# Existence of Hölder Continuous Young Measure Solutions to Coercive Non-Monotone Parabolic Systems in Two Space Dimensions

**Jens Frehse, Maria Specovius-Neugebauer**

**no. 441**

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereichs 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

Bonn, März 2009

# Existence of Hölder Continuous Young Measure Solutions to Coercive Non-Monotone Parabolic Systems in Two Space Dimensions

Jens Frehse[a] * and Maria Specovius-Neugebauer[b] *
[a]*Institute of Applied Mathematics, University of Bonn*;
[b]*Fachbereich Mathematik, University of Kassel*

Dedicated to Vsevolod A. Solonnikov

**Abstract**

We consider parabolic systems $u_t - \operatorname{div}(a(\nabla u)) = f$ in two space dimensions where the elliptic part is derived from a potential and is coercive, but not monotone. With natural assumptions on the data we obtain the existence of a long time Hölder continuous solution in the sense of Young measures.

**Keywords**: Nonlinear parabolic systems, Hölder continuity, Young measure solutions, convergence of finite elements

**AMS classification**: 35K55, 35K50

## 1 Introduction

In a recent paper [7] we have presented an a priori estimate in Morrey spaces for systems of evolution equations under coerciveness and entropy conditions *without* assuming monotonicity or ellipticity. This method is used here to show the existence of long time Hölder continuous weak solutions to a class of parabolic system in two space variables where only coercivity and the existence of a potential for the second order part is needed. The price for the lack of an ellipticity/monotonicity condition is that we have to accept weak solutions in the sense of Young measures (cf. Section 5). Nevertheless it is of interest that Hölder continuity is achieved although the special second order operator need not be elliptic.

*This research was supported by the DFG, SFB 611 "Singuläre Phänomene und Skalierung in mathematischen Modellen" and the SFB/TR 30 "Prozessintegrierte Herstellung funktional gradierter Strukturen auf der Grundlage thermomechanisch gekoppelter Phänomene"

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. We consider a parabolic system of the form

$$\begin{aligned} u_t - \sum_{i=1,2} D_i a_i(t,x,\nabla u) = f \qquad &\text{in } [0,T^0]\times\Omega\,,\\ u=(u_1,\ldots,u_N)\,,\quad a_i=(a_i^1,\ldots,a_i^N)\,,\qquad &\nabla=\nabla_x \end{aligned} \tag{1}$$

with homogeneous Dirichlet boundary condition and initial value condition

$$u(0)=u_0, \quad u_0\in H_0^1(\Omega), \tag{2}$$

where $H_0^1(\Omega)$ denotes the usual Sobolev spaces of $L^2$-functions such that $\nabla u \in L^2$, and the traces on the boundary vanish. For technical reasons, we need the coefficient functions defined on a slightly larger domain than $\Omega$. More precisely, we fix a domain $\Omega_1 \subset \mathbb{R}^2$ with $\overline{\Omega}\subset\Omega_1$. As in [7] we need that the coefficient functions are derived from a potential $A$ with specific properties, i.e. there exists a function $A:[0,T^0]\times\Omega_1\times\mathbb{R}^{N\times 2}\to\mathbb{R}$ such that

- $\dfrac{\partial}{\partial\eta_i^\nu}A(t,x,\eta)=a_i^\nu(t,x,\eta)$, a. e. in $(t,x)$
- $\dfrac{\partial}{\partial t}A(t,x,\eta)=A_t(t,x,\eta)$ exists for all $\eta\in\mathbb{R}^{N\times 2}$ and a.e. in $(t,x)$,
- $A$, $A_t$, $a_i^\nu$ satisfy the Caratheodory condition, this means measurability with respect to $(t,x)$ for all $\eta$ and continuity with respect to $\eta$ a. e. in $(t,x)$
- the following growth and coerciveness conditions hold true a.e. in $t$ and $x$:

$$A(t,x,\eta)+|A_t(t,x,\eta)|+\sum_{i=1,2}|a_i(t,x,\eta)|^2\le C_0|\eta|^2+K\,, \tag{3}$$

$$\sum_{i=1,2}\sum_{\nu=1}^N a_i^\nu(t,x,\eta)\eta_i^\nu\ge\alpha_0|\eta|^2-K,\quad A(t,x,\eta)\ge\alpha_1|\eta|^2-K \tag{4}$$

with positive constants $C_0$, $\alpha_0$, $\alpha_1$ and $K$ (the letter $K$ is reserved for constants which need not to be specified, they can change from line to line).

For regular solutions $u$, we may test the equation (1) with the function $u_t\varphi$ where $\varphi$ is a sufficiently smooth function and obtain

$$\begin{aligned} &\int_{t_1}^{T}\int\Big[u_t^2\varphi - A\varphi_t+\sum_{i=1,2}a_iu_tD_i\varphi - A_t\varphi\Big]\,dx\,dt\\ &\qquad+\int A\,dx\Big|_{t_1}^{T}=\int_{t_1}^{T}\int f u_t\varphi\,dx\,dt\,. \end{aligned} \tag{5}$$

Equation (5) can be called "local energy conservation" or, with an inequality sign $\leq$, "entropy condition", the integrals on the left hand side are finite also for functions $u \in L^2(H_0^1(\Omega))$ such that $u_t \in L^2(L^2(\Omega))$. (Here $L^r(V) = L^r\big(0,T^0;V\big)$ is the $L^r$ space of $V$-valued functions on $[0,T^0]$, where $V$ is any Banach-space.) In [7] we derived from (5) and natural regularity conditions for the data a *Morrey condition* for $\nabla u$

$$\operatorname{ess\,sup}\Big\{ \int\limits_{B_R(x_0)} |\nabla u|^2\,dx \,\Big|\, 0 \leq t \leq T^0\,,\ x_0 \in \Omega\Big\} \leq K R^{2\alpha} \tag{6}$$

which implies that the solution $u$ of (1) is contained in the Hölder space $C^{\alpha/2}([0,T^0]\times\Omega)$ in the case of two space dimensions. The method can be applied for space dimensions $\geq 3$, too, however, this implies only a slight improvement of the Sobolev imbedding exponent.

Since (5) is not known a priori by a solution $u \in L^2(H_0^1)$, $u_t \in L^2(L^2)$, the result from [7] is only an a priori estimate. In order to obtain existence results, one has to find approximations of (1) with smooth solutions, but with a structure which allows to repeat the method of proof in [7]. In particular, the structure of an Euler operator for the second order part $\sum_{i=1,2} D_i a_i$ has to be preserved.

A singular pertubation of (1) does not look promising, since it is not clear how the special technique of weighted estimates used in [7] can be carried over then. The best way to approximate equation (1) seems to be the *finite element method* using continuous linear spline functions $u_h$. The finite element setting is exposed in next chapter. We obtain a sequence of continuous piecewise linear functions $u_h$ such that

$$u_h \rightharpoonup u \text{ weakly in } L^2\big(H_0^1\big), \quad \dot u_h \rightharpoonup u \text{ weakly in } L^2(L^2), \tag{7}$$

$$\sup_t \|u_h(t,\cdot)\|_{C^\alpha(\overline{\Omega})} + \sup_x \|u_h(\cdot,x)\|_{C^{\alpha/2}[0,T^0]} \leq K, \tag{8}$$

as $h$ tends to 0. The functions $u_h$ solve (1) approximately and satisfy a discrete analog of (6) uniformly for $0 < h < h_0$. The discrete Morrey estimate is the main difficulty to prove, it is elaborated in chapter 3 and 4.

From (8) we conclude that the limit $u$ is contained in $C^{\alpha/2}(C^\alpha)$. The limit $u$ satisfies equation (1) in the sense of Young measures, see the explanations and Theorem 5.1 in chapter 5.

Typical examples for our result are generated with the potentials

$$A(t,x,\nabla u) = \mu_1|\nabla u|^2 + \mu_2|\operatorname{div} u|^2 + H(|\det \nabla u|)\,,$$

$H$ convex, $|H(\xi)| \leq K + K|\xi|$, $|H'| \leq K$, $\mu_1,\mu_2 > 0$ or

$$A(\eta) = \frac{(|\eta|^2-1)^2}{1+|\eta|^2}.$$

There is a vast literature on regularity of parabolic systems, see the bibliography in [7, 13, 16, 11]. The regularity theory in the scalar case for nonlinear parabolic equations has been treated in the classical book of Ladyzhenskaja, Solonnikov and Uralceva [9], where Hölder continuity of scalar weak solutions is obtained for arbitrary space dimension and more general nonlinearities with coercive non-monotone spatial principal part. In [10] an example for a two dimensional parabolic system is presented, which has a nonconvex (in fact, quasiconvex) potential $A$ and a solution nowhere better than Lipschitz. For the non-monotone case which naturally leads to Young measure solutions consult [12, 3].

