Regularity Results for Three Dimensional Isotropic and Kinematic Hardening Including Boundary Differentiability

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Abstract

For flat Dirichlet boundary we prove that the first normal derivatives of the stresses and internal parameters are in $L^\infty(0,T;L^{1+\delta})$ and in $L^\infty(0,T;H^{\frac{1}{2}-\delta})$ up to the boundary.

This regards solutions of elastic-plastic flow problems with isotropic or kinematic hardening with von Mises yield function.

We show that the elastic strain tensor $\varepsilon(u)$ of three dimensional plasticity with isotropic hardening is contained in the space $L^\infty(0,T;L^6_{loc})$ and in $L^\infty(0,T;L^{4-\delta})$ up to the flat Dirichlet boundary. We obtain related results concerning traces of $\varepsilon(u)$.

In the case of kinematic hardening we present a simple proof of the $L^\infty(0,T;H^1_{loc})$ inclusion of the elastic strain tensor.

Keywords: plasticity with hardening, boundary differentiability, regularity of solutions

MSC(2000): 74C05, 35B65, 35K85

1 Introduction

We consider problems of plasticity with isotropic and kinematic hardening.

Let $\Omega \subset \mathbb{R}^n$ be an open connected, bounded subset with Lipschitz boundary $\partial\Omega$. We further assume that $\partial\Omega = \Gamma_D \dot{\cup} \Gamma_N$, where Γ_D has positive $(n-1)$-dimensional Hausdorff measure.

For a vector valued differentiable function $u : \mathbb{R}^n \to \mathbb{R}^n$ we define a second order tensor field $\varepsilon(u)$ by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\intercal)\,.$$

This tensor field is called the (linearized) strain tensor of the displacement field u.

In the small strain theory of linear elasticity for every deformation (given by the displacement field u) we have a stress field σ. Here σ is a symmetric second order tensor field, thus $\sigma : \Omega \to \mathbb{R}^{n\times n}_{\text{sym}}$.

The strain field $\varepsilon(u)$ and the stress field σ are linked by a linear relation

$$A\sigma = \varepsilon(u)\,, \qquad u\big|_{\Gamma_D} = 0\,.$$

$A \in L^\infty(\Omega, \mathbb{R}^{n\times n\times n\times n})$ is a symmetric, uniform elliptic fourth order tensor field, i.e., there exists $\alpha > 0$ such that for all $\tau \in \mathbb{R}^{n\times n}$

$$A\tau : \tau \geq \alpha|\tau|^2 \tag{1.1}$$

holds.
For a given applied body force density $f : \Omega \to \mathbb{R}^n$ and exterior surface force $g : \Gamma_N \to \mathbb{R}^n$, there holds the balance of forces in Ω

$$\begin{aligned} -\operatorname{div}\sigma &= f \text{ in } \Omega \\ \sigma\cdot\vec{n} &= g \text{ on } \Gamma_N\,. \end{aligned} \tag{1.2}$$

For the plasticity model we introduce a hardening parameter $\xi : \Omega \to \mathbb{R}^m$, and the yield function $\mathcal{F}(\sigma,\xi) : \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^m \to \mathbb{R}$.
The yield function $\mathcal{F}$ models the hardening behaviour of the material and is assumed to be continuous and convex.

In this paper we will consider the von Mises yield criterion only, the method of proof, however is much more general.
For $M \in \mathbb{R}^{n\times n}$ we denote by $M_D := M - \frac{1}{n}\operatorname{tr}(M)Id$ the deviator of M.

Let $\kappa > 0$, we define

$$\mathcal{F}(\sigma,\xi) = |\sigma_D| - (\kappa+\xi) \qquad \text{isotropic hardening } (\xi\in\mathbb{R}) \tag{1.3a}$$
$$\mathcal{F}(\sigma,\xi) = |\sigma_D - \xi_D| - \kappa \qquad \text{kinematic hardening } (\xi\in\mathbb{R}^{n\times n}_{\text{sym}})\,. \tag{1.3b}$$

We assume for the applied forces

$$f, \dot{f} \in L^\infty(0,T; L^n(\Omega,\mathbb{R}^n)) \tag{1.4a}$$
$$p, \dot{p} \in L^\infty(0,T; L^\infty(\partial\Omega,\mathbb{R}^n))\,. \tag{1.4b}$$

We define the set

$$\mathcal{M} := \left\{(\sigma,\xi) \in L^2(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^m)) | \mathcal{F}(\sigma,\xi) \leq 0 \text{ a.e. in } \Omega\times\mathbb{R}^m\times[0,T]\right\}.$$

In the following let $m = 1$ in the case of isotropic and $m = n\times n$ in the case of kinematic hardening.

We suppose the usual **safe load condition** (cf. Johnson [Joh78]). There exists an element $(\sigma^0, \xi^0) \in W^{1,\infty}\Big(0,T; L^\infty(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}) \times L^\infty(\Omega, \mathbb{R}^m)\Big)$ such that

$$\left.\begin{aligned} F(\sigma^0, \xi^0) &\le -\delta_0 < 0 \\ -\operatorname{div}\sigma^0 &= f \text{ in } \Omega\times[0,T] \\ \sigma^0\cdot\vec{n} &= p \text{ on } \Gamma_N\times[0,T] \\ (\sigma^0,\xi^0)(0) &= 0 \text{ in } \Omega\times\{t=0\} \end{aligned}\right\} \tag{1.5}$$

We abbreviate $v = \frac{\partial}{\partial t}u(x,t)$. We define the basic problem of plasticity with hardening as:

Definition 1.1 (basic Problem) *Find* $\sigma \in L^2(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{sym}))$, $\xi \in L^2(0,T; L^2(\Omega, \mathbb{R}^m))$ *such that* $\dot\sigma, \dot\xi \in L^2(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{sym})), L^2(0,T; L^2(\Omega, \mathbb{R}^m))$ *exists and ,* $v \in L^2(0,T; H^1_{\Gamma_D}(\Omega, \mathbb{R}^n))$ *such that for all* $(\tau,\eta)\in\mathcal{M}$ *a.e. in* $[0,T]$*:*

$$\int_\Omega \big(A\dot\sigma - \varepsilon(v)\big) : \big(\tau - \sigma\big)\,\mathrm{d}x + \int_\Omega \dot\xi\cdot(\eta-\xi)\,\mathrm{d}x \ge 0 \tag{1.6}$$

$$\begin{aligned} &\langle\sigma, \nabla w\rangle = \langle f, w\rangle + \int_{\Gamma_N} pw\,\mathrm{d}\Gamma\mathrm{d}s \quad \forall\, w\in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n) \text{ a.e. with respect to } t \\ &(\sigma,\xi)(0) = 0 \text{ in } \overline{\Omega}\times\{t=0\} \\ &\mathcal{F}(\sigma,\xi) \le 0 \text{ a.e. in } \Omega\times[0,T] \end{aligned} \tag{1.7}$$

Under mild regularity assumptions it can be shown, that the basic problem (1.6) &(1.7) is equivalent to the pointwise a.e. equations

$$\begin{aligned} \varepsilon(\dot u) &= A\dot\sigma + \dot\lambda\frac{\sigma_D}{|\sigma_D|} \\ 0 &= \dot\xi - \dot\lambda \end{aligned} \tag{1.8}$$

in isotropic hardening and

$$\begin{aligned} \varepsilon(\dot u) &= A\dot\sigma + \dot\lambda\frac{\sigma_D - \xi_D}{|\sigma_D-\xi_D|} \\ 0 &= \dot\xi - \dot\lambda\frac{\sigma_D - \xi_D}{|\sigma_D-\xi_D|} \end{aligned} \tag{1.9}$$

in kinematic hardening.

Remark: We have chosen the initial condition in (1.7) for simplicity, although the more realistic $(\sigma,\xi)(0) = (\sigma_0,\xi_0)\in\mathcal{M}$ is obviously covered by our methods.

With the approximation of the basic problem in section 2, it is simple to show (see [FL08]), that the basic problem 1.1 satisfies the associated flow rule of plasticity with hardening

$$\dot{\Pi} = \dot{\lambda} \frac{\partial}{\partial \sigma} \mathcal{F}(\sigma, \xi) \tag{1.10a}$$

$$\dot{\xi} = -\dot{\lambda} \frac{\partial}{\partial \xi} \mathcal{F}(\sigma, \xi), \tag{1.10b}$$

where $\dot{\Pi}$ denotes the *plastic strain* and $\dot{\lambda}$ is an non negative multiplier.
For the rest of this paper we will assume the elastic compliance tensor A to be sufficiently smooth (for simplicity constant).
The existence of solutions of (1.6) &(1.7) and the fact that the displacement velocity v lies in $L^2(0, T; H^1_{\Gamma_D}(\Omega, \mathbb{R}^n))$ was first shown by Johnson [Joh78], see also Löbach [Löb08].

In the case, that A is sufficiently smooth and some stricter assumptions[1] on the body- and surface forces (1.4), we have $\sigma, \xi \in L^2(H^1_{loc})$.[Ser94],[Löb08].
Futhermore, Seregin proved in the case (1.3b), that the strains $\varepsilon(u)$ (not $\varepsilon(\dot{u})$!) are contained in the space $L^\infty(H^1_{loc})$. From this and Sobolev imbedding one obtains for $n = 3$ that the displacement u is Hölder continuous with exponent $\frac{1}{2}$ in spatial direction, and in addition, since $\varepsilon(\dot{u}) \in L^2$, u is locally Hölder continuous in time with Hölder exponent $\frac{1}{2}$. All this concerns Seregin's result in the case of kinematic hardening.

Recently, the authors obtained in [FL08] the Hölder continuity up to the boundary in the case of isotropic and kinematic hardening with another method, however only in two space dimensions.
It seems that the analogue of Seregins result i.e. $\varepsilon(u) \in L^\infty(H^1_{loc})$ is not yet known in the case of isotropic hardening. We are not able to fill this gap, but at least in section 3 we achieve $\varepsilon(u) \in L^\infty(L^6_{loc})$ for $n = 3$ in the case of isotropic hardening and related results in sections 12 & 14.

Differentiability of the stresses and hardening parameters (full derivatives) up to the boundary is not yet known. The first result seems to be the present paper: in section 5-7 we prove that $D_n\sigma \in L^{1+\delta}$ for the normal derivatives of the stresses and hardening parameter in a portion of the boundary, where $\partial\Omega$ coincides with the hyperplane $\{x \in \mathbb{R}^n \,|\, x_n = 0\}$. In fact we have a slightly better result then $D_n\sigma \in L^{1+\delta}$, since there hold additional Morrey estimates cf. theorem 6.1 for the details.
Our proof of $L^{1+\delta}$-regularity of the stresses contains additional regularity information. In section 5 we prove

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega |D_n\sigma|^2 x_n \,\mathrm{d}x \leq K\,. \tag{1.11}$$

[1]The proof of Seregin [Ser94] needs less regularity for the body- and surface forces than Löbach [Löb08]. In Seregin's case we further need $f \in C([0, T]; H^1_{loc}(\Omega, \mathbb{R}^n))$ in the case isotropic hardening. In Löbach's case we need $Df, \triangle f \in L^\infty(0, T; L^n(\Omega, \mathbb{R}^n))$ for kinematic- and isotropic hardening.

This can be refined with a new plate filling technique which yields an additional Morreyspace refinement of (1.11), see section 7 for the details. This allows us to conclude $L^\infty(H^{1+\delta,1})$ rather than $L^\infty(H^{1-\delta,1})$ estimates.
In section 9 we present an alternative proof of the result of Knees, which works also for the case of isotropic hardening and even for Neumann boundary conditions (the latter is not elaborated).
Concerning **fractional differentiability** of the stresses the first result at the boundary in the kinematic case is due to Alber and Nesenenko [AN08]. These authors show for kinematic hardening that $\sigma, \xi \in H^{\frac{1}{3}-\delta}$. Furthermore D. Knees proved in [Kne08] $H^{\frac{1}{2}-\delta}$ in a flat situation for kinematic hardening.
We prove $\sigma, \xi \in \mathcal{N}^{\frac{1}{2},2}$ near the flat Dirichlet boundary, by the imbedding theorem for Nikol'skii spaces we obtain $\sigma, \xi \in H^{\frac{1}{2}-\delta}$. The tangential derivatives of σ, ξ have more regularity and are in $L^\infty(L^2)$.
In the kinematic case the regularity results on the stresses imply $D_\tau \varepsilon(u) \in L^\infty(L^2)$ for the tangential derivatives and $\varepsilon(u) \in L^\infty(H^{\frac{1}{2}-\delta})$ for the normal derivatives. In [AN08] Alber & Nesenenko first proved $u \in H^{\frac{4}{3}-\delta}$ in the kinematic case near the boundary.
Regarding the isotropic case, in section 10 we obtain $\nabla u \in L^\infty(L^{4-\delta})$ near the boundary which implies Hölder continuity of the displacements u with exponent $\alpha \leq \frac{1}{4}$ in space direction and Hölder exponent $\beta < \frac{1}{3}$ in time direction.

