of identity (Murat and Simon) or the speed method (Sokolowski and Zolesio), we refer the reader for example to [9, 21, 25, 31, 33] and the references therein.
The output functionals from Subsections 2.1, 2.2 correspond to functionals of domain integral type while that from Subsection 2.3 are of boundary integral type. Throughout this paper, we shall focus on shape functionals of domain integral type, that is

\[(3.1) \quad J(D) = \int_D j(x, u(x), \nabla u(x))dx.\]

However, our analysis will be quite similar in case of shape functionals of boundary integral type, but formulae will change.

The state function \(u\) is supposed to satisfy the Dirichlet problem for the Poisson equation

\[(3.2) \quad -\Delta u = f \text{ in } D, \quad u = g \text{ on } \partial D.\]

Herein, we assume that \(f, g: D \to \mathbb{R}\) and

\(j = j(x, y, z): D \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\)

are sufficiently smooth functions, where \(D \subset \mathbb{R}^n\) denotes the hold all. Note that the Dirichlet data in (3.2) have to be understood as the trace \(g|_{\partial D}\).

For sake of convenience we present briefly the derivation of the Hadamard representation of the shape gradient to (3.1), (3.2). To that end, we are going to use the following notation for the partial derivatives of \(j\)

\[(3.3) \quad j_y(x, y, z) = \frac{\partial j}{\partial y}(x, y, z) \in \mathbb{R}, \quad j_z(x, y, z) = \frac{\partial j}{\partial z}(x, y, z) \in \mathbb{R}^n.\]

Notice that the subsequent calculations can be found in [11, 13] in full detail.

For a sufficiently smooth domain perturbation field \(V: D \to \mathbb{R}^n\) (to be specified below) we can define the perturbed domain \(D_\varepsilon[V]\) as

\[D_\varepsilon[V] := \{(I + \varepsilon V)(x): x \in D\}, \quad \varepsilon > 0.\]

Then, the shape derivative of the functional (3.1) in the direction \(V\) is defined as the limes

\[dJ(D)[V] = \lim_{\varepsilon \to 0} \frac{J(D_\varepsilon[V]) - J(D)}{\varepsilon}.\]

For \(f \in C(\overline{D})\) the directional shape derivative of the domain integral \(\int_D f(x)dx\) is the boundary integral

\[\nabla \left(\int_D f(x)dx\right)[V] = \int_{\partial D} \langle V, \mathbf{n}\rangle f(x)d\sigma_x.\]
Thus, by the product rule we obtain (see also (3.3))

\[
\frac{dJ}{\partial D}[V] = \int_{\partial D} \langle \nabla \phi, \nabla u \rangle d\sigma_x + \int_D j_y(x, u, \nabla u) du[V] + \langle j_z(x, u, \nabla u), \nabla du[V] \rangle dx.
\]

Here, \( du[V] \) is the local shape derivative, defined point-wise by

\[
du(x)[V] = \lim_{\varepsilon \to 0} \frac{u_{\varepsilon}[V](x) - u(x)}{\varepsilon}, \quad x \in D \cap D_{\varepsilon}[V].
\]

The local shape derivative is a measure for the sensitivity of the solution \( u \) with respect to the domain perturbation \( V \). According to [11, 19, 26], it satisfies the following Dirichlet problem for the Laplacian:

\[
\Delta du[V] = 0 \text{ in } \Omega, \quad du[V] = \langle V, \nabla \rangle \partial \frac{(g - u)}{\partial n} \text{ on } \Gamma.
\]

We introduce the adjoint state function \( p \) according to

\[
-\Delta p = j_y(\cdot, u, \nabla u) - \text{div} j_z(\cdot, u, \nabla u) \text{ in } D, \quad p = 0 \text{ on } \partial D,
\]

and apply Green’s second formula to the second term of the right hand side in (3.4). Thus, in view of (3.5) and (3.6), we derive the Hadamard representation of the shape gradient

\[
\frac{dJ}{\partial D}[\kappa] = \int_{\partial D} \langle \nabla \phi, \nabla u \rangle \frac{\partial (g - u)}{\partial n} + \left( \langle j_z(x, g, \nabla u), \nabla \rangle - \frac{\partial p}{\partial n} \right) \frac{\partial (g - u)}{\partial n} d\sigma_x.
\]

We observe that the shape gradient lives completely on the varying boundary, involving the Dirichlet-to-Neumann maps (often called the Steklov-Poincaré operator) of \( u \) and \( p \). Particularly, as observed first by Hadamard [16], the shape gradient is a functional defined on the varying boundary.

