# On Boundary Regularity for the Stress in Problems of Linearized Elasto-Plasticity

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# ON BOUNDARY REGULARITY FOR THE STRESS IN PROBLEMS OF LINEARIZED ELASTO-PLASTICITY

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Abstract. We investigate regularity properties of the stress tensor near the boundary for models of elasto-plasticity in arbitrary dimension. Focusing on special geometries, namely on balls and infinite strips, we obtain $L^2$-estimates for the tangential derivatives of the stress tensor near the boundary. We indicate why these estimates may fail for more general domains. In addition, we establish $L^2$-estimates for (the trace of) the stress tensor on the boundary.

## 1. Introduction

We study regularity properties of solutions to problems arising from mechanics of linearized elastic plastic materials (this part of continuum mechanics is usually called *linearized elasto-plasticity*). Before we formulate results regarding mathematical analysis for some of these problems we briefly characterize what we mean by linearized elastic plastic response of materials and we end up the presentation with the formulations of relevant boundary value problems for the Prandtl-Reuss and Hencky model of plasticity. Then we provide an overview of known results concerning theoretical analysis of problems of linearized elasto-plasticity. We also introduce notation, define suitable approximations of the original problem and summarize results concerning mathematical properties of these approximations. Boundary regularity is then investigated for the Hencky model of plasticity although the analysis for the Prandtl-Reuss model can be treated in a similar manner. In Section 2, we establish results on uniform (approximation independent) estimates of tangential derivatives in special geometries, and we indicate in Appendix why these estimates are not proved for more general domains. Section 3 is devoted to the uniform estimates for $\boldsymbol{\sigma}$ on the boundary again in special geometries. Finally, for sake of completeness, interior regularity estimates are proved in Appendix. Moreover, some version of the Korn-type inequality and the trace theorem in infinite domains are established in Appendix as well.

1.1. **Linearized Elasto-Plasticity.** Let us consider a cylindrical steel bar with the cross-section $A$ subjected to a cyclic loading consisting of tension and compression along the axis of the bar. Assuming that the load $F$ is uniformly distributed over the end cross-sections of the bar then the stress $\sigma$ equals $F/A$. The associated

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strain $\varepsilon$ is equal to $(\Delta\ell)/\ell$, where $\ell$ is the original length of the bar and $\Delta\ell$ is its extension. It reveals (and these investigations go back to Hooke (1678)) that for small deformations the ration between $\sigma$ and $\varepsilon$ is a constant of a given material. This constant $E$ is called Young's modulus. Thus, for a bar under small cyclic loadings we have $E = \sigma/\varepsilon$. Such a linearized elastic response holds up to a certain values of the stress and strain. Complicated responses happen once this critical point in the stress-strain plane is overcome; these responses are connected with the phenomena such as nonlinear elasticity, yielding, (kinematic and isotropic) hardening, softening, fatigue, to name a few. Except for nonlinear elastic response, the other phenomena are associated with undergoing microstructural (entropy producing, irreversible) changes in the body. Similar results as in the tension and compression tests concern torsion or bending as well.

Linearized elasto-(perfect) plasticity idealizes this (in general very complicated) response of the material as follows. If the value of the stress is below a critical value $\kappa$ (called the yield stress) the material responds as a linearized elastic solid. Once the value of the yield stress $\kappa$ is attained during the loading process then without any change of stress the strain increases and this type of response is called perfect plasticity. Upon unloading the material responds as a linearized elastic material and we assume (as it is mostly done in linearized elasto-plasticity) that this elastic response is the same independently of the state of the body in which the unloading takes place. Due to microstructural irreversible changes that took place when the yield condition was activated, this unloading process leads to a stress-free state that is different from the initial (stress free) state. Rajagopal (1995) and Rajagopal and Srinivasa (2004) call this stress-free configuration *natural (or preferred) configuration* associated to the body. Thus, we can view (see Rajagopal and Srinivasa (2004) for details) linearized elasto-plasticity as a class of elastic responses from a corresponding class of evolving natural configurations (as said above in the context of traditional elasto-plasticity it is assumed that these elastic response functions are identical). From this point of view it is natural to split the deformation into the sum[1] of plastic and elastic response. How the class of underlining natural configuration evolves (or how to characterize the constitutive relation for the response relevant to plasticity) is determined by the maximization of the rate of entropy production. For the sake of completeness we repeat here this procedure in the context of small deformations (and refer to Rajagopal and Srinivasa (2004) and Kratochvíl et al. (2004) for the complete treatment in the context of large deformations).

