Boundary Differentiability for the Solution to Hencky's Law of Elastic Plastic Plane Stress

Heribert Blum, Jens Frehse

no. 435

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereichs 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

Bonn, Dezember 2008

Boundary differentiability for the solution to Hencky's law of elastic plastic plane stress

H. Blum[1] J. Frehse[2]

Summary

We analyze a plane stress model of Hencky type concerning regularity properties of the stresses up to the Dirichlet boundary. For curved boundary, we obtain that the first order tangential derivatives and certain normal derivatives of the stresses are in L^2. For straight boundary we have square integrable fractional derivatives of the stresses of order $\frac{1}{2}$ in normal direction.

Key words: elastic-plastic deformation, Hencky model, boundary differentiability
Subject classification: Primary: 74C05, Secondary: 35B65, 35J85, 74G60, 49N60

1. Introduction

An important open problem in the basic theory of elasto-plasticity is the question of boundary differentiability of the stresses for problems governed by Hencky's law. Let $\Omega \subset \mathbf{R}^3$ be a bounded Lipschitz domain and $\Gamma \subset \partial\Omega$. The dual variational principle for this problem reads as follows; Find $\sigma \in (L^2(\Omega; \mathbf{R}^3 \times \mathbf{R}^3)$ such that

$$J(\sigma) := \frac{1}{2} \int_\Omega \sigma A \sigma \, dx = \min\,, \tag{1}$$

where the compliance tensor A satisfies the ellipticity condition

$$\tau : A\tau \geq \lambda_0 |\tau|^2\,, \tag{2}$$

for symmetric 3×3-matrices τ , and σ satisfies the side conditions

(i) $\sigma = \sigma^T$, and

(ii) $\mathrm{div}\sigma = -f$, on Ω ,

(iii) $\sigma \cdot \nu = q$ on $\Gamma_N = \partial\Omega \setminus \Gamma$,

(iv) $|\sigma_D| \leq \mu$.

[1]Fakultät für Mathematik, Technische Universität Dortmund, Vogelpothsweg 87, D-44221 Dortmund, Germany
[2]Institut für Angewandte Mathematik, Universität Bonn, Beringstr. 6, D-53115 Bonn, Germany

((2) can be replaced by an integral version.) Equations (ii) and (iii) are understood in the weak sense, i.e.

$$(\sigma, \nabla\phi) = (f, \phi) + \int_{\Gamma_N} q\phi\, do\,, \quad \phi \in H^1_\Gamma\,,$$

where H^1_Γ consists of all functions in $H^1(\Omega, \mathbf{R}^3)$ such that the trace vanishes on Γ.

Here, $f \in L^2(\Omega; \mathbf{R}^3)$ is the volume force, $q \in L^\infty(\Gamma_N; \mathbf{R}^3)$ is the boundary traction, and $\mu > 0$ is the bound for the deviatoric part

$$\sigma_D = \sigma - \frac{1}{3}\mathrm{tr}(\sigma)\mathrm{I}$$

of the stress tensor.

It is known (see [15], [3]) that the stresses are locally in H^1 while the strains (c.f. (6) in the limit case $p \to \infty$) are only measures ([18]). In the general n-dimensional case, the displacements are in $L^{\frac{n}{n-1}}$ ([18]), even in $L^{\frac{n}{n-1}+\delta}$ ([9]). A remarkable discovery concerning boundary differentiability has been done by Seregin [16]. He showed that the approximations which he used for proving H^1_{loc}-regularity of σ cannot be estimated in H^1 uniformly up to the boundary, in general. (In fact, this concerns the normal derivative.) The limiting stress in Seregin's example, however, is contained in H^1 up to the boundary. In this context we mention [6] where the n-dimensional Hencky problem is studied near portions of the boundary which are circles or n-dimensional balls. It is proven that the first tangential derivatives are in L^2, and it is shown that the method of proof does not work for an *ellipse* as boundary curve. Concerning fractional differentiaility, Knees [11] showed that the stresses are in $\mathcal{H}^{1/2-\delta}$ up to the boundary of the basic domain, where $\mathcal{H}^{1/2-\delta}$ refers to the Nikolskii space (see 4. in the list below). This is related to the result of Repin and Seregin [14] concerning $O(h^{1/2-\delta})$-convergence of related approximations. The only result concerning L^2-inclusion of the first derivatives of the stresses up to the boundary is contained in [8], where it is shown that in the case of a circle as basic domain the tangential derivatives of σ are in L^2.

In this paper we consider a two-dimensional version of (1), the so-called plane-stress model. In this case, the basic domain is of the form $Z := \Omega \times [-d, d]$, where $\Omega \subset \mathbf{R}^2$ has a smooth boundary $\partial\Omega$. The force f depends only on the variables $x_1, x_2 \in \Omega$, the third component of f vanishes,

$$f = (f_1, f_2, 0)\,, \quad f_i(x) = f_i(x_1, x_2)\,,\, i = 1, 2\,. \tag{3}$$

We assume Neumann boundary conditions on $\Omega \times \{\pm d\}$ which, for small d, guarantee that

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0\,, \tag{4}$$

i.e. (b) holds on this part of the boundary.

For the rest of the boundary, we may assume mixed boundary conditions for the *existence* part, for the *differentiability* up to the boundary we succeed, of course, only in a part where pure Neumann or Dirichlet conditions hold. To be more precise, we may consider that $\partial\Omega = \Gamma_D \cup \Gamma_N$ and that homogeneous Dirichlet conditions are posed on $\Gamma_D \times [-d, d]$ and a Neumann condition is posed on $\Gamma_N \times [-d, d]$.

In this paper, we prove for the plane stress model under consideration

1. H^1_{loc}-differentiability of the stresses σ for the limiting problem $p = \infty$, using, with modifications, the dual approach of Bensoussan-Frehse [4], see Theorem 10.15.

2. L^2-estimates and existence of the tangential derivatives up to the Dirichlet boundary. Thus, in our setting, one gets rid of the restriction in [8] that the basic domain is a circle, see Theorem 5 here.

3. In the case that x_2 is the normal direction we have L^2-estimates and L^2-existence of $D_2\sigma_{12}$, $D_2\sigma_{22}$ up to the boundary.

4. In the present paper, L^2-estimates for the normal derivative $D_2\sigma_{11}$ are missing. At least we can show the following: In the case that x_2 is the normal direction the missing component has at least a fractional derivative of order 1/2 in L^2, more precisely $\sigma_{11} \in \mathcal{H}^{1/2}$ which means that
$$\sup_{h_0>h>0} h \int_{U\cap\Omega} |D^h\sigma|^2\,dx \le K\,,$$
see our Theorem 6. Thus, at least, for this problem one can get rid of the δ in the results of Repin and Knees, stating $\sigma \in \mathcal{H}^{1/2-\delta}$ up to the boundary.

5. For future studies on the Prandtl-Reuss model we found it useful, to state also a part of the differentiability results of the Rothe approximation of the Prandtl-Reuss model.

The first order necessary condition for the minimum problem (1) can be simply understood by the so-called Norton-Hoff-approximation which reads:

Minimize
$$J_p = \int_Z \{\frac{1}{2}\sigma A\sigma + \frac{\mu^{-p}}{p}|\sigma_D|^p\}\,d\hat{x}\,, \quad d\hat{x} = dx_1 dx_2 dx_3 \tag{5}$$
on the set
$$V_{f,q} := \{\sigma \in L^p_{\text{sym}} \,|\, (\sigma,\nabla\phi) = (f,\phi) + \int_{\Gamma_N} q\phi\,do\ ,\ \phi \in H^1_\Gamma(Z,\mathbf{R}^3)\}\,.$$
The corresponding Euler equation then reads as
$$\int \{\sigma A\tau + \mu^{-p}|\sigma_D|^{p-2}\sigma_D\tau\}\,dx = 0\,,$$
for all $\tau \in L^2_{\text{sym}}$, such that $(\tau,\nabla\phi) = 0$ for all $\phi \in H^1_\Gamma(Z,\mathbf{R}^3)$. There exists a displacement field $u \in H^{1,p}_\Gamma$ such that
$$\frac{\nabla u + \nabla u^T}{2} = A\sigma + |\sigma_D|^p\sigma_D\,, \tag{6}$$
c.f. the discussion in Section 2.

Under the above assumption on plane stresses and under the additional assumption $(A\sigma)_{i3} = 0\,,\ i = 1,2$, the Euler equation in fact reduces to a problem:

Find $\sigma = \sigma(x_1,x_2)\,,\ \sigma = \sigma^T = (\sigma_{ik})^3_{i,k=1}\,,\ \sigma_{i3} = 0\,,$ and $u = u(x_1,x_2)$ such that
$$\begin{pmatrix} D_1u_1 & \frac{D_1u_2+D_2u_1}{2} & 0 \\ \frac{D_1u_2+D_2u_1}{2} & D_2u_2 & 0 \\ 0 & 0 & D_3u_3 \end{pmatrix} = A\sigma + \mu^{-p}|\sigma_D|^{p-2}\sigma_D\,, \tag{7}$$
where
$$\begin{aligned} |\sigma_D|^2 &= |\sigma_{11} - \frac{\text{tr}(\sigma)}{3}|^2 + |\sigma_{22} - \frac{\text{tr}(\sigma)}{3}|^2 + |\frac{\text{tr}(\sigma)}{3}|^2 + 2|\sigma_{12}|^2 \\ &= \sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{12}^2 + (\text{tr}(\sigma))^2[\frac{3}{9} - \frac{2}{3}] \\ &= \sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{12}^2 + \frac{(\text{tr}(\sigma))^2}{3}\,. \end{aligned}$$

We observe that obviously

$$|\sigma_D|^2 \geq 2\sigma_{12}^2 + \frac{1}{3}\sigma_{11}^2 + \frac{1}{3}\sigma_{22}^2 \,. \tag{8}$$

The relation (7) consists of 4 nontrivial scalar equations, σ_{11} , σ_{22} , σ_{12} , are solutions of the Euler equations of the minimum problem

$$\int_\Omega \{\frac{1}{2} \sum_{m,n=1}^{2} \sigma_{ik} a_{ikmn} \sigma_{mn} + \frac{\mu^{-p}}{p} |\sigma_D|^p\} \, dx \,, \quad dx = dx_1 dx_2 \,. \tag{9}$$

In the limit case $p \to \infty$ this is a Hencky type problem in two space dimensions, however, with a modified definition of the deviatoric part σ_D of the stress tensor.

The Euler equation to this problem reads

$$\int_\Omega [\tau A\sigma + \mu^{-p} |\sigma_D|^{p-2} (\sigma_D \cdot \tau_D)] \, dx = 0 \,, \tag{10}$$

for all $\tau \in L^p_{\text{sym}}$ satisfying $\text{div}\tau = 0$, such that $\tau_{i3} = 0$.