It is obvious that it suffices to consider only boundary variations. In the sequel we will employ boundary variations in normal direction, that is \( V = \kappa \cdot n : \partial D \to \mathbb{R}^n \).

Then, the shape gradient becomes

\[
\frac{dJ}{\partial D}[\kappa] = \int_{\partial D} \kappa \left\{ j(x, g, \nabla u) + \left( \langle j_z(x, g, \nabla u), \nabla \rangle - \frac{\partial p}{\partial n} \right) \frac{\partial (g - u)}{\partial n} \right\} d\sigma_x.
\]

For the second order shape calculus we require that the boundary variation satisfies \( V \in C^{2,1}(\partial D) \). Thus, it suffices to assume that \( D \in C^{5,1} \) and \( \kappa \in C^{2,1}(\partial D) \) since \( n \in C^{2,1}(\partial D) \).
We employ second order variations of the type
\[ \partial D_{\varepsilon_1, \varepsilon_2} [\kappa_1, \kappa_2] = \{ x + \kappa_1(x)n(x) + \kappa_2(x)n(x) : x \in \partial D \}. \]

Then the second order shape derivative, the shape Hessian, is a symmetric bilinear form on pairs of boundary perturbation fields \((\kappa_1, \kappa_2)\), defined by
\[
d^2 J(D)[\kappa_1, \kappa_2] = \lim_{\varepsilon \to 0} \frac{d J(D_\varepsilon[\kappa_2])[\kappa_1] - d J(D)[\kappa_1]}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{d J(D_\varepsilon[\kappa_1])[\kappa_2] - d J(D)[\kappa_2]}{\varepsilon}.
\]

Explicit calculations of the shape Hessian in terms of polar coordinates can be found for example in [10, 12, 13]. Nevertheless, for our purpose existence and Lipschitz continuity of the shape Hessian will be completely sufficient.

With the above shape derivatives at hand we arrive at a second order Taylor expansion for the perturbed shape functional
\[
J(D_\varepsilon[\kappa]) = J(D) + \varepsilon d J(D)[\kappa] + \varepsilon^2 d^2 J(D)[\kappa, \kappa] + O(\varepsilon^3).
\]

4. A Class of Stochastic Domains

We obtain a family of stochastic domains from the preceding shape calculus by admitting random fields \(V(x, \omega) = \kappa(x, \omega)n(x)\) as boundary variations. We fix an unperturbed nominal domain \(D \subset \mathbb{R}^n, n = 2, 3,\) with boundary manifold \(\partial D \in C^{3,1}\).

We denote by \(n(x)\) the exterior unit normal vector to \(\partial D\) at the boundary point \(x \in \partial D\). Then for sufficiently small \(\varepsilon_0 > 0\) and for some scalar function \(\kappa(x) \in C^{2,1}(\partial D, \mathbb{R})\) with \(\|\kappa\|_{C^{2,1}(\partial D)} \leq 1\), the family of surfaces \(\partial D_\varepsilon = \{ x + \varepsilon \kappa(x)n(x) : x \in \partial D \}\) belongs to \(C^{2,1}\) since \(n \in C^{2,1}(\partial D)\).