2 Approximation

We will approximate the problem (1.6) by a sequence of penalized problems.
We define a viscoplastic type potential G_μ as follows,

$$G_\mu(\sigma, \xi) = \frac{1}{2\mu}\big(\mathcal{F}(\sigma, \xi)\big)_+^2$$

where $(a)_+ = \begin{cases} a & \text{if } a \geq 0 \\ 0 & \text{else} \end{cases}$.
In the case of isotropic and kinematic hardening with von Mises yield criterion we have

$$G'_\mu(\sigma, \xi) := \nabla_{(\sigma,\xi)} G_\mu(\sigma, \xi) \in L^2 \times L^2 \,.$$

If we set $\Sigma = (\sigma, \xi)$ and $\Pi = (\pi_s, \pi_\xi)$, we can derive the flow rule for the penalized problem of plasticity with hardening:

$$\dot{\pi}_\sigma = \dot{\lambda}\frac{\partial}{\partial \sigma}\mathcal{F}(\sigma, \xi)$$
$$\dot{\pi}_\xi = \dot{\lambda}\frac{\partial}{\partial \xi}\mathcal{F}(\sigma, \xi),$$

where $\dot{\lambda} \geq 0$ is a multiplier with

$$\dot{\lambda} = 0 \text{ if } \mathcal{F}(\sigma, \xi) < 0\,.$$

Due to the differentiability and convexity of $G'_\mu(\sigma, \xi)$, it's derivative $G'_\mu(\sigma, \xi)$ is a monotone operator. The monotonicity of $G'_\mu(\sigma, \xi)$ yields a generalized *principle of maximum plastic dissipation*

$$\dot{\Pi} : (\Xi - \Sigma) \leq 0 \qquad \forall\, \Xi = (\tau, \eta) \,:\, \mathcal{F}(\Xi) \leq 0\,.$$

The penalized problem reads:

Definition 2.1 *Find* $\big((\sigma_\mu, \xi), v_\mu\big) \in H^1(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{sym} \times \mathbb{R}^m)) \times L^2(0,T; H^1_{\Gamma_D}(\Omega, \mathbb{R}^n))$ *such that for a.e.* $t \in [0,T]$

$$\begin{pmatrix} \varepsilon(v_\mu) \\ 0 \end{pmatrix} = \begin{pmatrix} A\dot{\sigma}_\mu \\ \dot{\xi}_\mu \end{pmatrix} + \begin{pmatrix} G'_\mu(\sigma_\mu, \xi)_1 \\ G'_\mu(\sigma_\mu, \xi)_2 \end{pmatrix} \tag{2.1}$$

with the balance of forces

$$\begin{aligned} &\langle \sigma_\mu, \nabla w \rangle = \langle f, w \rangle + \int_{\Gamma_N} g w \,\mathrm{d}\Gamma \mathrm{d}s \quad \forall\, w \in H^1_{\Gamma_D}(\Omega, \mathbb{R}^n) \ \textit{a.e. with respect to } t \\ &(\sigma_\mu, \xi_\mu)(0) = 0 \ \textit{in } \overline{\Omega} \times \{t = 0\}\,. \end{aligned} \tag{2.2}$$

We have the estimates independent of μ

$$\|\sigma_\mu\|_{L^\infty(L^2)},\ \|\xi_\mu\|_{L^\infty(L^2)} \leq C \tag{2.3a}$$

$$\|\dot{\sigma}_\mu\|_{L^\infty(L^2)},\ \|\dot{\xi}_\mu\|_{L^\infty(L^2)} \leq C \tag{2.3b}$$

$$\|\varepsilon(\dot{u}_\mu)\|_{L^\infty(L^2)} \leq C \tag{2.3c}$$

$$\|\nabla\sigma_\mu\|_{L^\infty(L^2)},\ \|\nabla\xi_\mu\|_{L^\infty(L^2)} \leq C_{\Omega_0}, \quad \forall\, \Omega_0 \subset\subset \Omega\,. \tag{2.3d}$$

For (2.3d) additional regularity for $\sigma_\mu\big|_{t=0}$ and $\xi_\mu\big|_{t=0}$ is required:

$$\begin{aligned} &\nabla\sigma_\mu(0) \in L^2(L^2_{loc}) \\ &\nabla\xi_\mu(0) \in L^2(L^2_{loc})\,, \end{aligned} \tag{2.4}$$

and

$$Df, \triangle f \in L^\infty(0,T; L^n(\Omega, \mathbb{R}^n))\,.$$

These estimates yield the convergence to $\big((\sigma, \xi), v\big)$ solution of (1.6), as the penalty parameter μ tends to zero. For the details see Löbach [Löb08]. In [FL08] the convergence of λ_μ (for example $\lambda_\mu = \big(|\sigma_D| - (\kappa + \xi)\big)_+$ in the case of isotropic hardening) to a plastic multiplier λ (cf. equation (1.10)) was shown.

For $v_\mu = \dot{u}_\mu$, we have due to imbedding theorems and Korn's inequality in L^2, in the case of $n = 3$ dimensions

$$\|v_\mu\|_{L^\infty(L^6)} \leq C \tag{2.5}$$

provided the domain Ω has Lipschitz boundary (otherwise (2.5) holds only in the interior of Ω).

From now on, for the sake of clarity we omit the subscript μ for the penalty parameter.

3 Higher integrability of the strain in isotropic hardening

In the case of isotropic hardening with von Mises yield criterion the penalized equation (2.1) reads:

$$\varepsilon(v) = A\dot{\sigma} + \frac{1}{\mu}\big(|\sigma_D| - (\kappa+\xi)\big)_+ \frac{\sigma_D}{|\sigma_D|} \tag{3.1a}$$

$$0 = \dot{\xi} - \frac{1}{\mu}\big(|\sigma_D| - (\kappa+\xi)\big)_+ \,. \tag{3.1b}$$

Theorem 3.1 *Let $n = 3$, and f, $\dot{f} \in L^\infty(L^\infty)$, $Df, \triangle f \in L^\infty(L^3)$ and assume the safe load condition* (1.5) *holds true, then for every subset $\Omega_0 \subset\subset \Omega$*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_0} |\varepsilon(u)|^6 \,\mathrm{d}x \leq K_{\Omega_0} \tag{3.2}$$

uniformly as $\mu \to 0$.

proof The use the fact, that the assumptions imply $L^\infty(H^1_{loc})$-regularity for σ . Let $\chi_L := \chi_L(t,\cdot)$ be the characteristic function of the set where $|\varepsilon(u)(t)| \leq L$. We integrate the penalty approximation (3.1a) with respect to t and multiply the integrated equation by $\tau\chi_L\varepsilon(u)|\varepsilon(u)|^4$ evaluated at t. The integrated equation (3.1b) is multiplied by $\tau\chi_L|\varepsilon(u)|^5$, where τ is a localization function.
This yields

$$\begin{aligned}\int_\Omega \chi_L|\varepsilon(u)(t)|^6\tau^2 \,\mathrm{d}x \leq \int_\Omega \chi_L(t)|\varepsilon(u)(0)| \cdot |\varepsilon(u)(t)|^5\tau^2 \,\mathrm{d}x + \int_\Omega |A\sigma\Big|_{s=0}^{s=t} \tau^2\chi_L(t)|\varepsilon(u)(t)|^5 \,\mathrm{d}x \\ + \int_\Omega \frac{1}{\mu}\left(\int_0^t \big(|\sigma_D| - (\kappa+\xi)\big)_+ \frac{\sigma_D}{|\sigma_D|}\,\mathrm{d}s\right) \tau^2\chi_L\varepsilon(u)(t)|\varepsilon(u)(t)|^4 \,\mathrm{d}x\end{aligned} \tag{3.3}$$

and

$$\int_\Omega \xi\Big|_{s=0}^{s=t}\tau^2\chi_L(t)|\varepsilon(u)(t)|^5 \,\mathrm{d}x = \frac{1}{\mu}\int_\Omega \left(\int_0^t \big(|\sigma_D| - (\kappa+\xi)\big)_+ \,\mathrm{d}s\right) \chi_L(t)|\varepsilon(u)(t)|^5 \,\mathrm{d}x \,. \tag{3.4}$$

We subtract (3.4) from (3.3) and use the fact that

$$\frac{1}{\mu}\int_\Omega \left(\int_0^t \big(|\sigma_D|-(\kappa+\xi)\big)_+ \frac{\sigma_D}{|\sigma_D|}\,\mathrm{d}s\right)\tau^2\chi_L\varepsilon(u)(t)|\varepsilon(u)(u)|^4\,\mathrm{d}x$$
$$\leq \frac{1}{\mu}\int_\Omega \left(\int_0^t \big(|\sigma_D|-(\kappa+\xi)\big)_+\,\mathrm{d}s\right)\tau^2\chi_L\sum_{i,k=1}^{3}|\varepsilon_{ik}(u)(t)||\varepsilon(u)(u)|^4\,\mathrm{d}x$$
$$\leq \frac{1}{\mu}\int_\Omega \left(\int_0^t \big(|\sigma_D|-(\kappa+\xi)\big)_+\,\mathrm{d}s\right)\tau^2\chi_L|\varepsilon(u)(t)|^5\,\mathrm{d}x\,. \quad (3.5)$$

Hence the right hand side of (3.4) dominates the last term and we conclude

$$\int_\Omega \chi_L(t)|\varepsilon(u)(t)|^6\,\mathrm{d}x \leq \int_\Omega \chi_L(t)\left(|\varepsilon(u)(0)|+|A\sigma(0)|+|\xi(0)|\right)|\varepsilon(u)(t)|^5\tau^2\,\mathrm{d}x$$
$$+\int_\Omega \chi_L(t)\Big(|A\sigma(t)|+|\xi(t)|\Big)|\varepsilon(u)(t)|^5\tau^2\,\mathrm{d}x\,. \quad (3.6)$$

Since $\sigma,\xi\in H^1_{loc}$, uniformly as $\mu\to 0$ we may estimate the right hand side by

$$K\left(\int_\Omega \chi_L(t)\tau^2|\varepsilon(u)(t)|^6\,\mathrm{d}x\right)^{5/6}.$$

By passing to the limit $L\to\infty$, the theorem is proved. □

Remark : With a similar argument, in the case $n\geq 4$ one obtains $\varepsilon(u)\in L^\infty(0,T;L^{\frac{2n}{n-2}}_{loc})$ and for $n=2$ $\varepsilon(u)\in L^\infty(0,T;L^q_{loc})$ for all $q<\infty$.

4 Interior Hölder continuity of the displacements in isotropic hardening

From theorem 3.1 and Sobolev's imbedding theorem we conclude the Hölder continuity of the displacements u in spatial direction in *three space dimensions* with exponent $\frac{1}{2}$. By simple estimates, the Hölder continuity can be extended to the time direction.