Concerning notation, we continue to denote by σ and τ as symmetric 3×3-matrices with the property $\sigma_{i3} = \tau_{i3} = 0$. Then

$$\tau_D = \begin{pmatrix} \frac{2}{3}\tau_{11} - \frac{1}{3}\tau_{22} & \tau_{12} & 0 \\ \tau_{12} & \frac{2}{3}\tau_{22} - \frac{1}{3}\tau_{22} & 0 \\ 0 & 0 & -\frac{1}{3}(\tau_{11} + \tau_{22}) \end{pmatrix} , \tag{11}$$

and σ_D analogously. Furthermore, $A\sigma$ remains to be a 3×3-matrix.

From (10) one can derive an equation for the symmetric 2×2-matrix

$$\sigma' = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} , \quad \tau' = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} .$$

Using the fact that $\sigma_D : \tau_D = \hat{\sigma} : \tau'$, with

$$\hat{\sigma} = \begin{pmatrix} \frac{2\sigma_{11} - \sigma_{22}}{3} & \sigma_{12} \\ \sigma_{21} & \frac{2\sigma_{22} - \sigma_{11}}{3} \end{pmatrix} , \tag{12}$$

it reads with $A' = (a_{ikmn})^2_{i,k,m,n=1}$,

$$\int_\Omega [\tau' A' \sigma' + \mu^{-p} |\sigma_D|^{p-2} (\hat{\sigma}\tau')] \, dx = 0 \,, \tag{13}$$

for all $\tau' \in L^p_{\text{sym}}(\mathbf{R}^2 \times \mathbf{R}^2)$ such that

$$(\tau', \nabla\phi) = 0$$

for all $\phi \in H^{1,\frac{p}{p-1}}_{\Gamma_N}(\Omega, \mathbf{R}^2)$.

From (13) there follows the existence of a function $u \in H^{1,\frac{p}{p-1}}(\Omega; \mathbf{R}^2)$ such that

$$\begin{aligned} \frac{1}{2}(\nabla u + \nabla u^T, \tau') &= \int_\Omega [\tau' A' \sigma' + \mu^{-p} |\sigma_D|^{p-2} \hat{\sigma}\tau'] \, dx \\ &= \int_\Omega [\tau A\sigma + \mu^{-p} |\sigma_D|^{p-2} \sigma_D \tau_D] \, dx \,, \end{aligned} \tag{14}$$

for all $\tau' \in L^p_{\mathrm{sym}}(\mathbf{R}^2 \times \mathbf{R}^2)$, resp. for all $\tau \in L^p_{\mathrm{sym}}(\mathbf{R}^3 \times \mathbf{R}^3)$ such that

$$\tau = \begin{pmatrix} \tau' & 0 \\ 0 & 0 \end{pmatrix} .$$

Here and in the following, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$.

For uniform estimates as $p \to \infty$ we work with the **safe load condition**:

There exists a stress function

$$\sigma_0 \in V_{f,q}$$

such that

$$|\sigma_{0,D}| \leq \mu - \delta_0 \,, \tag{15}$$

for some $\delta_0 > 0$.

By convexity, equation (7), except the equation for $D_3 u_3$, is equivalent to the property that $\sigma_{11}, \sigma_{22}, \sigma_{12}$ minimize the functional in (9).

The derivation of regularity properties of the solution of the Norton-Hoff-problem is a difficult and interesting problem for itself. For the classical Norton-Hoff-approximation to Hencky's problem (1), we would like to mention Steinhauer's thesis [17] which contains the remarkable theorem that in the case of 2 dimensions, for $2 < p < \infty$, the stresses σ are contained in $C^{\alpha}_{\mathrm{loc}}(\Omega)$ and the displacements u are in $C^{1+\beta}_{\mathrm{loc}}(\Omega)$ for some $\alpha, \beta > 0$. Later, but independently, the same result for the displacements u was found by Bildhauer and Fuchs [5].

2. First estimates

In the following, we rescale σ, A, and u, such that we may choose $\mu = 1$ without loss of generality.

For our analysis we need an additional approximation. Define

$$\beta_L(\sigma) = \begin{cases} \frac{1}{p}|\sigma|^p, & \text{for } |\sigma| \leq L\,, \\ \frac{1}{2}L^{p-2}|\sigma|^2 + (\frac{1}{p} - \frac{1}{2})L^p, & \text{for } |\sigma| \geq L\,. \end{cases}$$

It is easy to see that β_L is continuously differentiable and that

$$\beta_L'(\sigma)\tau := \frac{\mathrm{d}}{\mathrm{d}t}\beta_L(\sigma + t\tau)_{|t=0} = \begin{cases} |\sigma|^{p-2}\sigma\tau\,, & \text{for } |\sigma| < L\,, \\ L^{p-2}\sigma\tau\,, & \text{for } |\sigma| > L\,. \end{cases} \tag{16}$$

Using the abbreviation

$$[\sigma]_L := \min\{|\sigma|, L\}$$

this reads $\beta_L'(\sigma) = [\sigma]_L^{p-2}\sigma$.

Remark. The function at the right hand side is uniformly continuous. Hence the formula can be extended for $|\sigma| = L$.

Furthermore, β_L is convex. For this, we prove the monotonicity property

$$\Big(\beta_L'(\sigma) - \beta_L'(\tilde{\sigma})\Big) \cdot (\sigma - \tilde{\sigma}) \geq 0\,. \tag{17}$$

Clearly, (17) holds if both tensors $\sigma, \tilde{\sigma}$ satisfy $|\sigma| < L, |\tilde{\sigma}| < L$ or $|\sigma| > L, |\tilde{\sigma}| > L$. We have to analyze the case $|\sigma| < L, |\tilde{\sigma}| > L$:

$$\begin{aligned}\left(\beta_L'(\sigma) - \beta_L'(\tilde{\sigma})\right) \cdot (\sigma - \tilde{\sigma}) &= |\sigma|^{p-2}\sigma \cdot (\sigma - \tilde{\sigma}) - L^{p-2}\tilde{\sigma} \cdot (\sigma - \tilde{\sigma}) \\ &\geq \frac{1}{2}|\sigma|^p - \frac{1}{2}|\sigma|^{p-2}|\tilde{\sigma}|^2 + \frac{1}{2}L^{p-2}|\tilde{\sigma}|^2 - \frac{1}{2}L^{p-2}|\sigma|^2 \\ &= \frac{1}{2}(|\sigma|^{p-2} - L^{p-2})(|\sigma|^2 - |\tilde{\sigma}|^2)\,.\end{aligned}$$

Since $|\sigma| \leq L \leq |\tilde{\sigma}|$ we obtain the inequality (17). The case $|\sigma| = L$ and/or $|\tilde{\sigma}| = L$ follows by continuous extension.

For later usage we remark that for $|\sigma| \neq L$ we have the representation

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\beta_L(\sigma + t\tau)_{|t=0} &= \beta_L''(\sigma)\tau \cdot \tau \\ &= \begin{cases} |\sigma|^{p-2}\tau \cdot \tau + (p-2)|\sigma|^{p-4}(\sigma \cdot \tau)^2\,, & \text{for } |\sigma| < L\,, \\ L^{p-2}\tau \cdot \tau\,, & \text{for } |\sigma| > L\,. \end{cases}\end{aligned}$$

To prove existence of a solution we approximate the Norton-Hoff problem or problem (9) by the truncated Norton-Hoff problem

$$J_L = \int_\Omega \{\frac{1}{2}\sigma A\sigma + \beta_L(\sigma_D)\}dx = \min!\,, \quad \sigma \in V\,, \tag{18}$$

where

$$V = \{\sigma \in H(\mathrm{div}, \Omega) \,|\, \sigma = \sigma^T\,, -\mathrm{div}\sigma = f \text{ weakly}, \sigma \cdot \nu = q \text{ on } \Gamma_N\}$$

and p is fixed.

Theorem 1. *Let the coerciveness condition (2) for A be satisfied. Let $f \in L^2, q \in L^\infty(\partial\Omega)$. Furthermore, let the safe load condition (15) hold and let $p > 1$. Then there exists a unique minimizer $\sigma^L = \sigma_p^L \in V$ of J_L and we have*

$$\|\sigma\|_2 \leq K\,, \quad \|\beta_L'(\sigma_D^L)\sigma_D^L\|_{p/(p-1)} \leq K_p\,, \quad \|\sigma_D^L\|_p \leq K_p\,,$$

uniformly as $L \to \infty$.

Proof.

(i) Existence and uniqueness is an obvious consequence of the coerciveness in L^2 and the strict convexity.

(ii) By comparison

$$J_L(\sigma^L) \leq J_L(\sigma_0) \leq K$$

uniformly in L and p, where σ_0 comes from the safe load condition. This implies

$$\|\sigma^L\|_2 \leq K \quad \text{and} \quad \frac{1}{p}\int_\Omega [\sigma_D^L]_L^{p-2}|\sigma_D^L|^2\,dx \leq K$$

uniformly in p.

By routine analysis – optimization and monotonicity arguments – one has that

$$\sigma^L \to \sigma = \sigma(p)\,, \quad \sigma(p) \to \sigma\,, \quad \text{strongly in } L^2\,,\ \text{as } L \to \infty\,,\ \text{resp. } p \to \infty\,,$$

where $\sigma(p)$ is the solution of (5) in our setting, c.f., e.g., [18], [4]. The existence of the deformations $u = u^L \in H^{1,2}_\Gamma$ in (14) follows via an argument using orthogonal decomposition in Hilbert spaces and Korn's inequality. Thereafter one may use Theorem 2 below for justifying the $\nabla u + \nabla u^T \in L^{\frac{p}{p-1}}$ for $u = u(p)$. □

It is useful to analyze also the Rothe approximation of the Prandtl-Reuss-law. For formulating the problem, we consider time steps $t = mh$, $m \in \mathbf{N}$, with some step size $h > 0$ and look for approximate displacement velocities $v = \dot{u}$ and stresses $\hat{\sigma}(t,\cdot)$ such that

$$\begin{aligned} \frac{\nabla v + \nabla^T v}{2} &= \frac{1}{h} A(\sigma(t,\cdot) - \sigma(t-h,\cdot)) + |\sigma_D(t,\cdot)|^{p-2}\sigma_D(t,\cdot)\,,\ t = mh\,, \\ \sigma(0,\cdot) &= \sigma_a \in L^2(\Omega)\,, \quad |\sigma_{a,D}| \le 1\,, \end{aligned} \tag{19}$$

and (i)-(iv) holds for $\sigma = \sigma(t,\cdot)$. For the continuous Prandtl-Reuss-Norton-Hoff problem, we simply replace (19) by the equation

$$\frac{\nabla v + \nabla^T v}{2} = A\dot{\sigma}(t,\cdot) + |\sigma_D(t,\cdot)|^{p-2}\sigma_D(t,\cdot)\,,\ t \ge 0\,.$$

For having reasonable applications, in this case a dependence of the boundary force with respect to t is assumed.