We list some *frequently used notation*:

The expressions $(\cdot,\cdot)_\Xi$, $\|\cdot\|_\Xi$ denotes the scalar product and norm, respectively, in $L^2(\Xi)$ where $\Xi \subset \mathbb{R}^2$ (the integration is performed with respect to the spatial variables only), we omit the subscript $\Xi$, if no confusion arises. We also omit the domain of integration in our calculations if it is obvious. For vector valued functions $u$ in $L^2$ we mainly write $u^2 (= \sum_{i=1}^N u_i^2)$ instead of $|u|^2$.

Further we use the notation $G \lesssim F$ to indicate that $G \leq K\,F$, where $K$ is a generic constant.

The expressions $D_i$ as well as $\nabla^m$ always refer to spatial derivatives, while the partial derivative with respect to $t$ is indicated either by the subscript $t$ or a dot: $u_t = \frac{\partial}{\partial t} u = \dot{u}$.

For $R > 0$, $x_0 \in \mathbb{R}^2$, we denote by $B_R = B_R(x_0)$ the open disk around $x_0$ with radius $R$ and by $Q_R$ the parabolic cylinder $[T - R^2, T] \times B_R(x_0)$.

We specify the assumptions on the data $u_0$ and $f$, we extend $u_0$ by zero to a function in $H_0^1(\Omega_1)$, we assume that $f \in L^2(L^2(\Omega_1))$ and $u_0$ and $f$ satisfy the following Morrey conditions for

$$\int_{B_R(x_0)} |\nabla u_0|^2 \, dx \lesssim R^{2\gamma}\,, \qquad \text{for all } B_R(x_0) \subset \Omega_1. \tag{9}$$

$$\iint_{Q_R} f^2 \, dx \lesssim R^{2\gamma}\,, \qquad \text{for all } Q_R \subset [0, T^0] \times \Omega_1. \tag{10}$$

## 2 Finite element approximation

Let $\Omega \subset \Omega_1 \subset \mathbb{R}^2$ be specified as in Section 1. We choose a set $\mathcal{T}_h$ of closed triangles such that $\Omega_h := \bigcup_{\Delta_h \in \mathcal{T}_h} \subset \Omega_1$ with the additional properties:

1. *diam* $\Delta_h \leq h$ *for all* $\Delta_h \in \mathcal{T}_h$.

2. *If* $\Delta_h \cap \Delta_h'$ *consists of exactly one point* $P$, *then* $P$ *is a corner of* $\Delta_h$ *and* $\Delta_h'$.

3. *If* $\Delta_h \cap \Delta_h'$ *consists of more than one point, then the intersection is a common edge of* $\Delta_h$ *and* $\Delta_h'$.

4. *There is a constant* $\kappa$ *such that each* $\Delta_h \in \mathcal{T}_h$ *contains a circle of radius* $\kappa h$.

5. *The set* $\Omega_h := \bigcup_{\Delta_h \in \mathcal{T}_h}$ *approximates* $\Omega$, *i.e.* $\operatorname{dist}(\partial\Omega, \partial\Omega_h) = O(h)$.

The corners of the triangles $\Delta_h$ are called *nodal points*. The *discrete neighborhood of a nodal point* $P \notin \partial\Omega_h$ is the set of nodal points

$$\mathcal{N}_h(P) = \{Q \in \Omega_h \big| Q \text{ is corner of a triangle } \Delta_h \in \mathcal{T} \text{ having } P \text{ as a corner } \} .$$

We use the following special finite element space

$$V_h = \{v_h \in H_0^1(\Omega_h) \cap C(\bar{\Omega}_h) \,\Big|\, v_h|_{\Delta_h} \text{ is linear for all } \Delta_h \in \mathcal{T}_h\}$$

To each nodal point $P \notin \partial\Omega_h$ we associate a basis function $w_h^P \in V_h$ defined by

$$w_h^P(P) = 1\,, \qquad w_h^P(Q) = 0 \text{ for all nodal points } Q \neq P. \tag{11}$$

Clearly, the set

$$\{w_h^P \big| P \text{ is a nodal point } \notin \partial\Omega_h\}$$

forms a basis of $V_h$. With $[V_h]^N$ we denote the space of vector functions whose components are in $V_h$. The functions $w_h \in [V_h]^N$, such that $N-1$ components vanish and the remaining component is a basis function of type (11), give rise to a basis $\mathcal{B}_h$ of $[V_h]^N$.

We choose a *finite element approximation* of (1) in the following way: Find

$$u_h(t,x) = \sum_{w \in \mathcal{B}_h} c_w(t) w(x), \tag{12}$$

such that

$$(\dot{u}_h, \varphi_h) + \sum_{i=1}^{2} \big(a_i(t,\cdot,\nabla u_h), D_i\varphi_h\big) = (f, \varphi_h) \tag{13}$$

for all $\varphi_h \in [V_h]^N$, a. e. with respect to $t$,

$$u_h(0,\cdot) = u_{0h}(\cdot)\,. \tag{14}$$

Due to (9) we may assume for the approximation $u_{0h} \in [V_h]^N$

$$u_{0h} \to u_0 \text{ strongly in } H^1(\Omega_1) \tag{15}$$

$$\int_{B_R(x_0)} |\nabla u_{0h}|^2 \, dx \le K R^{2\gamma} \textit{ uniformly as } h \to 0, \tag{16}$$

$$4h \le R \le R_0, \quad x_0 \in \Omega\,.$$

The system (13) is equivalent to the system of ordinary differential equations

$$\sum_{w \in \mathcal{B}_h} \dot{c}_w(t)(w, \hat{w}) + \sum_{i=1}^{2} \Big(a_i(t,\cdot, \sum_{w \in \mathcal{B}_h} c_w D_i w), \hat{w}\Big) = (f(\cdot,t), \hat{w})$$

for all $\hat{w} \in \mathcal{B}_h$, a. e. with respect to $t$. Since the matrix $\big((w,\hat{w})\big)_{w,\hat{w} \in B_h}$ is nonsingular, the theory of ordinary differential equations for absolutely continuous functions gives us the existence of a global solution $u_h : [0,T^0] \to [V_h]^N$ of the finite element approximation.

In fact, local solvability follows via Peano's theorem in the setting of absolute continuous functions and the global solvability follows via the extension argument from the discrete energy equation

$$\frac{1}{2}(\dot{u}_h, \dot{u}_h) + \frac{d}{dt}\int_{\Omega_h} A(t,\cdot,\nabla u_h)\,dx - \int_{\Omega_h} A_t(t,\cdot,\nabla u_h)\,dx = \big(f(t,\cdot),\dot{u}_h\big)\,. \tag{17}$$

From the energy equality (17) we obtain the following estimates for the solutions $u_h$ of (13) uniformly as $h \to 0$:

$$\operatorname*{ess\,sup}_{0\le t\le T} \int_{\Omega_h} |\nabla u_h|^2\,dx + \int_0^T\int_{\Omega_h} \dot{u}_h^2\,dx\,dt \le K_T \tag{18}$$

This follows from (17) by integrating from 0 to $t$ or 0 to $T$, using the growth and coerciveness assumptions for $A$ and $a_i$, combined with Gronwall's inequality and the $L^2$-assumptions on $f$ and $u_{0h}$. This procedure is standard and is not elaborated here.

We need some tools from the theory of finite elements. To formulate the result we introduce the notation

$$\|w\|' = \Big(\sum_{\Delta_h\in\mathcal{T}_h}\int_{\Delta_h} |w|^2\,dx\Big)^{1/2}.$$

For $v \in C(\Omega_h)$, we use the Lagrange interpolation $I_h v := \sum_P v(P) w_h^P$, with $w_h^P$ as in (11). Clearly, if $\operatorname{supp} v$ is compact in $\Omega_h$, then $\operatorname{supp} I_h v$ is only slightly larger, i.e. $\operatorname{supp} I_h v \subset \{x \mid \operatorname{dist}(x,\operatorname{supp} v) \le h\}$.