To specify random boundary variations, we consider a probability space \((\Omega, \Sigma, P)\) of admissible boundary perturbation fields \(\kappa \in C^{2,1}(\partial D) \otimes L^2(\Omega, dP)\). With the random field \(\kappa \in C^{2,1}(\partial D) \otimes L^2(\Omega, dP)\) and a fixed perturbation parameter \(\varepsilon > 0\) which is sufficiently small, we associate boundaries \(\partial D_\varepsilon(\omega)\) through the parametric representation
\[
(4.1) \quad \gamma_\varepsilon : \partial D \times \Omega \to \mathbb{R}^n, \quad \gamma_\varepsilon(x, \omega) := x + \varepsilon \kappa(x, \omega)n(x)
\]
where
\[
(4.2) \quad \|\kappa(\cdot, \omega)\|_{C^{2,1}(\partial D)} \leq 1 \quad \text{for } P\text{-almost all } \omega \in \Omega,
\]
which we assume in what follows. A realization of the stochastic domain \(D_\varepsilon(\omega)\) is the interior of the boundary manifold
\[
\partial D_\varepsilon(\omega) := \{ \gamma_\varepsilon(x, \omega) : x \in \partial D \}, \quad \omega \in \Omega.
\]
The assumption (4.2) implies in particular that the domain $D_\varepsilon(\omega)$ does almost surely not degenerate if $0 \leq \varepsilon < \varepsilon_0$.

Due to $\kappa \in C^{2,1}(D) \otimes L^2(\Omega, dP)$ the mean field

$$E_\kappa(x) := \int_\Omega \kappa(x, \omega) dP(\omega) = E(\kappa(x, \omega)), \quad x \in \partial D,$$

and the two-point correlation

$$\text{Cor}_\kappa(x, y) := \int_\Omega \kappa(x, \omega) \kappa(y, \omega) dP(\omega)$$

$$= E(\kappa(x, \omega) \kappa(y, \omega)), \quad x, y \in \partial D,$$

of the boundary variation $\kappa(x, \omega)$ under consideration are point-wise finite. Here, the notation $E(\cdot)$ denotes the expectation with respect to the probability measure $P(\omega)$. For what follows we assume that the expectation and the two-point correlation of $\kappa(x, \omega)$ are given. Without loss of generality we assume that the perturbation field $\kappa(x, \omega)$ is centered, i.e., that

$$E_\kappa(x) = 0.$$

5. **Stochastic Shape Functionals**

We consider now a shape functional with respect to the stochastic domain $D_\varepsilon(\omega)$, that is

$$J(D_\varepsilon(\omega)) = \int_{D_\varepsilon(\omega)} j(x, u(x, \omega), \nabla u(x, \omega)) dx,$$

where also the state function $u(\omega)$ depends on the stochastic variable through the boundary value problem

$$-\Delta u(\omega) = f \text{ in } D(\omega), \quad u(\omega) = g \text{ on } \partial D(\omega).$$

Our aim is the deterministic, approximate computation of the expectation $E_J := E(J(D_\varepsilon(\omega)))$ and of the variance $\text{Var}_J := \text{Var}(J(D_\varepsilon(\omega)))$ of the random solution $J(D_\varepsilon(\omega))$ of (5.1). Since the random shape functional’s dependence on the boundary variation $\kappa$ is nonlinear, we are going to derive next approximations for these deterministic quantities under the assumption of sufficiently small boundary perturbations, i.e., provided that the perturbation amplitude $\varepsilon > 0$ in (4.1) is sufficiently small. The key issue is the Taylor expansion (3.8) which will be used to linearize the problem.
Lemma 1. Assume a random normal variation \( V(x, \omega) := \kappa(x, \omega)n(x) \) that satisfy (4.2). Then, for sufficiently small \( \varepsilon > 0 \), the random shape functional \( J(D_\varepsilon(\omega)) \) (5.1), (5.2) admits the asymptotic expansion

\[
J(D_\varepsilon(\omega)) = J(D) + \varepsilon dJ(D)[\kappa(\omega)] + \frac{\varepsilon^2}{2} d^2J(D)[\kappa(\omega), \kappa(\omega)] + \mathcal{O}(\varepsilon^3)
\]