Let us assume that a body occupies a set $\Omega \subset \mathbb{R}^d$. Since we assume a priori that considered deformations are small, the initial, current and preferred (natural) configurations coincide and the deformation gradient is well approximated by the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ defined as

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)\,,$$

where $\boldsymbol{u}$ is the displacement. We also assume that the density is uniform (and the balance of mass is automatically fulfilled) and the inertia effects in the balance of

[1] In the context of large deformations the sum is "replaced" by a composition of two deformation tensors; one is relevant to instantaneous elastic (reversible) response of the body, the other to microstructural (irreversible) changes, see Rajagopal and Srinivasa (2004).

linear momentum are neglectable, which leads to the equation

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } [0,T]\times\Omega, \tag{1.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\boldsymbol{f}$ denotes the density of given external body forces and $T > 0$ is a given time.

Even in a purely mechanical context considered here we require that the reduced thermomechanical identity (RTI in short) holds. Since for the linearized strain the symmetric part of the velocity gradient equals to $\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})$, RTI takes the form

$$\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\psi}^* = \xi\,, \tag{1.2}$$

where $\psi^*$ is the Helmholtz potential and $\xi$ denotes the rate of entropy production; if $\xi \geq 0$ then the second law of thermodynamics is satisfied. At this point we focus on the appropriate constitutive equations for $\xi$ and $\psi^*$.

As stated above, the linearized elasto-plasticity is characterized by the decomposition of the linearized strain $\boldsymbol{\varepsilon}(\boldsymbol{u})$ into the elastic part $\boldsymbol{e}_{\mathrm{el}}$ and the plastic part $\boldsymbol{e}_{\mathrm{p}}$, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{e}_{\mathrm{el}} + \boldsymbol{e}_{\mathrm{p}}\,. \tag{1.3}$$

The Helmholtz potential is supposed to be a function of $\boldsymbol{e}_{\mathrm{el}}$. This means that

$$\psi^* = \psi^*(\boldsymbol{e}_{\mathrm{el}}) = \psi^*(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{e}_{\mathrm{p}})\,. \tag{1.4}$$

Inserting (1.4) with $\boldsymbol{e}_{\mathrm{el}} = \boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{e}_{\mathrm{p}}$ into (1.2) we obtain

$$\left(\boldsymbol{\sigma} - \frac{\partial\psi^*}{\partial\boldsymbol{e}_{\mathrm{el}}}\right)\cdot\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \frac{\partial\psi^*}{\partial\boldsymbol{e}_{\mathrm{el}}}\cdot\dot{\boldsymbol{e}}_{\mathrm{p}} = \xi\,. \tag{1.5}$$

If the response of the material does not dissipate any energy, i.e., if the material responds elastically, then $\xi = 0$, $\boldsymbol{e}_{\mathrm{p}} = \boldsymbol{0}$ and[2]

$$\boldsymbol{\sigma} = \frac{\partial\psi^*}{\partial\boldsymbol{e}_{\mathrm{el}}}\,. \tag{1.7}$$

In elasto-plasticity, requiring that (1.7) holds, (1.5) simplifies to

$$\frac{\partial\psi^*}{\partial\boldsymbol{e}_{\mathrm{el}}}\cdot\dot{\boldsymbol{e}}_{\mathrm{p}} = \xi\,. \tag{1.8}$$

