Large Data Existence Result for Unsteady Flows of Inhomogeneous Heat-Conducting Incompressible Fluids

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no. 430

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereichs 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

Bonn, Dezember 2008

LARGE DATA EXISTENCE RESULT FOR UNSTEADY FLOWS OF INHOMOGENEOUS HEAT-CONDUCTING INCOMPRESSIBLE FLUIDS

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ABSTRACT. We consider unsteady flows of inhomogeneous, incompressible, shear-thickening and heat conducting fluids where the viscosity depends on the density, the temperature and the shear rate, and the heat conductivity depends on the temperature and the density. For any values of initial total mass and initial total energy we establish the long-time existence of weak solution to flows at an arbitrary bounded domain with Lipschitz boundary.

1. INTRODUCTION

We are interested in understanding the mathematical properties of unsteady flows of incompressible, inhomogeneous, shear thickening, heat-conducting fluids described in terms of the density ϱ, the velocity $\boldsymbol{v} = (v_1, v_2, v_3)$, the pressure p and the temperature θ through the following set of equations (without any difficulty we could include given sources of linear momentum and energy into the equations)

$$\begin{aligned} &\varrho_{,t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0, \qquad\qquad \operatorname{div}\boldsymbol{v} = 0, \\ &(\varrho\boldsymbol{v})_{,t} + \operatorname{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v})) = -\nabla p, \\ &(\varrho\theta)_{,t} + \operatorname{div}(\varrho\theta\boldsymbol{v}) - \operatorname{div}\boldsymbol{q}(\varrho,\theta,\nabla\theta) = \mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v}))\cdot\mathbf{D}(\boldsymbol{v}). \end{aligned} \tag{1.1}$$

These equations, and all functions involved in their descriptions as well, are considered in $Q := (0,T)\times\Omega$, where

$$\begin{aligned} &\Omega\subset\mathbb{R}^3 \text{ is an open, connected, bounded set with} \\ &\text{Lipschitz boundary } \partial\Omega, \text{ and } T\in(0,\infty). \end{aligned} \tag{1.2}$$

For simplicity, we first restrict ourselves to the boundary conditions

$$\boldsymbol{v} = \mathbf{0} \quad\text{ and }\quad \boldsymbol{q}\cdot\boldsymbol{n} = 0 \quad\text{ on } [0,T]\times\partial\Omega. \tag{1.3}$$

The initial density is supposed to be bounded and the initial total energy is integrable, i.e.,

$$\varrho(0,\cdot) = \varrho_0 \in L^\infty(\Omega), \qquad \varrho(|\boldsymbol{v}|^2/2+\theta)(0,\cdot) = \varrho_0(|\boldsymbol{v}_0|^2/2+\theta_0)\in L^1(\Omega), \tag{1.4}$$

1991 *Mathematics Subject Classification.* 35Q35, 35D05, 76D03.

Key words and phrases. generalized Navier-Stokes-Fourier system, inhomogeneous fluid, incompressible fluid, shear-rate dependent viscosity, heat-conducting fluid, temperature dependent viscosity, weak solution, long-time existence, large data.

This research has been performed during the stay of J. Frehse at Jindřich Nečas Center for Mathematical Modeling, project LC06052 financed by MSMT..

J. Málek's contribution is a part of the research project MSM 0021620839 financed by MSMT; the support of GACR 201/08/0315 and GACR 201/06/0352 is also acknowledged.

M. Růžička has been supported by DFG Forschergruppe "Nonlinear Partial Differential Equations: Theoretical and Numerical Analysis".

and

$$
\begin{aligned}
&0 < \varrho_* \le \varrho_0(x) \le \varrho^* < +\infty && \text{for almost all (a.a.) } x \in \Omega\,,\\
&0 < \theta_* \le \theta_0(x) && \text{for a.a. } x \in \Omega\,,
\end{aligned}
\tag{1.5}
$$

where ϱ_*, ϱ^* and θ_* are constants. We shall write in short *Problem* $(\mathcal{P})$ to denote the problem to find ϱ, $\boldsymbol{v}$, p and θ satisfying (1.1)–(1.5).

As follows from (1.1) we assume that the deviatoric (traceless) part S of the Cauchy stress depends on the density, the temperature and the symmetric part of the velocity gradient $\mathsf{D}(\boldsymbol{v})$, and the thermal flux $\boldsymbol{q}$ is a function of the density, the temperature and its gradient. In order to have a clearer picture about the admissible structure of these functions we may think that S and $\boldsymbol{q}$ behave as (for $\varrho > 0$, $\theta > 0$, $\mathsf{D} \in \mathbb{R}^{3\times 3}$ symmetric)

$$
\begin{aligned}
&\mathsf{S}(\varrho,\theta,\mathsf{D}) = \nu(\varrho,\theta,|\mathsf{D}|^2)\mathsf{D} \sim \mu(\varrho,\theta)\big(\epsilon + |\mathsf{D}|^2\big)^{(r-2)/2}\mathsf{D}\,, && r \in (1,\infty)\,,\\
&\boldsymbol{q}(\varrho,\theta,\nabla\theta) = k(\varrho,\theta)\nabla\theta \sim \kappa(\varrho)\theta^\beta\nabla\theta = \frac{\kappa(\varrho)}{\beta+1}\nabla\theta^{\beta+1}\,, && \beta \in \mathbb{R}\,,
\end{aligned}
\tag{1.6}
$$

where $\epsilon \in [0,1]$, and for all $\varrho > 0$ and $\theta > 0$ it holds

$$
\begin{aligned}
&0 < m_* \le \mu(\varrho,\theta) \le m^* < +\infty \quad \text{and} \quad 0 < \kappa_* \le \kappa(\varrho) \le \kappa^* < +\infty\\
&(m_*,\ m^*,\ \kappa_*,\ \kappa^* \text{ are constants})\,.
\end{aligned}
\tag{1.7}
$$

Thus, in particular, for all $\varrho > 0$, $\theta > 0$, and $\mathsf{D}, \mathsf{B} \in \mathbb{R}^{3\times 3}$ symmetric

$$
\begin{aligned}
&\mathsf{S}(\varrho,\theta,\mathsf{D})\cdot\mathsf{D} \ge m_*\big(\epsilon + |\mathsf{D}|^2\big)^{(r-2)/2}|\mathsf{D}|^2 \ge 0\,, \qquad (m_* > 0)\\
&|\mathsf{S}(\varrho,\theta,\mathsf{D})| \le m^*\big(\epsilon + |\mathsf{D}|^2\big)^{(r-2)/2}|\mathsf{D}|\,,\\
&\big(\mathsf{S}(\varrho,\theta,\mathsf{D}) - \mathsf{S}(\varrho,\theta,\mathsf{B})\big)\cdot\big(\mathsf{D} - \mathsf{B}\big) \ge 0\,,
\end{aligned}
\tag{1.8}
$$

and for all $\varrho > 0$, $\theta > 0$, $\nabla\theta \in \mathbb{R}^3$

$$
\begin{aligned}
&\boldsymbol{q}(\varrho,\theta,\nabla\theta)\cdot\nabla\theta \ge \kappa_*\,\theta^\beta\,|\nabla\theta|^2 = \frac{4\kappa_*}{(\beta+2)^2}|\nabla\theta^{(\beta+2)/2}|^2 \ge 0\,, \qquad (\kappa_* > 0)\\
&|\boldsymbol{q}(\varrho,\theta,\nabla\theta)| \le \kappa^*\,\theta^\beta\,|\nabla\theta|\,.
\end{aligned}
\tag{1.9}
$$

The aim of this paper is to establish the following result.

Theorem 1.1. *Let* S *and* $\boldsymbol{q}$ *be continuous functions of the form* (1.6) *satisfying* (1.8) *and* (1.9) *with*

$$
r \ge 11/5 \qquad \text{and} \qquad \beta > -\min\{\tfrac{2}{3}, \tfrac{3r-5}{3(r-1)}\}\,. \tag{1.10}
$$

Then, for any set of data Ω, T, ϱ_0, $\boldsymbol{v}_0$, θ_0 *satisfying* (1.2), (1.4) *and* (1.5)*, there is a weak solution to Problem* $(\mathcal{P})$ *in the sense of Definition 2.1.*

Only for simplicity, we omitted external forces and external sources of energy in the governing equations (1.1).

There have been many studies concerning the mathematical analysis of time-dependent flows of inhomogeneous, incompressible fluids with the viscosity depending on or independent of $|\mathsf{D}(\boldsymbol{v})|^2$ and depending on or independent of the density. We discuss the most important earlier contributions in the next section. The main novelty of this paper consists in including the changes due to heat conduction into the model. We are not aware any such study in available sources.

The paper is organized in the following way: In the next section, after we introduce the appropriate function spaces we provide the precise definition of the notion of weak solution to Problem $(\mathcal{P})$. We also relate our result to earlier studies dealing with large-data existence theory for inhomogeneous incompressible fluids. Then in Section 3, we establish the so-called stability of the considered problem with respect to weakly converging (approximate) solutions. (We formally derive uniform estimates for sequences ϱ^n, $\boldsymbol{v}^n$ and θ^n that are assumed to solve apriori a perturbed problem $(\mathcal{P}_n)$. The derived estimates generate subsequences weakly converging to $(\varrho, \theta, \boldsymbol{v})$. The goal is to show that the weak limits ϱ, $\boldsymbol{v}$ and θ form a solution to Problem $(\mathcal{P})$.) This is definitely not a complete proof as we do not construct appropriate approximations here. Instead of going into the details of their constructions we prefer to provide references to studies where the approximations to problems similar to Problem $(\mathcal{P})$ are introduced and their existence is proved. We believe that the interested reader can modify these procedures with some effort. There are three important tools used within the proof of weak stability of Problem $(\mathcal{P})$: (i) Div-Curl Lemma developed by Tartar [21] and extended to time-space setting by Feireisl (see [10], [11], and also Lemma 2.1 below), (ii) renormalized solution for transport equations due to Lions and DiPerna (see [9] and [17]), and (iii) the Integration by parts formula as stated and proved in Section 4 (it has a clear origin in Frehse and Růžička [13]). The last tool seems to be a new conquest of this study. The final section includes possible extensions and concluding remarks.

