Local Derivations on Algebras
of Measurable Operators

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Abstract

The paper is devoted to local derivations on the algebra $S(\mathcal{M}, \tau)$ of τ-measurable operators affiliated with a von Neumann algebra $\mathcal{M}$ and a faithful normal semi-finite trace τ. We prove that every local derivation on $S(\mathcal{M}, \tau)$ which is continuous in the measure topology, is in fact a derivation. In the particular case of type I von Neumann algebras they all are inner derivations. It is proved that for type I finite von Neumann algebras without an abelian direct summand, and also for von Neumann algebras with the atomic lattice of projections, the condition of continuity of the local derivation is redundant. Finally we give necessary and sufficient conditions on a commutative von Neumann algebra $\mathcal{M}$ for the existence of local derivations which are not derivations on algebras of measurable operators affiliated with $\mathcal{M}$.

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1. Introduction

The study of derivations on algebras of unbounded operators and in particular on algebras of measurable operators affiliated with von Neumann algebras is one of the most attractive parts of the general theory of unbounded derivations on operator algebras.

Given an algebra A, a linear operator $D : A \to A$ is called a *derivation*, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (the Leibniz rule). Each element $a \in A$ implements a derivation D_a on A defined as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations*. If the element a, implementing the derivation D_a, belongs to a larger algebra B containing A, then D_a is called *a spatial* derivation on A.

If the algebra A is commutative, then it is clear that all inner derivations are trivial, i.e. identically zero. One of the main problem concerning derivations is to prove that a given derivation is inner or spatial, or to show the existence of non inner (resp. non spatial) derivations and in particular non zero derivations in the commutative cases.

In the paper [5] A. F. Ber, V. I. Chilin and F. A. Sukochev obtained necessary and sufficient conditions for the existence of non trivial derivations on regular commutative algebras. In particular it was proved that the algebra $L^0(0, 1)$ of all (equivalence classes of) complex measurable function on the $(0, 1)$ interval admits non trivial derivations. Independently A. G. Kusraev [12] by the methods of Boolean analysis gave necessary and sufficient conditions for the existence of non trivial derivations and automorphisms in extended f-algebras. In particular he also has proved the existence of non trivial derivations and automorphisms on the algebra $L^0(0, 1)$. It is clear that such derivations are discontinuous and non inner. We have conjectured in [1]-[3] that the existence of such "pathological" examples of derivations is closely connected with the commutative nature of these algebras. This was confirmed in the particular case of type I von Neumann algebras. Namely in [1]-[3] we have investigated and complety described derivations on the algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with a type I von Neumann algebra $\mathcal{M}$ and on its various subalgebras. Recently the above conjecture was also confirmed for the type I case in the paper [6] by a representation of measurable operators as operator valued functions. Another approach to similar problems in AW^*-algebras of type I was suggested in the recent paper [10].

In the paper [3] we have proved have the spatiality of derivations of the non com-

mutative Arens algebra $L^\omega(\mathcal{M}, \tau)$ associated with an arbitrary von Neumann algebra $\mathcal{M}$ and a faithful normal semi-finite trace τ. Moreover if the trace τ is finite then every derivation on $L^\omega(\mathcal{M}, \tau)$ is inner.

There exist various types of linear operators which are close to derivations [8],[9],[11],[13]. In particular R. Kadison [11] has introduced and investigated so called local derivations on von Neumann algebras and some polynomial algebras.

A linear operator Δ on an algebra A is called a *local derivation* if given any $x \in A$ there exists a derivation D (depending on x) such that $\Delta(x) = D(x)$. The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations [11] , [13]. In particular Kadison [11] has proved that each continuous local derivation from a von Neumann algebra $\mathcal{M}$ into a dual $\mathcal{M}$-bimodule is a derivation. Later this result was extended in [8] to a larger class of linear operators Δ from $\mathcal{M}$ into a normed $\mathcal{M}$-bimodule E satisfying the identity

$$\Delta(p) = \Delta(p)p + p\Delta(p) \qquad (1)$$

for every idempotent $p \in \mathcal{M}$.

It is clear that each local derivation satisfies (1) since given any idempotent $p \in \mathcal{M}$ we have $\Delta(p) = D(p) = D(p^2) = D(p)p + pD(p) = \Delta(p)p + p\Delta(p)$.

In [9] it was proved that every linear operator Δ on the algebra $M_n(R)$ satisfying (1) is automatically a derivation, where $M_n(R)$ is the algebra of $n \times n$ matrices over a unital ring R containing $1/2$.

The present paper is devoted to the study of local derivations on the algebra $S(\mathcal{M}, \tau)$ of all τ-measurable operators affiliated with a von Neumann algebra $\mathcal{M}$ and a faithful normal semi-finite trace τ. The main result (Theorem 2.2) presents an unbounded version of Kadison's result and it asserts that every local derivation on $S(\mathcal{M}, \tau)$ which is continuous in the measure topology automatically becomes a derivation. In particular in the case of type I von Neumann algebra $\mathcal{M}$ all such local derivations on $S(\mathcal{M}, \tau)$ are inner derivations (Corollary 2.3). We prove also that for type I finite von Neumann algebras without abelian direct summands as well as for von Neumann algebras with the atomic lattice of projections, the continuity condition on local derivations in Theorem 2.2 is redundant (Theorem 2.5 and Proposition 2.7 respectively).

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In section 3 we consider the problem of existence of local derivations which are not derivations on a class of commutative regular algebras, which include the algebras of measurable functions on a finite measure space (Theorem 3.5). As a corollary we obtain necessary and sufficient conditions for the existence of local derivation which are not derivations on algebras of measurable and τ-measurable operators affiliated with a commutative von Neumann algebra (Theorem 3.8).