Next, we assume that the constitutive equation for $\xi$ is of the form

$$\xi = \tilde{\xi}(\dot{\boldsymbol{e}}_{\mathrm{p}}) = \kappa|\dot{\boldsymbol{e}}_{\mathrm{p}}| \left(= \kappa\left(\dot{\boldsymbol{e}}_{\mathrm{p}}\cdot\dot{\boldsymbol{e}}_{\mathrm{p}}\right)^{1/2}\right) \qquad (\kappa > 0)\,. \tag{1.9}$$

Since the plastic part of the strain is relevant to fluid like behavior of the incompressible material it is reasonable to also assume that

$$\operatorname{tr}\boldsymbol{e}_{\mathrm{p}} = 0\,. \tag{1.10}$$

Applying the principle of maximization of the rate of entropy production, which means that we maximize $\tilde{\xi}$ w.r.t. $\dot{\boldsymbol{e}}_{\mathrm{p}}$ considering (1.8) and (1.10) as the constraints, the necessary condition for the extremum of this constrained maximization then

[2]For example, the isotropic homogeneous linearized elastic solid is characterized by the constitutive equation:

$$\psi^*(\boldsymbol{e}_{\mathrm{el}}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{e}_{\mathrm{el}})^2 + \mu\,\boldsymbol{e}_{\mathrm{el}}\cdot\boldsymbol{e}_{\mathrm{el}}\,. \tag{1.6}$$

Note that if $\boldsymbol{e}_{\mathrm{p}} = \boldsymbol{0}$ (1.7) leads to $\boldsymbol{\sigma} = \lambda\ \operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{u}) + 2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u})$.

leads to the equation (here $\gamma_1$ and $\gamma_2$ denote Lagrange multipliers associated to the constraints (1.8) and (1.10))

$$\frac{1+\gamma_1}{\gamma_1}\frac{\partial\tilde{\xi}}{\partial\dot{\boldsymbol{e}}_{\mathrm{p}}} = \frac{\partial\psi^*}{\partial\boldsymbol{e}_{\mathrm{el}}} - \frac{\gamma_2}{\gamma_1}\frac{\partial\,\mathrm{tr}\,\dot{\boldsymbol{e}}_{\mathrm{p}}}{\partial\dot{\boldsymbol{e}}_{\mathrm{p}}} = \boldsymbol{\sigma} - \frac{\gamma_2}{\gamma_1}\mathbf{I}\,, \tag{1.11}$$

where the last equality holds due to (1.7).

Hence, taking the scalar product of (1.11) with $\dot{\boldsymbol{e}}_{\mathrm{p}}$ and using (1.8) and (1.9) we deduce that $(1+\gamma_1)/\gamma_1 = 1$. Then, taking the trace of (1.11) and using the constraint (1.10) we obtain that $\gamma_2/\gamma_1 = \frac{1}{d}\,\mathrm{tr}\,\boldsymbol{\sigma}$. Thus, setting $\boldsymbol{\sigma}_D := \boldsymbol{\sigma} - \frac{1}{d}(\mathrm{tr}\,\sigma)\mathbf{I}$ (the deviatoric part of $\boldsymbol{\sigma}$) and using (1.9) and (1.11) we end up with the equation

$$\kappa\frac{\dot{\boldsymbol{e}}_{\mathrm{p}}}{|\dot{\boldsymbol{e}}_{\mathrm{p}}|} = \boldsymbol{\sigma}_D\,. \tag{1.12}$$