Theorem 4.1 *Under the conditions of theorem 3.1 the displacements u in problem* (1.1) *are Hölder continuous on $[0,T]\times\Omega_0$ with exponent $\frac{1}{2}$ for all $\Omega_0\subset\subset\Omega$.*

proof In view of the above remark it suffices to prove the Hölder Continuity in time direction. For the Hölder continuity in time direction we have to estimate the quantity

$$\big|u(t_1,x_0)-u(t_2,x_0)\big| \leq \big|u(t_1,x_0)-\overline{u}_{R,x_0}(t_1)\big| + \big|u(t_2,x_0)-\overline{u}_{R,x_0}(t_2)\big|$$
$$+\left|\fint_{B_R(x_0)}\big[u(t_2,x)-u(t_1,x)\big]\,\mathrm{d}x\right| \quad (4.1)$$
$$=A_0+B_0+C_0$$

where $\overline{u}_{R,x_0}(t_i)$ is the meanvalue of $u(t_i, x)$ extended over $B_R(x_0)$.
Since u is locally Hölder continuous in x-direction with exponent $\frac{1}{2}$, we have

$$|A_0| + |B_0| \le KR^{\frac{1}{2}}, \quad R \le R_0, \tag{4.2}$$

a.e. with respect to t_1, t_2. The term C_0 is estimated using the time derivative of u:

$$\begin{aligned} C_0 = \left| \fint_{B_R(x_0)} \int_{t_1}^{t_2} \dot{u}\,\mathrm{d}t\,\mathrm{d}x \right| &\le \int_{t_1}^{t_2} \left(\fint_{B_R(x_0)} |\dot{u}|^6\,\mathrm{d}x \right)^{\frac{1}{6}} |B_R(x_0)|^{\frac{5}{6}}\,\mathrm{d}t \\ &\le K|t_2 - t_1| R^{-3} \left(R^3\right)^{\frac{5}{6}} = K|t_1 - t_2| R^{-\frac{1}{2}} \end{aligned} \tag{4.3}$$

We have used the inclusion $\nabla\dot{u} \in L^\infty(L^2)$ and Sobolev imbedding theorem to conclude $\dot{u} \in L^\infty(L^6)$. From equations (4.1)-(4.3) we obtain

$$\left|u(t_1, x_0) - u(t_2, x_0)\right| \le KR^{\frac{1}{2}}|t_1 - t_2|R^{-\frac{1}{2}} + KR^{\frac{1}{2}}\,.$$

The optimal choice of R is $R = |t_1 - t_2|$. This yields

$$\left|u(t_1, x_0) - u(t_2, x_0)\right| \le K|t_1 - t_2|^{\frac{1}{2}}, \quad |t_1 - t_2| \le R_0^2\,.$$

This proves the theorem. □

5 Differentiability at the boundary. First results

For simplicity we discuss the differentiability of the stress σ and hardening parameter ξ in a neighborhood of a **boundary point** x_0, where $\partial\Omega$ is flat, i.e. $\exists R > 0$ such that

$$\partial\Omega \cap B_R(x_0) = \left\{x \in \mathbb{R}^n | x_n = 0\right\} \cap B_R(x_0) \tag{5.1a}$$

$$\Omega_R := \Omega \cap B_R(x_0) = \left\{x \in \mathbb{R}^n | x_n \ge 0\right\} \cap B_R(x_0) \tag{5.1b}$$

In this section, for the sake of mathematical insight, we consider the case of n space dimensions.
In the case of zero boundary conditions for u, the existence of tangential derivatives $D_\tau\sigma, D_\tau\xi \in L^\infty(L^2(\Omega_R))$ is proven analogously to the interior differentiability [Ser94]. One applies a difference quotient operator D^h_τ to the penalized equation (2.1)&(2.2), $D^h_\tau g(x) = \frac{1}{h}\big(g(x + h\vec{e}_\tau) - g(x)\big)$ where $\vec{e}_\tau$ is a unit vector in tangential direction, and one uses $\zeta^2(D^h_\tau\sigma, D^h_\tau\xi)$ as a test function, with $\operatorname{supp}\zeta \subset B_R(x_0), \zeta \equiv 1$ on $B_r(x_0), r < R$. We may assume that $\zeta(x_1, \dots, x_n)$ is constant for $0 \le x_n \le \delta_0, x_1 \dots, x_{n-1}$ fixed, say $\zeta(x_1, \dots, x_n) = \zeta_0(x_1, \dots, x_{n-1})\zeta_1(x_n)$. Then, after integration (also with respect to t), the right hand side of (2.1)&(2.2) gives , a nice definite term which, at the end, gives a uniform bound for $D^h_\tau\sigma$ in $L^\infty(L^2(\Omega_R))$ as $h \to 0, \mu \to 0$.
The left hand side is treated in the usual way:

$$L_0 := \int_0^T\!\!\int \zeta^2 D^h_\tau\nabla\dot{u} : D^h_\tau\sigma\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\!\!\int \left[\zeta^2 D^h_\tau\dot{u} D^h_\tau f - D^h_\tau\dot{u} : \nabla\zeta^2 D^h_\tau\sigma\right]\,\mathrm{d}x\,\mathrm{d}t$$

There are no boundary terms since $D^h_\tau \dot{u} = 0$ at $\partial\Omega \cap B_R$. From [Löb08] we know, that $\nabla \dot{u}$ is uniformly bounded in $L^\infty(L^2)$. (One has to adapt Johnson's proof [Joh78] who uses another penalization to the setting considered here.) Hence $D^h_\tau \nabla \dot{u}$ is uniformly bounded. This argumend also works for *Neumann boundary* conditions with zero surface force ($\sigma \cdot \vec{n} = 0$). Thus we obtain

Theorem 5.1 *Let u, σ be a solution of* (2.1)*&*(2.2)*. Assume the geometric situation* (5.1) *and let the hypotheses of theorem 3.1 be satisfied. Then the tangential derivatives $D_\tau\sigma$ exist and are, for $r < R$ uniformly bounded in $L^\infty(L^2(B_r(x_0) \cap \Omega))$, as the penalty parameter μ tends to 0.*

We consider this a corollary to [Ser94].
The first results concerning fractional differentiability of the stresses in normal direction were presented by Alber & Nesenenko [AN08] and Knees [Kne08] for the case of kinematic hardening.
We present a completely different approach fo obtaining some information about the normal derivatives of the stresses.

Theorem 5.2 *Under the assumptions of theorem 5.1 the stress σ and the hardening parameter ξ of the isotropic or kinematic hardening problem satisfies*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_R} |D_n\sigma|^2 x_n \,\mathrm{d}x \le K_r \tag{5.2a}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_R} |D_n\xi|^2 x_n \,\mathrm{d}x \le K_r \tag{5.2b}$$

uniformly as $\mu \to 0$.

proof We apply the difference operator D^h_n in normal direction to equation (2.1) & (2.2) and use the function $x_n\zeta^2(x')\big(D^h_n\sigma, D^h_n\xi\big)(x)$ as a test function. $x' = (x_1, \ldots, x_{n-1})$, ζ is a localization function such that $\operatorname{supp}\zeta \subset B_R(x_0)$, $\zeta \equiv 1$ on $B_r(x_0)$ and $\zeta(x', x_n) = const$ in the interval $(0 \le x_n \le \delta_0)$, for fixed $x' = (x_1, \ldots, x_{n-1})$ with some $\delta_0 > 0$.
recall: $x_{0n} = 0$.
On the right hand side the penalty term can be dropped due to convexity, further we obtain nice non negative definite terms such as

$$\frac{1}{2}\int_\Omega x_n\zeta^2 D^h_n\sigma A D^h_n\sigma \,\mathrm{d}x\Big|_0^T \tag{5.3a}$$

$$\frac{1}{2}\int_\Omega x_n|\zeta D^h_n\xi|^2 \,\mathrm{d}x\Big|_0^T \tag{5.3b}$$

On account of the equation, they are estimated by

$$\begin{aligned}\sum_{i,k=1}^{n} \int\limits_0^T\!\!\int\limits_\Omega x_n D_n^h D_i \dot{u}_k D_n^h \sigma_{ik} \zeta^2 \,\mathrm{d}x\,\mathrm{d}t &= -\sum_{i,k=1}^{n} \int\limits_0^T\!\!\int\limits_\Omega x_n D_n^h \dot{u}_k D_n^h f_k \zeta^2 x_n \,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\sum_{\substack{k=1\\ i\neq n}}^{n} \int\limits_0^T\!\!\int\limits_\Omega x_n D_n^h \dot{u}_k D_n^h \sigma_{ik} D_i \zeta^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\sum_{k=1}^{n} \int\limits_0^T \int\limits_{\Omega\cap B_R} D_n^h \dot{u}_k D_n^h \sigma_{nk} \zeta^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= A+B+C\,. \end{aligned} \tag{5.4}$$

For the penalty approximations, i.e. $\mu > 0$, the boundary term C is defined.
The term A is uniformly bounded since we have a $L^\infty(L^2)$ bound for $\nabla \dot{u}$ and f is sufficiently smooth.
The integrands of B are estimated by

$$K_\delta |D_n^h \dot{u}_k|^2 |\nabla\zeta|^2 x_n + \delta |D_n^h \sigma_{ik}|^2 \zeta^2 x_n \tag{5.5}$$

and the first term (with factor K_δ) is uniformly bounded, and the second is absorbed for small δ.
Finally, the integrand of C contains the factor

$$D_n^h \sigma_{nk} = \frac{1}{h}\int_0^h D_n \sigma_{nk}(x+sn)\,\mathrm{d}s\,.$$

We have $D_n\sigma_{nk} = f_k - \sum_{i=1}^{n-1} D_i \sigma_{ik}$ and for this term we have an L^2 bound up to the boundary $B_r \cap \partial\Omega$ since it contains only tangential derivatives of σ_{ik} Hence the term C is bounded uniformly, too.

Thus we may pass to the limit $h \to 0$ and obtain the theorem. □

Corallary 5.3 *In the kinematic case* (1.3b) *we have*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_r} |D_n \nabla u|^2 x_n \,\mathrm{d}x + \int_{\Omega_r} |D_\tau \nabla u|^2 \,\mathrm{d}x \le K,\ \tau = 1,\ldots,n-1 \tag{5.6}$$

uniformly as $\mu \to 0$.

This is derived from theorem 5.2 with the methods of [Joh78, Ser94, FL08] ,cf. also section 11 at the end of this paper, i.e. integrating equation (2.1) with respect to t and eliminating the penalty term using ξ.

In the case of isotropic hardening, (5.6) is not yet known, we work with a substitute.

Corallary 5.4

$$\operatorname*{ess\,sup}_{t\in[0,T]}\left\{\int_{\Omega_r}|\sigma|^2 x_n^{\delta-1}\,\mathrm{d}x+\int_{\Omega_r}|\xi|^2 x_n^{\delta-1}\,\mathrm{d}x+\int_{\Omega_r}|\varepsilon(u)|^2 x_n^{\delta-1}\,\mathrm{d}x\right\}\le K_\delta \tag{5.7}$$

for any $\delta>0$, uniformly as $\mu\to 0$.

proof Let $\varphi\in C_0^\infty(B_{r'}(x_0)), \varphi=1$ on $\Omega_r, r<r'<R$. $\varphi(x_1,\dots,x_{n-1},\gamma)=Const$ for $0\le\gamma\le\delta_r, x_1,\dots,x_{n-1}$ fixed. We have

$$\begin{aligned}\delta\int_{\Omega_{r'}}|\sigma|^2 x_n^{\delta-1}\varphi\,\mathrm{d}x &= \int_{\Omega_{r'}}|\sigma|^2 D_n x_n^\delta\varphi\,\mathrm{d}x\\ &= -2\int_{\Omega_{r'}}D_n\sigma:\sigma x_n^\delta\varphi\,\mathrm{d}x-\int_{\Omega_{r'}}|\sigma|^2 x_n^\delta D_n\varphi\,\mathrm{d}x\\ &\le\frac{\delta}{2}\int_{\Omega_{r'}}|\sigma|^2 x_n^{\delta-1}\varphi\,\mathrm{d}x+K_\delta\int_{\Omega_{r'}}|D_n\sigma|^2 x_n^{\delta+1}\varphi\,\mathrm{d}x+K\end{aligned} \tag{5.8}$$

and the corollary is proved as far as it concerns σ. The estimate for ξ is done analogously. The estimate for $\varepsilon(u)$ follows via a generalized argument with weight $x_n^{\delta-1}$ similarly to the proofs in section 3. Equation (2.1) is integrated with respect to t and the resulting first equation is multiplied by $\varepsilon(u)x_n^{\delta-1}$. This implies

$$\int_{\Omega_{r'}}|\varepsilon(u)|^2 x_n^{\delta-1}\,\mathrm{d}x\le\int_{\Omega_{r'}}\varepsilon(u):A\sigma x_n^{\delta-1}\,\mathrm{d}x+\text{ Penalty term}$$

The second equation of (2.1) is multiplied with $\varepsilon(u)x_n^{\delta-1}$ (kinematic case) or $|\varepsilon(u)|x_n^{\delta-1}$ (isotropic case) and we use it to dominate or eliminate the penalty term. □

6 L^1-estimates of the normal derivatives of the stresses at the boundary and refinements

From theorem 5.2 we conclude by Hölders inequality for $\theta\in(0,1)$ arbitrarily near to 0,

$$\begin{aligned}\int_{\Omega_r}|D_n\sigma|^\theta\,\mathrm{d}x &= \int_{\Omega_r}|D_n\sigma|^\theta x_n^{\theta/2}x_n^{-\theta/2}\,\mathrm{d}x\\ &\le\int_{\Omega_r}|D_n\sigma|^2 x_n+\int_{\Omega_r}x_n^{-\theta/(2-\theta)}\,\mathrm{d}x\\ &\le K_\theta\,.\end{aligned} \tag{6.1}$$

This follows since the negative exponent of x_n satiesfies $\frac{\theta}{2-\theta} < \theta < 1$. Thus

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_r} |D_n\sigma|^\theta \,\mathrm{d}x \le K_\theta \,. \tag{6.2}$$

However, one can do better. In section 7 we will prove the stronger estimate

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_r} |D_n\sigma|^2 x_n^{1-\delta} \,\mathrm{d}x \le K \,. \tag{6.3}$$

In the above geometrical setting the constant δ depends on the quotient of the largest and lowest eigenvalue of the quadratic form associated to the tensor A. It is useful to keep an additional Morrey condition in (6.3).