Here we have already dropped the equation for $D_3 v_3$ and consider (19) as an equation for 2×2-matrices. The solution $\sigma = \sigma(t,\cdot)$, for given $\sigma(t-h,\cdot)$, of (19) can be considered as the minimizer of

$$\frac{1}{2h}\int \sigma A \sigma\, dx - \frac{1}{h}\int \sigma A \sigma(t-h,\cdot)\, dx + \frac{\mu^{-p}}{p}\int |\sigma_D|^p\, dx\,. \tag{20}$$

Similar as in the Hencky-Norton-Hoff case above we approximate the penalty term by $\int \beta_L(\sigma_D)\, dx$.

For the Rothe problem, the assumptions for f, q and the safe load condition are replaced by

$$f \in C((0,T), L^2)\,, \quad q \in C((0,T), L^\infty(\Gamma_N))\,, \tag{21}$$

$$\sigma_0 \in C((0,T), L^\infty) \text{ satisfies (15) for } t = mh\,. \tag{22}$$

Then the following is easy to see.

Corollary 1. *Under the assumptions (21), (22), (2), there exists a solution $\sigma(t,\cdot)$ of (19), (20), $t = mh$, $m \in \mathbf{N}$, such that the estimates of Theorem 1 hold, namely*

$$\|\sigma(t)\|_2 \le K_h\,, \quad \|\beta'(\sigma_D(t))\sigma_D(t)\|_{p/(p-1)} + \|\sigma_D(t)\|_p \le K_{h,p}\,.$$

3. Uniform L^1-estimate for the strain tensor

To obtain a uniform L^1-estimate for the strain tensor ε we start from the identity

$$\frac{\nabla u + \nabla u^T}{2} = A\sigma + \beta_L'(\sigma_D) \tag{23}$$

in the truncated Norton-Hoff approximation (18). The approach is in analogy to the corresponding theorem in Temam's book [18]. We test by $\sigma = \sigma^L - \sigma_0$ where σ_0 comes from the safe load condition. By the balance of forces and the above equation we get

$$\begin{aligned} 0 &= (\frac{\nabla u + \nabla u^T}{2}, \sigma^L - \sigma_0) \\ &= \int_\Omega \sigma^L A(\sigma^L - \sigma_0) + \beta_L'(\sigma_D^L)(\sigma_D^L - \sigma_{0,D})\, dx =: B_1 + B_2\,. \end{aligned}$$

B_1 is uniformly bounded as L and p tend to ∞. In what follows we drop the index L in the notation. Moreover, we set $\mu = 1$ without loss of generality. For B_2 we get the estimate

$$\begin{aligned} K \geq B_2 &= \int_{|\sigma_D|<L} |\sigma_D|^{p-2}\sigma_D(\sigma_D - \sigma_{0,D}) + \int_{|\sigma_D|\geq L} L^{p-2}\sigma_D(\sigma_D - \sigma_{0,D}) \\ &\geq \frac{1}{2}\int_{|\sigma_D|<L} |\sigma_D|^{p-2}(|\sigma_D|^2 - |\sigma_{0,D}|^2) \\ &\qquad\qquad + \frac{1}{2}\int_{|\sigma_D|\geq L} L^{p-2}(|\sigma_D|^2 - |\sigma_{0,D}|^2) \qquad (24) \\ &= \frac{1}{2}\int_{1\leq|\sigma_D|<L} |\sigma_D|^{p-2}(|\sigma_D|^2 - |\sigma_{0,D}|^2) \\ &\qquad\qquad + \frac{1}{2}\int_{|\sigma_D|\leq 1} |\sigma_D|^{p-2}(|\sigma_D|^2 - |\sigma_{0,D}|^2) \\ &\qquad\qquad + \frac{1}{2}\int_{|\sigma_D|\geq L} L^{p-2}(|\sigma_D|^2 - |\sigma_{0,D}|^2) \qquad (25) \\ &=: B_{12} + B_{22} + B_{32}\,. \end{aligned}$$

Clearly, $B_{22} \geq -\frac{1}{2}|\Omega|$, since the integrand is ≤ 1. On account of the safe load condition,

$$|\sigma_D|^2 - |\sigma_{0,D}|^2 \geq \alpha_0 > 0 \quad \text{when } |\sigma_D| \geq 1\,.$$

Hence

$$B_{12} \geq \frac{\alpha}{2}\int_{|\sigma_D|\leq L} |\sigma_D|^{p-2}\, dx \quad \text{and} \quad B_{32} \geq \frac{\alpha_0}{2}\int_{|\sigma_D|>L} L^{p-2}\, dx\,,$$

where the constant K comes from the safe load condition. Since B_{22} is bounded we have arrived at estimates for

$$\int_{|\sigma_D|\leq L} |\sigma_D|^{p-2}\, dx \qquad \text{and} \qquad \int_{|\sigma_D|>L} L^{p-2}\, dx\,.$$

Inspecting (24) again we finally obtain the following result.

Theorem 2. *Under the assumptions of Theorem 1 the minimizers σ^L of (18) satisfy*

$$\int_{|\sigma_D|\leq L} |\sigma_D^L|^p\, dx + \int_{|\sigma_D|>L} L^{p-2}|\sigma_D^L|^2\, dx \, < K\,,$$

uniformly as $L \to \infty$ and $p \to \infty$, where $\mu = 1$ without loss of generality.

Corollary 2. *We have the following estimates (with the simplified notation $u = u^L, \sigma = \sigma^L$)*

$$\|\nabla u^L + \nabla(u^L)^T\|_{\frac{p}{p-1}} \leq K \quad \text{and} \quad \|u^L\|_2 \leq K\,,$$

uniformly in L and p. Moreover we get, for fixed p, the bound

$$\|\nabla u^L\|_{\frac{p}{p-1}} \leq K_p\,,$$

uniformly in L.

Proof. We use the estimate

$$|\nabla u + \nabla u^T|^{\frac{p}{p-1}} \le K|A\sigma|^{\frac{p}{p-1}} + K|\beta'(\sigma_D)|^{\frac{p}{p-1}} .$$

For the second summand we know that

$$|\beta'(\sigma_D)|^{\frac{p}{p-1}} \le \begin{cases} |\sigma_D|^p , & |\sigma_D| \le L \\ L^{\frac{p-2}{p-1}}|\sigma_D|^{\frac{p}{p-1}} , & |\sigma_D| > L \end{cases} .$$

Hence

$$\int_{|\sigma_D|\le L} |\beta'(\sigma_D)|^{\frac{p}{p-1}} \le K$$

and

$$\begin{aligned} \int_{|\sigma_D|>L} |\beta'(\sigma_D)|^{\frac{p}{p-1}} &= \int_{|\sigma_D|>L} L^{p-2+\frac{p-2}{p-1}}|\sigma_D|^{\frac{p}{p-1}} \\ &\le \int_{|\sigma_D|>L} L^{p-2}|\sigma_D|^{\frac{p}{p-1}+\frac{p-2}{p-1}} = \int_{|\sigma_D|>L} L^{p-2}|\sigma_D|^2 . \end{aligned}$$

The inequality $\|u\|_2 \le K$ follows via the (two-dimensional) embedding theorem presented in Temam's book,

$$\|\nabla u + \nabla u^T\|_1 \ge c\|u\|_2 .$$

□

By almost the same proof if is easy to see

Corollary 3. *The analogue of Theorem 2 and Corollary 2 hold for the Rothe approximation for fixed h, namely*

$$\int_{|\sigma_D|\le L} |\sigma_D^L(t)|^p \, dx + \int_{|\sigma_D|\ge L} L^{p-2}|\sigma_D^L(t)|^2 \, dx \le K_h ,$$

and

$$\|(\nabla u + \nabla u^T)(t)\|_{\frac{p}{p-1}} \le K_h \quad \text{and} \quad \|u(t)\|_2 \le K_h ,$$

uniformly in L and p. Moreover, for fixed p, we get

$$\|\nabla u^L(t)\|_{\frac{p}{p-1}} \le K_{p,h} ,$$

uniformly in L, $t = mk \, , m \in \mathbf{N} \, , t \le T$.

4. Interior differentiability of the stresses σ^L

The interior differentiability of the stresses and the strain tensor follows via the primal formulation (eliminating σ and obtaining a uniformly elliptic equation in u. Students can find the proof in the appendix to the preprint of the paper). Nevertheless we present the proof via the dual approach since it is useful to be introduced to looking on the problem in this way.

Proposition 1. *Let the same hypotheses be satisfied as in Theorem 1 and let $f \in H^{2,2}_{\text{loc}}(\Omega)$. Then $\sigma^L \in H^1_{\text{loc}}$.*

Proof. Again we drop the dependency on L, setting $u = u^L, \sigma = \sigma^L$. Let $D_j^h \tau$, $h \neq 0$, denote the usual difference quotients:

$$D_j^h \tau(x) = \frac{1}{h}(\tau(x + he_j) - \tau(x)) ,$$

e_j being the j-th unit vector. Let ζ be a localization function. We test the Euler equation for σ by $-D_j^{-h}(\zeta^2 D_j^h \sigma)$ and obtain, using also partial summation

$$\begin{aligned} -\frac{1}{2}&(\nabla u + \nabla u^T, D^{-h}(\zeta^2 D^h \sigma)) \\ &= (D^h \sigma A, \zeta^2 D^h \sigma) + (D_j^h \beta'(\sigma_D)), \zeta^2 D_j^h \sigma_D) . \end{aligned} \tag{26}$$

The left hand side L_0 of (26) is rewritten

$$\begin{aligned} L_0 &= (D^h \nabla u, \zeta^2 D^h \sigma) = (\nabla(\zeta^2 D^h u), D^h \sigma) - (\nabla \zeta^2 D^h u, D^h \sigma) \\ &= (\nabla(\zeta^2 D^h u), D^h f) - (\nabla \zeta D^h u, \zeta^2 D^h \sigma) = G_0 + F_0 . \end{aligned}$$

Since

$$G_0 = -(\zeta^2 D^h u, D^h \mathrm{div} f)$$

we get

$$|G_0| \leq \|D^h u\|_2 \|\nabla^2 f\|_2 \leq K_L \quad (h \to 0) .$$

Further, since $\nabla u \in L^2_{\mathrm{loc}}$ for fixed L, due to (23) and Korn's inequality we have

$$|F_0| \leq \varepsilon \int_\Omega \zeta^2 |D^h \sigma|^2 \, dx + K_\varepsilon \int_{\Omega_0} |\nabla u|^2 \, dx .$$