This equation however holds only if $\dot{\boldsymbol{e}}_{\mathrm{p}} \neq \mathbf{0}$. If so, then (1.12) implies the so-called von Mises yield condition

$$|\boldsymbol{\sigma}_D| = \kappa. \tag{1.13}$$

It also follows from (1.12) that $|\boldsymbol{\sigma}_D| < \kappa$ implies $\dot{\boldsymbol{e}_{\mathrm{p}}} = \mathbf{0}$ and the material responds purely elastically. One can thus summarize all these observations into the following compact Kuhn-Tucker form

$$\dot{\boldsymbol{e}}_{\mathrm{p}} = \phi\boldsymbol{\sigma}_D \quad \text{with} \quad \phi \geq 0\,, \quad |\boldsymbol{\sigma}_D| - \kappa \leq 0 \quad \text{and} \quad \phi\,(|\boldsymbol{\sigma}_D| - \kappa) = 0\,, \tag{1.14}$$

where $\phi = |\dot{\boldsymbol{e}}_{\mathrm{p}}|/\kappa = |\dot{\boldsymbol{e}}_{\mathrm{p}}|/|\boldsymbol{\sigma}_D|$.

There are other activation criteria that may be obtained by considering anisotropic elastic response and that are connected with names such as Rankine, Saint-Venant, Tresca, etc. Since we deal in our theoretical analysis only with von Mises condition (1.13) we skip details concerning the other conditions here.

1.2. **Formulations of boundary value problems.** Let us assume that the domain $\Omega$ that is occupied by the body is an open (possibly unbounded) $\mathcal{C}^2$-domain with two parts of boundary $\partial\Omega_1$ and $\partial\Omega_2$. We assume that the displacement is given on $\partial\Omega_1$ and the normal traction is given on $\partial\Omega_2$. We also prescribe the initial stress and assume that (1.7) and (1.14) hold. If we summarize all these relations we obtain a mixed boundary value problems for the Prandtl-Reuss model of elasto-plasticity (see Prandtl (1924); Reuss (1930)):

To find $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{e}_{\mathrm{el}}, \boldsymbol{e}_{\mathrm{p}}) : [0,T]\times\Omega \to \mathbb{R}^{d\times d}\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}$ such that

$$\begin{aligned}
&-\mathrm{div}\,\boldsymbol{\sigma} = \boldsymbol{f}, \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{e}_{\mathrm{el}} + \boldsymbol{e}_{\mathrm{p}}, \quad \boldsymbol{\sigma} = \frac{\partial\psi^*(\boldsymbol{e}_{\mathrm{el}})}{\partial\boldsymbol{e}_{\mathrm{el}}} \qquad \text{in } [0,T]\times\Omega,\\
&\dot{\boldsymbol{e}}_{\mathrm{p}} = |\dot{\boldsymbol{e}}_{\mathrm{p}}|\frac{\boldsymbol{\sigma}_D}{\kappa} \quad \text{with} \quad |\boldsymbol{\sigma}_D| \leq K \text{ and } |\dot{\boldsymbol{e}}_{\mathrm{p}}|(|\boldsymbol{\sigma}_D| - \kappa) = 0\,,\\
&\boldsymbol{u} = \boldsymbol{u}_0 \text{ on } [0,T]\times\partial\Omega_1, \quad \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_{\boldsymbol{n}} \text{ on } [0,T]\times\partial\Omega_2\,,\\
&\boldsymbol{\sigma}(0,x) = \boldsymbol{\sigma}_0(x) \text{ in } \Omega.
\end{aligned} \tag{1.15}$$

Another popular model of elastoplasticity is the Hencky model that is fomally obtained from (1.15) by replacing $\dot{\boldsymbol{e}}_{\mathrm{p}}$ by $\boldsymbol{e}_{\mathrm{p}}$. Thus, the mixed boundary value problem for the Hencky model (see Hencky (1924)) of elasto-plasticity reads as:

$$
\text{To find } (\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{e}_{\mathrm{el}}, \boldsymbol{e}_{\mathrm{p}}) : \Omega \to \mathbb{R}^{d\times d} \times \mathbb{R}^{d} \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \text{ such that}
$$