Theorem 6.1 *Let $\delta \in (0,1)$ be the number in (6.2). Under the assumptions of theorem 5.2 we have the estimate*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in[0,T]} \int_{[0,r]} \left(\operatorname*{ess\,sup}_{y'\in B_{2r}} \int_{B_{2r}} |D_n\sigma|^q |x'-y'|^{-p} \,\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \right) \mathrm{d}x_n \le K \\
&\operatorname*{ess\,sup}_{t\in[0,T]} \int_{[0,r]} \left(\operatorname*{ess\,sup}_{y'\in B_{2r}} \int_{B_{2r}} |D_n\xi|^q |x'-y'|^{-p} \,\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \right) \mathrm{d}x_n \le K
\end{aligned} \tag{6.4}$$

uniformly as $\mu \to 0$, provided that

$$q < 1 + \frac{\delta}{2-\delta},\ p < 1 - \frac{\delta}{2-\delta}\,.$$

Here $y' = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$, $B_{2r} = B_{2r}(x'_0) \supset \mathbb{R}^{n-1}$, $x'_0 = (x_{01}, \ldots, x_{0n-1})$.

This means that (6.2) is refined to the case $\theta > 1$, with an additional weight. Since there is the variable singularity y', theorem 6.1 is more than the statement $D_n\sigma \in L^\infty(L^{1+\delta_0, 3-p})$.
Remark The vectors y' in (6.4) depend on t and x_n. A priori it is not clear whether $y'(t, x_n)$ is a measurable function with respect ot (t, x_n). However with the usual Filippov-type argument from optimal control it is possible to find a selection and that the suprema are obtained a.e. and $|x'-y'|$ is measurable.
proof Let $B'_r := \{x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \big| \, |x'-x'_0| < r\}$
$I_r := (0 \le x_n \le r)$
Let $1 \le q < 2$, by Hölders inequality

$$\begin{aligned}
\int_{I_r\times B'_r} |D_n\sigma|^q |x'-y'|^{-p} \,\mathrm{d}x &= \int_{I_r\times B'_r} |D_n\sigma|^q x_n^{q(1-\delta)/2} x_n^{-q(1-\delta)/2} |x'-y'|^{-p} \,\mathrm{d}x \\
&\le \int_{I_r\times B'_r} |D_n\sigma|^2 x_n^{1-\delta} + \int_{I_r\times B'_r} x_n^{-q(1-\delta)/(2-q)} |x'-y'|^{-2p/(2-q)} \,\mathrm{d}x\,.
\end{aligned} \tag{6.5}$$

For the last integral we apply Fubini's theorem and we have a bound, if

$$\frac{q(1-\delta)}{2-q} < 1 \quad \text{and} \quad \frac{2p}{2-q} < 2\,.$$

This is the case if

$$q < \frac{2}{2-\delta} = 1 + \frac{\delta}{2-\delta} \text{ and } p < 2 - q = 1 - \frac{\delta}{2-\delta}\,.$$

Interchanging σ by ξ we achieve the same result for the hardening parameter. $\square$

7 An anisotropic Morrey estimate for the normal derivative of the stresses near the boundary

We still have to prove inequality (6.3). For doing this we establish an anisotropic Morrey estimate for $\dot\sigma, \dot\xi$ and $\nabla\dot u$. We asssume the geometric situation exposed in section 5. Let $dx' := \mathrm{d}x_1 \dots \mathrm{d}x_{n-1}$, $\Omega'_\rho := \{x' \in \mathbb{R}^{n-1} \,|\, (x', 0) \in \partial\Omega \cap B_\rho\}$.

Theorem 7.1 *Let $\big((\sigma,\xi),u\big)$ be a solution of* (2.1) *in $B_R \cap \Omega$ with $u\big|_{\partial\Omega\cap B_R} \equiv 0$ and assume that the data satisfies the smoothness assumptions* (1.4) *&* (1.5) *. Then there exist constants $\delta > 0$ and $C > 0$ such that*

$$\int_0^T\!\!\int_0^r\!\!\int_{\Omega'_{R_1}} (|\dot\sigma|^2 + |\dot\xi|^2 + |\nabla\dot u|^2)dx'\,\mathrm{d}x_n\,\mathrm{d}t \le Cr^\delta \tag{7.1}$$

uniformly as the penalty parameter $\mu \to 0$, for all r such that

$$[0,r] \times \Omega'_{R_1} \subset B_{\frac{R}{2}}(x_0) \cap \Omega,\ 0 < r \le r_0\,.$$

Theorem 7.1 tells us, that the integrals of $|\dot\sigma|^2 + |\dot\xi|^2 + |\nabla\dot u|^2$ over the strip $[0,r] \times \Omega'_{R_1}$ where R_1 is fixed and $r \to 0$ variable, tends to zero in a controlled way.
A very related statement can be found in [FL08].
proof (i): Let $\zeta_0 = \zeta_0(x_1,\dots,x_{n-1})$ a Lipschitz continuous localization function such that $\zeta_0 \equiv 1$ on $B_{R_1}(x'_0)$ and $\operatorname{supp}\zeta_0 \in B_R \cap \{x \in \mathbb{R}^n \,|\, x_n = 0\}$, and let $\zeta_r = \zeta_r(x_n)$ be a Lipschitz continuous function such that $\zeta_r \equiv 1$ on $[0,r]$, $|\nabla\zeta_r| \le \frac{1}{r}$ and $\zeta_r = 0$ on $(2r,\infty)$.
We use the function $\zeta_0^2\zeta_r^2(\dot\sigma,\dot\xi)$ as test function in (2.1) and obtain

$$\frac{1}{2}\int \big[A\dot\sigma : \dot\sigma + |\dot\xi|^2\big]\zeta_0^2\zeta_r^2\,\mathrm{d}x\Big|_0^T \le \mathbf{T} \tag{7.2}$$

where

$$\mathbf{T} = \int_0^T\!\!\int \nabla\dot u : \dot\sigma\zeta_0^2\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

The integration $\int$ is done over $\operatorname{supp}(\zeta_0, \zeta_r)$. We rewrite $\mathbf{T}$ using $\dot{u}\big|_{\partial\Omega} = 0$, $-\operatorname{div}\sigma = f$ and partial integration:

$$\mathbf{T} = \mathbf{T_1} + \mathbf{T_2} + \mathbf{T_3} \tag{7.3a}$$

$$\mathbf{T_1} = -\int_0^T\!\!\int \dot{u}\cdot\dot{f}\zeta_0^2\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t \tag{7.3b}$$

$$\mathbf{T_2} = -2\int_0^T\!\!\int \dot{u}\cdot\dot{\sigma}\nabla\zeta_0\zeta_0\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t \tag{7.3c}$$

$$\mathbf{T_3} = -2\int_0^T\!\!\int \dot{u}\cdot\dot{\sigma}\nabla\zeta_r\zeta_r\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t\,. \tag{7.3d}$$

We estimate

$$|\mathbf{T_1}|+|\mathbf{T_2}| \le K\left(\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega_R'} |\dot{u}|^2\,\mathrm{d}x'\,\mathrm{d}x_n\,\mathrm{d}t\right)^{\frac{1}{2}} K\left(\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega_R'} \big(|\dot{\sigma}|^2+|f|^2\big)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \tag{7.4}$$

and use Poincarés inequality to estimate

$$\int_0^r |\dot{u}|^2\,\mathrm{d}x_n \le Kr^2\int_0^r |D_n\dot{u}|^2\,\mathrm{d}x_n\,.$$

This yields

$$|\mathbf{T_1}|+|\mathbf{T_2}| \le Kr^{\frac{1}{2}} \tag{7.5}$$

since $\nabla\dot{u}, \sigma, f$ are bounded in L^2. The term $\mathbf{T_3}$ is estimated similarly, however we take into account, that $\nabla\zeta_r = 0$ on $[0, r]$. We obtain

$$\begin{aligned}\mathbf{T_3} &\le \frac{K}{\epsilon_0}\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega_R'} |\dot{\sigma}|^2\zeta_0^2\,\mathrm{d}x_n\,\mathrm{d}x'\,\mathrm{d}t + \epsilon_0\frac{1}{r^2}\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega_R'} |\dot{u}|^2\zeta_0^2\,\mathrm{d}x_n\,\mathrm{d}x'\,\mathrm{d}t\\ &\le \frac{K}{\epsilon_0}\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega_R'} |\dot{\sigma}|^2\zeta_0^2\,\mathrm{d}x_n\,\mathrm{d}x'\,\mathrm{d}t + \underbrace{\epsilon_0\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega_R'} |\nabla\dot{u}|^2\zeta_0^2\,\mathrm{d}x_n\,\mathrm{d}x'\,\mathrm{d}t}_{=:\mathbf{T_0}}\,.\end{aligned} \tag{7.6}$$

Thus we arrive at the inequality

$$\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega_R'} \big(|\dot{\sigma}|^2+|\dot{\xi}|^2\big)\zeta_r^2\zeta_0^2\,\mathrm{d}x'\,\mathrm{d}x_n\,\mathrm{d}t \le \frac{K}{\epsilon_0}\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega_R'} \big(|\dot{\sigma}|^2+|\dot{\xi}|^2\big)\zeta_0^2\,\mathrm{d}x'\,\mathrm{d}x_n\,\mathrm{d}t + Kr^{\frac{1}{2}} + \mathbf{T_0}\,. \tag{7.7}$$

Here we have used the ellipticity condition for the elastic compliance tensor A and we have absorbed the integral with factor ϵ_0 in front.
(ii): We start to estimate the term $\mathbf{T_0}$. In the kinematic case, we use the pair

$$\big(\varepsilon(\dot{u}), \varepsilon(\dot{u})\big)\zeta_0^2\chi_{[r,2r]}$$

as test function with $\chi_{[r,2r]} = \chi_{[r,2r]}(x_n)$ the characteristic function of the interval $[r, 2r]$. We also use this test function without $\chi_[0, 2r]$.
In the isotropic case we choose

$$\big(\varepsilon(\dot u), |\varepsilon(\dot u)|\big)\zeta_0^2\chi_{[r,2r]}$$

as test function, respectively with $\chi_{[0,2r]}$.
In the kinematic case, the components coming from the penalty term cancel out, in the isotropic case the sum of the components coming from the penalty term is less equal than zero. Thus we obtain using Hölder's inequality

$$\int_0^T\int_r^{2r}\int_{\Omega'_R} |\varepsilon(\dot u)|^2\zeta_0^2 \,\mathrm{d}x\,\mathrm{d}t \le K\int_0^T\int_r^{2r}\int_{\Omega'_R} \big(|\dot\sigma|^2 + |\dot\xi|^2\big)\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t \tag{7.8a}$$

$$\int_0^T\int_0^{2r}\int_{\Omega'_R} |\varepsilon(\dot u)|^2\zeta_0^2 \,\mathrm{d}x\,\mathrm{d}t \le K\int_0^T\int_0^{2r}\int_{\Omega'_R} \big(|\dot\sigma|^2 + |\dot\xi|^2\big)\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t \tag{7.8b}$$