Finally, we see by a convexity argument that

$$(D_j^h([\sigma_D]_L^{p-2} \sigma_D), D_j^h \sigma) \geq 0 . \tag{27}$$

Thus we arrive at the inequality

$$\begin{aligned} c_0 \int_\Omega \zeta^2 |D^h \sigma|^2 \, dx &\leq (D^h \sigma A, \zeta^2 D^h \sigma) \\ &\leq (D^h \sigma A, \zeta^2 D^h \sigma) + (D_j^h([\sigma_D]_L^{p-2} \sigma_D), \zeta^2 D_j^h \sigma_D) \\ &\leq |G_0| + |F_0| \leq K_L + \varepsilon \int_\Omega \zeta^2 |D^h \sigma|^2 \, dx + K_\varepsilon \int_{\Omega_0} |\nabla u|^2 \, dx . \end{aligned}$$

Thus we have found a uniform bound for $\|\zeta^2 D^h \sigma^L\|_2$ as $h \to 0$, for L, p fixed, and the proposition follows. □

Remark. We note the following useful identity: Since $u, \sigma^L, |\sigma_D^L|, [\sigma_D^L]_L^{p-2}$ etc. are in H^1, we may in (26) pass to the limit $h \to 0$. From the equation

$$\begin{aligned} -(\zeta^2 D^h u^L, D^h \mathrm{div}\, f) &- (\nabla \zeta D^h u^L, \zeta^2 D^h \sigma^L) \\ &= (D^h \sigma^L A, \zeta^2 D^h \sigma^L) + (D^h [\sigma]_D^L|_L^{p-2}, \zeta^2 D^h \sigma_D^L) \end{aligned}$$

we get in the limit

$$\begin{aligned} &-(\zeta^2 D u^L, D \mathrm{div}\, f) - (\nabla \zeta D u^L, \zeta^2 D \sigma^L) \\ &= (D \sigma^L A, \zeta^2 D \sigma^L) + \int D([\sigma_D^L]_L^{p-2} \sigma_D^L) \cdot \zeta^2 D \sigma_D^L \, dx \\ &= (D \sigma^L A, \zeta^2 D \sigma^L) + (p-2) \int_{|\sigma_D| \leq L} |D|\sigma_D^L||^2 |\sigma_D|^{p-2} \zeta^2 \, dx \\ &\quad + \int [\sigma_D^L]_L^{p-2} |D \sigma_D|^2 \zeta^2 \, dx . \end{aligned} \tag{28}$$

Again, we have a similar proposition for the Rothe approximation:

Proposition 2. *Under the conditions of Corollary 1 and the additional assumptions $f \in L^2(0, T, H^{2,2}_{\text{loc}}(\Omega))$ and $\sigma(0) \in H^1_{\text{loc}}(\Omega)$, the inclusion $\sigma^L(t, \cdot) \in H^1_{\text{loc}}$ holds for fixed L and h for the solution of the Rothe equation (19).*

5. H^1_{loc}-differentiability for the Norton-Hoff-approximation and the Hencky-problem

In this section we show an H^1-estimate for σ^L on interior subdomains Ω_0 uniformly for L and $p \to \infty$. We follow the proof in Bensoussan-Frehse ([3]) and ([4]).

Theorem 3. *Let $f \in H^{2,2}_{\text{loc}}(\Omega)$ and let the assumptions of Theorem 1 be satisfied. Then, for $\sigma = \sigma^L$ there holds the estimate*

$$\int_{\Omega_0} |\nabla \sigma^L|^2 \, dx + (p-2) \int_{|\sigma_D| \le L} |D|\sigma^L_D||^2 |\sigma_D|^{p-2} \zeta^2 \, dx \\ + \int [\sigma^L_D]_L^{p-2} |D\sigma_D| \zeta^2 \, dx \le K_{\Omega_0} ,$$

uniformly as $L, p \to \infty$.

Corollary 4. *In the limit case $L = \infty$ there holds*

$$\int_{\Omega_0} |\nabla \sigma|^2 \, dx + (p-2) \int_{\Omega_0} |\nabla|\sigma_D||^2 |\sigma_D|^{p-2} \, dx + \int_{\Omega_0} |\sigma|^{p-2} |\nabla \sigma_D|^2 \, dx \le K_{\Omega_0},$$

uniformly as $p \to \infty$.

Proof. We test the Euler equation for $u = u(L, p)$ by $-D_j^{-h}(\zeta^4 D_j^h \sigma)$ and conclude

$$-(\nabla u, D_j^{-h}(\zeta^4 D_j^h \sigma)) = (D_j^h \sigma \zeta^4, A D_j^h \sigma) + \int_{\Omega_0} D_j^h(\beta'(\sigma_D)) \zeta^4 D_j^h \sigma_D \, dx \tag{29}$$

The left hand side of (29) is rewritten

$$R_0 := -(\nabla u, D_j^{-h}(\zeta^4 D_j^h \sigma)) = (D_j^h \nabla u, \zeta^4 D_j^h \sigma) \\ = -(D_j^h u, \operatorname{div}(\zeta^4 D_j^h \sigma)) =: E_1 + E_2 ,$$

where

$$E_1 = -(D_j^h u, \nabla \zeta^4 D_j^h \sigma) \quad \text{and} \quad E_2 = (D_j^h u, \zeta^4 D_j^h f) .$$

We now pass to the limit $h \to 0$ which is possible since $u \in H^2_{\text{loc}}, \sigma^L \in H^1_{\text{loc}}$, and $\Delta f \in L^2, f \in L^2$. Then the left hand side of (29) converges to

$$R_0^\infty = -(D_j u, \nabla \zeta^4 D_j \sigma) + (D_j u, \zeta^4 D_j f) ,$$

for $h \to \infty$. As concerns the right hand side of (29), the first summand converges to $(D_j \sigma, \zeta^4 A D_j \sigma)$, the second summand satisfies

$$\liminf \int_{\Omega_0} D_j^h(\beta_L'(\sigma_D)) \zeta^4 D_j^h \sigma_D \, dx \ge \int_{|\sigma_D| \ne L} D_j(\beta_L'(\sigma_D)) \zeta^4 D_j \sigma_D \, dx .$$

This follows via Fatou's lemma from the fact that the integrand is nonnegative due to monotonicity and that, for $h \to 0$, $D_j^h \sigma$ converges a.e. for a subsequence to the limit $D_j \sigma$

and $D_j^h(\beta_L'(\sigma_D)\sigma_D)$ to $D_j(\beta'(\sigma_D)\sigma_D)$, the latter for $|\sigma_D| \neq L$. The points where $|\sigma_D^L| = 0$ are just left off due to monotonicity.

Thus we arrive at the inequality

$$\int_{\Omega_0} D_j\sigma A D_j\sigma\zeta^4\,dx + \int_{\Omega_0} D_j(\beta_L(\sigma_D))D_j\sigma_D^L\zeta^4\,dx \\ \leq -(D_ju, \nabla\zeta^4 D_j\sigma) + (D_ju, \zeta^4 D_jf) = G + H\,. \tag{30}$$

Now we have to use detailed index notation. For the first term G on the right of (30) we get, with summation convention for $i, k = 1, 2$,

$$-G = (D_ju_k, D_i\zeta^4 D_j\sigma_{ik}) = \\ (D_ju_k + D_ku_j, (D_i\zeta^4)D_j\sigma_{ik}) - (D_ku_j, D_i\zeta^4 D_j\sigma_{ik}) = G_1 + G_2\,. \tag{31}$$

For the second term on the right we get

$$|G_2] = |(u_j, D_k(D_i\zeta^4 D_j\sigma_{ik}))| \\ \leq |(u_j, D_kD_i\zeta^4 D_j\sigma_{ik})| + |(u_j, D_i\zeta^4 D_j \mathrm{div} f_i)|\,.$$

Since $\|u\|_{L^2} \leq K$ uniformly (cf. Corollary 2) we estimate the first term by

$$K\|\zeta^2 D_j\sigma_{ik}\|_{L^2};$$

the second is bounded since $f \in H^2$.

Let us now estimate the term G_1. For this term we get

$$G_1 = \int (A\sigma)_{kj} D_i\zeta^4 D_j\sigma_{ik} + \int \beta'(\sigma_D)_{jk} D_i\zeta^4 D_j\sigma_{ik} \\ = G_{11} + G_{12}\,. \tag{32}$$

By Hölder's inequality, since $\|\sigma\|_{L^2} \leq K$ and $0 \leq \zeta \leq 1$, we get

$$|G_{11}| \leq K\|\zeta^2 D_j\sigma\|_{L^2}$$

and, in view of (16),

$$|G_{12}| \leq K_\delta \int [\sigma_D]_L^{p-2}|\sigma_D|_{jk}^2|\nabla\zeta|^2\,dx + \delta\int [\sigma_D]_L^{p-2}|D_j\sigma_{ik}|^2\zeta^6\,dx \tag{33}$$

The first summand on the right of (33) is bounded uniformly by Theorem 2. The second summand is estimated using (8) with σ replaced by $D_j\sigma$, which states that

$$|D_j\sigma| \leq K|D_j\sigma_D|\,,$$

where $k(\cdot) \in L^\infty$. With this we obtain

$$\delta\int [\sigma_D^L]_L^{p-2}|D_j\sigma_{ik}|^2\zeta^6\,dx \leq \delta\int [\sigma_D^L]_L^{p-2}(|D_j\sigma_D|^2 + K)\zeta^6\,dx\,.$$

This latter term is absorbed, for small $\delta > 0$, by the corresponding term at the left hand side of (30). In fact, we calculate

$$\int D_j(\beta'(\sigma_D))D_j\sigma_D\zeta^6\,dx \\ = \int_{|\sigma_D|\leq L}\{|\sigma_D|^{p-2}|D_j\sigma_D|^2 + (p-2)|\sigma_D|^{p-2}|D_j|\sigma_D||^2\}\zeta^6\,dx \\ + \int_{|\sigma_D|\geq L} L^{p-2}|D_j\sigma_D|^2\zeta^6\,dx \\ = \int_\Omega [\sigma_D]_L^{p-2}|D_j\sigma_D|^2\zeta^6\,dx + \int_{|\sigma_D|\leq L}(p-2)|\sigma_D|^{p-2}(D_j|\sigma_D|^2)\zeta^6\,dx\,.$$

Thus, we arrive at the inequality

$$\int_\Omega D_j\sigma A D_j\sigma\zeta^4\,dx + \frac{1}{2}\int_\Omega [\sigma_D]_L^{p-2}|D_j\sigma_D|^2\zeta^2\,dx \\ + (p-2)\int_{|\sigma_D|\le L} |\sigma_D|^{p-2}(D_j|\sigma_D|^2)\zeta^2\,dx \le K\,.$$

□

Again we have

Corollary 5. *Under the assumptions of Proposition 2 the analogue statement of Theorem 3 holds for the Rothe approximation.*

Thus, since the inequality in Theorem 3 and in Corollary 5 is uniform, we have H^1_{loc}-differentiability for the Hencky and the Rothe-Hencky problem.