for \( P \)-a.e. \( \omega \in \Omega \). Herein, the functionals \( J(D) \) and \( dJ(D)[\kappa(\omega)] \) are defined with respect to the nominal domain \( D \), that is

\[
J(D) = \int_D j(x, u(x), \nabla u(x)) dx
\]

and

\[
dJ(D)[\kappa(\omega)] = \int_{\partial D} \kappa(\omega) \left\{ j(x, g, \nabla u) + \left( \langle j_z(x, g, \nabla u), n \rangle - \frac{\partial p}{\partial n} \right) \frac{\partial (g - u)}{\partial n} \right\} d\sigma_x,
\]

invoking the deterministic boundary value problems

\[
-\Delta u = f \text{ in } D, \quad u = g \text{ on } \partial D,
\]

and

\[
-\Delta p = j_y(\cdot, u, \nabla u) - \text{div} j_z(\cdot, u, \nabla u) \text{ in } D, \quad p = 0 \text{ on } \partial D.
\]

Proof. We apply the Taylor expansion (3.8) for an arbitrary, fixed realization \( \kappa(\cdot, \omega) \), \( \omega \in \Omega \). Under the assumption (4.2) we arrive at (5.3).

\[ \square \]

6. Computing the statistics

Based on the Taylor expansion (5.3) of the random shape functional \( J(D_\varepsilon(\omega)) \) (5.1), (5.2) we derive next two deterministic expressions for its second order statistics, i.e., the mean field and the variance.

Lemma 2. The expectation \( E_J \) satisfies

\[
E_J = J(D) + \mathcal{O}(\varepsilon^2)
\]

with \( J(D) \) being the deterministic shape functional (5.4), (5.6).

Proof. If \( \|\kappa(\cdot, \omega)\|_{C^2(D)} \leq 1 \) almost surely, we insert the Taylor expansion (5.3) and arrive at

\[
E(J(D_\varepsilon(\omega))) = E(J(D) + \varepsilon dJ(D)[\kappa(\omega)] + \mathcal{O}(\varepsilon^2))
\]

\[= J(D) + \varepsilon E(dJ(D)[\kappa(\omega)]) + \mathcal{O}(\varepsilon^2).\]
Abbreviating

\[ h(x) := j(x, g(x), \nabla u(x)) \]

we have thus to show that

\[ E(dJ(D)[k(\omega)]) = \int_{\Omega} \int_{\partial D} k(\omega) h(x) d\sigma_x dP(\omega) = 0. \]

In view of (4.4), this assertion follows by applying Fubini’s theorem

\[ E(dJ(D)[k(\omega)]) = \int_{\partial D} \left[ \int_{\Omega} k(\omega) dP(\omega) \right] h(x) d\sigma_x = 0, \]

which completes the proof. \(\square\)

**Lemma 3.** For \( \varepsilon > 0 \) sufficiently small, the variance \( \text{Var}_J \) of the random shape functional satisfies

\[ \text{Var}_J = \varepsilon^2 \text{Var}(dJ(D)[k(\omega)]) + O(\varepsilon^3) \]

with \( dJ(D)[k(\omega)] \) given by (5.5), (5.7).

**Proof.** We expand both terms on the right hand side of

\[ \text{Var}_J = E(\left(J(D_\varepsilon(\omega))\right)^2) - E(\left(J(D_\varepsilon(\omega))\right))^2. \]

On the one hand, we get

\[ E(\left(J(D_\varepsilon(\omega))\right)^2) = E\left(\left[J(D) + \varepsilon dJ(D)[k(\omega)] + \frac{\varepsilon^2}{2} d^2 J(D)[k(\omega), k(\omega)] + O(\varepsilon^3)\right]^2\right) \]

\[ = J(D)^2 + \varepsilon^2 E(\left(dJ(D)[k(\omega)]\right)^2) + 2\varepsilon J(D) E(\left(dJ(D)[k(\omega)]\right)) \]

\[ + \varepsilon^2 J(D) E(\left(d^2 J(D)[k(\omega), k(\omega)]\right)) + O(\varepsilon^3). \]