2. Definition of solution and Former studies

We shall use the following notation for relevant Banach spaces of functions defined on $\Omega \subset \mathbb{R}^3$ and on $Q = (0,T) \times \Omega$. For any $q \in [1,\infty]$, $L^q := L^q(\Omega)$ denotes the Lebesgue spaces with the norm $\|\cdot\|_q$ and $W^{1,q} := W^{1,q}(\Omega)$ is used to denote the Sobolev spaces with the norm $\|\cdot\|_{1,q}$. We denote by $W_0^{1,q} := W_0^{1,q}(\Omega)$ the closure of $C_0^\infty(\Omega)$ functions in the norm of $W^{1,q}$. If X is a Banach spaces of scalar functions, then X^3, X^4 or $X^{3\times 3}$, $X^{4\times 4}$ denotes the space of vector- or tensor-valued functions so that each their component belongs to X. Further, we use the following notation for the spaces of function with zero divergence and their duals ($r' = r/(r-1)$)

$$W^{1,q}_{0,\mathrm{div}} := \{\boldsymbol{v} \in W_0^{1,q}(\Omega)^3; \operatorname{div} \boldsymbol{v} = 0\}\,, \quad W^{-1,q'} = (W_0^{1,q})^*\,,\, W^{-1,q'}_{\mathrm{div}} = (W^{1,q}_{0,\mathrm{div}})^*\,.$$

Also, L^q_{div} denotes the closure of $W^{1,q}_{0,\mathrm{div}}$ in $L^q(\Omega)^3$. The symbols $L^q(0,T;X)$ and $C([0,T];X)$ denote the standard Bochner spaces. We write (a,b) instead of $\int_\Omega a(x)b(x)\,dx$ whenever $ab \in L^1(\Omega)$, and use the brackets $\langle a,b\rangle$ to denote the duality pairing for $a \in X^*$ and $b \in X$. We use $C([0,T];L^q_{\mathrm{weak}})$ and $C([0,T];L^q_{\mathrm{div,weak}})$ to denote the spaces of functions $\rho \in L^\infty(0,T;L^q)$, or $\boldsymbol{v} \in L^\infty(0,T;L^q_{\mathrm{div}})$, satisfying $(\rho(t),z) \in C([0,T])$ for all $z \in L^{q'}$, or $(\boldsymbol{v}(t),\boldsymbol{\varphi}) \in C([0,T])$ for all $\boldsymbol{\varphi} \in L^{q'}_{\mathrm{div}}$.

Definition 2.1. *Let* Ω, T, ϱ_0, $\boldsymbol{v}_0$, θ_0 *satisfy* (1.2), (1.4) *and* (1.5). *Assume that* $\mathbf{S}$ *and* $\boldsymbol{q}$ *satisfy* (1.8) *and* (1.9) *with*

$$r \geq 11/5 \qquad \text{and} \qquad \beta > -\min\left\{\tfrac{2}{3}, \tfrac{3r-5}{3(r-1)}\right\}. \tag{2.1}$$

We say that $(\varrho, \boldsymbol{v}, \theta)$ is a weak solution to Problem $(\mathcal{P})$ if

$$0 < \varrho_* \le \varrho(t,x) \le \varrho^* \qquad \text{for a.a. } (t,x) \in Q\,, \tag{2.2}$$

$$\begin{aligned} &\varrho \in C([0,T]; L^q) \quad \text{ for all } q \in [1,\infty) \\ &\varrho_{,t} \in L^{5r/3}(0,T;(W^{1,5r/(5r-3)})^*)\,, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\boldsymbol{v} \in C([0,T]; L^2_{div,weak}) \cap L^r(0,T;W^{1,r}_{0,div}) \\ &(\varrho\boldsymbol{v})_{,t} \in L^{r'}(0,T;W^{-1,r'}_{div})\,, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\theta \in L^\infty(0,T;L^1(\Omega)) \quad \text{ and } \quad \theta(t,x) \ge \theta_* > 0 \quad \text{ for a.a.} \\ &\theta^{\frac{\beta-\lambda+1}{2}} \in L^2(0,T;W^{1,2}) \quad \text{ for all } \lambda \in (0,1)\,, \\ &(\varrho\theta)_{,t} \in L^1(0,T;(W^{1,q})^*) \quad \text{ with } q \text{ sufficiently large} \end{aligned} \tag{2.5}$$

and $(\varrho, \boldsymbol{v}, \theta)$ fulfill the following weak formulations

$$\begin{aligned} &\int_0^T \langle \varrho_{,t}, z\rangle - (\varrho\boldsymbol{v}, \nabla z)\, dt = 0 \\ &\qquad\qquad \text{for all } z \in L^s(0,T;W^{1,s}) \text{ with } s = 5r/(5r-3) \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\int_0^T \langle (\varrho\boldsymbol{v})_{,t}, \boldsymbol{\varphi}\rangle - (\varrho\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{\varphi}) + \big(\mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{\varphi})\big)\, dt = 0 \\ &\qquad\qquad \text{for all } \boldsymbol{\varphi} \in L^r(0,T;W^{1,r}_{0,div}) \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\int_0^T \langle (\varrho\theta)_{,t}, h\rangle - (\varrho\theta\boldsymbol{v}, \nabla h) + (k(\varrho,\theta)\nabla\theta, \nabla h)\ dt = \int_0^T \big(\mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{v})h\big)\, dt \\ &\qquad\qquad \text{for all } h \in L^\infty(0,T;W^{1,q}) \text{ with } q \text{ sufficiently large} \end{aligned} \tag{2.8}$$

and the initial conditions are attained in the following sense[1]

$$\begin{aligned} &\lim_{t\to 0+} \|\varrho(t) - \varrho_0\|_q + \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 = 0 \qquad \text{for any } q \in [1,\infty) \\ &\lim_{t\to 0+} (\varrho(t), \theta(t)) = (\varrho_0, \theta_0). \end{aligned} \tag{2.9}$$

Since we consider in (2.7) only divergenceless test functions, the pressure does not appear in the considered definition of weak solution. One can however identify the pressure p from (2.7) aposteriori. To this end, we take $\boldsymbol{\varphi}$ of the form $\boldsymbol{\varphi} = \chi_{(0,t)}\boldsymbol{\psi}$ in (2.7), $\chi_{(0,t)}$ is the characteristic function of $(0,t)$ and $\boldsymbol{\psi} \in W^{1,r}_{0,\mathrm{div}}$, and compare the result with two auxiliary Stokes problems (with homogeneous Dirichlet boundary conditions and pressures having zero mean values)

$$\operatorname{div}\boldsymbol{u}^1 = 0\,, \quad -\Delta\boldsymbol{u}^1 + \nabla p^1 = \operatorname{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v} - \mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v})))\,,$$

and

$$\operatorname{div}\boldsymbol{u}^2 = 0\,, \quad -\Delta\boldsymbol{u}^2 + \nabla p^2 = (\varrho\boldsymbol{v})(t) - \varrho_0\boldsymbol{v}_0 \in L^2(\Omega)^3 \hookrightarrow W^{-1,2}(\Omega)^3\,.$$

[1]The properties regarding the attainment of the initial condition for the temperature can be strengthened. Using the concept of renormalized solution it is possible to show that for example $\|\sqrt{\theta(t)} - \sqrt{\theta_0}\|_2 \to 0$ as $t \to 0+$. We skip the proof of this statement here, referring to [5] for details. On the other hand, it follows from (2.9) that $\lim_{t\to 0+} \varrho(|\boldsymbol{v}|^2/2+\theta)(t,\cdot) = \varrho_0(|\boldsymbol{v}_0|^2/2+\theta_0)$, see $(1.4)_2$ for a comparison.

As the result, we obtain the existence of p of the form[2]

$$p = p^1 + (p^2)_{,t} \quad \text{where } p^1 \in L^{r'}(Q) \text{ and } p^2 \in L^\infty(0,T;L^2)\,. \tag{2.10}$$

Thus, due to the presence of the time derivative $(p^2)_{,t}$ in the above form for p, we do not know if p is an integrable function on Q.

Long-time and large-data existence theory for inhomogeneous incompressible fluids were investigated in several contributions. The first group of results concerns Newtonian fluids ($r = 2$). The results established by Antontsev and Kazhikov in the seventies are summarized in the monograph Antontsev et al. [1]. P.-L. Lions in the first chapter of his book [17] provides a detail exposition that includes several important extensions. In particular, using the concept of renormalized solution he establishes new convergence and continuity properties of the density that may vanish at some parts of Ω investigating herewith the models where the viscosity depends on the density. The another group of results deals with (isothermal flows of incompressible inhomogeneous) fluids where the viscosity depends on the shear rate ($r \neq 2$). We would like to mention the paper by Fernández-Cara et al. [12] where the existence of weak solution is proved for $r \geq \frac{12}{5}$ (using the fact that in such range of r for models where the viscosity is not changing with the density $\boldsymbol{v}_{,t} \in L^2(Q)$). In the spatially periodic setting this result has been improved by Guillén-González [15] and Retelsdorf [19] using higher differentiability method for the models that can be viewed as the power-law perturbation of the Newtonian model with the constant viscosity. Frehse and Růžička [13] treating the problem with no-slip boundary conditions with a viscosity that depends on both the density and the shear-rate established the existence result for $r > 11/5$. Theorem 1.1 slightly improves the result established in [13] in two directions: (i) the limiting case $r = 11/5$ is included, and (ii) the relationship between the stress tensor $\mathbf{S}$ and $\mathbf{D}(\boldsymbol{v})$ does not require the existence of potential to $\mathbf{S}$ (needed in [13]). Of course, the fact that we consider the full thermodynamic model for inhomogeneous incompressible fluids is the essential novelty of Theorem 1.1. For the sake of completeness we provide also basic references to analogous models for homogeneous power-law type fluids: regarding the isothermal case a summary of the results prior to 2006 can be found in [18]. The most recent results are established in [8] ($r > 6/5$) and [4] ($r > 8/5$ including also the dependence of the viscosity on the pressure, and considering Navier's slip boundary conditions). The thermal flows of incompressible homogeneous fluids are analyzed in [3] ($r = 2$) and [5] ($r > 9/5$ and the viscosity depending on the pressure, temperature and the shear-rate).

For the later reference we also state Div-Curl Lemma as formulated and reproved in [11], see [21] for the original work, and [10] for the available reference. We formulate the lemma in the form suitable to our framework. For $\boldsymbol{a} = (a_0, a_1, a_2, a_3)$ we set

$$\mathrm{Div}_{t,x}\,\boldsymbol{a} := (a_0)_{,t} + \sum_{i=1}^{3} (a_i)_{,x_i} \quad \text{and} \quad \mathrm{Curl}_{t,x}\,\boldsymbol{a} := \nabla_{t,x}\boldsymbol{a} - (\nabla_{t,x}\boldsymbol{a})^T\,,$$

where $\nabla_{t,x}\boldsymbol{a} := (\boldsymbol{a}_{,t}^T, \boldsymbol{a}_{,x_1}^T, \boldsymbol{a}_{,x_2}^T, \boldsymbol{a}_{,x_3}^T)\,.$

[2]The properties of the pressures p^1 and p^2 thus follow from $W_0^{1,q}$-solvability of the Stokes problems. To our knowledge, this holds for domains with Lipschitz boundary at least in the range $r \in [11/5, 3)$ analyzed in detail in this paper, see [2]. In general, C^1 or C^2 domains should suffice for such results, see [14], [22] or [8].