2. Continuous local derivations on the algebra $S(\mathcal{M}, \tau)$

Let H be a Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Consider a von Neumann algebra $\mathcal{M}$ in $B(H)$ with a faithful normal semi-finite trace τ. Denote by $P(\mathcal{M})$ the lattice of projections from $\mathcal{M}$.

Recall that a linear subspace $\mathcal{D}$ in H is said to be *affiliated* with $\mathcal{M}$ (and denoted $\mathcal{D}\eta\mathcal{M}$) if $u(\mathcal{D}) \subseteq \mathcal{D}$ for each unitary u from the commutant $\mathcal{M}' = \{y \in B(H) : xy = yx, \forall x \in \mathcal{M}\}$ of the algebra $\mathcal{M}$.

A linear operator x acting in H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with $\mathcal{M}$ (denoted $x\eta\mathcal{M}$) if $\mathcal{D}(x)\eta\mathcal{M}$ and $ux(\xi) = xu(\xi)$ for all $u \in \mathcal{M}'$ and $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra $\mathcal{M}$, if $\mathcal{D}\eta\mathcal{M}$ and there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} = \mathbf{1} - p_n$ is a finite projection in $\mathcal{M}$ for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in $\mathcal{M}$.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to $\mathcal{M}$ if $x\eta\mathcal{M}$ and its domain $\mathcal{D}(x)$ is strongly dense in H.

A linear subspace $\mathcal{D}$ in H is called τ-*dense* in H if $\mathcal{D}\eta\mathcal{M}$ and given any $\varepsilon > 0$ there exists a projection $p \in \mathcal{M}$ such that $p(H) \subset \mathcal{D}$ and $\tau(p^{\perp}) < \varepsilon$.

A linear operator x with the domain $\mathcal{D}(x) \subset H$ is said to be τ-*measurable* with respect to $\mathcal{M}$ if $x\eta\mathcal{M}$ and its domain $\mathcal{D}(x)$ is τ-dense in H.

Denote by $S(\mathcal{M})$ and $S(\mathcal{M}, \tau)$ respectively the sets of all measurable and τ-measurable operators affiliated with $\mathcal{M}$ and consider on $S(\mathcal{M}, \tau)$ the topology of convergence in measure (or briefly *measure topology*) t_{τ} which is given by the following family of neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(\mathcal{M}, \tau) : \exists e \in P(\mathcal{M}) \mid \tau(e^{\perp}) \leq \delta, xe \in \mathcal{M}, \|xe\|_{\mathcal{M}} \leq \varepsilon\},$$

where ε, δ are positive numbers.

It is well-known [14] that $S(\mathcal{M}, \tau)$ is a complete metrizable topological $*$-algebra with respect to the measure topology t_τ.

Lemma 2.1. *The algebra $S(\mathcal{M}, \tau)$ is semiprime, i.e. $aS(\mathcal{M}, \tau)a = \{0\}$ for $a \in S(\mathcal{M}, \tau)$ implies $a = 0$.*

Proof. Let $a \in S(\mathcal{M}, \tau)$ and $aS(\mathcal{M}, \tau)a = \{0\}$, i.e. $axa = 0$ for all $x \in S(\mathcal{M}, \tau)$. In particular for $x = a^*$ we have $aa^*a = 0$ and hence $a^*aa^*a = 0$, i.e. $|a|^4 = 0$. Therefore $a = 0$. The proof is complete. ∎

We are now in position to prove the main result of this section.

Theorem 2.2. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semifinite trace τ. Then every t_τ-continuous linear operator Δ on the algebra $S(\mathcal{M}, \tau)$ satisfying the identity (1) is a derivation on $S(\mathcal{M}, \tau)$. In particular any t_τ-continuous local derivation is a derivation on the algebra $S(\mathcal{M}, \tau)$.*

Proof. Given two orthogonal projections $p, \ q \in P(\mathcal{M})$ we have

$$\Delta(p) + \Delta(q) = \Delta((p + q)^2) = \Delta(p + q)(p + q) + (p + q)\Delta(p + q) =$$

$$= \Big[\Delta(p)p + p\Delta(p)\Big] + \Big[\Delta(q)q + q\Delta(q)\Big] + \Big[\Delta(p)q + p\Delta(q) + \Delta(q)p + q\Delta(p)\Big] =$$

$$= \Delta(p) + \Delta(q) + \Big[\Delta(p)q + p\Delta(q) + \Delta(q)p + q\Delta(p)\Big].$$

Therefore

$$\Delta(p)q + p\Delta(q) + \Delta(q)p + q\Delta(p) = 0. \tag{2}$$

Set $D_{P(\mathcal{M})} = \left\{ \sum_{k=1}^{n} \alpha_k p_k : \alpha_k \in \mathbb{R}, p_k \in P(\mathcal{M}), p_k p_l = 0, k \neq l, k, l = \overline{1, n}, n \in \mathbb{N} \right\}$.