$$
\begin{aligned}
&-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}, \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{e}_{\mathrm{el}} + \boldsymbol{e}_{\mathrm{p}}, \quad \boldsymbol{\sigma} = \frac{\partial \psi^*(\boldsymbol{e}_{\mathrm{el}})}{\partial \boldsymbol{e}_{\mathrm{el}}} \qquad \text{in } \Omega,\\
&\boldsymbol{e}_{\mathrm{p}} = |\boldsymbol{e}_{\mathrm{p}}|\frac{\boldsymbol{\sigma}_D}{\kappa} \quad \text{with} \quad |\boldsymbol{\sigma}_D| \le K \text{ and } |\boldsymbol{e}_{\mathrm{p}}|(|\boldsymbol{\sigma}_D| - \kappa) = 0\,,\\
&\boldsymbol{u} = \boldsymbol{u}_0 \text{ on } \partial\Omega_1, \quad \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_{\boldsymbol{n}} \text{ on } \partial\Omega_2.
\end{aligned} \tag{1.16}
$$

Before we provide a variational formulation of (1.15) and (1.16) we recall several facts concerning primar and dual formulations in linearized elasticity. In what follows, we thus assume that $\boldsymbol{e}_{\mathrm{p}} = \mathbf{0}$ (and consequently $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{e}_{\mathrm{el}}$) and that the stress and the strain are given by an invertible mapping

$$
\boldsymbol{\sigma} = \mathsf{B}(x, \boldsymbol{\varepsilon}) := \frac{\partial \psi^*(x, \boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{\varepsilon} = \mathsf{B}^{-1}(x, \boldsymbol{\sigma}) = \mathsf{A}(x, \boldsymbol{\sigma}) =: \frac{\partial \psi(x, \boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}. \tag{1.17}
$$

If the relations in (1.17) are linear and the material is homogeneous in the sense that the response is the same at all $x \in \Omega$ (1.17) simplifies to

$$
\boldsymbol{\sigma} = \mathsf{B}\boldsymbol{\varepsilon} \qquad \text{and} \qquad \boldsymbol{\varepsilon} = \mathsf{A}\boldsymbol{\sigma} \qquad (\mathsf{B} = \mathsf{A}^{-1})\,. \tag{1.18}
$$

Recall that the invertibility of $\mathsf{A}$ and the symmetry of $\mathsf{A}\boldsymbol{\sigma}$ lead to the observation that $\mathsf{A}$ has to satisfy

$$
A_{ijkh} = A_{ijhk} = A_{jikh} = A_{khij} \qquad (i, j, k, h = 1, \ldots, d) \tag{1.19}
$$

and that $\mathsf{B}$ is consequently given by

$$
B_{ijkh} A_{khlm} = \delta_{il}\delta_{jm}.
$$

It is also worth remarking that the linear structure stated in (1.18) gives the relation between the corresponding Helmoltz potential $\psi^*(\boldsymbol{\varepsilon})$ and its dual potential $\psi(\boldsymbol{\sigma})$, namely,

$$
2\psi(\boldsymbol{\sigma}) = \mathsf{A}\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = \mathsf{B}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} = 2\psi^*(\boldsymbol{\varepsilon}). \tag{1.20}
$$

This (and Hooke's relation as well) underlies certain symmetric role of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in the linearized elasticity. Recently, Rajagopal and Srinivasa (2007) develop the elasticity for general $\psi$ of the form $\psi = \tilde{\psi}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})$ and showed that one can incorporate a very complicated nonlinear hysteretic response within such an implicit constitutive framework.

A reason why the explicit form $(1.18)_1$ has been preferable is obvious: inserting $(1.18)_1$ into (1.1), we see that it immediately leads to an elliptic partial differential equation of second order (assuming that $\mathsf{B}$ is positively definite) for which the theory is well known (see Korn (1907); Lichtenstein (1921) for the proof of the existence of a classical solution and Duvaut and Lions (1976); Nečas and Hlaváček (1980) for the formulation in Sobolev spaces). In addition, the regularity of the solution for these models is known and one can simply apply the theory developed for elliptic partial differential equations of second order.