Lemma 2.1. *Let $Q \subset \mathbb{R}^4$ be a bounded set. Let p, q, ℓ, $s \in (1,\infty)$ be such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{\ell}$. Assume that $\{\boldsymbol{a}^n\}$ and $\{\boldsymbol{b}^n\}$ satisfy*

$$\boldsymbol{a}^n \rightharpoonup \boldsymbol{a} \quad \textit{weakly in } L^p(Q)^4 \quad \textit{and} \quad \boldsymbol{b}^n \rightharpoonup \boldsymbol{b} \quad \textit{weakly in } L^q(Q)^4\,, \tag{2.11}$$

and

$$\begin{aligned}&\mathrm{Div}_{t,x}\,\boldsymbol{a}^n \ \textit{ and } \ \mathrm{Curl}_{t,x}\,\boldsymbol{b}^n \ \textit{ are precompact}\\ &\textit{in } W^{-1,s}(Q) \textit{ and } W^{-1,s}(Q)^{4\times 4}\,,\ \textit{respectively.}\end{aligned} \tag{2.12}$$

Then

$$\boldsymbol{a}^n \cdot \boldsymbol{b}^n \rightharpoonup \boldsymbol{a}\cdot\boldsymbol{b} \quad \textit{weakly in } L^\ell(Q)\,, \tag{2.13}$$

where $\cdot$ represents the scalar product in $\mathbb{R}^4$.

3. Proof of Theorem 1.1

We restrict ourselves to the essential part of the proof here. For this purpose, we introduce Problems $(\mathcal{P}_n)$ that will differ from the Problem $(\mathcal{P})$ by a perturbation $\boldsymbol{f}^n$, vanishing in $L^{r'}(0,T;W^{-1,r'}_{0,\mathrm{div}})$, added to the balance of linear momentum $(1.1)_2$. Assuming that a large-data solution to $(\mathcal{P}_n)$ exists, we shall formally derive apriori estimates for $(\varrho^n,\boldsymbol{v}^n,\theta^n)$ that will be uniform with respect to $n\in\mathbb{N}$. Our goal will be to show that a weak limit of $(\varrho^n,\boldsymbol{v}^n,\theta^n)$ in function spaces relevant to the uniform estimates solves Problem $(\mathcal{P})$. The missing part of the proof, which is a construction of an appropriate approximation scheme that would result in a problem similar to Problem $(\mathcal{P}_n)$, is skipped here and we refer the reader to earlier papers for the same (for example, a combination of the approximation in [13] for the *isothermal* problem and the approximation in [4] for the incompressible *homogeneous* heat-conducting, shear-rate dependent fluids is a possible choice). For the technical reasons (different structure of the interpolation and embedding inequalities) we investigate the case $r\in[11/5,3)$, and leave the modifications needed in the analysis of the case $r\geq 3$ to the reader.

3.1. **Definition of the Problems** $(\mathcal{P}_n)$**.** Let $(\boldsymbol{f}^n,\varrho_0^n,\boldsymbol{v}_0^n,\theta_0^n)$ be sufficiently regular functions such that

$$\boldsymbol{f}^n \to \boldsymbol{0} \quad \text{strongly in } L^{r'}(0,T;W^{-1,r'})\,, \tag{3.1}$$

$$\varrho_0^n \to \varrho_0 \quad \text{strongly in } L^\infty\,, \tag{3.2}$$

$$\boldsymbol{v}_0^n \to \boldsymbol{v}_0 \quad \text{strongly in } L^2_{\mathrm{div}}\,, \tag{3.3}$$

$$\theta_0^n \to \theta_0 \quad \text{strongly in } L^1\,. \tag{3.4}$$

We define Problem $(\mathcal{P}_n)$ analogously to Problem $(\mathcal{P})$ making the following modifications: we put $\boldsymbol{f}^n$ on the right hand side of the equation $(1.1)_2$ and take the initial values for $(\varrho^n,\boldsymbol{v}^n,\theta^n)$ near the initial values for Problem $(\mathcal{P})$. We also assume that sufficiently regular long-time and large-data solution to such a perturbed problem can be established and our objective is to study the limit $n\to\infty$. Thus, in particular we assume that $(\varrho^n,\boldsymbol{v}^n,\theta^n)$ fulfill

$$\begin{aligned}&\operatorname{div}\boldsymbol{v}^n = 0 \ \text{ and } \int_0^T \langle \varrho^n_{,t}, z\rangle - (\varrho^n\boldsymbol{v}^n,\nabla z)\,dt = 0\\ &\qquad\qquad \text{for all } z\in L^s(0,T;W^{1,s}) \text{ with } s=5r/(5r-3)\,,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\int_0^T \big\langle(\varrho^n\boldsymbol{v}^n)_{,t},\boldsymbol{\varphi}\big\rangle - \big(\varrho^n\boldsymbol{v}^n\otimes\boldsymbol{v}^n,\nabla\boldsymbol{\varphi}\big) + \big(\mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)),\mathbf{D}(\boldsymbol{\varphi})\big)\,dt\\ &= \int_0^T \big\langle\boldsymbol{f}^n,\boldsymbol{\varphi}\big\rangle\,dt \qquad\qquad \text{for all } \boldsymbol{\varphi}\in L^r(0,T;W^{1,r}_{0,\mathrm{div}})\,,\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}&\int_0^T \big\langle(\varrho^n\theta^n)_{,t},h\big\rangle - \big(\varrho^n\theta^n\boldsymbol{v}^n,\nabla h\big) + \big(k(\varrho^n,\theta^n)\nabla\theta^n,\nabla h\big)\,dt\\ &= \int_0^T \big(\mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)),\mathbf{D}(\boldsymbol{v}^n)h\big)\,dt\\ &\qquad\qquad \text{for all } h\in L^\infty(0,T;W^{1,q}) \text{ with } q \text{ sufficiently large}\,,\end{aligned}\tag{3.7}$$

and attains the initial conditions $\varrho_0^n, \boldsymbol{v}_0^n$ and θ_0^n in an appropriate way. Note that in our understanding (3.5) is equivalent to

$$\begin{aligned}&\int_{t_1}^{t_2}(\varrho^n,z_{,t}) + (\varrho^n\boldsymbol{v}^n,\nabla z)\,d\tau = (\varrho^n,z)(t_2) - (\varrho^n,z)(t_1)\\ &\qquad\qquad \text{for all } z \text{ smooth, and a.a. } t_1,\ t_2\text{: } 0\le t_1\le t_2\le T.\end{aligned}\tag{3.8}$$

A similar remark concerns (3.6) and (3.7) as well.

3.2. **Uniform estimates.** We shall derive estimates for $(\varrho^n,\boldsymbol{v}^n,\theta^n)$ that are uniform with respect to $n\in\mathbb{N}$. Without specifying in detail what are the precise properties of $(\varrho^n,\boldsymbol{v}^n,\theta^n)$ these estimates are not rigorous - they can be however obtained rigorously for a suitable designed approximation scheme (as would be for example the scheme obtained by putting together approximations used in [13] and [4]).

As $\operatorname{div}\boldsymbol{v}^n = 0$, (3.5) is a transport equation for ϱ^n. By the method of characteristics (cf. [1]), one then concludes, using also $(1.5)_1$, that

$$0 < \varrho_* \le \varrho^n(t,x) \le \varrho^* < +\infty \qquad \text{for a.a } (t,x)\in Q\,. \tag{3.9}$$

Next, taking $z = |\boldsymbol{v}^n|^2$, $t_1 = 0$ and $t_2 = t$ in (3.8) we obtain

$$\int_0^t (\varrho^n,|\boldsymbol{v}^n|^2_{,t}) + (\varrho^n\boldsymbol{v}^n,\nabla|\boldsymbol{v}^n|^2)\,d\tau = (\varrho^n,|\boldsymbol{v}^n|^2)(t) - (\varrho^n_0,|\boldsymbol{v}^n_0|^2)\,. \tag{3.10}$$

Inserting $\boldsymbol{\varphi} = \boldsymbol{v}^n\chi_{[0,t]}$ into (3.6) ($\chi_{[a,b]}$ stands for the characteristic function of the interval $[a,b]$) and using (3.10) we obtain the identity

$$\tfrac12(\varrho^n,|\boldsymbol{v}^n|^2)(t) + \int_0^t \big(\mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)),\mathbf{D}(\boldsymbol{v}^n)\big)\,d\tau = \int_0^t \langle\boldsymbol{f}^n,\boldsymbol{v}^n\rangle\,d\tau + \tfrac12(\varrho^n_0,|\boldsymbol{v}^n_0|^2)\,. \tag{3.11}$$

Using $(1.8)_1$ this leads to

$$\begin{aligned}&(\varrho^n,|\boldsymbol{v}^n|^2)(t) + \int_0^t \big(\mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)),\mathbf{D}(\boldsymbol{v}^n)\big)\,d\tau + \tfrac{m_*}{2}\int_0^t \|\mathbf{D}(\boldsymbol{v}^n)\|_r^r\,d\tau\\ &\qquad \le (\varrho^n_0,|\boldsymbol{v}^n_0|^2) + C\int_0^T \|\boldsymbol{f}^n\|^{r'}_{-1,r'}\,d\tau \le C(\varrho^*,\|\boldsymbol{v}_0\|_2) \le M\,,\end{aligned}\tag{3.12}$$

where M denotes a positive constant (depending on the size of data) that maximizes all the estimates. Using (3.9), it follows from (3.12) that

$$\sup_{t\in[0,T]}\|\boldsymbol{v}^n(t)\|_2^2 + \sup_{t\in[0,T]}\|(\varrho^n|\boldsymbol{v}^n|^2)(t)\|_1 \le M\,. \tag{3.13}$$

It also follows from (3.12), using Korn's inequality and the estimates $(1.8)_{1,2}$, that

$$\begin{aligned} 0 \le \int_0^T \big(\mathbf{S}(\varrho^n, \theta^n, \mathbf{D}(\boldsymbol{v}^n)), \mathbf{D}(\boldsymbol{v}^n)\big)\, dt + \int_0^T \|\nabla \boldsymbol{v}^n\|_r^r \, dt \le M\,, \\ \int_0^T \|\mathbf{S}(\varrho^n, \theta^n, \mathbf{D}(\boldsymbol{v}^n))\|_{r'}^{r'}\, dt \le M\,. \end{aligned} \tag{3.14}$$

The standard interpolation between $L^\infty(0,T;L^2)$ and $L^r(0,T;L^{\frac{3r}{3-r}})$ (for simplicity we deal in the sequel with the case $r < 3$ as the case $r \ge 3$ can be treated easier with slightly different inequalities) implies that

$$\int_0^T \|\boldsymbol{v}^n\|_{5r/3}^{5r/3}\, dt \le M \quad \text{ and also } \quad \int_0^T \|\varrho^n\, \boldsymbol{v}^n\|_{5r/3}^{5r/3}\, dt \le M\,. \tag{3.15}$$