(iii) We need some type of Korn's inequality for $\varepsilon(\dot u)$ on the domain $[0, 2r] \times \Omega'_R$. The difficulty is, that the constants have to be uniform as $r \to 0$. To see what we can expect we rewrite

$$\begin{aligned}\int_0^T\int_0^{2r}\int_{\Omega'_R} |\varepsilon(\dot u)|^2\zeta_0^2\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_0^{2r}\int_{\Omega'_R}\Big\{\frac12\sum_{i,k=1}^{n}|D_i\dot u_k|^2 + \frac12\sum_{i,k=1}^{n-1}D_i\dot u_kD_k\dot u_i \\ &\quad + \frac12\sum_{k=1}^{n-1}D_n\dot u_kD_k\dot u_n + \frac12|D_n\dot u_n|^2\Big\}\zeta_0^2\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t \\ &=: U_1 + U_2 + U_3 + U_4\end{aligned} \tag{7.9}$$

For the term U_2 we use tangential partial integration.(To justify this operation one can apply tangential mollification operations to $D_i\dot u_k$) This yields

$$U_2 = \int_0^T\int_0^{2r}\int_{\Omega'_r}\left(\sum_{i=1}^{n-1}D_i\dot u_i\right)^2\zeta_0^2\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t - \text{ pollution terms,} \tag{7.10}$$

where the pollution terms can be estimated by

$$\int_0^T\int_0^{2r}\int_{\Omega'_r}\big(|\dot u|^2|\nabla\zeta_0|^2 + |\dot u||\nabla\dot u||\nabla\zeta_0|^2\big)\,\mathrm{d}x\,\mathrm{d}t \le Kr\,. \tag{7.11}$$

The last inequality follows from Poincarés inequality applied to $\dot u$ as in the begining of the proof.
The difficulty is the term U_3. By partial integration

$$\begin{aligned}U_3 &= -2\sum_{k=1}^{n-1}\int_0^T\int_0^{2r}\int_{\Omega'_R}\big(D_nD_k\dot u_i\dot u_n\zeta_0^2 + 2D_n\dot u_k\dot u_n\zeta_0D_k\zeta_0\big)\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t \\ &= -\sum_{k=1}^{n-1}\int_0^T\int_0^{2r}\int_{\Omega'_R}\big\{D_n\dot u_nD_k\dot u_k\zeta_r^2 + 2D_k\dot u_k\dot u_nD_n\zeta_r\zeta_r\big\}\,\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t - \text{pollution terms.}\end{aligned}$$

(7.12)

(For justification, here it also suffices to use tangential mollification of $D_i\dot u_k, D_n u_i$.) The pollution terms can be estimated as in (7.11). From (7.9) and the representation of $\dot u_i$ and from (7.10) we obtain

$$
\begin{aligned}
\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}|\varepsilon(\dot u)|^2\zeta_r^2\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t \ge{}& \int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}\left\{\frac12|\nabla\dot u|^2+\frac12\left(\sum_{k=1}^{n-1}D_k\dot u_k\right)^2+\frac12|D_n\dot u_n|^2-D_n\dot u_n\sum_{k=1}^{n-1}D_k\dot u_k\right\}\zeta_0^2\zeta_r^2\mathrm{d}x\mathrm{d}t\\
&-2\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}\sum_{k=1}^{n-1}D_k\dot u_k\dot u_n D_n\zeta_r\zeta_r\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t-Kr\\
\ge{}& \int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}\left\{\frac12|\nabla\dot u|^2\zeta_r^2-2\sum_{k=1}^{n-1}D_k\dot u_k\dot u_n D_n\zeta_r\zeta_r\right\}\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t-Kr\\
={}&: W_1\;+\;W_2-Kr\,.
\end{aligned}
\tag{7.13}
$$

We estimate the second integral from below using Poincarés and Youngs inequality. We take into account that $\operatorname{supp} D_n\zeta_r=[r,2r]$ and $|D_n\zeta_r|\le\frac1r$.

$$
W_2\ge-\frac{1}{\epsilon_0}\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega'_R}|\nabla\dot u|^2\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t-K\epsilon_0\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}|\nabla\dot u|^2\zeta_0^2\,. \tag{7.14}
$$

Using (7.13), (7.14) we have

$$
\left(\frac12-K\epsilon_0\right)\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}|\nabla\dot u|^2\zeta_0^2\zeta_r^2\,\mathrm{d}x\mathrm{d}t\le\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}|\varepsilon(\dot u)|^2\zeta_0^2\,\mathrm{d}x\mathrm{d}t+\left(K\epsilon_0+\frac{1}{\epsilon_0}\right)\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega'_R}|\nabla\dot u|^2\zeta_0^2\,\mathrm{d}x\mathrm{d}t+Kr\,. \tag{7.15}
$$

This means that $\int_0^T\!\int_0^{2r}\!\int_{\Omega'_R}|\nabla\dot u|^2$ is dominated by $K\int_0^T\!\int_0^{2r}\!\int_{\Omega'_R}|\varepsilon(\dot u)|^2+$ a pollution term Kr + a hole filling term containing $\int_r^{2r}$.

(iv) We use (7.15) to estimate the term T_0 in (7.6). This yields

$$
T_0\le K\epsilon_0\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}|\varepsilon(\dot u)|^2\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t+K\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega'_R}|\nabla\dot u|^2\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t+Kr\,. \tag{7.16}
$$

From (7.7),(7.16) and (7.8) we obtain

$$
\begin{aligned}
\int_0^T\!\!\int_0^{r}\!\!\int_{\Omega'_R}\big(|\dot\sigma|^2+|\dot\xi|^2\big)\zeta_0^2\zeta_r^2\,\mathrm{d}x'\,\mathrm{d}x_n\,\mathrm{d}t\le{}& Kr^{1/2}+K\int_0^T\!\!\int_r^{2r}\!\!\int_{\Omega'_R}|\nabla\dot u|^2\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t\\
&+K\epsilon_0\int_0^T\!\!\int_0^{2r}\!\!\int_{\Omega'_R}\big(|\dot\sigma|^2+|\dot\xi|^2\big)\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
\tag{7.17}
$$

which implies, for small ϵ_0

$$\int_0^T\int_0^{2r}\int_{\Omega'_R}\big(|\dot\sigma|^2+|\dot\xi|^2\big)\zeta_0^2\,\mathrm{d}x'\,\mathrm{d}x_n\,\mathrm{d}t\le Kr^{1/2}+K\int_0^T\int_r^{2r}\int_{\Omega'_R}\big(|\dot\sigma|^2+|\dot\xi|^2+|\nabla\dot u|^2\big)\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t\,. \tag{7.18}$$

We add the $\frac{1}{2K}$-fold of (7.8) to (7.18) and obtain

$$\int_0^T\int_0^{2r}\int_{\Omega'_R}\big(\frac12|\dot\sigma|^2+\frac12|\dot\xi|^2+\frac{1}{2K}|\varepsilon(\dot u)|^2\big)\zeta_0^2\zeta_r^2\,\mathrm{d}x\,\mathrm{d}t\le K\int_0^T\int_r^{2r}\int_{\Omega'_R}\big(|\dot\sigma|^2+|\dot\xi|^2+|\nabla\dot u|^2\big)\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t\,. \tag{7.19}$$

In (7.19), we replace the term with $|\varepsilon(\dot u)|^2$ using (7.15) and obtain finally the **plate filling inequality**

$$\int_0^T\int_0^{r}\int_{\Omega'_R}\big(|\dot\sigma|^2+|\dot\xi|^2+|\nabla\dot u|^2\big)\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t\le\int_0^T\int_r^{2r}\int_{\Omega'_R}\big(|\dot\sigma|^2+|\dot\xi|^2+|\nabla\dot u|^2\big)\zeta_0^2\,\mathrm{d}x\,\mathrm{d}t+Kr^{1/2}\,. \tag{7.20}$$

Now we are able to prove theorem 7.1.
(v) We apply a plate filling step: From

$$\int_0^T\int_0^{2r}\int_{\Omega'_R}W\,\mathrm{d}x\,\mathrm{d}t\le\int_0^T\int_r^{2r}\int_{\Omega'_R}W\,\mathrm{d}x\,\mathrm{d}t+Kr^{1/2}$$

we conclude

$$\int_0^T\int_0^{r}\int_{\Omega'_R}W\,\mathrm{d}x\,\mathrm{d}t\le\frac{K}{1+K}\int_0^T\int_0^{2r}\int_{\Omega'_R}W\,\mathrm{d}x\,\mathrm{d}t+Kr^{1/2}$$

and the statement of theorem 7.1 is derived via an iteration argument. Compare the proof in [FL08] for a recent paper on this subject. □

Corallary 7.2

$$\int_0^T\int_0^{r}\int_{\Omega'_R}\Big||\dot\sigma|^2+|\dot\xi|^2+|\nabla\dot u|^2\Big|\,x_n^{-\delta_1}\,\mathrm{d}x'\,\mathrm{d}x_n\,\mathrm{d}t\le K \tag{7.21}$$

for some $\delta_1<\delta$ *.*

proof We estimate the integral in (7.21) by

$$2\sum_{j=N}^{\infty}(2^j)\int_0^T\int_0^{2^{-j}}\int_{\Omega'_r}\Big(|\dot\sigma|^2+|\dot\xi|^2+|\nabla\dot u|^2\Big)\,\mathrm{d}x'\mathrm{d}x_n\le 2\sum_{j=N}^{\infty}(2^j)^{\delta_1-\delta}\le K\,.$$

□

Now, we are able to prove the basic estimate

Theorem 7.3 *Under the assumptions of theorem 7.1, we have for* $\bar{\delta} < \frac{\delta}{2}$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_0^r \int_{\Omega'_{R_1}} \big(|D_n\sigma|^2 + |D_n\xi|^2\big) x_n^{1-\bar{\delta}} \,\mathrm{d}x'\mathrm{d}x_n \le K \quad 0 \le r \le r_0 \tag{7.22}$$

uniformly as $\mu \to 0$.

proof We follow the proof of theorem 5.1, but replace the weight x_n by

$$\tilde{x}_{n\bar{\delta}} = \frac{x_n}{\delta_1 + x_n^{\bar{\delta}}},$$

where $\delta_1 > 0$ is an additional parameter which tends to 0. This is necessary at the beginning of the proof in order to ensure that the integrals are defined.
We arrive at the analougue of inequality (5.4)

$$\frac{1}{2}\int_0^T\int_\Omega \tilde{x}_{n\bar{\delta}}\zeta^2 \big[AD_n^h\sigma : D_n^h\sigma + |D_n^h\xi|^2\big]^2 \,\mathrm{d}x\,\mathrm{d}t \le \tilde{A} + \tilde{B} + \tilde{C} \tag{7.23}$$

where $\tilde{A}, \tilde{B}, \tilde{C}$ are defined as A, B, C in (5.4) with x_n replaced by $\tilde{x}_{n\bar{\delta}}$. In particular, this means that the factor $1 = D_n x_n$ in the integral C is replaced by

$$D_n \frac{x_n}{\delta_1 + x_n^{\bar{\delta}}} = (1-\bar{\delta})\frac{x_n^{\bar{\delta}}}{(\delta_1 + x_n^{\bar{\delta}})^2} + \frac{\delta_1}{(\delta_1 + x_n^{\bar{\delta}})^2} \ge 0\,.$$

The terms $\tilde{A}$ and $\tilde{B}$ with the new weight $\tilde{x}_{n\bar{\delta}}$ are treated as in theorem 5.1. The integrands are estimated by Youngs's inequality and the term $\epsilon_0|D_n^h\sigma|^2\zeta^2\tilde{x}_{n\bar{\delta}}$ which is absorbed by the righthand side of (7.23). The remaining parts of $\tilde{A}, \tilde{B}$ are bounded as $h \to 0, \delta_1 \to 0, \mu \to 0$ since $\nabla\dot{u} \in L^2$. The term $\tilde{C}$ is rewritten using $-\operatorname{div}\sigma = f$ as in the proof of theorem 5.1: Thus we obtain

$$|\tilde{C}| \le \sum_{k=1}^{n}\sum_{i=1}^{n-1} \int_0^T\int_{\Omega_R} |D_n^h\dot{u}_k||D_i^h\sigma_{ik}|D_n\tilde{x}_{n\bar{\delta}}\zeta^2 \,\mathrm{d}x\,\mathrm{d}t + K\,. \tag{7.24}$$

We now pass to the limit $h \to 0$ which is admissible since $\dot{u} \in L^\infty(H^1), D_i\sigma \in L^\infty(L^2)$ for $i = 1, \ldots, n-1$. Thereafter, we estimate

$$\lim_{h\to 0} |\tilde{C}| \le K \int_0^T\int_{\Omega_R} |\nabla\dot{u}|^2 |D_n\tilde{x}_{n\bar{\delta}}|^2\zeta^2 \,\mathrm{d}x\,\mathrm{d}t + K\,. \tag{7.25}$$

Recall that the tangential estimates of $D_\tau\sigma$ are in L^2. Now, we use that

$$|D_n\tilde{x}_{n\bar{\delta}}| \le x_n^{-\bar{\delta}}$$

uniformly as $\delta_1 \to 0$. Hence the righthand side of (7.4) remains bounded due to corollary 7.2. Thus the statement follows from (7.23) and Fatous Lemma. $\square$

8 $L^\infty(L^{4-\delta})$-estimates for the stresses at the Dirichlet boundary

Again, we consider the geometric situation described in section 5 with a flat portion of the boundary $\partial\Omega$. While the property of $D_n\sigma$ holds in any dimension, we restrict now to the case of three space dimensions.