It is however well known that (weak formulations of) mixed boundary value problems of the linearized elasticity can be reformulated equivalently (again either in terms of $\boldsymbol{\varepsilon}(\boldsymbol{u})$ or in terms of $\boldsymbol{\sigma}$) by incorporating methods of calculus of variations. Indeed, if we define the set of all admissible displacements as

$$
\mathcal{D}_{el} := \{\boldsymbol{u} \in W^{1,2}(\Omega)^d;\ \boldsymbol{u} = \boldsymbol{u}_0 \text{ on } \partial\Omega_1\},
$$

and the set of all admissible stresses as

$$\mathcal{F}_{el} := \{\boldsymbol{\sigma} \in L^2(\Omega)^{d\times d};\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^T,\ -\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f},\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{f}_{\boldsymbol{n}} \text{ on } \partial\Omega_2\},$$

then (weak formulations of) mixed boundary value problems of the linearized elasticity is equivalent to (note that $\boldsymbol{\varepsilon} := \boldsymbol{\varepsilon}(\boldsymbol{u})$)

$$\text{(1.21)} \qquad \text{Find } \boldsymbol{u} \in \mathcal{D}_{el} \text{ that minimizes:} \qquad \frac{1}{2}\int_\Omega \mathsf{B}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} - \boldsymbol{f}\cdot\boldsymbol{\varepsilon}\,dx - \int_{\partial\Omega_2} \boldsymbol{f}_{\boldsymbol{n}}\cdot\boldsymbol{u}\,dS\,,$$

or to

$$\text{(1.22)} \qquad \text{Find } \boldsymbol{\sigma} \in \mathcal{F}_{el} \text{ that minimizes:} \qquad \frac{1}{2}\int_\Omega \mathsf{A}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\,dx - \int_{\partial\Omega_1} \boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{u}_0\,dS\,.$$

For the proof of equivalence we refer to Duvaut and Lions (1976) or to Temam (1985). Note that (1.21) is called primary formulation (of relevant problem) and (1.22) is called the dual formulation.

At this point, we return to the boundary value problems 1.15 and 1.16 of linearized elasto-plasticity by generalizing the dual formulation (1.22). We first observe that in the linearized elasto-(perfect) plasticity the Helmholtz potential is given through (compare it with $(1.20)_1$)

$$\text{(1.23)} \qquad \psi_{pl}(\boldsymbol{\sigma}) := \begin{cases} \frac{1}{2}\mathsf{A}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} & \text{if } |\boldsymbol{\sigma}_D| \le \kappa, \\ +\infty & \text{if } |\boldsymbol{\sigma}_D| > \kappa. \end{cases}$$

Such a formulation reflects the requirement that the magnitude of $\sigma_D$ cannot overcome the critical yield stress $\kappa$ and consequently such a possibility is penalized by infinite amount of energy.

Thus, defining a new set of admissible stresses as

$$\mathcal{F} := \{\boldsymbol{\sigma} \in L^2(\Omega)^{d\times d};\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^T,\ -\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f},\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{f}_n \text{ on } \partial\Omega_2,\ |\boldsymbol{\sigma}_D| \le \kappa\},$$

we can reformulate the Hencky model (1.16) (and generalize (1.22)) in the following way:

$$(\mathcal{H}) \qquad \text{Find } \boldsymbol{\sigma} \in \mathcal{F} \text{ that minimize:} \qquad \frac{1}{2}\int_\Omega \mathsf{A}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\,dx - \int_{\partial\Omega_1} \boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{u}_0\,dS.$$

Analogously, we can reformulate the Prandtl-Reuss model (1.15):

$$(\mathcal{PR}) \qquad \text{Find } \boldsymbol{\sigma} \in \mathcal{F} \text{ that minimize:} \qquad \frac{1}{2}\int_\Omega \mathsf{A}\dot{\boldsymbol{\sigma}}\cdot\dot{\boldsymbol{\sigma}}\,dx - \int_{\partial\Omega_1} \dot{\boldsymbol{\sigma}}\boldsymbol{n}\cdot\dot{\boldsymbol{u}}_0\,dS.$$