As a consequence of the estimates (3.9), $(3.14)_1$ and (3.15) we observe, by Hölder's inequality, that

$$\int_0^T \Big|(\varrho^n \boldsymbol{v}^n \otimes \boldsymbol{v}^n, \nabla \boldsymbol{v}^n)\Big|\, dt \le M \iff r \ge \frac{11}{5}\,, \tag{3.16}$$

which gives the restriction for r in Theorem 1.1. Finally, using these estimates it follows from (3.5) and (3.6) that

$$\int_0^T \|(\varrho^n \boldsymbol{v}^n)_{,t}\|_{W^{-1,r'}_{\mathrm{div}}}^{r'}\, dt \le M \quad \text{ and } \quad \int_0^T \|\varrho^n_{,t}\|_{(W^{1,5r/(5r-3)})^*}^{5r/3}\, dt \le M\,. \tag{3.17}$$

Regarding the estimates concerning the temperature we first take $h = -(\theta^n(t,\cdot) - \theta_*)^-$, where z^- denotes the negative part of z (i.e., $z^- = 0$ if $z \ge 0$ and $z^- = -z$ if $z < 0$), as a test function in (3.7). As the right hand side of (3.7) is nonnegative we then conclude from a minimum principle argument (cf. [16]), using $(1.5)_2$ as well, that

$$\theta^n(t,x) \ge \theta_* \quad \text{ for a.a. } (t,x) \in Q\,. \tag{3.18}$$

Taking $h = 1$ in (3.7), and using $(3.14)_1$ and also (3.9), leads to

$$\sup_{t\in[0,T]} \|(\varrho^n\theta^n)(t)\|_1 + \sup_{t\in[0,T]} \|\theta^n(t)\|_1 \le M\,. \tag{3.19}$$

Next, taking $h = -(\theta^n)^{-\lambda}$ with $0 < \lambda < 1$ in (3.7) (note that by (3.18) we know that $\|(\theta^n)^{-\lambda}\|_{\infty,Q} \le M$) we conclude that[3]

$$\begin{aligned} \int_0^T \|(\theta^n)^{\frac{\beta-\lambda-1}{2}}\nabla\theta^n\|_2^2\, dt \le M \quad &\Longrightarrow \quad \int_0^T \|(\theta^n)^{\frac{\beta-\lambda+1}{2}}\|_{1,2}^2\, dt \le M \\ &\Longrightarrow \quad \int_0^T \|\theta^n\|_{3(\beta-\lambda+1)}^{\beta-\lambda+1}\, dt \le M\,. \end{aligned} \tag{3.20}$$

The standard interpolation of the last estimate with $(3.19)_2$ (valid for $\beta > -\frac{2}{3}$, which is one of the restriction on β from Theorem 1.1) leads to

$$\int_0^T \|\theta^n\|_s^s\, dt \le M \text{ for all } s \in \big[1, \tfrac{5}{3} + \beta\big)\,. \tag{3.21}$$

[3] To establish the first implication in (3.20) we use (3.19) and the inequality $\|\theta^{\frac{\beta-\lambda+1}{2}}\|_2 \le c\big(\|\theta\|_1^{\frac{\beta-\lambda+1}{2}} + \|\nabla(\theta^{\frac{\beta-\lambda+1}{2}})\|_2\big)$ that holds if $\frac{\beta-\lambda+1}{2} > 0$ (and can be easily proved by a contradiction argument), and the inequality $\|\theta^{\frac{\beta-\lambda+1}{2}}\|_2 \le c$ valid for $\frac{\beta-\lambda+1}{2} \le 0$. The second implication in (3.20) follows from the continuous embedding of $W^{1,2}$ into L^6.

Since $(1.9)_2$ holds, we have

$$|k(\varrho^n,\theta^n)\nabla\theta^n| \le \kappa^*|\theta^n|^\beta|\nabla\theta^n| = \kappa^*(\theta^n)^{\frac{\beta-\lambda-1}{2}}|\nabla\theta^n|(\theta^n)^{\frac{\beta+\lambda+1}{2}}\,.$$

As the sequence $(\theta^n)^{\frac{\beta-\lambda-1}{2}}|\nabla\theta^n|$ is uniformly bounded, by $(3.20)_1$, in $L^2(Q)$, and $(\theta^n)^{\frac{\beta+\lambda+1}{2}}$ is bounded, by (3.21), in $L^{\frac{2}{\beta+\lambda+1}(\frac{5+3\beta}{3}-\delta)}(Q)$ ($\delta>0$ arbitrarily small), we observe that

$$\int_Q |k(\varrho^n,\theta^n)\nabla\theta^n|^m\,dx\,dt \le M \quad \text{ for all } m \text{ fulfilling } 1\le m<\frac{5+3\beta}{4+3\beta}\,. \tag{3.22}$$

Further, making use of (3.9) and (3.19), we have for $\gamma>1$

$$\begin{aligned}\int_0^T \|\varrho^n\boldsymbol{v}^n\theta^n\|_\gamma^\gamma dt &\le \varrho^*\int_0^T \|\boldsymbol{v}^n\|^\gamma_{\frac{3r}{3-r}}\|\theta^n\|^\gamma_{\frac{3r\gamma}{(3+\gamma)r-3\gamma}}\,dt\\ &\le C\int_0^T \|\boldsymbol{v}^n\|^\gamma_{1,r}\|\theta^n\|_1^{(1-\alpha)\gamma}\|\theta^n\|^{\alpha\gamma}_{3(\beta-\lambda+1)}\,dt\\ &\le C\int_0^T \|\boldsymbol{v}^n\|^\gamma_{1,r}\|\theta^n\|^{\alpha\gamma}_{3(\beta-\lambda+1)}\,dt\,,\end{aligned} \tag{3.23}$$

where $\alpha\in[0,1]$ fulfills

$$\begin{aligned}&\frac{(3+\gamma)r-3\gamma}{3r\gamma} = \frac{1-\alpha}{1}+\frac{\alpha}{3(\beta-\lambda+1)}\\ \iff\quad &\alpha = \frac{(\beta-\lambda+1)}{3(\beta-\lambda)+2}\,\frac{2r\gamma-3r+3\gamma}{r\gamma}\,.\end{aligned} \tag{3.24}$$

Applying the Hölder inequality to (3.23) we obtain

$$\begin{aligned}&\int_0^T \|\varrho^n\boldsymbol{v}^n\theta^n\|_\gamma^\gamma dt \qquad (3.25)\\ &\le C\left(\int_0^T \|\theta^n\|^{\beta-\lambda+1}_{3(\beta-\lambda+1)}\,dt\right)^{\frac{\alpha\gamma}{\beta-\lambda+1}}\left(\int_0^T \|\boldsymbol{v}^n\|_{1,r}^{\frac{(\beta-\lambda+1)\gamma}{(\beta-\lambda+1)-\alpha\gamma}}\,dt\right)^{\frac{(\beta-\lambda+1)-\alpha\gamma}{\beta-\lambda+1}} \le M,\end{aligned}$$

where the bound (uniform w.r.t. n) follows from $(3.14)_1$ and (3.20) provided that

$$\frac{(\beta-\lambda+1)\gamma}{(\beta-\lambda+1)-\alpha\gamma} = r\,. \tag{3.26}$$

Using the formula for α given in (3.24), we see that (3.26) holds if $\gamma=\frac{r(3\beta-3\lambda+5)}{3\beta+2r-3\lambda+5}$, and $\gamma>1$ if $\beta>-\frac{3r-5}{3(r-1)}$, which is the other restriction on β from Theorem 1.1. To summarize, we have observed that

$$\int_0^T \|\varrho^n\boldsymbol{v}^n\theta^n\|_\gamma^\gamma dt \le M \quad \text{ with } \gamma = \frac{r(3\beta-3\lambda+5)}{3\beta+2r-3\lambda+5}\,. \tag{3.27}$$

Note, that $\gamma > 1$ if $\beta > -\frac{3r-5}{3(r-1)}$. Finally, we conclude from (3.7) and the estimates (3.14), (3.22) and (3.27) that[4]

$$\|(\varrho^n\theta^n)_{,t}\|_{L^1(0,T;(W^{1,q})^*)} \le M \quad \text{for } q \text{ sufficiently large.} \tag{3.29}$$

3.3. **Weak convergences.** The uniform estimates (3.9), (3.13)–(3.15), (3.17)–(3.21) and Alaoglu-Bourbaki Theorem imply the existence of subsequences[5] selected from $\{\varrho^n\}$, $\{\boldsymbol{v}^n\}$ and $\{\theta^n\}$ and a triple $(\varrho, \boldsymbol{v}, \theta)$ such that

$$\begin{aligned} &\varrho^n \rightharpoonup \varrho \quad \text{weakly in } L^q(Q) \text{ for any } q \in [1,\infty) \text{ and *-weakly in } L^\infty(Q)\,, \\ &\quad 0 < \varrho_* \le \varrho(t,x) \le \varrho^* < \infty \quad \text{for a.a. } (t,x) \in Q\,, \end{aligned} \tag{3.30}$$

$$\varrho^n_{,t} \rightharpoonup \varrho_{,t} \quad \text{weakly in } L^{5r/3}(0,T;(W^{1,5r/(5r-3)})^*)\,, \tag{3.31}$$

$$\begin{aligned} &\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \quad \text{weakly in } L^r(0,T;W^{1,r}_{0,\mathrm{div}}) \text{ and } L^{5r/3}(Q)^3 \\ &\qquad \text{and *-weakly in } L^\infty(0,T;H)\,, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\theta^n \rightharpoonup \theta \quad \text{weakly in } L^q(Q) \text{ for any } q \in [1, \tfrac{5+3\beta}{3})\,, \\ &\quad \theta(t,x) \ge \theta_* > 0 \quad \text{for a.a. } (t,x) \in Q\,. \end{aligned} \tag{3.33}$$

Also, there exists $\overline{\varrho\boldsymbol{v}} \in L^{5r/3}(Q)^3$, $\overline{\mathbf{S}} \in L^{r'}(Q)^{3\times 3}$ and $\overline{\theta^\alpha} \in L^2(0,T;W^{1,2})$ such that[6]

$$\varrho^n\boldsymbol{v}^n \rightharpoonup \overline{\varrho\boldsymbol{v}} \quad \text{weakly in } L^{5/3}(Q)^3\,, \tag{3.34}$$

$$\mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in } L^{r'}(Q)^{3\times 3}\,,\ \overline{\mathbf{S}} = \overline{\mathbf{S}}^T, \overline{\mathbf{S}} \text{ being traceless}\,, \tag{3.35}$$

$$(\theta^n)^\alpha \rightharpoonup \overline{\theta^\alpha} \quad \text{weakly in } L^2(0,T;W^{1,2}) \quad \text{for } \alpha \in (0,(\beta+1)/2] \cap \mathbb{Q}\,. \tag{3.36}$$