Theorem 8.1 *Let $n = 3$. Under the conditions of 6.1 and the geometric setting (5.1), we have for all $\delta_1 > 0$ near to 0*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_r} |\sigma|^{4-\delta_1}\,\mathrm{d}x \le K_{\delta_1} \tag{8.1}$$

uniformly as $\mu \to 0$ for isotropic and kinematic hardening.

Remark Theorem 8.1 also follows via the estimate of the fractional derivatives of order $\frac{1}{2}$ of σ and ξ , see section 9. We found it useful to present an alternative method of proof.

proof From theorem 7.3 and corollary 5.4 we obtain by Hölders inequality

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in[0,T]} \iint\limits_{[0,r]}\!\!\!\!\!\!\!\!\quad_{B_r'} |D_n\sigma||\sigma|\,\mathrm{d}x_1\,\mathrm{d}x_2 dx_3\\ &\le \operatorname*{ess\,sup}_{t\in[0,T]}\left[\int_{[0,r]}\int_{B_r'} |D_n\sigma|^2 x_n^{1-\delta_0}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 + \int_{[0,r]}\int_{B_r'} |\sigma|^2 x_n^{\delta_0-1}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right] \le K_{\delta_0}\end{aligned} \tag{8.2}$$

for all $\delta_0 > 0$ and B_r' as in the proof of theorem 6.1. From Gagliardo's lemma we conclude

$$\operatorname*{ess\,sup}_{t,x_n} \int_{B_r'} |\sigma|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \le K + \int_{[0,r]}\int_{B_r'} D_n|\sigma|^2 dx_1 dx_2 dx_3 \le K + K_{\delta_0}\,. \tag{8.3}$$

On the other hand, since the tangential derivatives are uniformly bounded in L^2 near the boundary we have in view of Sobolev's inequality

$$\int_{[0,r]} \left(\int_{B_r'} |\sigma|^{2p}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{1}{p}} \mathrm{d}x_3 \le K \int_{[0,r]\times B_r'} \left(|D_\tau\sigma|^2 + |\sigma|^2\right)\,\mathrm{d}x \tag{8.4}$$

for all $1 \le p < \infty$. We now use an argument which is frequently used in fluid-dynamics:

$$\begin{aligned}\int_{[0,r]\times B_r'} |\sigma|^2|\sigma|^{2\left(1-\frac{1}{p}\right)}\,\mathrm{d}x &\le \int_{[0,r]} \left(\int_{B_r'} |\sigma|^{2p}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{1}{p}} \left(\int_{B_r'} |\sigma|^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1-\frac{1}{p}} \mathrm{d}x_3\\ &\le \operatorname*{ess\,sup}_{x_3} \left(\int_{B_r'} |\sigma|^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1-\frac{1}{p}} \int_{[0,r]} \left(\int_{B_r'} |\sigma|^{2p}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{1}{p}}\,.\end{aligned}$$

(8.5)

The latter quantity is bounded due to (8.2) & (8.4). □

9 $\mathcal{N}^{\frac{1}{2},2}$-fractional differentiability of the stresses and hardening parameters in normal direction up to the boundary

Again, for the sake of mathematical insight we consider the case of n space dimensions and the geometric situation described in section 5, i.e. the boundary $\partial\Omega$ coincides with the hyperplane $\{x \in \mathbb{R}^n \big| \, x_n = 0\}$ in a neighbourhood $B_R(x_0)$ of a boundary point x_0, and $B_R(x_0) \cap \{x \in \mathbb{R}^n \big| \, x_n \geq 0\}$. $\Omega_r := B_r(x_0) \cap \Omega, \ r \leq R$.
We treat both kinematic and isotropic hardening.
Remember, that $\dot{u} = 0$ on $B_R \cap \partial\Omega$.

Theorem 9.1 *Let σ, ξ be the solution of* (2.1) *resp.* (1.6). *Under the regularity assumptions* (1.4)*&*(1.5) *and the geometric setting above there holds the estimate*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \sup_{0<h<r_0} \frac{1}{h} \int_{\Omega_r} |\sigma(x + h\vec{e}_n) - \sigma(x)|^2 \, \mathrm{d}x \leq K_{r_0,r} \tag{9.1a}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \sup_{0<h<r_0} \frac{1}{h} \int_{\Omega_r} |\xi(x + h\vec{e}_n) - \xi(x)|^2 \, \mathrm{d}x \leq K_{r_0,r} \tag{9.1b}$$

uniformly as $\mu \to 0$, in the isotropic and kinematic case.

Remark Theorem 9.1 states that $\sigma, \xi \in L^\infty(\mathcal{N}^{\frac{1}{2},2}(\Omega_r))$ up to the boundary, where $\mathcal{N}^{\frac{1}{2},2}$ denotes the Nikol'skii space for $p = 2, s = \frac{1}{2}$. By the imbedding theorems (see for example [KJF77, Ada75]) we have $\sigma \in L^\infty(H^{\frac{1}{2}-\delta}(\Omega_r))$.
In the kinematic case it has been known already that $\sigma, \xi \in L^\infty(H^{\frac{1}{2}-\delta})$. This was proved by Alber & Nesenenko [AN08] and Knees [Kne08] in the kinematic case. Their proof is different, another formulation of the problem is used. Let us note that, Knees works with a reflection argument.
proof of theorem 9.1 We apply the difference operators D_n^h to equation (2.1) & (2.2) and use the function $h\zeta^2\big(D_n^h\sigma, D_n^h\xi\big)$ as test function, where ζ is defined in section 5. As usual, on the right hand side of equation there arises the part coming from the penalty term which is dropped due to monotonicity and there remain the terms

$$\frac{h}{2}\int_{\Omega_R} \zeta^2 A D_n^h\sigma : D_n^h\sigma \, \mathrm{d}x\Big|_0^T + \frac{h}{2}\int_{\Omega_R} |\zeta D_n^h\xi|^2 \, \mathrm{d}x\Big|_0^T \tag{9.2}$$

which will give the estimate for (9.1).
On the left hand side we obtain by the divergence theorem and (1.2)

$$h\int_0^T\int_{\Omega_R} D_n^h\varepsilon(\dot u) : D_n^h\sigma\zeta^2\,\mathrm{d}x\,\mathrm{d}t = h\int_0^T\int_{\Omega_R}\left[D_n^h\dot u D_n^h f\zeta^2 - 2\sum_{i,k=1}^n D_n^h\dot u_k D_n^h\sigma_{ik}\zeta D_i\zeta\right]\mathrm{d}x\,\mathrm{d}t$$
$$- h\sum_{k=1}^n\int_0^T\int\limits_{\Omega_R\cap\{x_n=0\}} D_n^h\dot u_k D_n^h\sigma_{kn}\zeta^2\,\mathrm{d}x_1\ldots\,\mathrm{d}x_{n-1}\,\mathrm{d}t$$
$$= B_1 + B_2 + B_3\,. \tag{9.3}$$

The term B_1 is obviously bounded since f is smooth and $\nabla\dot u\in L^\infty(L^2)$ with uniform bound. The term B_2 is estimated by Hölder's inequality

$$|B_2| \le \delta_1 h\int_0^T\int_{\Omega_R}|D_n^h\sigma|^2\zeta^2\,\mathrm{d}x\,\mathrm{d}t + K_{\delta_1}h\int_0^T\int_{\Omega_R}|D_n^h\dot u|^2\,\mathrm{d}x\,\mathrm{d}t \tag{9.4}$$

and the first term estimating $|B_2|$ is absorbed by the terms in (9.2), using Gronwalls inequality, the second is bounded since $\nabla\dot u\in L^\infty(L^2)$ uniformly as $\mu\to 0$.
The difficulty is the term B_3 which we treat in the following way:
We may write

$$\begin{aligned} hD_n^h\dot u_k(x) &= \int_0^h D_n\dot u_k(x+s\vec e_n)\,\mathrm{d}s\\ D_n^h\sigma_{nk}(x) &= \frac{1}{h}\int_0^h D_n\sigma_{nk}(x+s\vec e_n)\,\mathrm{d}s\\ &= -\frac{1}{h}\int_0^h f_k\,\mathrm{d}s - \frac{1}{h}\int_0^h\sum_{i=1}^{n-1}D_i\sigma_{ik}(x+s\vec e_n)\,\mathrm{d}s\,. \end{aligned} \tag{9.5}$$

Define $B'_r := B_r(x_0)\cap\partial\Omega$. Since the tangential derivatives of σ are in $L^\infty(L^2)$ (cf. theorem 5.1), we conclude from (9.5)

$$\operatorname*{ess\,sup}_{t\in[0,T]}\int_{B'_r}|D_n^h\sigma_{nk}|^2\,\mathrm{d}x_1\ldots\,\mathrm{d}x_{n-1} \le \frac{1}{h}K + \operatorname*{ess\,sup}_{t\in[0,T]}\sum_{i=1}^{n-1}\frac{1}{h}\int_0^h\int_{B'_r}|D_i\sigma|^2\,\mathrm{d}x_1\ldots\,\mathrm{d}x_{n-1}\,\mathrm{d}s \le \frac{1}{h}K\,. \tag{9.6}$$

Furthermore

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in[0,T]}\int_{B'_r}|hD_n^h\dot u_k|^2\,\mathrm{d}x_1\ldots\,\mathrm{d}x_{n-1} &\le \operatorname*{ess\,sup}_{t\in[0,T]}\int_{B'_r}\left|\int_0^h D_n\dot u_k(x+s\vec e_n)\,\mathrm{d}s\right|^2\mathrm{d}x_1\ldots\,\mathrm{d}x_{n-1}\\ &\le \operatorname*{ess\,sup}_{t\in[0,T]}\int_{B'_r}h\int_0^h|D_n\dot u_k|^2\,\mathrm{d}s\,\mathrm{d}x_1\ldots\,\mathrm{d}x_{n-1}\\ &\le h\operatorname*{ess\,sup}_{t\in[0,T]}\int_{B'_r}|D_n\dot u_k|^2\,\mathrm{d}x\,.\end{aligned}$$

$$(9.7)$$

This implies

$$|B_3| \leq \sum_{k=1}^{n} \left(\int_0^T \int_{B'_r} \left(hD_n^h \dot{u}_k \right)^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \right)^{\frac{1}{2}} \left(\int_0^T \int_{B'_r} |D_n^h \sigma_{nh}|^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \right)^{\frac{1}{2}} \leq (hK)^{\frac{1}{2}} \cdot (\frac{1}{h} K)^{\frac{1}{2}}.$$

This proves the theorem. □

From the imbedding theorems for anisotropic Nikol'skii spaces [KJF77] we obtain

Corallary 9.2 *Under the assumptions of theorem 9.1 we have for* 3 *space dimensions*

$$\sigma, \xi \in L^\infty(L^{4-\delta}(\Omega_r))$$

and in 2 *space dimensions*

$$\sigma, \xi \in L^\infty(L^{6-\delta}(\Omega_r))$$

with uniform bounds as $\mu \to 0$.

proof

$n = 3$: We use the fact that the full first tangential derivatives are in L^2. We have to calculate the harmonic mean of the numbers $1, 1, \frac{1}{2}$ which is

$$\left[\frac{1}{3} (1 + 1 + \frac{2}{1}) \right]^{-1} = \frac{3}{4}.$$

By the Sobolev-Nikol'skii imbedding theorem we conclude $\sigma, \xi \in L^\infty(L^q)$ with

$$q = \frac{3 \cdot 2}{3 - 2 \cdot (3/4)} - \delta = 4 - \delta.$$

$n = 2$: Similarly $[\frac{1}{2}(1 + 2)]^{-1} = \frac{2}{3}$

$$q = \frac{2 \cdot 2}{2 - 2 \cdot (2/3)} - \delta = 6 - \delta.$$

□

10 Regularity properties of the displacements near the boundary

In this section, we confine ourselves to the case of 3 space dimensions. Both kinematic and isotropic hardening are treated. We consider a neighbourhood of the dirichlet boundary in a flat part.