Note that a function $\psi^*$ that would be dual to $\psi_{pl}$ from (1.23) cannot be written down explicitly as a function of $\boldsymbol{\varepsilon}$ in general, but only as a function of $\boldsymbol{e}_{\mathrm{el}}$. For further discussion see[3] Duvaut and Lions (1976); Prager and Hodge (1951), Anzellotti and Giaquinta (1980, 1982) (where the authors firstly recovered the displacements in a primal formulation for which they used variational integrals with convex functions

[3]Note also that in this case $(1.17)_1$ is still valid but $(1.17)_2$ must be represented as a multi-valued mapping that has the form

$$\boldsymbol{\varepsilon} = \mathsf{A}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_1,$$

where $\boldsymbol{\varepsilon}_1$ is an arbitrary matrix satisfying for all $\boldsymbol{\xi}$, $|\boldsymbol{\xi}_D| \le \kappa$ the relation $\boldsymbol{\varepsilon}_1\cdot(\boldsymbol{\xi}-\boldsymbol{\sigma}) \le 0$. Here, we refer to Rajagopal (2003), Rajagopal (2005) and Rajagopal and Srinivasa (2008) for more details regarding modeling of material responses within the context of implicit constitutive theory, and to Bulíček et al. (2009) and Málek (2008) for mathematical results for implicitely constituted fluid models.

of measures), Kohn and Temam (1983) (who firstly presented the adequate setting of the function spaces which are used in the dual formulation of Hencky's model) and Temam (1985) (where the existence of a solution of such a minimum problem and also the corresponding displacement $\boldsymbol{u}$ is established in details).

1.3. **A survey of results dealing with regularity of stress tensor in linearized elasto-plasticity.** We are interested in knowing if it is possible to prove that for some $\alpha > 0$ the quantity $\boldsymbol{\sigma}$, a unique solution of $(\mathcal{H})$, belongs to $W^{\alpha,2}(\Omega)^{d\times d}$, and if the analogous result holds for the Prantl-Reuss model $(\mathcal{PR})$ as well.

Interior regularity (it means $\boldsymbol{\sigma} \in W^{1,2}_{loc}(\Omega)^{d\times d}$) to $(\mathcal{H})$ has been established in Seregin (1990, 1992, 1996a) (for primal formulation) and Bensoussan and Frehse (1993) (for dual formulation). See also Bensoussan and Frehse (1996) for the extension of the same result to $(\mathcal{PR})$ (and Demyanov (2009) for an alternative proof). On the other hand, there are only few results for regularity of $\boldsymbol{\sigma}$ near the boundary: while the results proved in Frehse and Málek (1999); Knees (2006); Steinhauer (2003) and Blum and Frehse (2008) suggest that at least $L^2$ -regularity of the tangential derivatives of $\boldsymbol{\sigma}$ should hold up to the boundary and that the normal derivatives belongs to some fractional Sobolev space, the result of Seregin (1996b) who constructed a sequence of approximations for which some essential quantity explodes when approaching the limit problem $(\mathcal{H})$ gives a significant warning regarding the global (even fractional) regularity. We discuss both types of results in a more detail.