For $x = \sum_{i=1}^{n} \alpha_i p_i \in D_{P(\mathcal{M})}$ we have

$$\Delta(x^2) = \Delta\left(\left(\sum_{i=1}^{n} \alpha_i p_i\right)^2\right) = \Delta\left(\sum_{i=1}^{n} \alpha_i^2 p_i\right) = \sum_{i=1}^{n} \alpha_i^2 \Delta(p_i),$$

i.e.

$$\Delta(x^2) = \sum_{i=1}^{n} \alpha_i^2 \Delta(p_i). \tag{3}$$

Further we have

$$\Delta(x)x + x\Delta(x) = \Delta\left(\sum_{i=1}^{n} \alpha_i p_i\right)\left(\sum_{i=1}^{n} \alpha_i p_i\right) + \left(\sum_{i=1}^{n} \alpha_i p_i\right)\Delta\left(\sum_{i=1}^{n} \alpha_i p_i\right) =$$

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$$= \left(\sum_{i=1}^{n} \alpha_i^2 \Delta(p_i) p_i + \sum_{i \neq j} \alpha_i \alpha_j \Delta(p_i) p_j \right) + \left(\sum_{i=1}^{n} \alpha_i^2 p_i \Delta(p_i) + \sum_{i \neq j} \alpha_i \alpha_j p_i \Delta(p_j) \right) =$$

$$= \sum_{i=1}^{n} \alpha_i^2 \Big[\Delta(p_i) p_i + p_i \Delta(p_i) \Big] + \sum_{i \neq j} \alpha_i \alpha_j \Big[\Delta(p_i) p_j + p_i \Delta(p_j) \Big] =$$

$$= \sum_{i=1}^{n} \alpha_i^2 \Delta(p_i) + \sum_{i<j} \alpha_i \alpha_j \Big[\Delta(p_i) p_j + p_i \Delta(p_j) + \Delta(p_j) p_i + p_j \Delta(p_i) \Big],$$

and thus

$$\Delta(x)x + x\Delta(x) = \sum_{i=1}^{n} \alpha_i^2 \Delta(p_i) + \sum_{i<j} \alpha_i \alpha_j \Big[\Delta(p_i) p_j + p_i \Delta(p_j) + \Delta(p_j) p_i + p_j \Delta(p_i) \Big]. \quad (4)$$

Now (2) and (4) imply

$$\Delta(x)x + x\Delta(x) = \sum_{i=1}^{n} \alpha_i^2 \Delta(p_i) \quad (5)$$

and therefore from (3) and (5) we obtain that $\Delta(x^2) = \Delta(x)x + x\Delta(x)$ for all $x \in D_{P(\mathcal{M})}$. Since the set $D_{P(\mathcal{M})}$ is t_τ-dense in $S(\mathcal{M}, \tau)_{sa}$ and the operator Δ is t_τ-continuous, we have that $\Delta(x^2) = \Delta(x)x + x\Delta(x)$ for all $x \in S(\mathcal{M}, \tau)_{sa}$ (the self-adjoint operators from $S(\mathcal{M}, \tau)$).

Now let us show that this relation is valid for arbitrary operators from $S(\mathcal{M}, \tau)$. Consider $x \in S(\mathcal{M}, \tau)$ and let $x = x_1 + ix_2$, where $x_1, x_2 \in S(\mathcal{M}, \tau)_{sa}$. The identity

$$x_1 x_2 + x_2 x_1 = (x_1 + x_2)^2 - x_1^2 - x_2^2,$$

implies that

$$\Delta(x_1 x_2 + x_2 x_1) = \Delta(x_1)x_2 + x_1\Delta(x_2) + \Delta(x_2)x_1 + x_2\Delta(x_1).$$

Therefore

$$\Delta(x^2) = \Delta((x_1 + ix_2)^2) = \Delta(x_1^2) + i\Delta(x_1 x_2 + x_2 x_1) - \Delta(x_2^2),$$

i.e.

$$\Delta(x^2) = \Delta(x_1^2) + i\Delta(x_1 x_2 + x_2 x_1) - \Delta(x_2^2).$$

Further we have

$$\Delta(x)x + x\Delta(x) = \Delta(x_1 + ix_2)(x_1 + ix_2) + (x_1 + ix_2)\Delta(x_1 + ix_2) =$$

$$= \Big[\Delta(x_1)x_1 + x_1\Delta(x_1) \Big] + i\Big[\Delta(x_1)x_2 + x_1\Delta(x_2) + \Delta(x_2)x_1 + x_2\Delta(x_1) \Big] - \Big[\Delta(x_2)x_2 + x_2\Delta(x_2) \Big] =$$

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$$= \Delta(x_1^2) + i\Delta(x_1 x_2 + x_2 x_1) - \Delta(x_2^2),$$

i.e.

$$\Delta(x)x + x\Delta(x) = \Delta(x_1^2) + i\Delta(x_1 x_2 + x_2 x_1) - \Delta(x_2^2).$$

Comparing this relation with the above one we obtain

$$\Delta(x^2) = \Delta(x)x + x\Delta(x).$$

This means that Δ is a Jordan derivation on $S(\mathcal{M}, \tau)$ in the sense of [7]. In [7, Theorem 1] it is proved that any Jordan derivation on a semiprime algebra is a (associative) derivation. Thus Lemma 2.1 implies that the linear operator Δ is a derivation on $S(\mathcal{M}, \tau)$. The proof is complete. ∎

For type I von Neumann algebras the above result can be strengthened as follows

Corollary 2.3. *Let $\mathcal{M}$ be a type I von Neumann algebra with a faithful normal semi-finite trace τ. Then every t_τ-continuous linear operator Δ on $S(\mathcal{M}, \tau)$ satisfying* (1) (*in particular local derivation*) *is an inner derivation.*

Proof. By Theorem 2.2 Δ is a derivation on $S(\mathcal{M}, \tau)$. By [2, Corollary 4.5] every t_τ-continuous derivation on $S(\mathcal{M}, \tau)$, with $\mathcal{M}$ of type I, is inner. Therefore Δ is an inner derivation on $S(\mathcal{M}, \tau)$. The proof is complete. ∎

Now let us show that for finite von Neumann algebras of type I without abelian direct summands the assertion of Corollary 2.3 is valid for local derivations Δ without assumption of t_τ-continuity.