6. Estimation of the tangential derivatives

6.1. Properties for L fixed

The boundary differentiability, i.e. $\sigma \in H^1$, for fixed L and p can be also done via estimating difference quotients of σ, similar as in the interior analysis. Since the precise treatment with flattening the boundary locally is somewhat tedious, we prefer to argue that the Euler equation of the approximation (18) is equivalent to a uniformly elliptic system in primal formulation, with global Lipschitz nonlinearities

$$-\sum_{i,k=1}^{2} D_i g_{ik}(\nabla u) = \tilde{f}\,. \tag{34}$$

As already remarked in Section 4, an elementary proof is found in the appendix of the preprint to this paper.

By the theory of these systems we know that the second derivatives of u are bounded in L^2 up to the boundary, provided that we avoid neighborhood of boundary points where Dirichlet and Neumann boundary have nonempty intersection. Let us assume

$$\Gamma_D\,,\,\Gamma_N \in H^{3,\infty}(\partial\Omega)\,. \tag{35}$$

In the case of Neumann boundary, we need some regularity of the boundary force,

$$q \in H^{1,\infty}(\Gamma_N)\,. \tag{36}$$

Of course we could have argued like this also for the interior differentiability, but we preferred the dual approach in order to prepare the techniques.

Thus we state for the solution u^L, σ^L of the truncated Norton-Hoff model (18):

Theorem 4. *Under the assumptions of Theorem 3 and, in addition, the boundary regularity (35) und (36), there holds*

$$\nabla\sigma^L\,,\,\nabla^2 u^L \in L^2(U\cap\Omega)\,,$$

where U is an open subset such that either $U_0\cap\Gamma_N$ or $U_0\cap\Gamma_D$ is empty, $U \subset\subset U_0$.

6.2. Boundary estimates as $L \to \infty$, $p \to \infty$

Due to the preceding chapter we have $\nabla\sigma^L \in L^2$, $|\nabla\sigma_D^L|^2|\sigma_D^L|^{p-2} \in L^1$ up to the boundary of Ω, however we dot have uniform estimates yet. Let ψ be the mapping which flattens the boundary locally. It is defined in the following way.

Let $x_0 \in \partial\Omega$, $\psi : U(x_0) \to \mathbf{R}^2$ be a one-to-one mapping with $\psi \in H^{3,\infty}$, $\det \nabla\psi \neq 0$ such that $\psi(\partial\Omega \cap U) \subset (x_2 = 0)$, $\psi(\Omega \cap U) \subset (x_2 \geq 0)$, and $\psi(\mathcal{C}\Omega \cap U) \subset (x_2 \leq 0)$.

By the chain rule there holds

$$\nabla_y(w(\psi^{-1}(y))) = \nabla w_{|\psi^{-1}(y)} \nabla_y \psi^{-1}(y)\,,$$

and since $\psi(\psi^{-1}(y)) = y$, we have

$$\nabla(\psi(\psi^{-1}(y)) = \mathrm{Id},\ (\nabla\psi)(\psi^{-1}(y))\nabla\psi^{-1}(y) = \mathrm{Id},\ \nabla\psi^{-1}(y) = (\nabla\psi)^{-1}_{|\psi^{-1}(y)}\,.$$

(Here, $\nabla\psi$ is a row vector.)

Since $\psi : U \cap \partial\Omega \to V(0) \cap (x_2 = 0)$ and since $u_{|\partial\Omega} = 0$ we observe that

$$u(\psi^{-1}((y_1 + h, 0))) = u(\psi^{-1}((y_1, 0))) = 0\,,$$

for $(y_1, 0) \in V(0)$. From this we conclude

$$\frac{\partial}{\partial y_1}(u(\psi^{-1}(y_1, 0))) = 0 \qquad \text{and}$$

$$\frac{\partial}{\partial y_1}(\psi^{-1})_{|(y_1,0)} \cdot \nabla u(\psi^{-1}(y_1, 0)) = 0 \tag{37}$$

Since $\psi(\psi^{-1}(y_1, 0)) = (y_1, 0)$ we have

$$\nabla\psi_{|\psi^{-1}(y_1,0)} \frac{\partial}{\partial y_1}(\psi^{-1})_{|(y_1,0)} = (1, 0)\,, \qquad \text{and}$$

$$\frac{\partial}{\partial y_1}(\psi^{-1})_{|(y_1,0)} = (\nabla\psi)^{-1}(1, 0) = ((\nabla\psi)^{-1})_1\,.$$

We set $((\nabla\psi)^{-1})_1 = g$ and obtain from (37) that $g \cdot \nabla u = 0$ at $U \cap \partial\Omega$, with a nonvanishing smooth vector function g. The operator $g \cdot \nabla$ is the "tangential derivative".

With these notations we obtain

Theorem 5. *Let $u = u^L$, $\sigma = \sigma^L$ be the solution and assume the conditions of Theorem 1 and Theorem 4. Let U be an open subset such that $U \cap \partial\Omega \subset \Gamma_D$ (Dirichlet boundary) Then the integrals over the tangential derivatives*

$$\int_{U_0\cap\Omega} |g \cdot \nabla\sigma|^2\, dx,\ \int_{U_0\cap\Omega} |g \cdot \nabla\sigma|^2 [\sigma_D]_L^{p-2}\, dx, \quad \text{and} \quad \int_{U_0\cap\Omega\cap\{|\sigma_D|\leq L\}} |\sigma_D|^{p-2} |g \cdot \nabla|\sigma_D||^2\, dx \tag{38}$$

are uniformly bounded in $L^2(U_0 \cap \Omega)$, $U_0 \subset\subset U$, as $L, p \to \infty$.

Proof. We apply the operation $g \cdot \nabla$ to the Euler equation of σ^L in the set $U \cap \Omega$. The application of $g \cdot \nabla$ is admissible since we have shown that σ^L is in H^1, hence $u \in H^2$

for $L<\infty$. Further, we test the arising equation with $\zeta^2(g\cdot\nabla)\sigma^L$, where ζ^2 is a smooth localization function, $\zeta\equiv 1$ in $U_0\subset\subset U$. This yields

$$\sum_{i,k}(g\cdot\nabla D_i u_k^L,\zeta^2 g\cdot\nabla\sigma_{ik}^L)=(\zeta^2 g\cdot\nabla\sigma^L,A(g\cdot\nabla)\sigma^L)+(\zeta^2 g\cdot\nabla\beta_L'(\sigma_D^L),g\cdot\nabla\sigma_D^L)$$

On the right hand side, there occur terms which are positive definite and are estimated from below by

$$c_0\int\zeta^2|g\cdot\nabla\sigma^L|^2$$

and

$$\int_{|\sigma_D|\le L}\zeta^2|\sigma_D^L|^{p-2}|g\cdot\nabla\sigma_D^L|^2 \qquad (39)$$
$$+(p-2)\int_{|\sigma_D|\le L}\zeta^2|\sigma_D^L|^{p-2}|g\cdot\nabla|\sigma_D^L||^2+L^{p-2}\int_{|\sigma_D|>L}\zeta^2|g\cdot\nabla\sigma_D|^2$$

The difficulty is to handle the left hand side

$$\text{Left}=\sum_{i,k}(g\cdot\nabla D_i u_k^L,\zeta^2 g\cdot\nabla\sigma_{ik}^L)\,.$$

We approximate σ_{ik}^L by a smoother function $\tilde\sigma_{ik}$ such that $-\sum D_i\tilde\sigma_{ik}\to f_k$ in $L^2(\Omega)$. This is possible: We have $\sigma^L\in H^1(\Omega)$, hence it can be extended to a $H^1(\mathbf{R}^2)$-function by Calderon's extension theorem and be convoluted by a smooth mollifier. Then we perform integration by parts putting the derivative D_i on the right hand factor. In the case of Dirichlet boundary for u, no boundary term occurs since $g\cdot\nabla u=0$ at $U\cap\partial\Omega$ and $\nabla^2 u\in L^2$. We obtain

$$\text{Left}+o(1)=-\sum_{i,k}(g\cdot\nabla u_k^L,D_i(\zeta^2 g)\nabla\tilde\sigma_{ik})-\sum_{i,k}(g\cdot\nabla u_k^L,\zeta^2 g\nabla D_i\tilde\sigma_{ik})$$
$$-\sum_{i,k}(D_i g\cdot\nabla u_k^L,\zeta^2 g\nabla\tilde\sigma_{ik})=\tilde{\mathcal{B}}+\tilde{\mathcal{C}}+\tilde{\mathcal{D}}\,.$$

In the terms $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ we may remove the $\sim$-sign passing to the limit for the approximation $\tilde\sigma_{ik}$ of σ_{ik}. In the term $\tilde{\mathcal{C}}$ we temporarily put ∇ on the other terms, go to the limit $\tilde\sigma_{ik}\to\sigma_{ik}$, $-\sum_{i=1}^n D_i\tilde\sigma_{ik}\to f_k$. Finally, we put ∇ back to $-\sum_i D_i\sigma_{ik}=f_k$ and obtain

$$\text{Left}=-\sum_{i,k}(g\cdot\nabla u_k^L,D_i(\zeta^2 g)\nabla\sigma_{ik}^L)-\sum_{i,k}(g\cdot\nabla u_k^L,\zeta^2 g\nabla f_k)$$
$$-\sum_{i,k}(D_i g\cdot\nabla u_k^L,\zeta^2 g\nabla\sigma_{ik}^L)=\mathcal{B}+\mathcal{C}+\mathcal{D}\,.$$

The term $\mathcal{C}$ is easily estimated by a constant putting the ∇ off u_k to the other factors. In particular, a term like $-(u_k^L,D_j(g_j\zeta^2 g f_k))$ arises which remains bounded if $\nabla f_k\in L^2$ since $u_k^L\in L^2$ due to Corollary 2. The term $\mathcal{D}$ is rewritten by

$$\mathcal{D}=-\sum_{i,k,l}(D_i g_l D_l u_k^L,\zeta^2 g\cdot\nabla\sigma_{ik}^L)=-\sum_{i,k,l}(D_i g_l(D_l u_k^L+D_k u_l^L),\zeta^2 g\cdot\nabla\sigma_{ik}^L)$$
$$+\sum_{i,k,l}(D_i g_l D_k u_l^L,\zeta^2 g\cdot\nabla\sigma_{ik}^L)=\mathcal{D}_1+\mathcal{D}_2\,.$$

The term $\mathcal{D}_2$ is treated via partial integration, putting the derivative D_k onto the other factors. Similar to the treatment of the term $\mathcal{C}$, the operations are justified introducing once more approximations $\tilde\sigma_{ik}$. A boundary term does not occur since $u_l = 0$ on $U \cap \partial\Omega$. Thus we obtain

$$\mathcal{D}_2 = -\sum_{i,l}(D_i g_l u_l, \zeta^2 g \cdot \nabla f_i) - \sum_{i,k,l}(D_i g_l u_l, D_k(\zeta^2 g) \cdot \nabla \sigma_{ik}) = \mathcal{D}_{21} + \mathcal{D}_{22}\,.$$

The first term $\mathcal{D}_{21}$ is bounded uniformly since $\|u_l\|_{L^2} \le K$ uniformly and g and f are smooth. The term $\mathcal{D}_{22}$ has the form "integral of a smooth function multiplied by u_l times $D\sigma_{ik}$" plus a bounded term.