**Proposition 2.1** (cf [?, p. 105 ff], e.g.)

(i) (Properties of the interpolation operator) *For $v \in C(\Omega_h)$ with $v\big|_{\Delta_h} \in H^2(\Delta_h)$ for all $\Delta_h \in \mathcal{T}_h$, and $I_h v$ as above, the following estimates hold true:*

$$\|v - I_h v\|_{\Omega_h} \lesssim h^m \|\nabla^m v\|'_{\Omega_h}, m = 1,2 \tag{19}$$

$$\|\nabla(v - I_h v)\|_{\Omega_h} \lesssim h\|\nabla^2 v\|'_{\Omega_h} \tag{20}$$

$$\|I_h v\|_{\Omega_h} \lesssim \|v\|_{\Omega_h} + h\|\nabla v\|_{\Omega_h} + h^2\|\nabla^2 v\|'_{\Omega_h} \tag{21}$$

*with some constant not depending on $h \to 0$.*

(ii) *For all $v_h \in V_h$ there holds the so-called inverse inequality*

$$\|\nabla v_h\|_{\Omega_h} \lesssim \frac{1}{h}\|v_h\|_{\Omega_h}. \tag{22}$$

Observe that it is necessary to use the norm $\|\cdot\|'$ since neither $v \in H^2(\Omega_h)$ nor $I_h v \in H^2(\Omega_h)$ is required. The proposition can be applied to any subdomain $\Omega_0 \subset \Omega_h$, then we write $\|, \|_{\Omega_0}$ in the norms of the right hand side.

We also need a weighted variant of (20): Let $W = W^h$ be defined by

$$\begin{aligned} W^h(t,x) &= -\frac{T-t}{|x-x_0|^2 + h^2 + T - t}, \text{ then} \\ W_t^h(t,x) &= \frac{|x-x_0|^2 + h^2}{\left(|x-x_0|^2 + h^2 + T - t\right)^2}. \end{aligned} \tag{23}$$

**Proposition 2.2** *For all $v \in \mathcal{C}(\bar{\Omega})$, such that $v|_{\Delta_h} \in H^2(\Delta_h)$ for any triangle $\Delta_h \in \mathcal{T}_h$ the following inequality holds with a constant independent of $h$ and $v$:*

$$\|W_t^{-1/2}\nabla(v - I_h v)\| \lesssim h\|W_t^{-1/2}\nabla^2 v\|' .$$

**Proof:** Since (20) is proved by reducing the assertion to the triangles $\Delta_h \in \mathcal{T}_h$, it is enough to show

$$|W_t^{-1}(x)| \le \sup_{x\in\Delta_h} |W_t^{-1}(x)| \le BW_t^{-1}(x) \text{ for } x \in \Delta_h \,,$$

where $\Delta_h \in \mathcal{T}_h$ arbitrary. The first inequality is trivial, to show the second we assume $x_0 = 0$ for simplicity and consider the annuli

$$\hat{A}_k = \{kh \le |x| \le (k+2)h\} \;, \qquad k = 0, 1, \dots .$$

For $x \in \hat{A}_k$ we have by elementary calculations

$$\begin{aligned} &k^2h^2 + h^2 + 2(T-t) + \frac{(T-t)^2}{\big((k+h)^2+1\big)h^2} \\ &\quad\le W_t^{-1}(t,x) = |x|^2 + h^2 + 2(T-t) + \frac{(T-t)^2}{h^2 + |x|^2} \\ &\quad\le \big((k+2)^2+1\big)h^2 + 2(T-t) + \frac{(T-t)^2}{(k^2+1)h^2} \\ &\quad\le W_t^{-1}(t,x) + 4(k+1)h^2 + \frac{(T-t)^2}{h^2}\frac{4(k+1)}{\big((k+2)^h+1\big)(k^2+1)} \\ &\quad\le 5W_t^{-1}(t,x) \end{aligned}$$

hence $\sup_{x\in\hat{A}_k} W_t^{-1} \le 5W_t^{-1}(t,x)$.

## 3 Discrete Morrey estimates for truncation terms

In order to prove a uniform Morrey estimate for the finite element approximation we adopt the arguments of [7].

We need to use the discrete variant of the energy-equality (5) with $\varphi = \tau_R^2$, and $\varphi = W\tau_R^2$, where $R > 0$ is fixed and $\tau_R$ denotes a localization function of the form

$$\tau_R(t,x) = \psi(t)\zeta(|x-x_0|) \,,$$

where

$$\psi(t) = \begin{cases} 0, & t < T - 2R^2, t > T \\ R^{-2}(t - (T - 2R^2)), & T - 2R^2 \le t < T - R^2, \\ 1, & T - R^2 \le t \le T, \end{cases}$$

and $\zeta \in C_0^2(\mathbb{R})$ such that $\zeta(r) = 1$ for $0 \le r \le R$, $\zeta(r) = 0$ for $|r| \ge 2R$, $|\zeta'(r)| \lesssim R^{-1}$, $|\zeta''(r)| \lesssim R^{-2}$.

Since $\dot{u}_h \tau_R^2$, $\dot{u}_h W \tau_R^2$ and $(u_h - \bar{u}_h)\tau_R^2$ are not admissible test functions in the finite element equation (13), some additional arguments are necessary. We keep $x_0$ fixed during this section and set $B_R = B(x_0, R)$ with $R \ge 4h$. If $\varphi(t, \cdot) \in H_0^1(\Omega)$ a.e. in $t$, then $I_h\varphi \in V_h$ and the discretized equation (13) leads to the identity

$$\begin{aligned} &(\dot{u}_h, \varphi) + \sum_{i=1,2} \big(a_i^h, D_i\varphi\big) \\ &= \big(\dot{u}_h, \varphi - I_h\varphi\big) + \sum_{i=1,2} \big(a_i^h, D_i(\varphi - I_h\varphi)\big) + \big(f, I_h\varphi\big) \\ &=: T_1(\varphi) + T_2(\varphi) + T_3(\varphi)\,, \end{aligned} \tag{24}$$

with $a_i^h = a(.,.,\nabla u_h)$. Note that the application of the interpolation operator $I_h$ implies $B_{2R+h}$ as domain of integration in the $L^2$-scalar product for $T_j(\varphi)$.

**Proposition 3.1** *While testing with $\varphi = \tau_R^2 \dot{u}_h$, the following estimate holds true for the truncation terms:*

$$\begin{aligned} &|T_1(\varphi)| + |T_2(\varphi)| \le \\ &\big|\big(\dot{u}_h, \dot{u}_h\tau_R^2 - I_h(\dot{u}_h\tau_R^2)\big)\big| + \sum_i \big|\big(a_i^h, D_i(\dot{u}_h\tau_R^2 - I_h(\dot{u}_h\tau_R^2))\big)\big| \\ &\lesssim \|\dot{u}_h\|^2_{A_{2R,h}} + R^{-2}\|\nabla u_h\|^2_{A_{2R,h}} + 1 \end{aligned} \tag{25}$$

*a.e. with respect to $t$, where $A_{2R,h} = B_{2R+h} \backslash B_{R-h}$.*

**Remark.** Proposition 3.1 states that the truncation terms $T_1(\varphi)$ and $T_2(\varphi)$, with $\varphi = \tau_h^2 \dot{u}_h$, are estimated by *hole filling terms*, i. e. terms where the domain of integration has the *hole* $B_{R-h}$.

**Proof:** Since $\tau_R^2 = 1$ on $B_R$, we obtain

$$u_h\tau_R^2 - I_h(u_h\tau_R^2) = 0 \text{ for } x \in B_{R-h} \text{ and } x \notin B_{2R+h}\,,$$

hence the domain of integration reduces to $A_{2R,h} = B_{2R+h} \backslash B_{R-h}$ in both terms on the left hand side. Using the notations of (24), Hölder's inequality together with the interpolation estimate (19) leads to

$$|T_1(\dot{u}_h\tau_R^2)| \lesssim \|\dot{u}_h\|_{A_{2R,h}} h^2 \|\nabla^2(\dot{u}_h\tau_R^2)\|'_{A_{2R,h}}\,.$$

Due to the choice of the space $V_h$ we have $\nabla^2 \dot{u}_h = 0$ on each triangle $T \in \mathcal{T}_h$, hence

$$\|\nabla^2(\dot{u}_h\tau_R^2)\|' \lesssim \sum_{i,j} \|D_i\dot{u}_h D_j\tau_R^2\| + \|\dot{u}_h D_iD_j\tau_R^2\|\,.$$

Now the inverse inequality (22), the properties $|D_i\tau_R^2| \lesssim R^{-1}$, $|D_jD_i\tau_R^2| \lesssim R^{-2}$ and the relation $R \geq 4h$ implies

$$h^2\|\nabla^2(\dot{u}_h\tau_R^2)\|' \lesssim \frac{h^2}{R}\|\nabla\dot{u}_h\|_{B_{2R}\setminus B_R} + \frac{h^2}{R^2}\|\dot{u}_h\|_{B_{2R}\setminus B_R} \lesssim \frac{h}{R}\|\dot{u}_h\|_{A_{2R,h}}\,,$$

hence

$$|T_1(\dot{u}_h\tau_R^2)| \lesssim \|\dot{u}_h\|^2_{A_{2R,h}}\,. \tag{26}$$

In a similar manner, we can treat the second term on the left hand side of (25)

$$\begin{aligned}|T_2(\dot{u}_h\tau_R^2)| &\lesssim \sum_i \|a_i^h\|_{A_{2R,h}}\|D_i(\dot{u}_h\tau_R^2 - I_h(\dot{u}_h\tau_R^2))\| \lesssim \\ &\lesssim \left(\|R^{-1}\nabla u_h\|_{A_{2R,h}} + R^{-1}|A_{2R,h}|^{1/2}\right) Rh\|\nabla^2(\dot{u}_h\tau_R^2))\|'_{A_{2R,h}}\,.\end{aligned}$$