3.4. **Strong convergences of $\{\varrho^n\}$, $\{\boldsymbol{v}^n\}$ and $\{\theta^n\}$ and their consequences.** Starting with (3.9), (3.30), (3.31) and Aubin-Lions Lemma (see [20], Corollary 8.4) one obtains that

$$\varrho^n \to \varrho \quad \text{strongly in } C([0,T];W^{-1,5r/3})\,. \tag{3.37}$$

Next, we show by Div-Curl Lemma 2.1 that

$$\begin{aligned} &\varrho^n v_i^n \rightharpoonup \varrho v_i \quad \text{weakly in } L^q(Q) \text{ for all } q \in [1,5r/6] \\ &\big(\overset{(3.34)}{\Longrightarrow} \text{ for all } q \in [1,5r/3]\big)\,. \end{aligned} \tag{3.38}$$

We take $\boldsymbol{a}^n = (\varrho^n, \varrho^n v_1^n, \varrho^n v_2^n, \varrho^n v_3^n)$ and $\boldsymbol{b}^n = (v_i^n,0,0,0)$, $i \in \{1,2,3\}$ fixed. Clearly, by (3.30) and (3.34) $\boldsymbol{a}^n$ converges weakly to the limit $(\varrho, \overline{\varrho v_1}, \overline{\varrho v_2}, \overline{\varrho v_3})$ in $L^q(Q)^4$ for all $q \in [1,5r/3]$, and $\boldsymbol{b}^n \rightharpoonup (v_i,0,0,0)$ weakly in $L^{5r/3}(Q)^4$. Since

$$\begin{aligned} &\mathrm{Div}_{t,x}\,\boldsymbol{a}^n = (\varrho^n)_{,t} + \mathrm{div}(\varrho^n\boldsymbol{v}^n) = 0\,, \\ &\mathrm{Curl}_{t,x}\,\boldsymbol{b}^n = \begin{pmatrix} 0 & \nabla\boldsymbol{v}^n \\ -(\nabla\boldsymbol{v}^n)^T & \mathcal{O} \end{pmatrix} \quad (\mathcal{O} \text{ denotes zero } 3\times 3 \text{ matrix})\,, \end{aligned}$$

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[4]More precisely, it follows from the version of (3.7) which holds for almost all $\tau \in (0,T)$ and the estimates (3.14), (3.22) and (3.27) that for sufficiently large q and for almost all $\tau \in (0,T)$

$$\|(\varrho^n\theta^n)_{,t}(\tau)\|_{(W^{1,q})^*} = \sup_{\|h\|_{1,q}\le 1} \big|\langle(\varrho^n\theta^n)_{,t}(\tau),h\rangle\big| \le g^n(\tau), \tag{3.28}$$

where $\|g^n(\tau)\|_{L^1(0,T)} \le M < \infty$. Thus, integrating (3.28) w.r.t. τ over $(0,T)$ we obtain (3.29).

[5]We shall use the standard convention and denote these selected subsequences again through $\{\varrho^n\}$, $\{\boldsymbol{v}^n\}$, $\{\theta^n\}$. This comment concerns later selections as well.

[6]For $\alpha \in (0,\beta/2] \cap \mathbb{Q}$ we use (3.20) and (3.18) for the last convergence.

and $\nabla \boldsymbol{v}^n$ is bounded in $L^r(Q)^{3\times 3}$ which is compactly embedded into $W^{-1,r}(Q)$, Div-Curl Lemma 2.1 implies (3.38). From (3.31) and (3.38) we conclude that ϱ and $\boldsymbol{v}$ satisfy (2.6). This in turn implies, by using test functions of the form $\chi_{(t_1,t_2)}h$, $h \in W^{1,5r/(5r-3)}$ in (2.6), partial integration with respect to time and the density of $W^{1,5r/(5r-3)}$ in L^1 that $\varrho \in C([0,T]; L^\infty_{\text{weak}})$, i.e. for all $h \in L^1$ and all $0 \le t_0 \le T$ we have

$$\lim_{t\to t_0} (\varrho(t), h) = (\varrho(t_0), h)\,. \tag{3.39}$$

Using the concept of renormalized solution to the equation (3.5), it is possible to strengthen (3.37) and (3.39). Proceeding step by step as in Lions [17] one can observe that

$$\varrho^n \to \varrho \quad \text{strongly in } C([0,T]; L^q) \quad \text{for all } q \in [1,\infty) \quad \text{and a.e. in } Q\,, \tag{3.40}$$

and also,

$$\lim_{t\to 0+} \|\varrho(t) - \varrho_0\|_q = 0 \quad \text{for all } q \in [1,\infty)\,, \tag{3.41}$$

which is the first part of $(2.9)_1$.

Let us now establish the convergence properties of $\{\boldsymbol{v}^n\}$ and of related quantities. First, note that using (3.17) and (3.38), we get

$$(\varrho^n \boldsymbol{v}^n)_{,t} \rightharpoonup (\varrho \boldsymbol{v})_{,t} \quad \text{weakly in } L^{r'}(0,T; W^{-1,r'}_{\text{div}})\,. \tag{3.42}$$

Second, from (3.13), (3.32) and (3.40) we deduce that

$$\sqrt{\varrho^n}\boldsymbol{v}^n \rightharpoonup \sqrt{\varrho}\boldsymbol{v} \quad \text{weakly in } L^2(Q)^3\,. \tag{3.43}$$

Next, we notice that Aubin-Lions Lemma (see [20], Corollary 8.4), $(3.13)_2$, (3.9) and $(3.17)_1$, and also (3.38) imply that

$$\varrho^n \boldsymbol{v}^n \to \varrho \boldsymbol{v} \quad \text{strongly in } C(0,T; W^{-1,r'}_{\text{div}})\,. \tag{3.44}$$

From this and (3.32) follows that (as $n \to \infty$)

$$\begin{aligned} \int_0^T \big(\varrho^n \boldsymbol{v}^n, \boldsymbol{v}^n\big)\,dt &= \int_0^T \langle \varrho^n \boldsymbol{v}^n, \boldsymbol{v}^n\rangle_{W^{-1,r'}_{\text{div}}, W^{1,r}_{0,\text{div}}}\,dt \\ &\to \int_0^T \langle \varrho \boldsymbol{v}, \boldsymbol{v}\rangle_{W^{-1,r'}_{\text{div}}, W^{1,r}_{0,\text{div}}}\,dt = \int_0^T (\varrho \boldsymbol{v}, \boldsymbol{v})\,dt\,, \end{aligned} \tag{3.45}$$

i.e. the sequence $\|\sqrt{\varrho^n}\boldsymbol{v}^n\|_2$ converges to $\|\sqrt{\varrho}\boldsymbol{v}\|_2$. This together with (3.43) implies

$$\sqrt{\varrho^n}\boldsymbol{v}^n \to \sqrt{\varrho}\boldsymbol{v} \quad \text{strongly in } L^2(Q)^3\,, \tag{3.46}$$

and

$$\sqrt{\varrho^n}\boldsymbol{v}^n(t) \to \sqrt{\varrho}\boldsymbol{v}(t) \quad \text{strongly in } L^2(\Omega)^3 \quad \text{for almost all } t \in [0,T]\,. \tag{3.47}$$

Using (3.15), (3.40) and (3.46), we come to the conclusion that

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \text{strongly in } L^q(Q) \quad \text{for all } q \in [1, 5r/3) \quad \text{and a.e. in } Q\,. \tag{3.48}$$

Using this and (3.30) we easily check that for $r > 6/5$

$$\varrho^n \boldsymbol{v}^n \otimes \boldsymbol{v}^n \rightharpoonup \varrho \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{weakly in } L^{q'}(0,T; W^{-1,q'}) \quad \text{for } q \text{ sufficiently large.}$$

By density arguments we deduce from this, (3.30) and (3.32) that for $r \ge 11/5$

$$\varrho^n \boldsymbol{v}^n \otimes \boldsymbol{v}^n \rightharpoonup \varrho \boldsymbol{v} \otimes \boldsymbol{v} \quad \text{weakly in } L^{r'}(0,T; W^{-1,r'}_{\text{div}})\,. \tag{3.49}$$

Finally, we establish the strong convergence properties of $\{\theta^n\}$, again by means of Div-Curl Lemma 2.1. For this purpose, we take $\boldsymbol{a}^n = (\varrho^n\theta^n, Q_1^n, Q_2^n, Q_3^n)$ with $\boldsymbol{Q}^n := \varrho^n\theta^n\boldsymbol{v}^n + \kappa(\varrho^n,\theta^n)\nabla\theta^n$, and $\boldsymbol{b}^n = ((\theta^n)^\alpha, 0, 0, 0)$ with $\alpha \in (0, (\beta+1)/2)$ rather small. By (3.22), (3.27), (3.40) and (3.33) we observe that $\{\boldsymbol{a}^n\}$ converges weakly to $\boldsymbol{a}$ in $L^a(Q)$, for some $a > 1$ (near 1), where $a_1 = \varrho\theta$, and $\boldsymbol{b}^n \rightharpoonup (\overline{\theta^\alpha}, 0, 0, 0)$ in $L^b(Q)$ for b big enough so that $1/a + 1/b < 1$ (which is possible if α is small enough). Observing that for $1 < s < \frac{4}{3}$

$$\mathrm{Div}_{t,x}\,\boldsymbol{a}^n = (\varrho^n\theta^n)_{,t} + \mathrm{div}\,\boldsymbol{Q}^n = \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n))\cdot\mathbf{D}(\boldsymbol{v}^n) \subset L^1(Q) \hookrightarrow\hookrightarrow W^{-1,s}(Q)$$

and

$$\mathrm{Curl}_{t,x}\,\boldsymbol{b}^n = \begin{pmatrix} 0 & \nabla(\theta^n)^\alpha \\ -(\nabla(\theta^n)^\alpha)^T & \mathcal{O} \end{pmatrix} \subset L^2(Q)^{4x4} \hookrightarrow\hookrightarrow W^{-1,2}(Q)^{4x4}\,,$$

another application of Div-Curl Lemma 2.1 leads to the conclusion that

$$\varrho^n(\theta^n)^{\alpha+1} \rightharpoonup \varrho\theta\overline{\theta^\alpha} \quad \text{weakly in } L^{1+\epsilon}(Q) \quad \text{for some } \epsilon > 0\,. \tag{3.50}$$

Due to (3.40) and (3.21), (3.50) leads to

$$\varrho(\theta^n)^{\alpha+1} \rightharpoonup \varrho\theta\overline{\theta^\alpha} \quad \text{weakly in } L^{1+\tilde\epsilon}(Q) \quad \text{for some } \tilde\epsilon > 0\,. \tag{3.51}$$