Integrals of this type are bounded due to Lemma 1, below. As result we obtain the uniform estimate

$$|\mathcal{D}_{22}| \le K\,.$$

The term $\mathcal{D}_1$ is rewritten by using the representation of $\nabla u + \nabla u^T$ via Euler's equation,

$$\mathcal{D}_1 = -\sum_{i,k,l}(D_i g_l((A\sigma^L)_{lk} + \beta'_{lk}(\sigma^L_D)), \zeta^2 g \cdot \nabla \sigma^L_{ik})\,.$$

The term $\mathcal{D}_1$ can be estimated by

$$\begin{aligned} |\mathcal{D}_1| \le \varepsilon_0 \int [\sigma^L_D]^{p-2}_L |\zeta^2 g \cdot \nabla \sigma^L|^2 \\ + \varepsilon_0 \int |\zeta^2 g \cdot \nabla \sigma^L|^2\, dx + K_{\varepsilon_0} \int \beta'(\sigma^L_D)\sigma^L_D\, dx + K\,. \end{aligned} \tag{40}$$

The third summand in (40) is bounded due to Theorem 2. We further may estimate $|\nabla\sigma^L|$by $|\nabla\sigma^L_D|$ in the first two integrals, cf. (8) applied to $D\sigma_D$.

The term $\mathcal{B} + o(1) = -\sum_{i,k}(g \cdot \nabla u^L_k, D_i(\zeta^2 g)\nabla\tilde\sigma_{ik})$ is treated via partial integration in the following way. The operator $g \cdot \nabla$ is moved off u^L_k and the first order derivatives acting on $\tilde\sigma_{ik}$ are moved off. One of the lower order terms which occur is of the type

$$u^L_k \cdot \tilde\sigma_{ik} \text{ times products of derivatives of } \zeta^2 \text{ and } g\,.$$

These terms are fine since u^L_k and $\tilde\sigma_{ik}$ are estimated in L^2.

Furthermore, after passing to the limit $\tilde\sigma_{ik} \to \sigma^L_{ik}$ there occur terms of the type

$$u^L_k D\sigma^L_{ik} \cdot \text{ smooth function}$$

which are treated by Lemma 1. There remains the term

$$B' = -\int D_j u^L_k D_i(\zeta^2 g_j)(g \cdot \nabla)\tilde\sigma_{ik} + C\,,$$

where C is a term with integrand $D_j u^L_k \tilde\sigma_{ik}$ times a smooth function, which is treated via Lemma 1. We have

$$\begin{aligned} B' = -\sum_{i,k,j}(D_j u^L_k + D_k u^L_j, D_i(\zeta^2 g_j) g \cdot \nabla\tilde\sigma_{ik}) \\ + \sum_{i,k,j}(D_k u^L_j, D_i(\zeta^2 g) g \nabla\tilde\sigma_{ik}) = B_1 + B_2\,. \end{aligned}$$

We may pass to the limit to replace $\tilde\sigma$ by σ in B_1. The first term B_1 is rewritten using Euler's equation and we estimate as before in (40) using the regularity of g. This yields

$$B \le K + B' \le \varepsilon_0 \int [\sigma^L_D]^{p-2}_L \zeta^2 |(g \cdot \nabla)\sigma^L|^2\, dx + \varepsilon_0 \int \zeta^2 |g \cdot \nabla \sigma^L|^2\, dx + K + B_2\,.$$

The second term B_2 is estimated performing partial integration for the derivative D_k and we use the fact that $-\sum_{k=1}^n D_k\sigma_{ik} = f_i$. We pass to the limit $\tilde\sigma_{ik} \to \sigma_{ik}$, moving $g\cdot\nabla$ off $\tilde\sigma_{ik}$. Finally, the derivative which reached u in this process, is moved back.This yields an estimate for B_2.

Collecting our results we see that the good terms in (39) are estimated by ε_0 times the same ones plus a constant being uniformly bounded as $L\to\infty$ and $p\to\infty$. Thus the theorem is proved. $\square$

Lemma 1. *Let $u = u(\cdot, L, p)$ and $\sigma = \sigma(\cdot, L, p)$ be the solution of (18), and let g_0 be a Lipschitz-continuous function with support in $U = U(x_0), x_0 \in \partial\Omega\,, U\cap\Gamma_N = \emptyset$. Then we have*

$$\Big|\int g_0 D_i u_k \sigma_{rs}\,dx\Big| \le K\,,\quad \Big|\int g_0 u_k D_i\sigma_{rs}\,dx\Big| \le K\,, \tag{41}$$

uniformly as $p, L\to\infty$.

Proof. We clearly know that

$$\|u_l\|_{L^2}\le K\,,\quad \|\sigma_{rs}\|_{L^2}\le K\,,\quad \int [\sigma_D]_L^{p-2}|\sigma_D|^2\,dx \le K\,, \tag{42}$$

uniformly as $L, p\to\infty$.

a) If $i = k$, then
$$D_i u_k = D_i u_i = (A\sigma)_{ii} + [\sigma_D]_L^{p-2}(\hat\sigma)_{ii}\,,$$
with $\hat\sigma$ defined in (12), and (41) follows from (42), using also (8).

b) Thus we assume $i\ne k$. Without loss of generality we may assume $i = 1\,, k = 2$, since the proof can be repeated by permutation of the indices. We have the cases $(r,s) = (1,1)\,, (r,s) = (1,2)\,,$ and $(r,s) = (2,2)\,.$

For $(r,s) = (1,1)$ we have

$$\int g_0 D_1 u_2\sigma_{11}\,dx = -\int g_0 u_2 D_1\sigma_{11} - \int D_1 g_0 u_2\sigma_{11}\,dx\,.$$

The second summand is bounded due to (42) and the assumption on g_0. The first summand is rewritten as

$$-\int g_0 u_2 D_1\sigma_{11}\,dx = -\int g_0 u_2 f_1\,dx + \int g_0 u_2 D_2\sigma_{21}\,dx = E_1 + E_2\,.$$

E_1 is bounded due to the assumption that $f_1\in L^2$. The term E_2 underlies a partial integration (observe that $u_2 = 0$ on $\partial\Omega$) and we obtain

$$E_2 = -\int g_0 D_2 u_2\sigma_{21}\,dx - \int D_2 g_0 u_2\sigma_{21}\,dx = E_{12} + E_{22}$$

The term E_{12} is estimated as in a), the term E_{22} is bounded obviously due to (42) and the assumption on g_0.

For $(r,s) = (1,2)$ we have

$$\begin{aligned}\int g_0 D_1 u_2\sigma_{12}\,dx &= -\int g_0 u_2 D_1\sigma_{12}\,dx - \int D_1 g_0 u_2\sigma_{12}\,dx\\ &= -\int g_0 u_2 f_2\,dx + \int g_0 u_2 D_2\sigma_{22}\,dx - \int D_1 g_0 u_2\sigma_{12}\,dx\,.\end{aligned}$$

The first the the third summand are obviously bounded, the second is rewritten as

$$E_{13} = -\int g_0 D_2 u_2 \sigma_{22}\, dx\,,$$

plus a bounded term. E_{13} is treated similar as in a).

If $(r,s) = (2,2)$ we write

$$\begin{aligned}\int g_0 D_1 u_2 \sigma_{22}\, dx &= \int g_0 (D_1 u_2 + D_2 u_1)\sigma_{22}\, dx - \int g_0 D_2 u_1 \sigma_{22}\, dx \\ &= \int \Big(g_0 \sigma_{22} [(A\sigma)_{21} + [\sigma_D]_L^{p-2} (\hat{\sigma})_{21}] \Big)\, dx \\ &\quad + \Big\{ \int D_2 g_0 u_1 \sigma_{22}\, dx + \int g_0 u_1 D_2 \sigma_{22}\, dx \Big\}\,.\end{aligned}$$

The first summand is bounded uniformly due to te properties (42) of σ, the second summand analogously. The third summand is rewritten

$$\begin{aligned}\int g_0 u_1 D_2 \sigma_{22}\, dx &= \int g_0 u_1 (f_2 - D_1 \sigma_{21})\, dx \\ &= \int g_0 u_1 f_2\, dx + \int D_1 g_0 u_1 \sigma_{21}\, dx + \int g_0 D_1 u_1 \sigma_{21}\, dx\,.\end{aligned}$$

The first two summands are bounded uniformly as before, the third summand via the argument used in a).

This completes the proof of the lemma

□

For flat boundary, it is not hard to see that the proof of Theorem 5 works also for Neumann zero boundary for σ. We did not analyze the transformed case with curved boundary since Lemma 1 was proven only for Dirichlet boundary.

7. Nikolskii-$\mathcal{H}^{1/2}$-Differentiability of the stresses in normal direction

We consider the simple case of a flat boundary part, without loss of generality an interval I of the hyperplane $(x_2 = 0)$. Let $U \subset \mathbf{R}^2$ be an open set with $I \subset U \cap \partial\Omega$ and $U \cap \Omega \subset (x_2 \geq 0)$. In the case of Dirichlet boundary conditions in $U \cap \partial\Omega$, we prove that the normal derivatives of the stresses, i.e. those in x_2-direction, are contained in the Nikolskii space $\mathcal{H}^{1/2}$. In the case of Neumann boundary we are only able to treat the part where the boundary force vanishes.

We consider the case that

$$I \subset \Gamma_D \quad \text{or} \quad I \subset \Gamma_N \cap (f = 0)\,. \tag{43}$$

Theorem 6. *Let $u = u(p), \sigma = \sigma(p)$ be the solution of the Norton-Hoff problem. Assume the regularity condition (2), $f \in H^{2,2}(\Omega)$, and the safe load condition (15). Then, in the situation (43)*

$$\sup_{h_0 > h > 0} h \int_{U\cap\Omega} |D^h \sigma|^2\, dx \leq K\,,$$

uniformly as $p \to \infty$.