Here we also used the growth condition of $a_i$, and the same arguments as for (26) lead to

$$hR\|\nabla^2(\dot{u}_h\tau_R^2)\|'_{A_{2R,h}} \lesssim \|\dot{u}_h\|_{A_{2R,h}}$$

hence

$$|T_2(\dot{u}_h\tau_R^2)| \lesssim \|R^{-1}\nabla u_h\|^2_{A_{2R,h}} + 1 + \|\dot{u}_h\|^2_{B_{2R+h}} \tag{27}$$

and (26) together with (27) gives the assertion (25). □

The second series of estimates is gathered in the following proposition:

**Proposition 3.2** *Let $W(t,x)$ be defined as in (23), then with $\varphi = W\tau_R^2\dot{u}_h$ we have*

$$\begin{aligned}&|T_1(\varphi)| + |T_2(\varphi)| \leq \\ &\left|(\dot{u}_h, \dot{u}_hW\tau_R^2 - I_h(\dot{u}W\tau_R^2))\right| + \sum_i \left|\left(a_i^h, D_i(\dot{u}_hW\tau_R^2 - I_h(\dot{u}_hW\tau_R^2))\right)\right| \\ &\leq \varepsilon\|\nabla u_hW_t^{1/2}\|^2_{B_{2R+h}} + K\left(\|W_t^{1/2}\|^2_{B_{2R+h}} + \|\dot{u}_h\|^2_{B_{2R+h}}\right)\,.\end{aligned} \tag{28}$$

**Proof:** We start with various elementary estimates related to the weight $W$ and the localization function $\tau_R^2$, namely

$$h|\nabla W| \leq \frac{h}{(T-t+h^2+|x-x_0|^2)^{1/2}} \leq 1\,, \tag{29}$$

$$\begin{aligned}W_t^{-1}|\nabla W|^2 &\lesssim 1\,,\\ |D_iD_jW| &\lesssim (T-t+h^2+|x-x_0|^2)^{-1}, \\ h^2W_t^{-1}|D_iD_jW|^2 &\lesssim 1\end{aligned} \tag{30}$$

independent of $x, x_0, T, t$ and $h$. Furthermore, for $x \in B_{2R}\setminus B_R$, we get

$$\begin{aligned}W_t^{-1}|D_i(\tau_R^2)|^2 &\lesssim \frac{(T-t+|x-x_0|^2+h^2)^2}{|x-x_0|^2}\frac{1}{R^2} \lesssim 1 \\ W_t^{-1}|W|^2|D_iD_j\tau_R^2|^2 &\lesssim R^{-2} \text{ for } x \in B_{2R}\setminus B_R\,.\end{aligned} \tag{31}$$

Again applying the interpolation inequality (19), we obtain with (29) and $W \le 1$

$$
\begin{aligned}
|T_1(\dot u_h \tau_R^2 W)| &\le \|\dot u_h\|_{B_{2R+h}} h \|\nabla(\dot u_h \tau_R^2 W)\|\,, \\
&\lesssim \|\dot u_h\|_{B_{2R+h}} \Big( h\|\nabla \dot u_h\|_{B_{2R}} + \frac{h}{R}\|\dot u_h\|_{B_{2R}\setminus B_R} + \|\dot u_h\|_{B_{2R}} \Big) \\
&\lesssim \|\dot u_h\|^2_{B_{2R+h}}
\end{aligned}
\tag{32}
$$

independent of $h, R, x_0$ and $T$ as long as $R \ge 4h$.

Finally we have to estimate the term $T_2(\dot u_h \tau_R^2 W)$. Using Hölder's inequality, the growth condition for $a_i$ and the weighted interpolation inequality of 2.2 we get

$$
\begin{aligned}
&|T_2(\dot u_h \tau_R^2 W)| \le \\
&\le \sum_i \|a_i^h W_t^{1/2}\|_{B_{2R+h}} \|W_t^{-1/2} \nabla\big(\dot u_h W \tau_R^2 - I_h(\dot u_h W \tau_R^2)\big)\| \\
&\lesssim \Big( \|\nabla u_h W_t^{1/2}\|_{B_{2R+h}} + \|W_t^{1/2}\|_{B_{2R+h}} \Big)\, h\, \|W_t^{-1/2} \nabla^2(\dot u_h W \tau_R^2)\|'_{B_{2R}}\,.
\end{aligned}
\tag{33}
$$

Exploiting again $D_i D_j \dot u_h = 0$ on the interior of the triangles $\Delta_h$, we have

$$
\begin{aligned}
D_i D_j(\dot u_h W \tau_R^2) = (D_j \dot u_h)(D_i W)\tau_R^2 + (D_j \dot u_h) W (D_i \tau_R^2) + \\
+ \dot u_h \big( D_i W D_j \tau_R^2 + D_j W D_i \tau_R^2 + (D_i D_j W)\tau_R^2 + W(D_i D_j \tau_R^2)\big)\,.
\end{aligned}
\tag{34}
$$

Next we use (29)–(31) to obtain

$$
\begin{aligned}
& h\|W_t^{-1/2} \nabla^2(\dot u_h \cdot W \tau_R^2)\| \\
&\lesssim h\|\nabla \dot u_h\|_{B_{2R}} + h\|\nabla \dot u_h\|_{B_{2R}\setminus B_R} + 2\frac{h}{R}\|\dot u_h\|_{B_{2R}\setminus B_R} + \|\dot u_h\|_{B_{2R}} \\
&\lesssim \|\dot u_h\|_{B_{2R+h}}\,,
\end{aligned}
$$

now estimate (28) follows from (32) and (33). □

The proof of the Morrey estimate needs a third type of test functions, namely $\varphi = (u_h - \bar u_h)\tilde\tau_R^2$, where

$$
\begin{aligned}
\bar u_h &= \begin{cases} \frac{1}{4\pi R} \cdot \int\limits_{\partial B_{2R}} u_h \, dx & \text{if } B_{2R} \subset \Omega, \\ 0 & \text{else,} \end{cases} \\
\tilde\tau_R &= \zeta(x - x_0) \cdot \psi\big(t + (T - R^2)\big)\,.
\end{aligned}
\tag{35}
$$

Again we have to estimate the truncation terms which one has to accept since $(u - \bar u)\tilde\tau_R^2 \notin [V_h]^N$. This is done in the following proposition.

**Proposition 3.3** *With $\varphi = (u_h - \bar u_h)\tilde\tau_R^2$ in (24) the following estimate holds true:*

$$
\begin{aligned}
T_1(\varphi) + T_2(\varphi) &= \big|\big(\dot u_h, (u_h - \bar u_h)\tilde\tau_R^2 - I_h\big((u_h - \bar u_h)\tilde\tau_R^2\big)\big| + \\
&\quad + \Big|\Big(a_i^h, D_i\big[(u_h - \bar u_h)\tilde\tau_R^2 - I_h\big((u_h - \bar u_h)\tilde\tau_R^2\big)\big]\Big)\Big| \le \\
&\le R^2 \Big( \|\dot u_h\|^2_{A_{2R,h}} + \|\nabla u_h |W_t|^{1/2}\|^2_{A_{2R,h}} \Big)\,.
\end{aligned}
\tag{36}
$$

**Proof:** Arguing as in Proposition 3.1 with $\tilde{\tau}_R = 1$ on $B_R$, in the left-hand side of (36) there appear only integrals over $A_{2R,h}$. A variant of Poincaré's inequality, [7, Prop. 6.1] gives

$$\int_{A_{2R,h}} |u_h - \bar{u}_h|^2 \, dx \lesssim \int_{A_{2R,h}} |x - x_0|^2 |\nabla u_h|^2 \, dx, \tag{37}$$

if $A_{2R,h} \subset \Omega_h$. If $A_{2R,h} \cap \partial\Omega_h \neq \emptyset$, we extend $u_h$ by zero outside of $\Omega_h$ and replace $A_{2R,h}$ by $A_{4R}$ in the righthand side of (37). This works since $u_h$ vanishes on a sufficiently large subset of $A_{4R}$. Furthermore, we have the obvious inequality

$$R^{-2} + |x - x_0|^2 R^{-4} \lesssim |W_t| \text{ for } x \in A_{2R,h} \tag{38}$$

From Hölder's inequality and Proposition 2.2, it follows

$$\begin{aligned} T_1 &\leq \|\dot{u}_h\|_{A_{2R,h}} \cdot h\big(\|\nabla u_h \tilde{\tau}_R^2\|_{A_{2R,h}} + \|(u_h - \bar{u}_h) \cdot \nabla(\tilde{\tau}_R^2)\|\big)_{B_{2R}\setminus B_R} \\ &\lesssim R^2 \|\dot{u}_h\|_{A_{2R,h}} + \frac{h^2}{R^2} \|\nabla u_h \tilde{\tau}_R^2\|^2_{A_{2R,h}} + \frac{h^2}{R^4} \|u_h - \bar{u}_h\|^2_{B2R\setminus B_R} \\ &\lesssim R^2 \|\dot{u}\|^2_{A_{2R,h}} + h^2 \|\nabla u_h |W_t|^{1/2}\|^2_{A_{2R,h}} \\ &\lesssim R^2 \big(\|\dot{u}\|^2_{A_{2R,h}} + \|\nabla u_h |W_t|^{1/2}\|^2_{A_{2R,h}}\big) \,, \end{aligned} \tag{39}$$

for the last two inequalities we used (37), (38) and finally the condition $4h < R$.