Let $\mathcal{M}$ be a homogeneous von Neumann algebra of type $I_n, n \in \mathbb{N}$, with the center Z and with a faithful normal semi-finite trace τ. In this case $\mathcal{M}$ is $*$-isomorphic to the algebra $M_n(Z)$ of all $n \times n$ matrices over Z, and the algebra $S(\mathcal{M}, \tau)$ is $*$-isomorphic to the algebra $M_n(S(Z, \tau_Z))$ of all $n \times n$ matrices over the commutative algebra $S(Z, \tau_Z)$ of τ_Z-measurable operators with respect to Z, where τ_Z is the restriction of the trace τ onto Z. If $\{e_{i,j}, \ i, j = \overline{1, n}\}$ is the set of matrix units in $M_n(S(Z, \tau_Z))$ then each element $x \in M_n(S(Z, \tau_Z))$ is represented as

$$x = \sum_{i,j=1}^n \lambda_{i,j} e_{i,j}, \ \lambda_{i,j} \in S(Z, \tau_Z), \ i, j = \overline{1, n}.$$

Let $\delta : S(Z, \tau_Z) \to S(Z, \tau_Z)$ be a derivation. Setting

$$D_\delta(\sum_{i,j=1}^n \lambda_{i,j} e_{i,j}) = \sum_{i,j=1}^n \delta(\lambda_{i,j}) e_{i,j} \tag{6}$$

we obtain a linear operator D_δ on $M_n(S(Z, \tau_Z))$, which is a derivation on the algebra $M_n(S(Z, \tau_Z))$ (see [2]).

Now if $\mathcal{M}$ is an arbitrary finite von Neumann algebra of type I with the center Z, then there exists a family $\{z_n\}_{n \in F}$, $F \subseteq \mathbb{N}$, of orthogonal central projections in $\mathcal{M}$ such that $\sup_{n \in F} z_n = \mathbf{1}$ and $\mathcal{M}$ is $*$-isomorphic to the C^*-product of homogeneous von Neumann algebras $z_n \mathcal{M}$ of type I_n respectively $n \in F$, i.e.

$$\mathcal{M} \cong \bigoplus_{n \in F} z_n \mathcal{M}.$$

This implies that the algebra $S(\mathcal{M}, \tau)$ can be embedded as a subalgebra of the direct product of the algebras $S(z_n \mathcal{M}, \tau_n)$, where τ_n is the restriction of the trace τ onto the algebra $z_n \mathcal{M}$, $n \in F$ (see for details [2, Section 4]).

Consider a derivation D on the algebra $S(\mathcal{M}, \tau)$ and denote by δ its restriction on the center $S(Z, \tau_Z)$ of the algebra $S(\mathcal{M}, \tau)$. Then δ maps each $z_n S(Z, \tau_Z) \cong Z(S(z_n \mathcal{M}, \tau_n))$ into itself and hence it induces a derivation δ_n on $z_n S(Z, \tau_Z)$ for each $n \in F$.

Define as in (6) the derivation D_{δ_n} on the matrix algebra $M_n(z_n Z(S(\mathcal{M}, \tau))) \cong S(z_n \mathcal{M}, \tau_n)$ for each $n \in F$. Put

$$D_\delta(\{x_n\}_{n \in F}) = \{D_{\delta_n}(x_n)\}, \ \{x_n\}_{n \in F} \in S(\mathcal{M}, \tau). \tag{7}$$

In [2] it is proved that D_δ is a derivation on the algebra $S(\mathcal{M}, \tau)$, which is restricted to the center of $S(\mathcal{M}, \tau)$ coincides with δ (and thus with D). In [2, Lemma 4.3] it has been proved that an arbitrary derivation D on the algebra $S(\mathcal{M}, \tau)$ for the finite type I von Neumann algebra $\mathcal{M}$ can be uniquely decomposed into the sum

$$D = D_a + D_\delta \tag{8}$$

where D_a is an inner derivation on $S(\mathcal{M}, \tau)$ implemented by an element $a \in S(\mathcal{M}, \tau)$ and D_δ is the derivation defined as in (7).

Further we shall need the following technical result.

Lemma 2.4. *Every local derivation Δ on the algebra $S(\mathcal{M}, \tau)$ is necessarily $P(Z)$-homogeneous, i.e.*

$$\Delta(zx) = z\Delta(x)$$

for any central projections $z \in P(Z) = P(\mathcal{M}) \cap Z$, and for all $x \in S(\mathcal{M}, \tau)$.