By a well-known monotone operator argument (Minty's method)[7], we conclude from (3.51) that

$$\overline{\theta^\alpha} = \theta^\alpha \ \text{ a.e. in } Q\,. \tag{3.53}$$

Thus, it follows from (3.51) that $\{\varrho^{\frac{1}{1+\alpha}}\theta^n\}$ converges to $\{\varrho^{\frac{1}{1+\alpha}}\theta\}$ weakly in $L^{1+\alpha}(Q)$ and that $\|\varrho^{\frac{1}{1+\alpha}}\theta^n\|_{L^{1+\alpha}(Q)}$ converges to $\|\varrho^{\frac{1}{1+\alpha}}\theta\|_{L^{1+\alpha}(Q)}$. Consequently,

$$\varrho^{\frac{1}{1+\alpha}}\theta^n \to \varrho^{\frac{1}{1+\alpha}}\theta \quad \text{strongly in } L^{1+\alpha}(Q)\,.$$

This together with (3.21) and (3.33) implies that

$$\theta^n \to \theta \quad \text{strongly in } L^q(Q) \ \text{for all } q \in [1, 5/3+\beta) \quad \text{and a.e. in } Q\,. \tag{3.54}$$

As a consequence of (3.40), (3.54), (3.27) and (3.48) we conclude that

$$\varrho^n\theta^n \to \varrho\theta \quad \text{strongly in } L^q(Q) \ \text{for all } q \in [1, 5/3+\beta) \tag{3.55}$$

and (proceeding similarly as between the lines (3.23) and (3.27))

$$\varrho^n\theta^n\boldsymbol{v}^n \to \varrho\theta\boldsymbol{v} \quad \text{strongly in } L^1(Q)^3\,. \tag{3.56}$$

It also follows from the established strong convergence of $\{\theta^n\}$ and from (3.36) that

$$(\theta^n)^\alpha \rightharpoonup \theta^\alpha \quad \text{weakly in } L^2(0,T;W^{1,2}) \ \text{for all } \alpha \in (0, (\beta+1)/2)\,. \tag{3.57}$$

---

[7]Since z^α is a increasing continuous function for $\alpha > 0$ ($z \in \mathbb{R}^+$), it holds for $h > 0$, $h \in L^{1+\epsilon}(Q)$ that

$$0 \le \int_0^T \Big(\varrho[(\theta^n)^\alpha - h^\alpha], \theta^n - h\Big)dt = \int_Q \big[\varrho(\theta^n)^{\alpha+1} - \varrho h^\alpha\theta^n - \varrho(\theta^n)^\alpha h + \varrho h^{\alpha+1}\big]dx\,dt\,. \tag{3.52}$$

Taking limit $n \to \infty$ in (3.52), using (3.51) as well, we obtain easily

$$0 \le \int_0^T \Big(\varrho[\overline{\theta^\alpha} - h^\alpha], \theta - h\Big)dt \quad \text{valid for all } h > 0, h \in L^{1+\epsilon}(Q)\,.$$

Minty's argument then leads to (3.53).

Next, we notice that

$$\begin{aligned}\boldsymbol{q}(\varrho^n,\theta^n,\nabla\theta^n) &= k(\varrho^n,\theta^n)\nabla\theta^n \\ &= \frac{2}{\beta-\lambda+1}\big[\theta^n\big]^{\frac{1+\lambda-\beta}{2}}\,k(\varrho^n,\theta^n)\nabla\big(\theta^n\big)^{\frac{\beta-\lambda+1}{2}}\,.\end{aligned} \tag{3.58}$$

Taking $\lambda > 0$ small and q such that

$$(\beta+\lambda+1)\frac{q}{2} = \frac{5}{3}+\beta-\lambda\,, \tag{3.59}$$

we observe that due to $(1.9)_2$ and (3.21) the following uniform bound holds:

$$\int_0^T \|\big[\theta^n\big]^{\frac{1+\lambda-\beta}{2}}\,k(\varrho^n,\theta^n)\|_q^q\,dt \le C\int_0^T\int_\Omega \big(\theta^n\big)^{(\beta+\lambda+1)\frac{q}{2}}\,dx\,d\tau \le M\,. \tag{3.60}$$

Note that q defined in (3.59) fulfills $q > 2$ for $\lambda > 0$ small enough. Vitali's theorem, (3.60) and almost everywhere convergences for ϱ^n and θ^n established in (3.40) and (3.54) lead to

$$\big[\theta^n\big]^{\frac{1+\lambda-\beta}{2}}\,k(\varrho^n,\theta^n) \to \big[\theta\big]^{\frac{1+\lambda-\beta}{2}}\,k(\varrho,\theta) \quad \text{strongly in } L^2(0,T;L^2)\,. \tag{3.61}$$

Moreover, (3.57) implies that

$$\nabla(\theta^n)^{\frac{\beta-\lambda+1}{2}} \rightharpoonup \nabla\theta^{\frac{\beta-\lambda+1}{2}} \quad \text{weakly in } L^2(0,T;L^2)\,, \tag{3.62}$$

Starting from (3.58), and referring to (3.62), (3.61) and (3.22) we finally come to the conclusion that

$$\boldsymbol{q}(\varrho^n,\theta^n,\nabla\theta^n) \rightharpoonup \boldsymbol{q}(\varrho,\theta,\nabla\theta) \quad \text{weakly in } L^m(Q) \ \text{ for all } m\in\Big(1,\tfrac{5+3\beta}{4+3\beta}\Big)\,. \tag{3.63}$$

3.5. **Strong convergence of $\{\mathbf{D}(\boldsymbol{v}^n)\}$ and Limit $n\to\infty$ in (3.6) and in (3.7).** Since the requirement on our parameter r, namely $r \ge 11/5$, puts the Problem $(\mathcal{P})$ among the subcritical ones (see [18] for details), one expects that the Minty method or standard monotone operator technique should give the strong convergence of $\mathbf{D}(\boldsymbol{v}^n)$. We shall show that this is indeed the case relying however on the integration by parts formula established in Section 4. This formula represents the key tool in the argument.

Using (3.42), (3.49), (3.35), and (3.1), we can take limit $n\to\infty$ in (3.6) and obtain

$$\int_0^T \big\langle(\varrho\boldsymbol{v})_{,t},\boldsymbol{\varphi}\big\rangle - \big(\varrho\boldsymbol{v}\otimes\boldsymbol{v},\nabla\boldsymbol{\varphi}\big) + \big(\overline{\mathbf{S}},\mathbf{D}(\boldsymbol{\varphi})\big)\,dt = 0 \qquad \text{for all } \boldsymbol{\varphi}\in L^r(0,T;W^{1,r}_{0,\mathrm{div}})\,. \tag{3.64}$$

If we take $\boldsymbol{\varphi}(s,x) = \chi_{(t_0,t_1)}(s)\boldsymbol{\psi}(x)$ in (3.64), where $\chi_{(t_0,t_1)}$ denotes the characteristic function of the interval (t_0,t_1) and $\boldsymbol{\psi}\in W^{1,r}_{0,\mathrm{div}}$, it is possible to conclude (see for example [18], Sect. B.3.8 for similar arguments), using also (3.30) and (3.32), that $\varrho\boldsymbol{v}\in C([0,T],L^2_{\mathrm{div,weak}})$. Using in (3.6) test functions $\boldsymbol{\varphi}$'s of the form $\boldsymbol{\varphi}(s,x) = \chi_{(0,t)}(s)\boldsymbol{\psi}(x)$ and performing partial integration with respect to time in the first term, we obtain with the help of (3.44), (3.2), (3.3), (3.49), (3.35) and (3.1)

$$(\varrho(t)\boldsymbol{v}(t),\boldsymbol{\psi}) - (\varrho_0\boldsymbol{v}_0,\boldsymbol{\psi}) = \int_0^t \big(\varrho\boldsymbol{v}\otimes\boldsymbol{v},\nabla\boldsymbol{\psi}\big) - \big(\overline{\mathbf{S}},\mathbf{D}(\boldsymbol{\psi})\big)\,dt \quad \text{for all } \boldsymbol{\psi}\in W^{1,r}_{0,\mathrm{div}}\,.$$

This immediately implies

$$\lim_{t\to 0+}(\varrho(t)\boldsymbol{v}(t)-\varrho_0\boldsymbol{v}_0,\boldsymbol{\psi}) = 0 \quad \text{for all } \boldsymbol{\psi}\in L^2_{\mathrm{div}}\,. \tag{3.65}$$

Letting $n \to \infty$ in (3.11), neglecting the second nonnegative term and using (3.47), one easily concludes that

$$(\varrho, |\boldsymbol{v}|^2)(t) \leq (\varrho_0, |\boldsymbol{v}_0|^2) \quad \text{for a.a. } t \in [0,T]\,. \tag{3.66}$$

Since

$$\|\sqrt{\varrho(t)}(\boldsymbol{v}(t) - \boldsymbol{v}_0)\|_2^2 = (\varrho, |\boldsymbol{v}|^2)(t) - 2((\varrho\boldsymbol{v})(t), \boldsymbol{v}_0) + (\varrho(t), |\boldsymbol{v}_0|^2)\,, \tag{3.67}$$

we can conclude, using (3.66), (3.41) for the first term and (3.65), (3.39) for the other terms, that the right hand side of (3.67) tends to zero as $t \to 0+$. Consequently,

$$\lim_{t\to 0+} \|\sqrt{\varrho(t)}(\boldsymbol{v}(t) - \boldsymbol{v}_0)\|_2^2 = 0\,, \tag{3.68}$$

which implies (together with $(3.30)_2$) the second part of $(2.9)_1$ and (together with (3.39), (3.41)) the fact that

$$\lim_{t\to 0+} (\varrho, |\boldsymbol{v}|^2)(t) = (\varrho_0, |\boldsymbol{v}_0|^2)\,. \tag{3.69}$$

Note, that the couple $(\varrho, \boldsymbol{v})$ satisfies (2.6), which implies that also (4.5) is satisfied. Using (3.69) and Lemma 4.1 we conclude that there are $T_\delta \in (0,T]$ such that

$$T_\delta \to T \quad \text{as } \delta \to 0+\,, \tag{3.70}$$

$$\int_0^{T_\delta} \langle(\varrho\boldsymbol{v})_{,t}, \boldsymbol{v}\rangle - (\varrho\boldsymbol{v}\otimes\boldsymbol{v}, \boldsymbol{v})\,dt = \frac{1}{2}(\varrho, |\boldsymbol{v}|^2)(T_\delta) - \frac{1}{2}(\varrho_0, |\boldsymbol{v}_0|^2)\,, \tag{3.71}$$

and that in addition (3.47) holds for $t = T_\delta$. Next, taking $\boldsymbol{\varphi} = \boldsymbol{v}\chi_{(0,T_\delta)}$ in (3.64) we conclude, using (3.71) that

$$\frac{1}{2}(\varrho, |\boldsymbol{v}|^2)(T_\delta) + \int_0^{T_\delta} (\overline{\mathbf{S}}, \mathbf{D}(\boldsymbol{v}))\,dt = \frac{1}{2}(\varrho_0, |\boldsymbol{v}_0|^2)\,. \tag{3.72}$$