In the classical Hencky case the borderline case 1/2 of the fractional differentiability ([14], [11]) has not been achieved yet, but for the present problem it is possible.

Proof. Let $\zeta \in C_0^\infty(U)$ be a localization such that $\zeta = 1$ in $U_0(I)$, $U_0 \subset\subset U$. In the truncated Norton-Hoff approximation (18) ($L < \infty, p < \infty$) and in the case of Dirichlet boundary conditions on $U \cap (x_2 = 0)$ we choose $\zeta^2 D_2^h \sigma$ as a test function. (Note that $\sigma(x+he_2)$ and $\sigma(x)$ are defined for $x \in \Omega \cap U$.) In the case of Neumann boundary conditions we apply the difference quotient D_2^h to the Norton-Hoff equation and use $\sigma\zeta^2$ as a test function.

(i) Dirichlet case.

At the right hand side R of the resulting equation we rewrite and estimate the integrands

$$\sigma : AD_2^h\sigma = \frac{1}{2}D_2^h(\sigma : A\sigma) - hD_2^h\sigma : AD_2^h\sigma$$
$$\beta'(\sigma_D)D_2^h\sigma_D \le D_2^h\beta(\sigma_D)$$

due to the convexity of β. Thus we obtain

$$\begin{aligned} R &= \int [\zeta^2\sigma : AD_2^h\sigma + \zeta^2\beta'(\sigma_D)D_2^h\sigma_D]\,dx \\ &\le \int \zeta^2 D_2^h(\frac{1}{2}\sigma : A\sigma + \beta(\sigma_D))\,dx - h\int \zeta^2 D_2^h\sigma : AD_2^h\sigma\,dx \\ &= -h\int \zeta^2 D_2^h\sigma : AD_2^h\sigma\,dx - \int_{U\cap(x_2\ge h)} D_2^{-h}\zeta^2(\frac{1}{2}\sigma : A\sigma + \beta(\sigma_D))\,dx \\ &\quad -h^{-1}\int_0^h\int_{M_h}(\frac{1}{2}\sigma : A\sigma + \beta(\sigma_D))\zeta^2\,dx_1dx_2\,, \end{aligned}$$

where

$$M_h := \{x_1 \,|\, \exists x_2 \in (0,h) \text{ such that } (x_1,x_2) \in U\cap\Omega\}\,.$$

Thus, luckily, the boundary term has the same sign as the positively definite term containing A. On the left hand side we obtain

$$\sum_{i,k=1}^2 \int D_iu_kD_2^h\sigma_{ik}\zeta^2\,dx = -\sum_{i,k=1}^2\int u_k[D_2^hf_k\zeta^2 + D_2^h\sigma_{ik}D_i\zeta^2]\,dx\,.$$

In the case of Dirichlet boundary for u no boundary terms occur.

The term with the integrand $u_kD_2^hf_k\zeta^2$ is uniformly bounded for $p\to\infty$ and $0<h<h_0$, since $\|u_k\|_{L^2}$ is bounded and f_k is Lipschitz continuous. The last term is rewritten using

$$D_2^h\sigma_{ik}(x) = \frac{1}{h}\int_0^h D_2\sigma_{ik}(x+e_2t)\,dt =: \fint_0^h D_2\sigma_{ik}(x+e_2t)\,dt\,,$$

(e_2 unit vector in x_2-direction) as

$$\begin{aligned} -\int u_kD_2^h\sigma_{ik}D_i\zeta^2\,dx = \int D_2u_k\fint_0^h \sigma_{ik}(x+e_2t)\zeta^2\,dt\,dx \\ + \int u_k\fint_0^h \sigma_{ik}(x+e_2t)D_2\zeta^2\,dt\,dx\,. \end{aligned} \tag{44}$$

The last term in (44) is uniformly bounded since u_k, σ_{ik} are uniformly bounded in L^2. The remaining term is rewritten

$$\sum_k\int D_2u_k\fint_0^h \sigma_{ik}(x+e_2t)\zeta^2\,dt\,dx =$$

$$\sum_k \int (D_2 u_k + D_k u_2) \fint_0^h \sigma_{ik}(x+e_2 t)\zeta^2 \, dt \, dx$$
$$- \sum_k \int D_k u_2 \fint_0^h \sigma_{ik}(x+e_2 t)\zeta^2 \, dt \, dx \, .$$

Again, the last term is simple to estimate, since we move the derivatives D_k on the left factor

$$\sum_k \int u_2 [D_k \fint_0^h \sigma_{ik}\zeta^2 + \fint_0^h \sigma_{ik} D_k \zeta^2] \, dt \, dx := T_3 \, .$$

Using $\|u\|_{L^2} \le K$, $-\sum_k D_k \sigma_{ik} = f_i$, $\|\sigma_{ik}\|_{L^2} \le K$, we obtain that T_3 is bounded.

So, there remains

$$\begin{aligned} \int (D_2 u_k + D_k u_2) \fint_0^h \sigma_{ik} D_i \zeta^2 \, dt \, dx &= \int (A\sigma)_{ik} \fint_0^h \sigma_{ik} D_i \zeta^2 \, dt \, dx \\ &+ \int |\sigma_D|^{p-2} (\hat{\sigma})_{ik} \fint_0^h \sigma_{ik} D_i \zeta^2 \, dt \, dx \, , \end{aligned}$$

where $\hat{\sigma}$ has been defined in (12). Note that we need to treat only the index $i = 1$ since $D_2\zeta^2$ has compact support in Ω, but this is not important.

We estimate for $i \neq k$

$$\begin{aligned} |\sigma_D|^{p-2} (\hat{\sigma})_{2k} \fint_0^h \sigma_{ik} \, dt &\le K|\sigma_D|^p + K|\sigma_D|^{p-2} \left(\fint_0^h \sigma_{ik} \, dx \right)^2 \\ &\le K|\sigma_D|^p + K \left(\fint_0^h |\sigma_{ik}| \, dt \right)^p \le K|\sigma_D|^p + K \fint_0^h |\sigma_D|^p \, dt \end{aligned}$$

and for $i = k$

$$\begin{aligned} |\sigma_D|^{p-2} (\hat{\sigma})_{2k} \fint_0^h \sigma_{ik} \, dt &\le K|\sigma_D|^p + K|\sigma_D|^{p-2} (\fint_0^h \sigma_{ik} \, dt)^2 \\ &\le K|\sigma_D|^p + K|\sigma_D|^{p-2} \left[(\fint_0^h |(\sigma_D)_{11}| \, dt)^2 + (\fint_0^h |(\sigma_D)_{22} \, dt)^2 \right] \\ &\le K|\sigma_D|^p + K \fint_0^h |(\sigma_D)_{11}|^p \, dt + K \fint_0^h |(\sigma_D)_{22}|^p \, dt \, . \end{aligned}$$

Due to this estimate also the last integral is uniformly bounded and, finally, we have a uniform estimate for the term

$$h \int D_2^h \sigma : A D_2^h \sigma \zeta^2 \, dx + h^{-1} \int_0^h \int_{M_h} (\frac{1}{2}\sigma : A\sigma + \beta(\sigma_D))\zeta^2 \, dx_1 \, .$$

(ii) Neumann zero-boundary

We arrive at the right hand side

$$R = \int [\zeta^2 \sigma : A D_2^h \sigma + \zeta^2 D_2^h \beta'(\sigma_D) : \sigma_D] \, dx$$

and estimate

$$\begin{aligned} h D_2^h \beta'(\sigma_D) : \sigma_D &= |\sigma_D(x+he_2)|^{p-2} \sigma_D(x+he_2) : \sigma_D - |\sigma_D|^p \\ &\le \frac{p-1}{p} |\sigma_D(x+he_2)|^p + \frac{1}{p}|\sigma_D|^p - |\sigma_D|^p \\ &= \frac{p-1}{p} \left[|\sigma_D(x+he_2)|^p - |\sigma_D|^p \right] \, . \end{aligned}$$

The first summand of R containing A is estimated as in the Dirichlet case, so we obtain

$$\begin{aligned} R &\le -h\int \zeta^2 D_2^h\sigma : AD_2^h\sigma\,dx + \int \zeta^2 D_2^h(\frac{1}{2}\sigma : A\sigma + \frac{p-1}{p}|\sigma_D|^p)\,dx \\ &= -h\int \zeta^2 D_2^h\sigma : AD_2^h\sigma\,dx - \int D_2^{-h}\zeta^2(\frac{1}{2}\sigma : A\sigma + \frac{p-1}{p}|\sigma_D|^p)\,dx \\ &\quad -\frac{1}{h}\int_0^h\int_{M_h}(\frac{1}{2}\sigma : A\sigma + \frac{p-1}{p}|\sigma_D|^p)\,dx\,. \end{aligned}$$

So the right hand side can be treated as in the Dirichlet case, the "boundary terms" with $\frac{1}{h}\int_0^h$ fortunately have the correct sign.

The left hand side is rewritten, using partial integration and exploiting the zero Neumann condition

$$\begin{aligned} L &= \sum_{i,k=1}^{2}\int \zeta^2 D_2^h D_i u_k \sigma_{ik}\,dx = -\sum_{i,k=1}^{2}\int D_2^h u_k D_i(\zeta^2\sigma_{ik})), dx \\ &= -\sum_{i,k=1}^{2}\int D_2^h u_k[\zeta^2 f_k + 2\zeta D_i\zeta\sigma_{ik}]\,dx \\ &= -\sum_{i,k=1}^{2}\int \fint_0^h D_2 u_k(x+e_2 t)\,dt[\zeta^2 f_k + 2\zeta D_i\zeta\sigma_{ik}]\,dx \\ &= -\sum_{i,k=1}^{2}\int \fint_0^h (D_2 u_k + D_k u_2)(x+e_2 t)\,dt[\zeta^2 f_k + 2\zeta D_i\zeta\sigma_{ik}]\,dx \\ &\quad +\sum_{i,k=1}^{2}\int \fint_0^h D_k u_2(x+e_2 t)\,dt[\zeta^2 f_k + 2\zeta D_i\zeta\sigma_{ik}]\,dx = B_1 + B_2 \end{aligned}$$

The term B_1 is essentially estimated as in the Dirichlet case; the term $D_2u_k + D_ku_2$ has to be expressed by the right hand side of the Euler equation.

For the term B_2 we have two cases. For the index $k = 1$ we may move the derivative D_1 off u_2 via partial integration. No boundary term occurs since D_2 is tangential. The resulting product

$$\fint_0^h u_k\,dt D_1[\zeta^2 f + 2\zeta D_i\zeta\sigma_{ik}]$$

can be estimated uniformly since we have estimates for u_k, ∇f, $D_1\sigma$ in L^2.