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Proof. Take $z \in P(Z)$ and $x \in S(\mathcal{M}, \tau)$. For the element zx by the definition of the local derivation Δ there exist a derivation D on $S(\mathcal{M}, \tau)$ such that $\Delta(zx) = D(zx)$. Since the projection z is central one has $D(z) = 0$ and therefore

$$\Delta(zx) = D(zx) = D(z)x + zD(x) = zD(x),$$

i.e. $\Delta(zx) = zD(x)$. Multiplying by z we obtain

$$z\Delta(zx) = z^2 D(x) = zD(x) = \Delta(zx)$$

i.e.

$$z^\perp \Delta(zx) = (\mathbf{1} - z)\Delta(zx) = 0.$$

Therefore by the linearity of Δ we have $z\Delta(x) = z\Delta(zx) + z\Delta(z^\perp x) = z\Delta(zx) = \Delta(zx)$ that is $z\Delta(x) = \Delta(zx)$. The proof is complete. $\blacksquare$

Theorem 2.5. *Let $\mathcal{M}$ be a finite von Neumann algebra of type I without abelian direct summands and let τ be a faithful normal semi-finite trace on $\mathcal{M}$. Then every local derivation Δ on the algebra $S(\mathcal{M}, \tau)$ is a derivation and hence can be represented as (8).*

Proof. Let $\{z_n\}_{n \in F}$, $F \subseteq \mathbb{N}$ be the family of orthogonal central projections in $\mathcal{M}$ with $\sup_{n \in F} z_n = \mathbf{1}$, such that $z_n \mathcal{M}$ is a homogeneous von Neumann algebra of type $\mathrm{I}_n, n \in F$. Since $\mathcal{M}$ does not have an abelian direct summand we have that $1 \notin F$.

Consider an arbitrary local derivation Δ on $S(\mathcal{M}, \tau)$. By Lemma 2.4 we have that

$$\Delta(z_n x) = z_n \Delta(x)$$

for each $n \in F$. This implies that Δ maps each $z_n S(\mathcal{M}, \tau) = S(z_n \mathcal{M}, \tau_n)$ into itself and hence induces a local derivation $\Delta_n = \Delta|_{z_n S(\mathcal{M}, \tau)}$ on the algebra $S(z_n \mathcal{M}, \tau_n) \cong M_n(Z(S(z_n \mathcal{M}, \tau_n)))$ for each $n \in F$. Since $n \neq 1$, [9, Theorem 2.3] implies that the operator Δ_n on the matrix algebra $M_n(Z(S(z_n \mathcal{M}, \tau_n)))$ is a derivation. Therefore $\Delta = \{\Delta_n\}_{n \in F}$ is also a derivation and by [2, Lemma 4.3] can be uniquely represented in form (8). The proof is complete. $\blacksquare$

Remark 2.6. In the latter theorem the condition on $\mathcal{M}$ to have no abelian direct summand is crucial, because in the case of abelian von Neumann algebras the picture is

completely different. Local derivations on algebras of τ-measurable operators affiliated with abelian von Neumann algebras will be considered in the next section.

Now let $\mathcal{M}$ be a von Neumann algebra with the atomic lattice of projections and with a faithful normal semi-finite trace τ. Then the von Neumann algebra $\mathcal{M}$ is a direct sum of type I_α factors $\mathcal{M}_\alpha = B(H_\alpha)$, where H_α is a Hilbert space with $\dim H_\alpha = \alpha$ and the algebra $S(\mathcal{M}, \tau)$ can be embedded as a subalgebra of the direct product of the algebras $S(\mathcal{M}_\alpha, \tau_\alpha)$, where τ_α is the restriction of the trace τ onto the algebra $\mathcal{M}_\alpha$. As in Theorem 2.5 by Lemma 2.4 we have that every local derivation maps each direct summand $S(\mathcal{M}_\alpha, \tau_\alpha) = B(H_\alpha)$ into itself. Therefore we obtain from [9, Corollary 3.8] (see also [13]) that every local derivation on the algebra $S(\mathcal{M}, \tau)$ is a derivation and hence by [1, Corollary 3.11] and [2, Theorem 4.4] is inner derivation.

Thus, we have proved the following result.

Proposition 2.7. *If M is a von Neumann algebra with the atomic lattice of projections and with a faithful normal semi-finite trace τ, then every local derivation on the algebra $S(\mathcal{M}, \tau)$ is inner derivation.*

3. Local derivations on commutative regular algebras

In this section we shall discuss the problem of existence of local derivations which are not derivations on the algebras $S(\mathcal{M})$ and $S(\mathcal{M}, \tau)$ in the case where the von Neumann algebra $\mathcal{M}$ is commutative. Following the approach of the paper [5] we shall consider this problem in a more general setting – on commutative regular algebras.

Let A be a commutative algebra with the unit $\mathbf{1}$ over the field $\mathbb{C}$ of complex numbers. We denote by ∇ the set $\{e \in A : e^2 = e\}$ of all idempotents in A. For $e, f \in \nabla$ we set $e \leq f$ if $ef = e$. With respect to this partial order, the lattice operation $e \vee f = e + f - ef$, $e \wedge f = ef$ and the complement $e^\perp = \mathbf{1} - e$, the set ∇ forms a Boolean algebra. A non zero element q from the Boolean algebra ∇ is called an *atom* if $0 \neq e \leq q$, $e \in \nabla$, imply that $e = q$. If given any nonzero $e \in \nabla$ there exists an atom q such that $q \leq e$, then the Boolean algebra ∇ is said to be *atomic*.

An algebra A is called *regular* (in the sense of von Neumann) if for any $a \in A$ there exists $b \in A$ such that $a = aba$.

Further, we shall always assume that A is a unital commutative regular algebra over $\mathbb{C}$, and that ∇ is the Boolean algebra of all its idempotents. In this case given any

element $a \in A$ there exists an idempotent $e \in \nabla$ such that $ea = a$, and if $ga = a, g \in \nabla$ then $e \leq g$. This idempotent is called the *support* of a and denoted by $s(a)$.

Suppose that μ is a strictly positive countably additive finite measure on the Boolean algebra ∇ of idempotent from A and consider the metric $\rho(a, b) = \mu(s(a - b))$, $a, b \in A$. From now on we shall *assume that (A, ρ) is a complete metric space* (cf. [4], [5]).