On the other hand, we find

\[ E(\left(J(D_\varepsilon(\omega))\right)^2) = \left(\left.J(D) + \varepsilon E(\left(dJ(D)[k(\omega)]\right)) + \frac{\varepsilon^2}{2} E\left(d^2 J(D)[k(\omega), k(\omega)]\right) + O(\varepsilon^3)\right)^2\right) \]

\[ = J(D)^2 + \varepsilon^2 E(\left(dJ(D)[k(\omega)]\right)^2) + 2\varepsilon J(D) E(\left(dJ(D)[k(\omega)]\right)) \]

\[ + \varepsilon^2 J(D) E(\left(d^2 J(D)[k(\omega), k(\omega)]\right)) + O(\varepsilon^3). \]
Subtracting both equations and observing \( E\left(dJ(D)[\kappa(\omega)]\right) = 0 \) yields the desired result. □

Using again the abbreviation (6.2) we shall compute
\[
E\left(dJ(D)[\kappa(\omega)]^2\right) = \int_\Omega \left[ \int_{\partial D} \kappa(x, \omega) h(x) d\sigma_x \right]^2 dP(\omega).
\]
This expression depends nonlinearly on the perturbation field \( \kappa(\omega) \). However, by applying once more Fubini’s theorem, we arrive at
\[
E\left(dJ(D)[\kappa(\omega)]^2\right) = \int_\Omega \int_{\partial D} \left[ \int_{\partial D} \kappa(x, \omega) \kappa(y, \omega) dP(\omega) \right] h(x) h(y) d\sigma_y d\sigma_x.
\]
Thus, from the two-point correlation of \( \kappa \) (cf. (4.3)) we derive the sought quantity via
\[
(6.5) \quad E\left(dJ(D)[\kappa(\omega)]^2\right) = \int_{\partial D} \int_{\partial D} \text{Cor}_\kappa(x, y) h(x) h(y) d\sigma_y d\sigma_x.
\]
Consequently, we need just a fast quadrature method on \( \partial D \times \partial D \) to approximate the variance (6.3) via (6.5).

Fast quadrature methods for tensor product domains can be derived by Smolyak’s construction [32]. That way one can realize complexity bounds that reflect essentially, i.e., up to a logarithmic factor, the cost of a quadrature rule on \( \partial D \). For particular quadrature schemes we refer the reader to [4, 5, 18, 22, 23, 24] and the references therein.

7. Numerical results

We consider the unit square \([0, 1]^2\) as nominal domain \( D \). Only the upper edge \( y = 1 \) is assumed to be uncertain, that is
\[
D_\varepsilon(\omega) = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1 + \varepsilon f(x, \omega)\}.
\]
We model \( f(x, \omega) \in C^{2,1}([0, 1]) \) as a cubic spline with random coefficients and \( f^{(i)}(0) = f^{(i)}(1) = 0, i = 0, 1, 2 \). That is, denoting by \( B_4 \) the centered cardinal B-spline on \([-2, 2]\), we have
\[
f(x, \omega) = \sum_{i=1}^N \alpha_i(\omega) B_4((N + 3)x - i - 1).
\]
The coefficients \( \alpha_i(\omega) \) are supposed to be equally distributed in \([-0.5, 0.5]\) and uncorrelated. We refer to Fig. 3 for an illustration of the choice \( N = 5 \).
The two-point correlation of $f$ is given by

$$\text{Cor}_f(s, t) = \frac{1}{12} \sum_{i=0}^{N} B_4((N + 3)s - i - 1)B_4((N + 3)t - i - 1).$$

This yields the following two-point correlation of $\kappa$:

$$\text{Cor}_\kappa(x, y) = \begin{cases} 
\text{Cor}_f(s, t), & x = (s, 1) \land y = (t, 1), \\
0, & \text{elsewhere}.
\end{cases}$$

A plot of the two-point correlation of $f$ for $N = 5$ is presented in Fig. 4. Notice that the present setup fulfills our smoothness assumptions on the stochastic boundary since the random boundary variations are $C^{2,1}$. 