By $(1.8)_3$ we get for all $\mathbf{B} \in L^r(Q)^{3\times 3}\,, \mathbf{B} = \mathbf{B}^T$,

$$0 \leq \int_0^{T_\delta} \big(\mathbf{S}(\varrho^n, \theta^n, \mathbf{D}(\boldsymbol{v}^n)) - \mathbf{S}(\varrho^n, \theta^n, \mathbf{B}), \mathbf{D}(\boldsymbol{v}^n) - \mathbf{B}\big)\,dt\,. \tag{3.73}$$

Using (3.11) with $t = T_\delta$, we can rewrite (3.73) for all $\mathbf{B} \in L^r(Q)^{3\times 3}\,, \mathbf{B} = \mathbf{B}^T$, as

$$\begin{aligned} 0 \leq &\frac{1}{2}\big((\varrho_0^n, |\boldsymbol{v}_0^n|^2) - (\varrho^n, |\boldsymbol{v}^n|^2)(T_\delta)\big) + \int_0^{T_\delta} \langle \boldsymbol{f}^n, \boldsymbol{v}^n\rangle\,dt \\ &- \int_0^{T_\delta} \big(\mathbf{S}(\varrho^n, \theta^n, \mathbf{D}(\boldsymbol{v}^n)), \mathbf{B}\big)\,dt - \int_0^{T_\delta} \big(\mathbf{S}(\varrho^n, \theta^n, \mathbf{B}), \mathbf{D}(\boldsymbol{v}^n) - \mathbf{B}\big)\,dt \end{aligned} \tag{3.74}$$

Letting $n \to \infty$ in (3.74) and using (3.1)–(3.3), (3.32), (3.35), (3.40), (3.54), $(1.8)_2$, the continuity of $\mathbf{S}$, Lebesgue Dominated Convergence Theorem and (3.47) we conclude that

$$\begin{aligned} 0 \leq \frac{1}{2}\big((\varrho_0, |\boldsymbol{v}_0|^2) - (\varrho, |\boldsymbol{v}|^2)(T_\delta)\big) - \int_0^{T_\delta} \big(\overline{\mathbf{S}}, \mathbf{B}\big)\,dt - \int_0^{T_\delta} \big(\mathbf{S}(\varrho, \theta, \mathbf{B}), \mathbf{D}(\boldsymbol{v}) - \mathbf{B}\big)\,dt \\ \text{for all } \mathbf{B} \in L^r(Q)^{3\times 3}\,, \quad \mathbf{B} = \mathbf{B}^T\,. \end{aligned} \tag{3.75}$$

This together with (3.72) implies that

$$0 \leq \int_0^{T_\delta} \big(\overline{\mathbf{S}} - \mathbf{S}(\varrho, \theta, \mathbf{B}), \mathbf{D}(\boldsymbol{v}) - \mathbf{B}\big)\,dt \quad \text{for all } \mathbf{B} \in L^r(Q)^{3\times 3}\,, \quad \mathbf{B} = \mathbf{B}^T\,. \tag{3.76}$$

Letting finally $\delta \to 0+$ and using (3.70) we easily obtain from (3.76) that

$$0 \le \int_0^T \big(\overline{\mathbf{S}} - \mathbf{S}(\varrho,\theta,\mathbf{B}), \mathbf{D}(\boldsymbol{v}) - \mathbf{B}\big)\, dt \quad \text{for all } \mathbf{B} \in L^r(Q)^{3\times 3}\,, \quad \mathbf{B} = \mathbf{B}^T\,. \tag{3.77}$$

Minty's method using $\mathbf{B} = \mathbf{D}(\boldsymbol{v}) \pm \lambda\mathbf{E}$, where $\lambda > 0$ and $\mathbf{E} \in L^r(Q)^{3\times 3}$ symmetric are arbitrary, then leads to the conclusion that

$$0 = \int_0^T \big(\overline{\mathbf{S}} - \mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v})), \mathbf{E}\big)\, dt \quad \text{for all } \mathbf{E} \in L^r(Q)^{3\times 3}\,, \quad \mathbf{E} = \mathbf{E}^T\,, \tag{3.78}$$

which implies

$$\overline{\mathbf{S}}(t,x) = \mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v}))(t,x) \quad \text{for a.a. } (t,x) \in Q\,. \tag{3.79}$$

This and (3.64) shows that (2.7) is satisfied for ϱ and $\boldsymbol{v}$.

Next we show that also (2.8) is satisfied. Using again $(1.8)_3$ and proceeding as above, it is simple to verify that

$$\lim_{n\to\infty} \int_0^T \big(\mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)) - \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})\big)\, dt = 0. \tag{3.80}$$

Setting $a^n(t,x) = a^n := \big(\mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)) - \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}))\big)\cdot \mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v}) \ge 0$, (3.80) says that $a^n \to 0$ strongly in $L^1(Q)$. Consequently, for any $h \in L^\infty(Q)$

$$\begin{aligned} 0 = \lim_{n\to\infty} \int_Q a^n\, h\, dx\, dt = {} & \lim_{n\to\infty} \int_Q \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{v}^n) h\, dx\, dt \\ & + \lim_{n\to\infty} \int_Q \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{v}) h\, dx\, dt \\ & - \lim_{n\to\infty} \int_Q \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{v}^n) h\, dx\, dt \\ & - \lim_{n\to\infty} \int_Q \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{v}) h\, dx\, dt\,. \end{aligned} \tag{3.81}$$

By (3.32), (3.40), (3.54), (3.35) and (3.79), we conclude from (3.81) that (we also use Lebesgue Dominated Convergence Theorem)

$$\begin{aligned} \lim_{n\to\infty} \int_Q \mathbf{S}(\varrho^n,\theta^n,\mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{v}^n) h\, dx\, dt = \int_Q \mathbf{S}(\varrho,\theta,\mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{v}) h\, dx\, dt \\ \text{for all } h \in L^\infty(Q)\,. \end{aligned} \tag{3.82}$$

Finally, letting $n \to \infty$ in (3.7), it is an easy matter to conclude, using above established convergence properties ((3.82), (3.55), (3.56) and (3.63) in particular), that the triplet ϱ, θ and $\boldsymbol{v}$ satisfies (2.8). Note that it follows directly from (3.7) and (3.28), as a consequence of the convergence results (3.82), (3.56) and (3.63) that $\int_0^T \langle z, h\rangle\, dt := \lim_{n\to\infty} \int_0^T \langle (\varrho^n\theta^n)_{,t}, h\rangle\, dt$ exists for all $h \in L^\infty(0,T;W^{1,q}(\Omega))$. And also, due to (3.55) we have $\lim_{n\to\infty} \int_0^T \langle (\varrho^n\theta^n)_{,t}, h\rangle\, dt = \int_0^T \langle (\varrho\theta)_{,t}, h\rangle\, dt$ for all $h \in \mathcal{D}(0,T;W^{1,q}(\Omega))$. So, $z = (\varrho\theta)_{,t}$. Finally, we take $h = 1\chi_{[0,t]}$ in (2.8) and observe that $\lim_{t\to 0+}(\varrho(t),\theta(t)) = (\varrho_0,\theta_0)$, which implies $(2.9)_2$.

The proof of Theorem 1.1 is complete.

4. Integration by parts formula

The aim of this section is to establish an important lemma that extends the validity of integration by parts formula to integrable functions. Several definitions precedes its formulation.

For any function z and for $h > 0$ we set

$$(\omega_h^+ * z)(t,x) := \frac{1}{h}\int_0^h z(t+\tau,x)\,d\tau \quad \text{and} \quad (\omega_h^- * z)(t,x) := \frac{1}{h}\int_{-h}^0 z(t+\tau,x)\,d\tau\,.$$

Introducing also the notation

$$D^h z := \frac{z(t+h,x)-z(t,x)}{h} \quad \text{and} \quad D^{-h} z := \frac{z(t,x)-z(t-h,x)}{h}$$

we observe that

$$(\omega_h^+ * z)_{,t} = D^h(z) \quad \text{and} \quad (\omega_h^- * z)_{,t} = D^{-h}(z) \tag{4.1}$$

For the later reference we mention a simple relation

$$\begin{aligned} -\int_{t_0}^{t_1} f(\tau) D^h g(\tau)\,d\tau = \int_{t_0}^{t_1} D^{-h} f(\tau) g(\tau)\,d\tau + \frac{1}{h}\int_{t_0}^{t_0+h} f(\tau-h)g(\tau)\,d\tau \\ - \frac{1}{h}\int_{t_1}^{t_1+h} f(\tau-h)g(\tau)\,d\tau\,, \end{aligned} \tag{4.2}$$

and for the couple $(\varrho, \boldsymbol{v})$ we define for $t > 0$

$$K(t) := \frac{1}{2}(\varrho(t), |\boldsymbol{v}(t)|^2) = \frac{1}{2}\int_\Omega \varrho(t,x)|\boldsymbol{v}(t,x)|^2\,dx\,.$$

Lemma 4.1. *For $r \geq 11/5$ assume that*

$$\begin{aligned} &\boldsymbol{v} \in L^r(0,T; W^{1,r}_{0,div}) \cap L^\infty(0,T; L^2(\Omega)^3) \\ &(\varrho \boldsymbol{v})_{,t} \in \big(L^r(0,T; W^{1,r}_{0,div})\big)^* \end{aligned} \tag{4.3}$$

$$\varrho \in L^\infty(Q) \cap C([0,T]; L^q) \qquad \text{for all } q \in [1,\infty)\,, \tag{4.4}$$

and the couple $(\varrho, \boldsymbol{v})$ is a weak solution to $\varrho_{,t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$, which means that for all $t_0, t_1 \in [0,T]$

$$\begin{aligned} &\int_{t_0}^{t_1} (\varrho(\tau), z_{,t}(\tau)) + (\varrho(\tau)\boldsymbol{v}(\tau), \nabla z(\tau))\,d\tau = (\varrho(t_1), z(t_1)) - (\varrho(t_0), z(t_0)) \\ &\text{for all } z \in L^s(0,T; W^{1,s}) \text{ with } s = 5r/(5r-3) \text{ and } z_{,t} \in L^{1+\delta}(0,T; L^{1+\delta})\,. \end{aligned} \tag{4.5}$$

Then the following formula holds for almost all t_0, $t_1 \in [0,T]$ $(t_0 < t_1)$[8]

$$\int_{t_0}^{t_1} \big\langle (\varrho \boldsymbol{v})_{,t}, \boldsymbol{v} \big\rangle - \big(\varrho \boldsymbol{v} \otimes \boldsymbol{v}, \nabla \boldsymbol{v}\big)\,dt = K(t_1) - K(t_0)\,. \tag{4.6}$$