To estimate the second term, we first note that $|W_t|^{-1/2} \lesssim R$ for $|x - x_0|^2 \leq 4R^2$, $R^2 \leq T - t \leq 4R^2$. Applying Hölder's inequality, then again the growth condition (3) and the Proposition 2.2 we gain

$$\begin{aligned} &\Big|\big(a_i^h, D_i\left[(u_h - \bar{u}_h)\tilde{\tau}_R^2 - I_h\big((u_h - \bar{u}_h)\tilde{\tau}_R^2\big)\right]\big)\Big| \\ &\lesssim R(\|\nabla u_h |W_t|^{1/2}\|_{A_{2R,h}} + \||W_t|^{1/2}\|_{A_{2R,h}}) \frac{h}{R} \||(W_t)|^{-1/2} \nabla^2\big((u_h - \bar{u}_h)\tilde{\tau}_R^2\big)\|'_{A_{2R,h}} \\ &\lesssim R^2 \|\nabla u_h |W_t|^{1/2}\|^2_{A_{2R,h}} + 1 + \frac{h^2}{R^2} \|\nabla u_h\|^2_{A_{2R,h}} + \frac{h^2}{R^4} \|u - \bar{u}_h\|^2_{A_{2R,h}} \,. \end{aligned}$$

The last two terms can be treated exactly as in (39), so that we end up with (36). □

Finally we must estimate

$$|T_3(\varphi)| \leq \|f\|_{B_{2R+h}} \|\varphi\|_{B_{2R+h}} \leq K_\varepsilon \|f\|^2_{B_{2R+h}} + \varepsilon \|I_h \varphi\|^2_{B_{2R+h}},$$

which comes down to estimate $I_h\varphi$ using (21). We carry out the calculations for $\varphi = W\tau_R^2 \dot{u}_h$, in the other cases the arguments run in a completely analogous way or even simpler. With

$$D_j(W\tau_R^2 \dot{u}_h) = D_j W \, \tau_R^2 \dot{u}_h + W D_j(\tau_R^2)\dot{u}_h + W\tau_R^2 D_j \dot{u}_h$$

relation (34) for the second derivatives and taking into account the bounds (29), (30) and

$$|W| + |\tau_R^2| + R|\nabla \tau_R^2| + R^2 |\nabla^2 \tau_R^2| \lesssim 1$$

we obtain

$$\begin{aligned}
\|W\tau_R^2\dot u_h\|_{B_{2R+h}} &\lesssim \|\dot u_h\|_{B_{2R}},\\
h\|\nabla(W\tau_R^2\dot u_h)\|_{B_{2R+h}} &\lesssim \frac{h}{R}\|\dot u_h\|_{B_{2R}} + h\|\nabla\dot u_h\|_{B_{2R}},\\
h^2\|\nabla^2(W\tau_R^2\dot u_h)\|_{B_{2R+h}} &\lesssim \frac{h}{R}\|\dot u_h\|_{B_{2R}} + \frac{h^2}{R}\|\nabla\dot u_h\|_{B_{2R}} + \left(\frac{h}{R}+1+\frac{h^2}{R^2}\right)\|\dot u_h\|_{B_{2R}}.
\end{aligned}$$

Now we again exploit $\frac{h}{R}\lesssim 1$ together with the inverse inequality (22) and arrive at the following bounds:

$$\begin{aligned}
&\|I_h(W\tau_R^2\dot u_h)\|_{B_{2R+h}} + \|I_h(\tau_R^2\dot u_h)\|_{B_{2R+h}} \lesssim \|\dot u_h\|_{B_{2R}}\\
&\|I_h((u_h-\bar u_h)\tilde\tau_R^2)\|_{B_{2R+h}} \lesssim \|u_h\|_{B_{2R}}.
\end{aligned}$$

## 4 Discrete Morrey Estimates for the Finite Element Solutions

In order to obtain uniform convergence for a subsequence of $u_h$ we can reproduce our proof from [7] and obtain a *hole filling* inequality

$$\iint\limits_{Q_R} \left\{|\dot u_h|^2 + |\nabla u_h|^2|W_t|\right\}\,dx \le K \iint\limits_{Q_{MR}\setminus Q_R} \left\{|\dot u_h|^2 + |\nabla u_h|^2|W_t|\right\}\,dx + KR^{2\gamma}. \tag{40}$$

In fact, in the proof here we can choose $M=8$, e.g. which is quite rough, by the way, since we deal with a Lipschitz boundary $\partial\Omega$, in [7, Sec. 6] we have dealt with slightly weaker assumptions on the boundary.

for $R\ge 4h$ and all parabolic cylinders $Q_R$, uniformly as $h\to 0$, here $W_t$ is defined as in (23). From (40) we derive the Morrey condition for $|\dot u_h|^2 + |\nabla u_h|^2|W_t|$ in a standard way via the hole filling argument, cf. [7, (23)] and [15].

We emphasize that in [7], the inequality (40) was derived by using the following assertions: $u\in L^\infty(H_0^1(\Omega))\cap H^1([0,T^0]\times\Omega)$, $u_t\in L^2([0,T^0]\times\Omega)$, $u$ satisfies the initial condition together with

$$\int_0^{T_0}\int_\Omega \Big[-u_t\varphi + \sum_{i=1}^n a_i(.,\nabla u)\cdot D_i\varphi\Big]\,dx\,dt = \int_0^{T_0}\int_\Omega f\varphi\,dx\,dt \tag{41}$$

and the entropy inequality (5), respectively, for the three special test functions $\varphi=\dot u\tau_R^2$, $\varphi=\dot uW\tau_R^2$ and $\varphi=(u-\bar u)\tau_R^2$. Other than this only the growth and coerciveness properties of the coefficients $a_i$ and their potential $A$ were needed in the argumentation.

If replace $u$ by $u_h$, there appear the additional truncation terms $T_i(\varphi)$, $i=1,2,3$ on the right hand sides of the weak equation (41) and the entropy inequality (5) for the approximations $u_h$. The corresponding estimates of Section 3 are the main part in treating the finite element approximations.

Otherwise the proof for (40) runs just along the same lines as in [7], therefore we confine us to a mere sketch of the proof here to avoid a complete repetition. We introduce the time intervals and the space-time cylinder

$$\mathscr{I}_R = [T - R^2, T], \quad \tilde{\mathscr{I}}_R = [T - 2R^2, T - R^2], \quad \mathscr{H}_R = \mathscr{I}_{\sqrt{2}R} \times (B_{2R} \setminus B_R).$$

1. *Bounds for* $\iint_{Q_R} |\nabla u_h|^2 |W_t|\, dx$: We choose $\varphi = \dot{u}_h W \tau_R^2$ in the basic equation (24) and proceed estimating as in the first step of Section 3 in our paper [7]. We arrive at the inequality

$$\begin{aligned}\iint |\nabla u_h|^2 |W_t| \tau_R^2 \, dx\, dt \lesssim \iint_{\tilde{\mathscr{I}}_R \times B_{2R}} |\nabla u_h|^2 R^{-2} \tau_R \, dx\, dt + \\ + K \iint_{\mathscr{H}_R} |\nabla u_h|^2 |W_t| \, dx\, dt + \iint \dot{u}_h^2 \tau_R^2 \, dx\, dt + R^{2\gamma} + \\ + \text{ truncation terms.}\end{aligned} \tag{42}$$

The truncation terms here can be estimated using Proposition 3.2 by

$$\varepsilon \int_{\mathscr{I}_{\sqrt{2}R}} \|\nabla u_h W_t^{1/2}\|^2_{B_{2R+h}} \, dt + K R^2 + K \int_{\mathscr{I}_{\sqrt{2}R}} \|\dot{u}_h\|^2_{B_{2R+h}} \, dt\,.$$

If any of the time intervals touches the value $t = 0$, we have to replace it by the corresponding intersection of the interval with $t > 0$ while the initial value appears on the right hand side. Note that the right hand side of (42) contains *good* terms, namely $KR^{2\gamma}$ (coming from the right hand side $f$ and the initial condition $u_0$) and the integral $\iint_{\mathscr{H}_R}$. The terms

$$\iint \dot{u}_h^2 \tau_R^2 \, dx\, dt \le \int_{\mathscr{I}_{\sqrt{2}R}} \|\dot{u}_h\|^2_{B_{2R+h}} \, dt$$

will be estimated in a second step, and we will deal with the *critical integral*

$$I_{crit} = \iint_{\tilde{\mathscr{I}}_R \times B_{2R}} |\nabla u_h|^2 \, R^{-2} \, dx\, dt \tag{43}$$

in a third step.