With a similar method as in the proof of theorem 3.1 we conclude from theorem 8.1

Theorem 10.1 *Under the assumption of theorem 5.2 we have for the strains for all $\delta_1 > 0$ near zero*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_r} |\nabla u|^{4-\delta_1} \, \mathrm{d}x \le K_{\delta_1} \tag{10.1}$$

uniformly as $\mu \to 0$.

We do not present the proof which is very similar to the one of theorem 3.1 .
Recall, that we treat both, the isotropic and kinematic case.

Theorem 10.2 *Under the assumptions of theorem 5.2 we have for the strains in the kinematic case*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_r} |D_\tau u|^2 \, \mathrm{d}x \le K \tag{10.2}$$

and

$$\operatorname*{ess\,sup}_{t\in[0,T]} \sup_{0<h<h_0} h \int |\nabla D_n^h u|^2 \, \mathrm{d}x \le K \,, \tag{10.3}$$

where D_τ denote the tangential derivatives and D_n^h the difference quotient in normal direction.

proof We integrate (2.1) with respect to t, eliminate the penalty term, apply the operation D_τ or D_n^h to each equation and use $D_\tau u$ or $\sqrt{h} D_n^h u$ as a testfunction. Then we obtain the statement of the theorem. $\square$

From theorem 10.1 we can derive the Hölder continuity of the displacements u at the boundary.

Theorem 10.3 *Let $n = 3$, under the assumptions of theorem 6.1 the displacements u in isotropic and kinematic hardening are uniformly Hölder continuous in Ω_r. The Hölder exponent α in space direction is any number $\alpha < \frac{1}{4}$, the Hölder exponent in time direction any number $\beta < \frac{1}{3}$.*

proof We use the fact, that

$$\nabla \dot{u} \in L^\infty(L^2) \tag{10.4}$$

and theorem 10.1. From (10.4) we conclude by Sobolev imbedding

$$\dot{u} \in L^\infty(L^6) \,. \tag{10.5}$$

From theorem 10.1 we conclude by Sobolev's theorem that u is Hölder continuous in space direction with exponent

$$\alpha < 1 - \frac{3}{4\delta} = \frac{1}{4} - \delta' .$$

This is the first statement of the theorem. For the Hölder continuity in time direction let $Q_s(y_0)$ be a cube in $\{x \in \mathbb{R}^3 | x_3 \geq 0\}$ with sidelength s and center y_0. We have

$$\begin{aligned} \big|u(y_0, t_1) - u(y_0, t_2)\big| &\leq \sum_{i=1,2} \big|u(y_0, t_i) - \frac{1}{s^3} \int_{Q_s(y_0)} u(t_i, y) dy\big| \\ &+ \frac{1}{s^3} \left| \int_{Q_s(y_0)} \big(u(y, t_1) - u(y, t_2)\big) dy \right| \\ &=: A_1 + A_2 + A_3 \end{aligned} \tag{10.6}$$

Due to Hölder continuity in spatial direction we have

$$A_i \leq K s^{1/4 - \delta_1}, \ i = 1, 2 .$$

For A_3 we have

$$\begin{aligned} A_3 &= \frac{1}{s^3} \left| \int_{t_1}^{t_2} \int_{Q_s(y_0)} \dot{u}(y, t) dy dt \right| \\ &\leq \frac{1}{s^3} \left| \int_{t_1}^{t_2} \left(\int_{Q_s(y_0)} \dot{u}^6 dy \right)^{1/6} s^{3 - 5/6} dt \right| \\ &\leq \frac{1}{\sqrt{s}} K |t_2 - t_1| \end{aligned} \tag{10.7}$$

where we have used (10.5). The optimal choice for s is if

$$s^{1/4 - \delta_1} = \frac{1}{\sqrt{s}} |t_2 - t_1| \text{ i.e. } |t_2 - t_1| = s^{\frac{3}{4} - \delta_1}$$

which implies

$$A_i \leq |t_2 - t_1|^{\frac{1}{3} - \delta'} .$$

This proves the theorem. $\square$

11 Differentiability of the strain tensor $\varepsilon(u)$ in kinematic hardening

In this section, we present a short proof of the $L^\infty(H^1_{loc})$ property of the strain tensor $\varepsilon(u)$ in the case of kinematic hardening. The the same techniques as in section 9 can be used to show $H^{\frac{1}{2} - \delta}$-boundary differentiability.

We have the almost everywhere equation for the penalized problem.

$$\varepsilon(\dot{u}) = A\dot{\sigma} + \frac{1}{\mu}\big(|\sigma_{\mu D} - \xi_{\mu D}| - \kappa\big)_+ \frac{\sigma_D - \xi_D}{|\sigma_D - \xi_D|} \tag{11.1a}$$

$$0 = \dot{\xi} - \frac{1}{\mu}\big(|\sigma_{\mu D} - \xi_{\mu D}| - \kappa\big)_+ \frac{\xi_D - \sigma_D}{|\sigma_D - \xi_D|} \tag{11.1b}$$

Let $\varepsilon(u)$ be the strain tensor of the penalized problem (11.1) with kinematic hardening.

Theorem 11.1 *Under the regularity assumptions* (1.4a),(1.4b) *and the safe load condition* (1.5), *for every subset $\Omega_0 \subset\subset \Omega$ we have the uniform estimate*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_0} |\nabla\varepsilon(u)|^2 \,\mathrm{d}x \le K_{\Omega_0,T} \tag{11.2}$$

as the viscosity coefficient μ tends to zero.

Remark: We emphasize that we <u>do not</u> state the differentiability for the displacement velocity $\dot{u}$!

Corallary 11.2 *Inequality* (11.2) *also holds for the limiting case $\mu = 0$.*

proof We integrate the sum of the equations (11.1a),(11.1b) with respect to t from 0 to s and thereafter use the function $D^{-h}(\zeta^2 D^{+h}\varepsilon(u))$ as a testfunction. Here ζ is a smooth localization function. Because of the sign-situation in (11.1a),(11.1b) the penalty term cancels and we obtain

$$\begin{aligned}\int_\Omega \big(\varepsilon(u)\big|_{t=s} - \varepsilon(u)\big|_{t=0}\big) D^{-h}(\zeta^2 D^{+h}(\varepsilon(u)))\,\mathrm{d}x \\ = \int_\Omega \big[\big(A\sigma\big|_{t=s} - A\sigma\big|_{t=0}\big) + \big(\xi\big|_{t=s} - \xi\big|_{t=0}\big)\big]\, D^{-h}(\zeta^2 D^{+h}\varepsilon(u))\,\mathrm{d}x\,.\end{aligned} \tag{11.3}$$

By partial summation and rearranging we achieve

$$\int_\Omega |\zeta D^{+h}\varepsilon(u)\big|_{t=s}|^2\,\mathrm{d}x = \int_\Omega \zeta^2 D^{+h}\varepsilon(u)\Big[D^{+h}(A\sigma)\big|_{t=s} - D^{+h}(A\sigma)\big|_{t=0} + D^{+h}\xi\big|_{t=s} - D^{+h}\xi\big|_{t=0}\Big]\,\mathrm{d}x\,. \tag{11.4}$$

(We drop $\varepsilon(u)(0)$ for simplicity.)
Passing to the limit $h \to 0$, we may replace the difference quotient by partial derivatives. By assumption the initial data $\sigma\big|_{t=0}$, $\xi\big|_{t=0}$ is in H^1_{loc}, we obtain for almost all t

$$\begin{aligned}\int_\Omega \zeta^2|\nabla\varepsilon(u)|^2\,\mathrm{d}x \le \int_\Omega \zeta^2|\nabla A\sigma|^2\,\mathrm{d}x + \int_\Omega \zeta^2|\nabla\xi|^2\,\mathrm{d}x \\ + \int_\Omega \zeta^2|\nabla A\sigma\big|_{t=0}|^2\,\mathrm{d}x + \int_\Omega \zeta^2|\nabla\xi\big|_{t=0}|^2\,\mathrm{d}x\,.\end{aligned} \tag{11.5}$$

This proofs the theorem. □

Theorem 11.3 *Assume the geometric setting of section 5, then the strain tensor satisfies*

$$\varepsilon(u) \in L^\infty(H^{\frac{1}{2}-\delta}) .$$

proof With the results and the techniques of section 9 the proof follows easily.

12 Trace properties of the strain tensor

In the case of kinematic hardening, Seregin [Ser94] obtained $\varepsilon(u) \in L^\infty\big(0,T;H^1_{loc}(\Omega)\big)$ which implies, due to imbedding theorems, for dimensions $n=3$

$$\operatorname*{ess\,sup}_{t,\alpha,j} \int\limits_{H^j_\alpha\cap\Omega_0} |\varepsilon(u)|^4 \,\mathrm{d}x' \le K_{\Omega_0,T} \quad t\in[0,T],\ \alpha\in\mathbb{R},\ j=1,\dots,3 \quad \Omega_0 \subset\subset \Omega \tag{12.1}$$

for the solution of (1.6). Here H^j_α is the hyperplane

$$H^j_\alpha := \big\{x=(x_1,x_2,x_3)\in\mathbb{R}^3 \,\big|\, x_j=\alpha\big\} \tag{12.2}$$

and

$$\mathrm{d}x' = \prod_{\substack{i=1\\ i\neq j}}^{3} dx_i \,.$$

One should fill the gap of regularity between the case of isotropic and kinematic hardening as much as possible, so from this point of view it is interesting that (12.1) holds also in the case of isotropic hardening.

Theorem 12.1 *Under the assumptions of theorem 3.1 the strain tensor $\varepsilon(u)$ of the isotropic hardening problem* (3.1) *satisfies the trace property* (12.1) *.*

We emphasize that the elasticity tensor A is independent of t.
proof We integrate equation (3.1a) and (3.1b) from 0 to t and test the resulting first equation with $\varepsilon(u)|\varepsilon(u)|^2(1+\delta|\varepsilon(u)|^4)^{-1}$ and the second with $|\varepsilon(u)|^3(1+\delta|\varepsilon(u)|^4)^{-1}$, evaluated at t . This yields (with $\varepsilon(u(t))=\varepsilon(t)$, $\varepsilon(0)=0$ for simplicity)

$$\frac{|\varepsilon(t)|^4}{1+\delta|\varepsilon(t)|^4} = A\sigma\big|_0^t\varepsilon(t)|\varepsilon(t)|^2\frac{1}{1+\delta|\varepsilon(t)|^4} + \xi\big|_0^t|\varepsilon(t)|^3\frac{1}{1+\delta|\varepsilon(t)|^4} + E_0 \tag{12.3}$$

with

$$E_0 = \frac{1}{\mu}\int_0^t \big(|\sigma_D(s)| - (\xi(s)+\kappa)\big)_+ \left(\frac{\sigma_D(s)}{|\sigma_D(s)|}\varepsilon(t) - |\varepsilon(t)|\right) \mathrm{d}s|\varepsilon(t)|^2\frac{1}{1+\delta|\varepsilon(t)|^4} \,. \tag{12.4}$$

Obviously $E_0 \le 0$. Thus (remember $(\sigma,\xi)(0)=0$)

$$\frac{|\varepsilon(t)|^4}{1+\delta|\varepsilon(t)|^4} \le \big(K\sigma(t)-\xi(t)\big)\frac{|\varepsilon(t)|^3}{1+\delta|\varepsilon(t)|^4} \,. \tag{12.5}$$