In the case $k = 2$ we represent D_2u_2 via the Euler equation by

$$(A\sigma)_{22} + |\sigma_D|^{p-1}\hat{\sigma}_{22}$$

and proceed as in the case $k = 1$.

The theorem is proved. $\square$

Remark: It is of interest that the proof gives also an estimate for $|\text{trace }\sigma|^2$ and the penalty term at the boundary.

References

[1] Anzellotti, G., Giaquinta, M.: Existence of the displacements field for an elasto-plastic body subject to Hencky's law and von Mises yield condition. manuscripta mathematica **32**, 101–136 (1980)

[2] Anzellotti, G., Giaquinta, M.: On the existence of the fields of stresses and displacements for an elasto-perfectly plastic body in static equilibrium. Journal de Mathématiques Pures et Appliquées **61**, 219–244 (1982)

[3] Bensoussan, A., Frehse, J.: Asymptotic behaviour of Norton-Hoff's law in plasticity theory and H^1 regularity. In: P.G. Ciarlet, J.L. Lions (eds.), Boundary Value Problems for PDE and Applications (Volume in honor of Prof. E. Magenes), RMA Res. Notes in Appl. Math. **29**, 3–25, Masson, Paris (1993)

[4] Bensoussan, A., Frehse J.: Regularity Results for Nonlinear Elliptic Systems and Applications. Springer Berlin, Applied Mathematical Sciences **151** (2002)

[5] Bildhauer, M., Fuchs, M.: Smoothness of weak solutions of the Ramberg/Osgood equations on plane domains. ZAMM **87(1)**, 70–76 (2007)

[6] Bulicek, M., Frehse, J., Malek, J., On boundary regularity of models of perfect elastoplasticity. to appear

[7] Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer Berlin-Heidelberg, Grundlehren der mathematischen Wissenschaften **219** (1976)

[8] Frehse, J., Málek, J.: Boundary regularity results for models of elasto-perfect plasticity. Math. Models Methods Appl. Sci. **9(9)**, 1307–1321 (1999)

[9] Hardt, R.M., Kinderlehrer, D.: Elastic plastic deformation. Appl. Math. Optim. **10**, 203–246 (1983)

[10] Hardt, R.M., Kinderlehrer, D.: Some regularity results in plasticity. Proc. of Symposia in Pure Math. **44**, 239–244 (1986)

[11] Knees, D.: Global regularity of the elastic fields of a power-law model on Lipschitz domains. Math. Methods Appl. Sci. **29(12)**, 1363–1391 (2006)

[12] Kohn, R., Temam, R.: Dual spaces of stresses and strains, with applications to Hencky plasticity. Appl. Math. Opt. **10**, 1–35 (1983)

[13] Löbach, D.: Interior stress regularity for the Prandtl Reuss and Hencky model of perfect plasticity using the Perzyna approximation, Bonner Math. Schr. **386**, Univ. Bonn (2007)

[14] Repin, S., Seregin, G.: Error estimates for stresses in the finite element analysis of the two-dimensional elasto-plastic problems. Int. J. Eng. Sci. **33(2)**, 255–268 (1995)

[15] Seregin, G.A.: On differential properties of extremals of variational problems arising in the theory of plasticity. Differ. Equations **26(6)**, 756–766 (1990)

[16] Seregin, G.A.: Remarks on regularity up to the boundary for solutions to variational problems in plasticity theory. J. Math. Sci. **93(5)**, 779–781 (1999)

[17] Steinhauer, M., On analysis of some nonlinear systems of partial differential equations of continuum mechanics, Bonner Math. Schr. **359**, Univ. Bonn (2003)

[18] Temam, R.: Mathematical Problems in Plasticity. Gauthier-Villars, Paris (1984)

Appendix

We present a proof, that the Euler equation (23) of the approximate dual variational problem (18), with L fixed, together with the condition $-\mathrm{div}\sigma = f$, leads to a uniformly elliptic equation.

Let M^n be the space of symmetric $n \times n$-matrices and $M_0^3 \subset M^3$ the space of matrices (m_{ik}) satisfying $m_{i3} = 0\,,\ i = 1, 2, 3$. For $m \in M^2$ define $m_D \in M^3$ by

$$m_D = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{3}\mathrm{tr(m)}\mathbf{I}\,.$$

Let $y \in M^2\,,\ \sigma \in M_0^3$ satisfy

$$\tau : y = \tau : A\sigma + [\sigma_D^L]_L^{p-2}\sigma_D : \tau_D\,, \quad \forall \tau \in M^2\,. \tag{45}$$

Then there exists a globally Lipschitz continuous function g such that

$$\begin{aligned} &\sigma = g(y) \\ &c|y| \le |g(y)| \le K|y|\,, \quad \big(g(y) - g(z)\big) : (y - z) \ge c_0|y - z|^2\,,\ y, z \in M^2\,. \end{aligned} \tag{46}$$

Proof. We can write the right hand side of (45) in the form $G(\sigma') = y$, where

$$\begin{pmatrix} \sigma' & 0 \\ 0 & 0 \end{pmatrix} = \sigma \in M_0^3\,,$$

so $G : M^2 \to M^2$. The mapping G is globally Lipschitz continuous and satisfies

$$G(\sigma') : \sigma' = (\sigma, A\sigma) + [\sigma_D^L]_L^{p-2}|\sigma_D|^2 \ge c_0|\sigma_D|^2 \ge c_1|\sigma'|^2\,,$$

hence

$$|G(\sigma')| \ge c_1|\sigma'|\,.$$

Furthermore, $|G(\sigma')| \le k|\sigma'|$ and there holds the monotonicity condition

$$\big(G(\sigma') - G(\tau')\big) : (\sigma' - \tau') \ge c|\sigma' - \tau'|^2\,.$$

Hence there is an inverse mapping $g : M^2 \to M^2$ such that

$$g(G(\sigma')) = \sigma'\,, \quad G(g(y)) = y\,.$$

Due to the above proved properties we have

$$c_0|y| \le g(y) \le c_1|y|\,,$$

and from the monotonicity property of G we conclude with $\sigma' = g(y)\,,\ \tau' = g(z)$,

$$(y - z) : \big(g(y) - g(z)\big) \ge c_0|g(y) - g(z)|^2\,. \tag{47}$$

From the global Lipschitz property of G we conclude

$$|G(\sigma') - G(\tau')|^2 \le K|\sigma' - \tau'|^2\,,$$

hence

$$|y-z|^2 \leq K|g(y)-g(z)|^2\,. \tag{48}$$

From (47) and (48) we obtain (46). □

Thus we see that the dual problem

$$\frac{\nabla u + \nabla u^T}{2} = G(\sigma')$$

leads to

$$\sigma' = g\left(\frac{\nabla u + \nabla u^T}{2}\right)$$

and hence to the elliptic equation

$$-\mathrm{div}\left(g\left(\frac{\nabla u + \nabla u^T}{2}\right)\right) = f\,. \tag{49}$$

This proves the equivalence of the dual problem (23) to (49) and allows us to apply standard elliptic theory.

Bestellungen nimmt entgegen:

Sonderforschungsbereich 611
der Universität Bonn
Poppelsdorfer Allee 82
D - 53115 Bonn

Telefon: 0228/73 4882
Telefax: 0228/73 7864
E-mail: astrid.link@iam.uni-bonn.de http://www.sfb611.iam.uni-bonn.de/

Verzeichnis der erschienenen Preprints ab No. 415

415. Weber, Hendrik: On the Short Time Asymptotic of the Stochastic Allen-Cahn Equation

416. Albeverio, Sergio; Ayupov, Shavkat A.; Kudaybergenov, Karim K.; Kalandarov, T. S.: Complete Description of Derivations on τ-compact Operators for Type I von Neumann Algebras

417. Bartels, Sören: Semi-Implicit Approximation of Wave Maps into Smooth or Convex Surfaces

418. Albeverio, Sergio; Ayupov, Shavkat A.; Kudaybergenov, Karim K.: Structure of Derivations on Various Algebras of Measurable Operators for Type I von Neumann Algebras

419. Albeverio, Sergio; Ayupov, Shavkat A.; Abdullaev, Rustam Z.: On an Algebra of Operators Related to Finite Traces on a von Neumann Algebra

420. Otto, Felix; Viehmann, Thomas: Domain Branching in Uniaxial Ferromagnets – Asymptotic Behavior of the Energy

421. Kurzke, Matthias: Compactness Results for Ginzburg-Landau Type Functionals with General Potentials

422. Hildebrandt, Stefan; von der Mosel, Heiko: Conformal Mapping of Multiply Connected Riemann Domains by a Variational Approach

423. Harbrecht, Helmut: On Output Functionals of Boundary Value Problems on Stochastic Domains

424. Griebel, Michael; Hegland, Markus: On the Numerical Determination of Maximum A-Posteriori Density Estimators Based on Exponential Families and Gaussian Process Priors with an Infinite Number of Parameters

425. Capella, Antonio; Otto, Felix: A Rigidity Result for a Perturbation of the Geometrically Linear Three-Well Problem

426. Albeverio, Sergio; Motovilov, Alexander K.; Shkalikov, Andrei A.: Bounds on Variation of Spectral Subspaces under J-Self-Adjoint Perturbations

427. Albeverio, Sergio; Günther, Uwe; Kuzhel, Sergii: J-Self-Adjoint Operators with C-Symmetries: Extension Theory Approach

428. Kurzke, Matthias; Melcher, Christof; Moser, Roger; Spirn, Daniel: Dynamics for Ginzburg-Landau Vortices under a Mixed Flow

429. Otto, Felix; Steiner, Jutta: The Concertina Pattern – From Micromagnetics to Domain Theory

430. Frehse, Jens; Málek, Josef; Ružička, Michael: Large Data Existence Result for Unsteady Flows of Inhomogeneous Heat-Conducting Incompressible Fluids

431. Croce, Roberto; Griebel, Michael; Schweitzer, Marc Alexander: Numerical Simulation of Bubble and Droplet Deformation by a Level Set Approach with Surface Tension in Three Dimensions

432. Frehse, Jens; Löbach, Dominique: Regularity Results for Three Dimensional Isotropic and Kinematic Hardening Including Boundary Differentiability

433. Arguin, Louis-Pierre; Kistler, Nicola: Small Perturbations of a Spin Glass System

434. Bolthausen, Erwin; Kistler, Nicola: On a Nonhierarchical Version of the Generalized Random Energy Model. II. Ultrametricity

435. Blum, Heribert; Frehse, Jens: Boundary Differentiability for the Solution to Hencky's Law of Elastic Plastic Plane Stress