Example 3.1. The most important example of a complete commutative regular algebra (A, ρ) is the algebra $A = L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ of all (classes of equivalence of) measurable complex functions on a measure space (Ω, Σ, μ), where μ is finite countably additive measure on Σ, and $\rho(a, b) = \mu(s(a - b)) = \mu(\{\omega \in \Omega : a(\omega) \neq b(\omega)\})$ (see for details [4, Lemma] and [5, Example 2.5]).

Remark 3.2. If (Ω, Σ, μ) is a general localizable measure space, i.e. the (not finite in general) measure μ has the finite sum property, then the algebra $L^0(\Omega, \Sigma, \mu)$ is a unital regular algebra, but $\rho(a, b) = \mu(s(a - b))$ is not a metric in general. But one can represent Ω as a union of pair-wise disjoint measurable sets with a finite measure and thus this algebra is a direct sum of commutative regular complete metrizable algebras from the above example.

Following [5] we call an element $a \in A$ *finitely valued* (respectively, *countably valued*) if $a = \sum\limits_{k=1}^{n} \alpha_k e_k$, where $\alpha_k \in \mathbb{C}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, ..., n$, $n \in \mathbb{N}$ (respectively, $a = \sum\limits_{k=1}^{\omega} \alpha_k e_k$, where $\alpha_k \in \mathbb{C}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, ..., \omega$, where ω is a natural number or ∞ (in the latter case the convergence of series is understood with respect to the metric ρ)). We denote by $K(\nabla)$(respectively $K_c(\nabla)$) the set of all finitely valued (respectively countably valued) elements in A. It is known that $\nabla \subset K(\nabla) \subset K_c(\nabla)$, both $K(\nabla)$ and $K_c(\nabla)$ are regular subalgebras in A, and moreover the closure of $K(\nabla)$ in (A, ρ) coincides with $K_c(\nabla)$ (see [5, Proposition 2.8]).

Now let D be a derivation on the given regular commutative algebra A. By [5, Proposition 2.3] (see also [4, Theorem]) we have that $s(D(a)) \leq s(a)$ for any $a \in A$, and $D|_\nabla = 0$. Therefore by the definition, each local derivation Δ on A satisfies the following two condition:

$$s(\Delta(a)) \leq s(a), \ \forall a \in A, \tag{9}$$

$$\Delta|_\nabla \equiv 0. \tag{10}$$

This means that (9) and (10) are necessary conditions for a linear operator Δ to be a local derivation on the algebra A. We are going to show that in fact these two condition are also sufficient.

First we recall some further notions from the paper [5].

Let B be a unital subalgebra in the algebra A. An element $a \in A$ is called:

– *algebraic with respect to* B, if there exists a polynomial $p \in B[x]$ (i.e. a polynomial on x with coefficients from B), such that $p(a) = 0$;

– *integral with respect to* B, if there exists a unitary polynomial $p \in B[x]$ (i.e. the coefficient of the largest degree of x in $p(x)$ is equal to $\mathbf{1} \in B$), such that $p(a) = 0$;

– *transcendental with respect to* B, if a is not algebraic with respect to B;

– *weakly transcendental with respect to* B, if $a \neq 0$ and for any non-zero idempotent $e \leq s(a)$ the element ea is not integral with respect to B.

Lemma 3.3. *Given any element $a \in A$ there exists an idempotent $e \in \nabla$ such that*

(i) ea is integral with respect to $K_c(\nabla)$, moreover in this case $ea \in K_c(\nabla)$;

(ii) $e^\perp a$ is weakly transcendental with respect to $K_c(\nabla)$, if $e \neq \mathbf{1}$.

Proof. Denote by ∇_{int} the set of all idempotents $e \in \nabla$ such that ea is integral with respect to $K_c(\nabla)$. By [5, Proposition 3.8] each integral element with respect to $K_c(\nabla)$ in fact belongs to $K_c(\nabla)$. Therefore $\nabla_{int} = \{e \in \nabla : ea \in K_c(\nabla)\}$. We set $e = \sup \nabla_{int}$. Since ∇ is a complete Boolean algebra of countable type [5, Proposition 2.7], there exists a countable family of mutually disjoint elements $\{e_k\}_{k \geq 1}$ in ∇ such that $\sup_{k \geq 1} e_k = e$ and given any $e' \in \nabla_{int}$ there exists $k \geq 1$ such that $e_k \leq e'$. It is clear that $e_k \leq e'$, $e' \in \nabla_{int}$ imply that $e_k \in \nabla_{int}$ and thus $e_k a \in K_c(\nabla)$. Therefore $ea = \sum_{k \geq 1} e_k a \in K_c(\nabla)$. Further since $s(a)^\perp a = 0 \in K_c(\nabla)$ we have that $s(a)^\perp \leq e$, i.e. $e^\perp \leq s(a)$ and hence $s(e^\perp a) = e^\perp$. Now let us show that if $e \neq \mathbf{1}$ then $e^\perp a$ is weakly transcendental with respect to $K_c(\nabla)$. Suppose the opposite, i.e. there exists a non-zero idempotent $q \leq e^\perp = s(e^\perp a)$ such that qa is integral with respect to $K_c(\nabla)$. This means that $q \in \nabla_{int}$, i.e. $q \leq e$. This is a contradiction with $0 \neq q \leq e^\perp$. Therefore $e^\perp a$ is weakly transcendental with respect to $K_c(\nabla)$. The proof is complete. ∎

The following Lemma is the crucial step for the proof of main results in this section.