We integrate over the set $H_\alpha^j \cap \Omega_0$ and use the fact, that

$$\operatorname*{ess\,sup}_{t,\alpha,j} \int\limits_{H_\alpha^j \cap \Omega_0} \left(|\sigma|^4 + |\xi|^4\right) \mathrm{d}x' \le K_{\Omega_0} \quad (\mu \to 0)$$

also in the isotropic case. Together with Hölder's inequality we obtain from (12.5)

$$\operatorname*{ess\,sup}_{t,\alpha,j} \int\limits_{H_\alpha^j \cap \Omega_0} \frac{|\varepsilon(t)|^4}{1+\delta|\varepsilon(t)|^4} \mathrm{d}x' \le K \operatorname*{ess\,sup}_{t,\alpha,j} \left(1 + \int\limits_{H_\alpha^j \cap \Omega_0} \left(|\sigma|^4 + |\xi|^4\right) \mathrm{d}x' \right)^{\frac{1}{4}} \left(\int\limits_{H_\alpha^j \cap \Omega_0} |\varepsilon(t)|^4 \left(1 + \delta|\varepsilon(t)|^4\right)^{-\frac{4}{3}} \mathrm{d}x' \right)^{\frac{3}{4}} \tag{12.6}$$

and

$$\operatorname*{ess\,sup}_{t,\alpha,j} \int\limits_{H_\alpha^j \cap \Omega_0} \frac{|\varepsilon(t)|^4}{1+\delta|\varepsilon(t)|^4} \,\mathrm{d}x' \le K_{\Omega_0} \tag{12.7}$$

uniformly for $t \in [0,T]$, $\alpha \in \mathbb{R}$, $j = 1, \ldots, n$, $\mu \to 0$, $\delta \to 0$. The theorem is proved. □

13 Plastic strain in isotropic hardening

We continue to give some contributions concerning regularity properties of the unknown variables $\sigma, \xi, \varepsilon(\dot{u}), \varepsilon(u)$ which are related to the $L^\infty(H^1_{loc})$-inclusions of σ and ξ. We obtain some reasonable additional information about $\xi, \dot{\xi}$ and the plastic strain $A\dot{\sigma} - \varepsilon(\dot{u})$.

Theorem 13.1 *Under the hypothesis of theorem 3.1, for all $\Omega_0 \subset\subset \Omega$, we have the estimates*

$$\int_0^T \int_{\Omega_0} |\dot{\xi}| \cdot |\xi|^5 \,\mathrm{d}x\,\mathrm{d}t \le K(T, \Omega_0) \tag{13.1a}$$

$$\int_0^T \int_{\Omega_0} |\varepsilon(\dot{u})_{ik} - (A\dot{\sigma})_{ik} - \gamma\dot{\xi}| \cdot |\varepsilon(u)_{ik} - (A\sigma)_{ik} - \gamma\xi|^5 \,\mathrm{d}x\,\mathrm{d}t \le K(T, \Omega_0, \Lambda_0) \tag{13.1b}$$

where $1 \le \gamma \le \Lambda_0$ or $-\Lambda_0 \le \gamma \le -1$, uniformly for $\mu \to 0$.

Remark

1. In the case of dimensions $n \ge 4$, the exponent 5 has to be replaced by $\frac{2n}{n-2} - 1$ and for $n = 2$ any exponent ≥ 1 may replace 5.

2. Inequality (13.1a) is a refinement of the fact, that $\xi \in L^\infty(L^6_{loc})$ which follows, in turn, from $\xi \in L^\infty(H^1_{loc})$.

3. The quantity $\varepsilon(\dot u) - A\dot\sigma$ is the plastic strain.

proof
(i) We multiply equations (3.1) with

$$\varphi = \left(\frac{\dot\xi}{1+|\dot\xi|} - 1\right)|\xi_L|^5\tau^2$$

where $\xi_L = \min\{|\xi|, L\}\,\mathrm{sign}\,\xi$ and $\tau = \tau(x)$ is a localization function in $W^{1,\infty}$. We observe that $\varphi \le 0$ and obtain

$$\begin{aligned}\int_0^T\int_{\Omega_0}\big(1+|\dot\xi|\big)^{-1}|\dot\xi|^2|\xi_L|^5\tau^2\,\mathrm{d}x\,\mathrm{d}t &\le \int_0^T\int_{\Omega_0}\dot\xi|\xi_L|^5\tau^2\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{\Omega_0}F_L(\xi)\tau^2\,\mathrm{d}x\bigg|_0^T\end{aligned}\tag{13.2}$$

where F_L is the primitive of $|\xi_L|^5$. There holds $|F_L(\xi)| \le |\xi|^6$, hence the right hand side of (13.2) is bounded due to the fact that $\xi \in L^\infty(0,T;H^1_{loc}(\Omega))$ and due to the initial condition $\xi\Big|_{t=0}$, uniformly as $\mu \to 0,\ L \to \infty$. We have

$$|\dot\xi|^2\big(1+|\dot\xi|\big)^{-1} = |\dot\xi| - 1 + \big(1+|\dot\xi|\big)^{-1}\tag{13.3}$$

and since the integrals

$$\left|\int_0^T\int_{\Omega_0}\Big\{\big(1+|\dot\xi|\big)^{-1} - 1\Big\}|\xi_L|^5\tau^2\,\mathrm{d}x\,\mathrm{d}t\right|$$

are uniformly bounded for $L \to \infty,\ \mu \to 0$, we remain with an estimate for

$$\int_0^T\int_{\Omega_0}|\dot\xi||\xi_L|^5\tau^2\,\mathrm{d}x\,\mathrm{d}t$$

which proves (13.1a).

(ii) For proving (13.1b) we multiply equation (3.1b) by γ and add the resulting equation to (3.1a) with index (i,k).
We obtain

$$\begin{aligned}\dot w := \varepsilon(\dot u)_{ik} - (A\dot\sigma)_{ik} - \gamma\dot\xi &\ge 0 \ \text{ if } \gamma \ge 1\\ \text{or } &\le 0 \ \text{ if } \gamma \le -1\,.\end{aligned}\tag{13.4}$$

Thereafter, we multiply (13.4) by

$$(\dot w \pm 1)(1+|\dot w|)^{-1}|w_L|^5\tau^2$$

with + or − according to the sign in (13.4). The rest of the proof is analogously to the first part concerning $\dot{\xi}$. $\square$

One would like to get rid of of the term $\gamma\xi$ in theorem 13.1. This is possible if we confine ourselves to Morrey space estimates:

Theorem 13.2 *Under the assumptions of theorem 13.1 we have the estimates*

$$\int_0^T\int_{B_R} |\dot{\xi}|\,\mathrm{d}x\,\mathrm{d}t \le KR^{\frac{5}{2}} \qquad B_R \subset \Omega_0 \subset\subset \Omega \tag{13.5a}$$

$$\int_0^T\int_{B_R} |\varepsilon(\dot{u})_{ik} - (A\dot{\sigma})_{ik}|\,\mathrm{d}x\,\mathrm{d}t \le KR^{\frac{5}{2}} \tag{13.5b}$$

$$\int_0^T\int_{B_R} |\dot{\lambda}|\,\mathrm{d}x\,\mathrm{d}t \le KR^{\frac{5}{2}}\,. \tag{13.5c}$$

Remark: Note that (13.5a)&(13.5b) do not follow from the inclusion $\varepsilon(\dot{u}), \dot{\sigma}, \dot{\xi} \in L^\infty(L^2)$.

proof (i) We test the equation (3.1b) with

$$\varphi = \left(\frac{\dot{\xi}}{|\dot{\xi}|+1} - 1\right)\tau^2 \le 0, \quad \operatorname{supp}\tau \subset B_{2R},\ \tau = 1 \text{ on } B_R$$

and obtain similar as in the proof of theorem 13.1 that

$$\begin{aligned}
\int_0^T\int_{B_{2R}} \frac{|\dot{\xi}|^2}{|\dot{\xi}|+1}\tau^2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{B_{2R}} \dot{\xi}\tau^2\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{B_{2R}} \xi(T)\tau^2\,\mathrm{d}x \\
&\le \left(\int_{B_{2R}} \xi^6(T)\,\mathrm{d}x\right)^{\frac{1}{6}} \left(KR^3\right)^{\frac{5}{6}} \\
&\le KR^{5/2}
\end{aligned} \tag{13.6}$$

and using (13.3)

$$\int_0^T\int_{B_r} |\dot{\xi}|\tau^2\,\mathrm{d}x\,\mathrm{d}t \le KR^{\frac{5}{2}} + KR^3 \tag{13.7}$$

statement (13.5b) follows.

(ii) Combining the part (ii) of the proof of theorem 13.1 and part (i) here, we obtain

$$\int_0^T\int |\varepsilon(\dot{u})_{ik} - (A\dot{\sigma})_{ik} - \gamma\dot{\xi}|\,\mathrm{d}x\,\mathrm{d}t \le KR^{5/2}\,. \tag{13.8}$$

From (13.5a) and (13.8) we obtain (13.5b). $\square$

14 Further properties of the displacement velocity in isotropic hardening

In the case of isotropic hardening yet we are <u>not</u> able to prove that $\varepsilon(u) \in L^\infty(H^1_{loc})$, in other words that

$$\int_0^t \varepsilon(\dot u)\,\mathrm{d}s \in L^\infty(H^1_{loc})\ ? \tag{14.1}$$

From this point of view, it is of interest, that at least

$$\int_0^t \varepsilon(\dot u)\frac{\sigma_D}{|\sigma_D|}\,\mathrm{d}s \in L^\infty(H^1_{loc})\,.$$

We consider again the penalized equation for isotropic hardening

$$\varepsilon(v) = A\dot\sigma + \frac{1}{\mu}\big(|\sigma_D| - (\kappa+\xi)\big)_+\frac{\sigma_D}{|\sigma_D|} \tag{14.2a}$$

$$0 = \dot\xi - \frac{1}{\mu}\big(|\sigma_D| - (\kappa+\xi)\big)_+\,. \tag{14.2b}$$

But now we take for the elastic compliance tensor A the inverse Lamé-Navier Operator, defined by

$$AB = \mu_0\,\mathrm{tr}(B)Id + \lambda_0 B_D\,,\quad \mu_0,\lambda_0 > 0\,. \tag{14.3}$$

Theorem 14.1 *Let* $\varepsilon(u),\sigma,\xi$ *be a solution of* (14.2)*, assume the safe load condition* (1.5)*, regularity on the data and* (14.3)*.*
Then, for every $\Omega_0 \subset\subset \Omega$*,*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_0}\left|\nabla\int_0^t \varepsilon(\dot u)\frac{\sigma_D}{|\sigma_D|}\,\mathrm{d}s\right|^2 \mathrm{d}x \le K_{T,\Omega_0} \tag{14.4}$$

uniformly as $\mu \to 0$*.*

proof We multiply the equation (14.2a) with $\zeta^2\frac{\sigma_D}{|\sigma_D|}$ and add the resulting equation to equation (14.2b) multiplied with ζ^2. This yields

$$\begin{aligned}\zeta^2(\varepsilon(\dot u)\frac{\sigma_D}{|\sigma_D|}) &= \zeta^2\Big(A\dot\sigma : \frac{\sigma_D}{|\sigma_D|}\Big) + \zeta^2\dot\xi \\ &= \zeta^2\lambda_0\frac{1}{2}\frac{\partial}{\partial t}|\sigma_D| + \zeta^2\dot\xi\end{aligned} \tag{14.5}$$

The penalty term cancels out. Here $\zeta = \zeta(x)$ is a localization function.
We have used that A is of the form (3.2), so

$$A\dot\sigma : \frac{\sigma_D}{|\sigma_D|} = \lambda_0\dot{\sigma_D}\frac{\sigma_D}{|\sigma_D|} = \lambda_0\frac{1}{2}\frac{\partial}{\partial t}|\sigma_D|\,. \tag{14.6}$$

We integrate equation (14.5) from 0 to t and obtain

$$\zeta^2 \int_0^t \varepsilon(\dot{u}) \frac{\sigma_D}{|\sigma_D|} \, \mathrm{d}s = \zeta^2 \left(\lambda_0 \frac{1}{2} |\sigma_D| \Big|_0^t + \xi \Big|_0^t \right)$$

Since the right hand side is uniformly bounded in $L^\infty(H^1_{loc})$, we obtain the theorem. $\square$

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