We do not want to repeat the arguments which lead to (42) in detail but want to give the hint, that the term $\iint |\nabla u_h|^2 |W_t| \tau_R^2 \, dx\, dt$ arises from

$$\begin{aligned}&\sum_{i=1,2} \iint a_i^h D_i(\dot{u}_h W \tau_R^2) \, dx\, dt \\ &= \iint \left[ \left( \frac{d}{dt} A(\cdot, \cdot, \nabla u_h) - A_t(., ., \nabla u_h) \right) \right] W \tau_R^2 \, dx\, dt + \\ &+ \sum_{i=1,2} \iint a_i^h \dot{u}_h D_i(W \tau_R^2) \, dx\, dt\end{aligned}$$

as lower estimate for

$$-\iint AW_t\tau_R^2\,dx\,dt+\text{ a pollution term containing }\tau_t\,.$$

The pollution term containing $\tau_t$ creates our *critical term*.

2. *Bounds for* $\iint \dot u_h^2\,dx\,dt$: We choose $\varphi=\dot u_h\tau_R^2$ in (24) and obtain

$$\begin{aligned}&\iint \dot u_h^2\tau_R^2\,dx\,dt+\iint \frac{d}{dt}A(\cdot,\cdot,\nabla u_h)\tau_R^2\,dx\,dt-\\&\quad-\iint A_t(\cdot,\cdot,\nabla u_h)\,\tau_R^2dx\,dt+2\sum_{i=1}^{2}\iint a_i^h\dot u_h\nabla\tau_R\tau_R\,dx\,dt=\\&=\iint f\dot u_h\tau_R^2\,dx\,dt+\text{ truncation terms }.\end{aligned}\tag{44}$$

The truncation terms are estimated in Proposition 3.1. The left hand side of (44) is treated exactly as in our paper [7, Formula (35)]. This leads to the estimate

$$\iint \dot u_h^2\tau_R^2\,dx\,dt\lesssim I_{crit}+\iint\limits_{\mathscr{H}_R}|\nabla u_h|^2|W_t|\,dx\,dt+\iint\limits_{\mathscr{H}_R}\dot u_h^2\,dx\,dt+R^{2\gamma},\tag{45}$$

where the term $R^{2\gamma}$ comes from pollution terms and the integral over $f$ in view of (10), and $I_{crit}$ is defined in (43).

Employing our estimate (25) we conclude from (45)

$$\iint \dot u_h^2\tau_R^2\,dx\,dt\lesssim I_{crit}+\iint\limits_{\tilde{\mathscr{I}}_R\times A_{2R,h}}\left\{|\nabla u_h|^2|W_t|+\dot u_h^2\right\}\,dx\,dt+R^{2\gamma}\tag{46}$$

uniformly as $h\to 0$, $R\ge h$, which completes the second step.

We combine (46) by multiplying (46) with a large constant and adding the result to inequality (42). This leads to the "pre-hole-filling inequality"

$$\begin{aligned}&\iint\left\{\dot u_h^2+|\nabla u_h|^2|W_t|\right\}\tau_R^2\,dx\,dt\lesssim I_{crit}+\\&\quad+\iint\limits_{\tilde{\mathscr{I}}_R\times A_{2R,h}}\left\{\dot u_h^2+|\nabla u_h|^2|W_t|\right\}\,dx\,dt+R^{2\gamma},\quad R\ge 4h\,.\end{aligned}$$

3. *Estimating the critical term via Cacciopoli's inequality:*
We intend to give a bound for the term $\iint_{\tilde{\mathscr{I}}_R\times B_{2R}}|\nabla u|^2R^{-2}\,dx\,dt$. To this end, we choose

$$\varphi=(u_h-\bar u_h)\tilde\tau_R^2$$

in the equation (24), where $\tilde\tau_R$ is defined as in (35) and $\bar u_h$ as in Proposition 3.3. From Proposition 3.3 we have

$$\begin{aligned}&\int \dot u_h(u_h-\bar u_h)\tilde\tau_R^2\,dx+\sum_{i=1,2}\int a_i^hD_i\big((u_h-\bar u_h)\tilde\tau_R^2\big)\,dx\\&\quad\lesssim\|f(t,\cdot)\|\|(u_h-\bar u_h)\tilde\tau_R^2)\|+R^2\big(\|\dot u_h\|_{A_{2R,h}}^2+\|\nabla u_h|W_t|^{1/2}\|_{A_{2R,h}}^2\big)\\&\quad\lesssim R^2\|f(t,\cdot)\|^2+R^{-2}\|u_h-\bar u_h\|_{B_{2R}}+R^2\big(\|\dot u_h\|_{A_{2R,h}}^2+\|\nabla u_h|W_t|^{1/2}\|_{A_{2R,h}}^2\big)\,.\end{aligned}\tag{47}$$

The left hand side of (47) is estimated as in [7], using the coerciveness condition for $a_i$ and, in particular, again Poincaré's inequality in a special form, namely

$$R^{-2}\int\limits_{B_{2R}} |u_h-\bar{u}_h|^2\,dx \lesssim R^{-2}\int\limits_{B_{2R}} |x-x_0|^2|\nabla u_h|^2\,dx$$

if $B_{2R}=B_{2R}(x_0)\subset\Omega$, and $B_{2R}$ has to be replaced by $B_{4R}$, if $B_{2R}\cap\complement\Omega\neq 0$. (Observe that $u_h$ can be extended by zero outside of $\Omega_h$ and $\bar{u}_h=0$ by definition in this case.) Further we can estimate

$$\begin{aligned}\Big|\int \dot{u}_h(u_h-\bar{u}_h)\tilde{\tau}_R^2\,dx\Big| &\le R^2\int\limits_{B_{2R}}\dot{u}_h^2\,dx+\frac{1}{R^2}\int\limits_{B_{2R}}|u_h-\bar{u}_h|^2\,dx\\ &\lesssim R^2\int\limits_{B_{2R}}\dot{u}_h^2\,dx+\frac{1}{R^2}\int\limits_{B_{2R}}|x-x_0|^2|\nabla u_h|^2\,dx,\end{aligned}$$

a.e. on $[T-4R^2,T-R^2]=:\hat{\mathscr{I}}_R$. Thus we obtain

$$\begin{aligned}I_{crit}\lesssim R^{-2}\iint a_i^h D_i u_h\tilde{\tau}_R^2\,dx\,dt+KR^2 &\lesssim \iint\limits_{\hat{\mathscr{I}}_R\times B_{2R}}\dot{u}_h^2\,dx\,dt\\ +R^{-4}\iint\limits_{\hat{\mathscr{I}}_R\times B_{4R}}|x-x_0|^2|\nabla u_h|^2\,dx\,dt &+\frac{1}{R^2}\text{ right hand side of (47)}\,.\end{aligned}$$

We take into account that $R^{-4}|x-x_0|^2\lesssim|W_t|$ and $\hat{\mathscr{I}}_R\times B_{4R}\subset Q_{4R}\setminus Q_R$. Then, for $R\ge 4h$, we arrive at

$$I_{crit}\lesssim\iint\limits_{Q_{4R}\setminus Q_{R-h}}\left\{\dot{u}_h^2+|\nabla u_h|^2|W_t|\right\}\,dx\,dt+KR^{2\gamma}\,.$$

Together with the results of the first and second step of this chapter we finally arrive at the hole filling inequality (40) and we have obtained the following

**Theorem 4.1** *Let $\Omega\subset\mathbb{R}^2$ be a Lipschitz domain. Assume the coefficient functions $a_i$ of problem* (1) *and their potential $A$ fulfill the conditions formulated in Section 1, in particular* (3), (4)*, and the Morrey conditions* (9) *and* (10) *hold for the data $f$ and $u_0$. Then the finite element approximations $u_h$ defined by* (13), (14) *satisfy* (8)*, in particular they are uniformly bounded in the Hölder-space $C^{\alpha/2}([0,T^0]\times(\overline{\Omega})$ as $h$ tends to* 0*.*