Figure 3. The random domain $D_\varepsilon(\omega)$.

Figure 4. The two-point correlation of $f$ ($N = 5$).
We consider the computation of the torsional rigidity of an infinite cylindric circular bar which is homogeneous and isotropic with cross section \( D_\varepsilon(\omega) \in \mathbb{R}^2 \) and shear modulus \( G = 1 \). According to Subsection 2.1, we have to compute the output functional

\[
J(D_\varepsilon(\omega)) = 2 \int_{D_\varepsilon(\omega)} u(x, \omega) dx
\]

with \( u(\omega) \) being defined by

\[-\Delta u(\omega) = 2 \text{ in } D_\varepsilon(\omega), \quad u(\omega) = 0 \text{ on } \partial D_\varepsilon(\omega).\]

We consider the case of \( N = 1, N = 5 \) and \( N = 10 \) B-splines. For each value \( \varepsilon_i := i/60, i = 1, 2, \ldots, 60 \), we sample randomly 250 times the torsion and compute the samples’ mean and empirical variance. The computation is performed by the finite element method with additive multilevel preconditioning. In Fig. 5 we visualized the approximate mean (cf. (6.1))

\[(7.1) \quad J(D) = 2 \int_D u(x) dx,\]

indicated by the bold line, and the sample average, indicated by the circles. Moreover the gray area is bounded by the curve \( J(D) \) and \( J(D) - c_N \varepsilon^2 \) \( (c_1 = 0.01, c_5 = 0.04, c_{10} = 0.08) \). The results validate that

\[
|J(D) - \mathbb{E}(J(D_\varepsilon(\omega)))| = \mathcal{O}(\varepsilon^2).
\]

The shape gradient in normal direction \( \kappa n \) of the torsional rigidity (7.1) is given by

\[
dJ(D)[\kappa] = \int_{\partial D} \kappa \left[ \frac{\partial \pi}{\partial n} \right]^2 d\sigma
\]

since the adjoint state satisfies \( p = u \) (see (3.6) and (3.7)). Thus, according to (6.2), (6.3), (6.5) the formula to approximate the variance reads as

\[
V(J) = \varepsilon^2 \int_0^1 \int_0^1 \text{Cor}_\varepsilon(s, t) \left[ \frac{\partial \pi}{\partial n}(s, 1) \right]^2 \left[ \frac{\partial \pi}{\partial n}(t, 1) \right]^2 dt ds.
\]

In Fig. 6 we plotted the sample variance versus the perturbation parameter \( \varepsilon \) (indicated by circles). The bold line corresponds to the deterministically computed variance \( \varepsilon^2 V(J) \). The gray area marks this curve \( \pm c_N \varepsilon \) \( (c_1 = 1.4 \cdot 10^{-4}, c_5 = 2.8 \cdot 10^{-4}, c_{10} = 3.6 \cdot 10^{-4}) \). The results confirm clearly the behaviour

\[
|\varepsilon^2 V(J) - \text{Var}(J(D_\varepsilon(\omega)))| = \mathcal{O}(\varepsilon).
\]
Figure 5. Mean of the torsional rigidity (top: $N = 1$, middle: $N = 5$, bottom: $N = 10$).
Figure 6. Variance of the torsional rigidity (top: $N = 1$, middle: $N = 5$, bottom: $N = 10$).
8. Conclusion

In the present paper we modeled and investigated output functionals of boundary value problems on stochastic domains. Under the assumption of a small perturbation amplitude, almost sure with respect to a probability measure on the space of admissible perturbations, we linearized the functional’s nonlinear dependence on the random perturbation by means of a second order shape calculus. Based on this linearization we computed, with leading order, deterministic expressions for the statistics of the random output functionals. By numerical experiments we validated and quantified the derived asymptotic expansions.

References


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