Lemma 3.4. *Each linear operator on the algebra A satisfying the conditions* (9)

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and (10) *is a local derivation on A.*

Proof. Let Δ be a linear operator on the algebra A which satisfies the conditions (9) and (10). Let us show that $\Delta|_{K_c(\nabla)} \equiv 0$. Since $\Delta|_\nabla \equiv 0$ it is clear that $\Delta|_{K(\nabla)} \equiv 0$. Further for $a, b \in A$ we have from (9)

$$\rho(\Delta(a), \Delta(b)) = \mu(s(\Delta(a) - \Delta(b))) = \mu(s(\Delta(a - b))) \leq \mu(s(a - b)) = \rho(a, b).$$

Thus implies that the linear operator Δ is uniformly continuous with respect to the metric ρ. Since $K(\nabla)$ is dense in $K_c(\nabla)$ we obtain that $\Delta|_{K_c(\nabla)} \equiv 0$.

Now take $a \in A$. By Lemma 3.3 there exists an idempotent $e \in \nabla$ such that $ea \in K_c(\nabla)$ and $e^\perp a$ is weakly transcendental with respect to $K_c(\nabla)$. Since $\Delta|_{K_c(\nabla)} = 0$ and $ea \in K_c(\nabla)$ we have

$$\Delta(a) = \Delta(ea) + \Delta(e^\perp a) = \Delta(e^\perp a).$$

In particular $s(\Delta(a)) = s(\Delta(e^\perp a)) \leq s(e^\perp a)$. Consider the trivial derivation $\delta \equiv 0$ on the regular subalgebra $K_c(\nabla)$ in A. Now [5, Proposition 3.7] implies that for the weakly transcendental element $e^\perp a$ with respect to the regular subalgebra $K_c(\nabla)$ and for the element $\Delta(a)$ in A with $s(\Delta(a)) \leq s(e^\perp a)$ there exists a unique derivation $\delta_1 : B \to A$ such that $\delta_1(e^\perp a) = \Delta(a)$, and $\delta_1|_{K_c(\nabla)} \equiv 0$, where B is the subalgebra in A generated by $K_c(\nabla)$ and the element $e^\perp a$. Now by [5, Theorem 3.1] the derivation δ_1 can be extended to a derivation $D : A \to A$ and it is clear that $D(e^\perp a) = \delta_1(e^\perp a) = \Delta(a)$. Further since $ea \in K_c(\nabla)$ and each derivation satisfies the conditions (9) and (10) we have as above that $D(ea) = 0$. Therefore

$$D(a) = D(ea) + D(e^\perp a) = D(e^\perp a) = \Delta(a),$$

i.e. for any $a \in A$ we have shown the existence of a derivation D on A such that $D(a) = \Delta(a)$. This means that Δ is a local derivation on A. The proof is complete. ∎

The following is the main result concerning the existence of local derivations on commutative regular algebras.

Theorem 3.5. *Let A be a unital commutative regular algebra over $\mathbb{C}$ and let μ be a finite strictly positive countably additive measure on the Boolean algebra ∇ of all idempotents of A. Suppose that A is complete in the metric $\rho(a, b) = \mu(s(a - b))$, $a, b \in A$. Then the following conditions are equivalent:*

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(i) $K_c(\nabla) \neq A$;

(ii) *the algebra A admits a non-zero derivation;*

(iii) *the algebra A admits a non-zero local derivation;*

(iv) *the algebra A admits a local derivation which is not a derivation;*

Proof. The implications (i) $\Leftrightarrow$ (ii) are proved in [5, Theorem 3.2]. The assertion (ii) $\Rightarrow$ (iii) is trivial because any derivation is a local derivation. In order to prove the implication (iii) $\Rightarrow$ (iv) we need the following Lemma

Lemma 3.6. *If D is a derivation on a commutative regular algebra A, then D^2 is a derivation if and only if $D \equiv 0$.*

Proof. Suppose that $D : A \to A$ is a derivation such that D^2 is also a derivation. The given any $a \in A$ we obtain from the Leibniz rule for D^2 and D respectively:

$$D^2(a^2) = 2aD^2(a)$$

and

$$D^2(a^2) = D(D(a^2)) = D(2aD(a)) = 2D(a)D(a) + 2aD(D(a)) = 2[D(a)]^2 + 2aD^2(a),$$

and therefore $[D(a)]^2 = 0$.

Since A is regular there exists an element $b \in A$ such that $D(a) = D(a)bD(a)$. Commutativity of A implies that $D(a) = [D(a)]^2 b = 0$, i.e. $D(a) = 0$ for all $a \in A$. The proof of Lemma 3.6 is complete. ∎

(iii) $\Rightarrow$ (iv). Since A admits a non-zero local derivation, clearly it admits a non-zero derivation (by the definition of local derivation). From [5, Theorem 3.2] this implies that $K_c(\nabla) \neq A$. Take an element $a \in A \backslash K_c(\nabla)$. By Lemma 3.3 above there exists an idempotent $e \in \nabla$ such that $ea \in K_c(\nabla)$ and the element $e^{\perp}a$ is weakly transcendental with respect to $K_c(\nabla)$ provided that $e \neq \mathbf{1}$. Since $a \notin K_c(\nabla)$, we have that $e \neq \mathbf{1}$, and hence the element $b = e^{\perp}a$ is indeed weakly transcendental with respect to $K_c(\nabla)$. By [5, Proposition 3.7, Theorem 3.1] as in Lemma 3.4 there exists a derivation D on A such that $D(b) = b$. Consider the linear map $\Delta = D^2$. Since D is a derivation on A, Δ satisfies the conditions (9) and (10) and by Lemma 3.4 Δ is a local derivation on A, and moreover $\Delta(b) = D(D(b)) = D(b) = b$, i.e. $\Delta \not\equiv 0$. By Lemma 3.6 Δ is not a derivation.