**Proof.** The hole filling inequality implies Morrey's condition

$$\iint\limits_{Q_R}|\dot{u}_h|^2+|\nabla u_h|^2dx\,dt\lesssim R^{2\beta},\quad R\ge 4h, \tag{48}$$

uniformly in $h$ and $R$, up to the boundary (read $Q_R$ as $Q_R\cap[0,T]\times\Omega_h$ near the boundary). Since $|W_t|\ge c_0R^{-2}$, $c_0>0$, on $B_R\setminus B_{R/4}$, after redefining $R$ we have

$$\iint\limits_{Q_R}|\nabla u_h|^2dx\,dt\lesssim R^{2+2\beta},$$

(see [7, Sect. 5] for the detailed argument). We inspect equation (44) once more and use the term $\int_{t_1}^{t_2} \frac{d}{dt} A(\cdot,\cdot,\nabla u_h)\tau_R^2 dx\, dt$ to create an estimate

$$\int A(\cdot,\cdot,\nabla u_h)\tau_R^2 dx|_{t=t_2} \leq \int A(\cdot,\cdot,\nabla u_h)\tau_R^2 dx|_{t=t_1} + \text{remainder}.$$

The remaining terms satisfy a Morrey condition to (48). Averaging $t_1$ on $[t_2 - R^2, t_2]$ leads to

$$\int_{B_R} |\nabla u_h|^2 dx|_T \lesssim \int A\tau_R^2 dx + R^{2\gamma} \lesssim R^{2\beta}.$$

Now we have to use a *discrete* version of Morrey's lemma, which holds true for functions from $V_h$, provided $R \geq 4h$. The proof can be done analogously as in the finite difference case, see [4] or using the "singularity"-estimates for the discrete Green function $G_h$ [6] together with the representation

$$|B_h| \int_{B_h} \tau_R(u_h - \bar{u}_h)dx = (\nabla[(u_h - \bar{u}_h)\tau_R], \nabla G_h).$$

These arguments give uniform Hölder continuity for $u_h$ in *space direction*, i.e. a bound in $L^\infty(0, T^0, C^\alpha(\overline{\Omega})$. Finally in view of this *and* the $L^2(L^2)$ bound for $\dot{u}_h$ we derive also uniform Hölder continuity of $u_h$ in time direction, i.e. (8) holds. □

# 5 Convergence of the Finite Element Approximation and Existence of a Weak Solution in the Sense of Young Measures

Due to the estimate (18) we may select a subsequence $(u_h|_{h\to 0})$ such that there exists $u \in L^2\big(H^1(\Omega)\big)$:

$$u_h \to u \text{ strongly in } L^2(L^2), \quad \nabla u_h \rightharpoonup \nabla u, \dot{u}_h \rightharpoonup u_t \text{ weakly in } L^2(L^2). \tag{49}$$

Furthermore, (18) implies $\nabla u \in L^\infty(L^2)$. Since $u_h(t) \in H_0^1(\Omega_h)$ for all $h$ and $\mathrm{dist}(\partial\Omega, \partial\Omega_h) \to 0$, it also follows that $u(t) \in H_0^1(\Omega)$ , a. e. with respect to $t \in [0, T^0]$. This holds without any assumption on $\partial\Omega$ - there is a proof using capacity methods [5]. A simpler proof uses that $\partial\Omega$ has the so called *segment property* [14] which is, after all, a mild assumption, cf. [1].
Finally, by Theorem 4.1, we have uniform convergence of the functions $u_h$, in particular, the estimate

$$\operatorname*{ess\,sup}_{0\leq t\leq T^0} \int_{B_R\cap\Omega} |\nabla u|^2\, dx \leq KR^\alpha, \text{ hence } u \in C^\alpha\big(\overline{\Omega}\big) \text{ a.e. in } t.$$

Unfortunately, due to the lack of monotonicity, we do not have $\nabla u_h \to \nabla u$ point-wise almost everywhere, so we cannot obtain that $a_i(.,\nabla u_h) \to a_i(.,.,\nabla u)$, and we do *not* obtain weak solutions in the usual way. Instead of this we have to confine us to solutions in the sense of Young measures. We use the setting as it is exposed by [8, 3, 12].

To this end we introduce the following notation: If $\nu$ is a probability measure on $\mathbb{R}^{N\times 2}$ and $g \in C(\mathbb{R}^{N\times 2})$, we set

$$\langle \nu, g\rangle = \int_{\mathbb{R}^{N\times 2}} g(\eta)d\nu(\eta),$$

if it exists.

**Definition 5.1** *We call $u \in L^\infty(H_0^1) \cap H^1([0,T^0]\times\Omega)$ with $u(0) = u_0$ a measure valued solution to problem (1), (2) if there exists a parametrized family $(\nu_{t,x})$ with $(t,x) \in [0,T^0]\times\Omega$ of probability measures on $\mathbb{R}^{N\times 2}$, such that the mapping*

$$[0,T^0]\times\Omega \ni (t,x) \to \langle \nu_{t,x}, a(t,x,\cdot)\rangle$$

*is Lebesgue measurable for any Caratheodory function a (i.e. a is measurable in $(t,x)$ and continuous in $\eta$), and the following identities hold:*

$$\langle \nu_{t,x}, id\rangle = \nabla u \text{ a.e. in } (t,x) \tag{50}$$

$$\iint\limits_{[0,T^0]\times\Omega} u_t\cdot\varphi + \langle \nu_{t,x}, a\rangle\nabla\varphi\,dx\,dt = \iint\limits_{[0,T^0]\times\Omega} f\cdot\varphi\,dx\,dt, \tag{51}$$

*for all $\varphi \in C_0^\infty((0,t)\times\Omega)$. The function is called a Young measure solution if there exists a $q > 0$ and a sequence $(u_k) \subset L^2(H^1)$ such that*

$$\int_A \Phi(\nabla u_k)d(x,y) \to \int_A \langle \nu_{t,x}, \Phi\rangle d(x,t)$$

*for any measurable set $A \subset [0,T^0]\times\Omega$ and $\Phi \in C(\mathbb{R}^{N\times 2})$ with $\Phi(\eta) \lesssim 1 + |\eta|^q$.*

Note that condition (51) implies that for $\varphi \in C_0^\infty(\Omega)$,

$$\frac{d}{dt}(\dot{u},\varphi) + (\langle \nu_{t,x}, a\rangle, \nabla\varphi) = (f,\varphi) \text{ a.e in } t.$$

We use the results of [2] to prove our main theorem.

**Theorem 5.1** *Let $\Omega$, $a_i$, $f$ and $u_0$ meet the requirements of theorem 4.1 and $T > 0$ arbitrary. Then there exists a function*

$$u \in L^\infty(H^1(\Omega)) \cap C^{\alpha/2}([0,T^0]\times\overline{\Omega}) \cap L^\infty(0,T^0,C^\alpha(\overline{\Omega}))$$

*with $u_t \in L^2(L^2(\Omega))$ such that $u$ solves the parabolic system (1) together with the initial condition (2) in the sense of Young measure.*

**Proof.** Let $(h_k) \to 0$ be a sequence of discretization parameters such that (49) holds for $h = h_k$. Applying Ball's variant for the fundamental theorem for Young measures [2] we obtain a family $\nu_{t,x}$ of probability measures on $\mathbb{R}^{N\times 2}$ such that (50) holds. Using the growth condition (3) for the coefficient functions $a_i$ we obtain that the sequence $a(\cdot,\cdot,\nabla u_h)$ is bounded in $L^2(L^2)$. Hence the main theorem in [2] together with remark [2, 3, p. 210] imply

$$a(\cdot,\cdot,\nabla u_h) \rightharpoonup \langle \nu_{t,x}, a\rangle \text{ in } L^2(L^2). \tag{52}$$

In order to show (51) we use the finite element equation (13) in the formulation (24) again:

$$\begin{aligned}\int_0^T \big\{(\dot u_h,\varphi) + (a_i^h, D_i\varphi) - (f,\varphi)\big\}\, dt = \\ = \int_0^T \big\{(\dot u_h, \varphi - I_h\varphi) + \big(a_i^h, D_i(\varphi - I_h\varphi)\big) - (f, \varphi - I_h\varphi)\big\}\, dt,\end{aligned} \tag{53}$$

where $\varphi \in C^1\big([0,T^0]\times(\Omega)\big)$, and $\varphi = 0$ on $\partial\Omega$ for all $t$. Since $\dot u_h, a_i^h$ are bounded in $L^2([0,T^0]\times\Omega)$, the right hand side of (53) tends to 0 as $h\to 0$, if we take into account that

$$\|\varphi_h - I_h\varphi\|_{L^2([0,T^0]\times\Omega)} + \|\nabla\varphi_h - \nabla I_h\varphi\|_{L^2([0,T^0])\times\Omega} = o(1)\,.$$

Owing to (49) and (52), the terms on the left hand side of (53) converge to

$$\int_0^T (u_t,\varphi)\, dt + \int_0^T\int_\Omega \int_{\mathbb{R}^{N\times 2}} a_i(\cdot,\cdot,\eta) d\nu_{x,t}(\eta)\, dx\, dt - \int_0^T (f,\varphi) dt,$$

and we obtain, that the weak limit $u$ of the finite element approximation satisfies the equation (51). Furthermore, by Theorem 4.1, $u\in C^{\alpha/2}$ with respect to $t$ and $u\in C^\alpha$ with respect to the spatial variable, while the initial condition is fulfilled due to (14) and (15). This finishes the proof of the theorem. □

## References


[1] S. Agmon, *Lectures on elliptic boundary value problems*, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965.