[8] Note that the statement (4.6) can be also written as

$$\int_{t_0}^{t_1} \big\langle (\varrho \boldsymbol{v})_{,t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}), \boldsymbol{v} \big\rangle\,dt = K(t_1) - K(t_0)\,.$$

Proof. Let t_0 and t_1 fulfill $0 < t_0 < t_1 < T$. Then for $h \in (0, \min\{T - t_1, t_0\})$ we set

$$L_h := \int_{t_0}^{t_1} \big\langle (\varrho \boldsymbol{v})_{,t}, \omega_h^+ * \omega_h^- * \boldsymbol{v} \big\rangle \, dt \,,$$

and we observe that to prove (4.6) is equivalent to show that

$$\lim_{h\to 0+} L_h = \int_{t_0}^{t_1} \big(\varrho \boldsymbol{v}, \frac{1}{2}\nabla |\boldsymbol{v}|^2\big) \, dt + \big[K(t_1) - K(t_0)\big] \,. \tag{4.7}$$

Integration by parts together with (4.1) results in

$$\begin{aligned} L_h = & - \int_{t_0}^{t_1} \big(\varrho \boldsymbol{v}, D^h(\omega_h^- * \boldsymbol{v})\big) \, dt \\ & + \big((\varrho \boldsymbol{v})(t_1), (\omega_h^+ * \omega_h^- * \boldsymbol{v})(t_1)\big) - \big((\varrho \boldsymbol{v})(t_0), (\omega_h^+ * \omega_h^- * \boldsymbol{v})(t_0)\big) \,, \end{aligned} \tag{4.8}$$

which implies

$$\lim_{h\to 0+} L_h = \lim_{h\to 0+} A_h + 2\big[K(t_1) - K(t_0)\big] \,, \tag{4.9}$$

where $A_h := - \int_{t_0}^{t_1} \big(\varrho \boldsymbol{v}, D^h(\omega_h^- * \boldsymbol{v})\big) \, dt$. Using the formula (4.2), we observe that

$$\lim_{h\to 0+} A_h = \lim_{h\to 0+} \int_{t_0}^{t_1} \big(D^{-h}(\varrho \boldsymbol{v}), \omega_h^- * \boldsymbol{v}\big) \, dt - 2\big[K(t_1) - K(t_0)\big] \,. \tag{4.10}$$

Then (4.9) and (4.10) lead to

$$\lim_{h\to 0+} L_h = \lim_{h\to 0+} B_h \tag{4.11}$$

where $B_h := \int_{t_0}^{t_1} \big(D^{-h}(\varrho \boldsymbol{v}), \omega_h^- * \boldsymbol{v}\big) \, dt$. Next, notice that

$$\begin{aligned} B_h &= \int_{t_0}^{t_1} \big(\varrho D^{-h} \boldsymbol{v}, \omega_h^- * \boldsymbol{v}\big) \, dt + \int_{t_0}^{t_1} \big((D^{-h}\varrho) \boldsymbol{v}(\cdot - h), (\omega_h^- * \boldsymbol{v})\big) \, dt \\ &= \int_{t_0}^{t_1} \big(\varrho, \frac{1}{2}|\omega_h^- * \boldsymbol{v}|^2_{,t}\big) \, dt + \int_{t_0}^{t_1} \big(\omega_h^- * (\varrho \boldsymbol{v}), \nabla(\boldsymbol{v}(\cdot - h) \cdot (\omega_h^- * \boldsymbol{v}))\big) \, dt \,, \end{aligned} \tag{4.12}$$

where we used (4.1) and the relation $D^{-h}\varrho = -\operatorname{div}(\omega_h^- * (\varrho \boldsymbol{v}))$, which follows from the fact that the couple $(\varrho, \boldsymbol{v})$ solves $\varrho_{,t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$ in a weak sense. Finally, inserting $z = \frac{1}{2}|\omega_h^- * \boldsymbol{v}|^2$ into (4.5) we conclude that

$$\begin{aligned} B_h = & \big(\varrho(t_1), \frac{1}{2}|\omega_h^- * \boldsymbol{v}|^2(t_1)\big) - \big(\varrho(t_0), \frac{1}{2}|\omega_h^- * \boldsymbol{v}|^2(t_0)\big) \\ & - \int_{t_0}^{t_1} \big(\varrho \boldsymbol{v}, \frac{1}{2}\nabla |\omega_h^- * \boldsymbol{v}|^2\big) \, dt + \int_{t_0}^{t_1} \big(\omega_h^- * (\varrho \boldsymbol{v}), \nabla(\boldsymbol{v}(\cdot - h) \cdot (\omega_h^- * \boldsymbol{v}))\big) \, dt \,. \end{aligned} \tag{4.13}$$

Taking limit $h \to 0$ in (4.13), and recalling (4.11) we obtain the assertion (4.7), for almost all t_0 and t_1 in $(0, T)$. This proves (4.6). □

5. Extensions and concluding remarks.

In this paper we established the existence of weak solution to unsteady flows of incompressible inhomogeneous heat-conducting fluids of a power-law type (sometimes people called these materials with such a polynomial dependence of the shear stress on the shear rate generalized Newtonian or modified Navier-Stokes fluids) without requiring any smallness of initial data or without restricting ourselves to a short-time interval. To our best knowledge, this is the first large data result for flows of incompressible fluids that includes both the thermal changes and inhomogeneity of material. Although the values of the power-law index r taken in the range $r \geq 11/5$ puts Problem $(\mathcal{P})$ into subcritical one, one needs to overcome certain technical difficulties in the proof. Following [13], we formulate and prove the integration by parts formula in Section 4 that reveals to be an appropriate tool to extend the standard monotone operator method to the considered setting. This allows us to re-prove the results for uniform temperature (isothermal case) under weaker assumptions on the the stress tensor (on one hand, no need to assume the existence of potential to the stress tensor, on the other hand, the possibility to include also the limiting case $r = 11/5$). Our aside intention was to present the proof as simple as possible. To reach this aim we apply Div-Curl Lemma for evolutionary problem as outlined and successfully incorporated in the analysis of compressible fluid models by Feireisl in [10] or [11], and we also relied on the theory for transport equation (based on the notion of renormalized solution) as presented by P.-L. Lions in [17]. We also restrict ourselves to thermally and mechanically isolated body, and assume that initial density and initial temperature are bounded by a positive constant from below, etc. The approach presented here might be applicable for the treatment of the following problems: (i) of the approach presented here seem to be possible: (i) following [17] one could treat the case $\varrho_* = 0$ (see $(1.5)_1$), (ii) following [3] or [4] one could replace the no-slip boundary condition by the Navier's one, (iii) following [7] one could also consider non-homogeneous Dirichlet boundary conditions for the temperature at some part of the boundary, and (iv) inspired by [6] one could also admit a suitable dependence of the heat conductivity coefficient on the shear rate.

6. Acknowledgement

Josef Málek appreciates several useful discussions with Miroslav Bulíček that resulted in a quite simple structure of the proof of Theorem 1.1.

References

[1] S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids*, Studies in Mathematics and its Applications, vol. 22, North-Holland Publishing Co., Amsterdam, 1990, Translated from the Russian.

[2] R. M. Brown and Z. Shen, *Estimates for the Stokes operator in Lipschitz domains*, Indiana Univ. Math. J. **44** (1995), no. 4, 1183–1206.

[3] M. Bulíček, E. Feireisl, and J. Málek, *Navier-Stokes-Fourier system for incompressible fluids with temperature dependent material coefficients*, Nonl. Anal. Real World Appl., doi:10.1016/j.nonrwa.2007.11.018, 2008.

[4] M. Bulíček, J. Málek, and K. R. Rajagopal, *Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity*, Indiana Univ. Math. J. **56** (2007), no. 1, 51–85.

[5] M. Bulíček, J. Málek, and K.R. Rajagopal, *Mathematical Analysis of Unsteady Flows of Fluids with Pressure, Shear-Rate and Temperature Dependent Material Moduli, that Slip at Solid Boundaries*, Preprint 2007-17 of the Jindřich Nečas Center for mathematical modeling, 2007.

[6] M. Bulíček, J. Málek, and K. R. Rajagopal, *Mathematical Results Concerning The Unsteady Flows of Chemically Reacting Incompressible Fluids*, Preprint 2008-007 of the Jindřich Nečas Center for mathematical modeling, 2008.

[7] L. Consiglieri, *Weak solutions for a class of non-Newtonian fluids with energy transfer*, J. Math. Fluid Mech. **2** (2000), no. 3, 267–293.

[8] L. Diening, M. Růžička, and J. Wolf, *Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity*, Preprint 08-02 at the Preprint Series of the Department of Mathematics, University of Freiburg, 2008.
[9] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), no. 3, 511–547.
[10] E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004.
[11] E. Feireisl and A. Novotný, *Singular limits in thermodynamics of viscous fluids*, Birkhäuser, Basel, 2008, to appear.
[12] E. Fernández-Cara, F. Guillén, and R. R. Ortega, *Some theoretical results for viscoplastic and dilatant fluids with variable density*, Nonlinear Anal. **28** (1997), no. 6, 1079–1100.
[13] J. Frehse and M. Růžička, *Non-Homogeneous Generalized Newtonian Fluids*, Math. Z., doi:10.1007/s00209-007-0278-1, 2007.
[14] G. P. Galdi, C. G. Simader, and H. Sohr, *On the Stokes problem in Lipschitz domains*, Ann. Mat. Pura Appl. (4) **167** (1994), 147–163.
[15] F. Guillén-González, *Density-dependent incompressible fluids with non-Newtonian viscosity*, Czechoslovak Math. J. **54(129)** (2004), no. 3, 637–656.
[16] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural′ceva, , Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.
[17] P. L. Lions, *Mathematical topics in fluid mechanics. Vol. 1*, Oxford Lecture Series in Mathematics and its Applications, vol. 3, The Clarendon Press Oxford University Press, New York, 1996, Incompressible models, Oxford Science Publications.
[18] J. Málek and K. R. Rajagopal, Mathematical issues concerning the Navier-Stokes equations and some of their generalizations, Handbook of Differential Equations, vol. 2, ch. 5, pp. 371–459, Handbook of Differential Equations, Elsevier B.V., 2005, pp. 371–459.
[19] K. Retelsdorf, *Existenze für inhomogene Nicht-Newtonsche Flüssigkeiten*, M.S. thesis, University Freiburg, 2002.
[20] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.
[21] L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV (Boston, Mass.), Res. Notes in Math., vol. 39, Pitman, Boston, Mass., 1979, pp. 136–212.
[22] J. Wolf, *Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity*, J. Math. Fluid Mech. **9** (2007), no. 1, 104–138.

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