(iv) $\Rightarrow$ (i). Let Δ be a local derivation on A, which is not a derivation. Then it is clear that Δ is not identically zero, i.e. $\Delta(a) \neq 0$ for an appropriate element $a \in A$. By definition there exists a derivation D on A such that $\Delta(a) = D(a) \neq 0$, i.e. D is a non-zero derivation on A. Therefore by [5, Theorem 3.2] we obtain that $K_c(\nabla) \neq A$. The proof of Theorem 3.5 is complete. ∎

The important special case of the last theorem is the following result concerning the regular algebra $L^0(\Omega, \Sigma, \mu)$ from the example 3.1.

Corollary 3.7. *Let (Ω, Σ, μ) be a finite measure space and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of all real or complex measurable functions on (Ω, Σ, μ). Following conditions are equivalent:*

(i) *the Boolean algebra of all idempotents from $L^0(\Omega)$ is not atomic;*

(ii) *$L^0(\Omega)$ admits a non-zero derivation;*

(iii) *$L^0(\Omega)$ admits a non-zero local derivation;*

(iv) *$L^0(\Omega)$ admits a local derivation which is not a derivation.*

Proof. This follows easily from Theorem 3.5, Example 3.1 and [5, Theorem 3.3]. ∎

It is well known that if $\mathcal{M}$ is a commutative von Neumann algebra with a faithful normal semifinite trace τ, then $\mathcal{M}$ is $*$-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex function on some localizable measure space (Ω, Σ, μ) and $\tau(f) = \int_\Omega f(t) d\mu(t)$ for $f \in L^\infty(\Omega, \Sigma, \mu)$. In this case the algebra $S(\mathcal{M})$ of all measurable operators affiliated with $\mathcal{M}$ may be identified with the algebra $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ of all measurable functions on (Ω, Σ, μ), while the algebra $S(\mathcal{M}, \tau)$ of τ-measurable operators from $S(\mathcal{M})$ coincides with the algebra

$$\{f \in L^0(\Omega) : \exists F \in \Sigma, \mu(\Omega \setminus F) < +\infty, \chi_F \cdot f \in L^\infty(\Omega)\}$$

of all totally τ-measurable functions on Ω, where χ_F is the characteristic function of the set F. In the trace τ is finite then $S(\mathcal{M}) = S(\mathcal{M}, \tau) \cong L^0(\Omega)$ is a commutative regular algebra. But if the trace τ is not finite the algebra $S(\mathcal{M}, \tau)$ is not regular. In this case considering Ω as a union of pairwise disjoint measurable sets with finite measures we obtain that $S(\mathcal{M})$ is a direct sum of commutative regular algebras (see Remark 3.2) and $S(\mathcal{M}, \tau)$ is a subalgebra of a direct sum of commutative regular algebras. Therefore from Lemma 2.4 and the above Corollary 3.7 we obtain the following solution

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of the problem concerning the existence of non trivial local derivations on algebras of measurable operator in the commutative case.

Theorem 3.8. *Let $\mathcal{M}$ be a commutative von Neumann algebra with a faithful normal semi-finite trace τ. The following conditions are equivalent:*

(i) *the lattice $P(\mathcal{M})$ of projections in $\mathcal{M}$ is not atomic;*

(ii) *the algebra $S(\mathcal{M})$ (respectively $S(\mathcal{M}, \tau)$) admits a non-inner derivation;*

(iii) *the algebra $S(\mathcal{M})$ (respectively $S(\mathcal{M}, \tau)$) admits a non-zero local derivation;*

(iv) *the algebra $S(\mathcal{M})$ (respectively $S(\mathcal{M}, \tau)$) admits a local derivation which is not a derivation.*

Remark 3.9. It should be noted that for general (non commutative) von Neumann algebras the above conditions are not equivalent but some implications are valid.

The implication (i) $\Rightarrow$ (ii) is not true in general because for a type I$_\infty$ von Neumann algebra $\mathcal{M}$ with the non atomic center Z the lattice $P(\mathcal{M})$ is not atomic but the algebras $S(\mathcal{M})$ and $S(\mathcal{M}, \tau)$ do not admit non inner derivations [2, Lemma 3.5 and Theorem 4.1].

The implication (ii) $\Rightarrow$ (i) is valid, because if we suppose the lattice $P(\mathcal{M})$ to be atomic then in view of [1, Corollary 3.1] and [2, Lemma 3.5 and Theorem 4.1] every derivation on the algebras $S(\mathcal{M})$ and $S(\mathcal{M}, \tau)$ is automatically Z-linear and hence it is inner, i.e. these algebras do not admit non inner derivations.

The condition (iii) is always fulfilled in the non commutative case, because every inner derivation which is implemented by a non central element is a non-zero derivation and hence it is a non-zero local derivation.

The implications (i) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (iv) are not true in general, since if we take a finite von Neumann algebra of type I$_n$, $(n \neq 1)$, with a faithful normal finite trace τ and with the non atomic center, then by Theorem 2.5 the algebra $S(\mathcal{M}) = S(\mathcal{M}, \tau)$ does not admit a local derivation which is not a derivation, but it admits non inner derivations of the form D_δ (see Section 2).

The implication (iv) $\Rightarrow$ (i) is true and follows from Proposition 2.7.

Finally, the implication (iv) $\Rightarrow$ (ii) is an open problem in the general case.

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