[2] J. M. Ball, *A version of the fundamental theorem for Young measures*, in PDEs and continuum models of phase transitions (Nice, 1988), vol. 344 of Lecture Notes in Phys., Springer, Berlin, 1989, pp. 207–215.

[3] S. Demoulini, *Young measure solutions for a nonlinear parabolic equation of forward-backward type*, SIAM J. Math. Anal., 27 (1996), pp. 376–403.

[4] J. Frehse, *Morrey space methods in the theory of elliptic difference equations*, in Numerische, insbesondere approximationstheoretische Behandlung von Funktionalgleichungen (Tagung, Math. Forschungsinst., Oberwolfach, 1972), Springer, Berlin, 1973, pp. 96–117. Lecture Notes in Math., Vol. 333.

[5] ——, *Capacity methods in the theory of partial differential equations*, Jahresber. Deutsch. Math.-Verein., 84 (1982), pp. 1–44.

[6] J. Frehse and R. Rannacher, *Eine $L^1$-Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente*, in Finite Elemente (Tagung, Univ. Bonn, Bonn, 1975), Inst. Angew. Math., Univ. Bonn, Bonn, 1976, pp. 92–114. Bonn. Math. Schrift., No. 89.

[7] J. Frehse and M. Specovius-Neugebauer, *Morrey estimates and hölder continuity for solutions to parabolic equations with entropy inequalities.* To appear in J. Reine Angew. Math.

[8] D. Kinderlehrer and P. Pedregal, *Weak convergence of integrands and the young measure representation*, SIAM J. Math. Anal., 23 (1992), pp. 1–19.

[9] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[10] S. Müller, M. O. Rieger, and V. Šverǎk, *Parabolic systems with nowhere smooth solutions*, Arch. Ration. Mech. Anal., 177 (2005), pp. 1–20.

[11] J. Naumann and J. Wolf, *Hölder continuity of weak solutions to parabolic systems with controlled growth non-linearities (two spatial dimensions)*, Matematiche (Catania), 55 (2000), pp. 125–144 (2001). Partial differential equations (Italian) (Pisa, 2000).

[12] M. Slemrod, *Dynamics of measure valued solutions to a backward-forward heat equation*, J. Dynam. Differential Equations, 3 (1991), pp. 1–28.

[13] J. Stará and O. John, *On some regularity and nonregularity results for solutions to parabolic systems*, Matematiche (Catania), 55 (2000), pp. 145–163 (2001). Partial differential equations (Italian) (Pisa, 2000).

[14] F. Stummel, *Über die Differenzenapproximation des Dirichletproblems für eine lineare elliptische Differentialgleichung zweiter Ordnung.*, Math. Ann., 163 (1966), pp. 321–339.

[15] K. O. Widmann, *Hölder continuity of solutions of elliptic systems*, Manuscripta Math, 5 (1971), pp. 299–308.

[16] J. Wolf, *Hölder continuity of weak solutions to certain nonlinear parabolic systems in two space dimensions*, in Applied nonlinear analysis, Kluwer/Plenum, New York, 1999, pp. 531–546.

Bestellungen nimmt entgegen:

Sonderforschungsbereich 611
der Universität Bonn
Poppelsdorfer Allee 82
D - 53115 Bonn

Telefon: 0228/73 4882
Telefax: 0228/73 7864
E-Mail: astrid.link@ins.uni-bonn.de http://www.sfb611.iam.uni-bonn.de/

**Verzeichnis der erschienenen Preprints ab No. 415**

415. Weber, Hendrik: On the Short Time Asymptotic of the Stochastic Allen-Cahn Equation

416. Albeverio, Sergio; Ayupov, Shavkat A.; Kudaybergenov, Karim K.; Kalandarov, T. S.: Complete Description of Derivations on $\tau$-compact Operators for Type I von Neumann Algebras

417. Bartels, Sören: Semi-Implicit Approximation of Wave Maps into Smooth or Convex Surfaces

418. Albeverio, Sergio; Ayupov, Shavkat A.; Kudaybergenov, Karim K.: Structure of Derivations on Various Algebras of Measurable Operators for Type I von Neumann Algebras

419. Albeverio, Sergio; Ayupov, Shavkat A.; Abdullaev, Rustam Z.: On an Algebra of Operators Related to Finite Traces on a von Neumann Algebra

420. Otto, Felix; Viehmann, Thomas: Domain Branching in Uniaxial Ferromagnets – Asymptotic Behavior of the Energy

421. Kurzke, Matthias: Compactness Results for Ginzburg-Landau Type Functionals with General Potentials

422. Hildebrandt, Stefan; von der Mosel, Heiko: Conformal Mapping of Multiply Connected Riemann Domains by a Variational Approach

423. Harbrecht, Helmut: On Output Functionals of Boundary Value Problems on Stochastic Domains

424. Griebel, Michael; Hegland, Markus: On the Numerical Determination of Maximum A-Posteriori Density Estimators Based on Exponential Families and Gaussian Process Priors with an Infinite Number of Parameters

425. Capella, Antonio; Otto, Felix: A Rigidity Result for a Perturbation of the Geometrically Linear Three-Well Problem

426. Albeverio, Sergio; Motovilov, Alexander K.; Shkalikov, Andrei A.: Bounds on Variation of Spectral Subspaces under J-Self-Adjoint Perturbations

427. Albeverio, Sergio; Günther, Uwe; Kuzhel, Sergii: J-Self-Adjoint Operators with C-Symmetries: Extension Theory Approach

428. Kurzke, Matthias; Melcher, Christof; Moser, Roger; Spirn, Daniel: Dynamics for Ginzburg-Landau Vortices under a Mixed Flow

429. Otto, Felix; Steiner, Jutta: The Concertina Pattern – From Micromagnetics to Domain Theory

430. Frehse, Jens; Málek, Josef; Ružička, Michael: Large Data Existence Result for Unsteady Flows of Inhomogeneous Heat-Conducting Incompressible Fluids

431. Croce, Roberto; Griebel, Michael; Schweitzer, Marc Alexander: Numerical Simulation of Bubble and Droplet Deformation by a Level Set Approach with Surface Tension in Three Dimensions

432. Frehse, Jens; Löbach, Dominique: Regularity Results for Three Dimensional Isotropic and Kinematic Hardening Including Boundary Differentiability

433. Arguin, Louis-Pierre; Kistler, Nicola: Small Perturbations of a Spin Glass System

434. Bolthausen, Erwin; Kistler, Nicola: On a Nonhierarchical Version of the Generalized Random Energy Model. II. Ultrametricity

435. Blum, Heribert; Frehse, Jens: Boundary Differentiability for the Solution to Hencky's Law of Elastic Plastic Plane Stress

436. Albeverio, Sergio; Ayupov, Shavkat A.; Kudaybergenov, Karim K.; Nurjanov, Berdach O.: Local Derivations on Algebras of Measurable Operators

437. Bartels, Sören; Dolzmann, Georg; Nochetto, Ricardo H.: A Finite Element Scheme for the Evolution of Orientational Order in Fluid Membranes

438. Bartels, Sören: Numerical Analysis of a Finite Element Scheme for the Approximation of Harmonic Maps into Surfaces

439. Bartels, Sören; Müller, Rüdiger: Error Controlled Local Resolution of Evolving Interfaces for Generalized Cahn-Hilliard Equations

440. Bock, Martin; Tyagi, Amit Kumar; Kreft, Jan-Ulrich; Alt, Wolfgang: Generalized Voronoi Tessellation as a Model of Two-dimensional Cell Tissue Dynamics

441. Frehse, Jens; Specovius-Neugebauer, Maria: Existence of Hölder Continuous Young Measure Solutions to Coercive Non-Monotone Parabolic Systems in Two Space Dimensions