The usual procedure how to obtain the desired regularity (or even how to establish the existence result) is to penalize the problem $(\mathcal{H})$ in a suitable way (this approach is used in Duvaut and Lions (1976) and Johnson (1976) for problems involving perfect-plasticity problem, and by Johnson (1978) and Hlaváček et al. (1988) for problems with hardening). One such a possibility is to replaced the Helmholtz potential $\Psi_{pl}$, see (1.23), by a penalized function $\Psi^p$ of the form (here a real parameter $p$ should guarantee the validity of von Mises yield condition $|\sigma_D| \le \kappa$ if $p \to \infty$)

$$\psi^p(\boldsymbol{\sigma}) := \frac{1}{2}\mathsf{A}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + \psi^p_{pen}(\boldsymbol{\sigma}_D), \tag{1.24}$$

Then one can similarly as in (1.17) look for the solution to (1.1) with relevant boundary condition and the constitutive relation of the form

$$\mathsf{A}\boldsymbol{\sigma} + \frac{\partial\psi^p_{pen}(\boldsymbol{\sigma}_D)}{\partial\boldsymbol{\sigma}_D} = \boldsymbol{\varepsilon}(\boldsymbol{u})\,. \tag{1.25}$$

Assuming that $\mathsf{A}$, one can insert (1.25) into (1.1) and apply the theory for elliptic PDE's (for a fixed $p$) and then investigate the behavior of soutions as $p \to \infty$. Among many, there are several kinds of penalization that are frequently used. The first one is the so-called[4] Norton-Hoff model (see Hoff (1954); Norton (1929) or Temam (1985))

$$\psi^p_{pen}(\boldsymbol{\sigma}_D) := \frac{1}{p\kappa^p}|\boldsymbol{\sigma}_D|^p. \tag{1.26}$$

Next, a very important general class of penalizations is the so-called Hohenemser-Prager model dated to 30's, see Lubliner (1990). If one is interested in approximating the von Mises relation (1.13) by the Hohenemser-Prager model, then one

[4]This model is also called the Ramberg-Osgood model (see Ramberg and Osgood (1943)).

arrives at the Perčina-Mises model (see Lubliner (1990) for details) for which the penalty term has the form

$$\psi^p_{pen}(\boldsymbol{\sigma}_D) := \frac{1}{2p}((|\boldsymbol{\sigma}_D| - \kappa)_+)^2. \tag{1.27}$$

Another possibility for approximation is to penalize the distance of $\boldsymbol{\sigma}$ from the set $\{\boldsymbol{\sigma};\ |\boldsymbol{\sigma}_D| \le \kappa\}$ (see Lubliner (1990) for details).

In the present paper we use the Perčina-Mises model (1.27). The main advantage of this approximation is the fact that one can use standard $L^2$ theory for elliptic PDEs to get $W^{1,2}$ regularity for the approximation from the very beginning. On the other hand, it is an interesting open problem for this approximation to get $W^{1,2}_{loc}(\Omega)$ estimates independently of $p$ for dimension $d \ge 5$ (see Appendix for details) although we already know that the limit solution of the Hencky problem satisfies such a property. This can be shown by using for example the Norton-Hoff approximation. Indeed, in case we consider (1.26), we immediately get $L^p$ estimates on $\boldsymbol{\sigma}_D$ which are used to estimate certain pollution terms while proving $W^{1,2}_{loc}$ estimates on $\boldsymbol{\sigma}$ independently of $p$. However, also the Norton-Hoff approximation (1.26) is connected with some open problems: it is not known whether for fixed $p$, the stress tensor $\boldsymbol{\sigma}$ is bounded; the regularity near the boundary is worse (for fixed $p$) than the regularity of the solution to the model (1.27) and it depends very essentially on $p$ and moreover blows up as $p \to \infty$.

There are only a few results dealing with regularity near the boundary. The first global result was established by Frehse and Málek (1999) where the Norton-Hoff approximation is used and the uniform ($p$-independent) estimates on tangential derivatives in whole $\Omega$ in case that $\Omega$ is a two-dimensional ball and if $\partial\Omega_2 = \emptyset$ and $\boldsymbol{u}_0 \equiv 0$ are established. If $d = 2, 3$, Knees (2006) (Theorem 3.6, page 1373) showed (by using the Norton-Hoff approximation) that for sufficiently smooth data the solution $\boldsymbol{\sigma}$ belongs for all $\delta > 0$ to the space $W^{\frac{1}{2}-\delta,2}(\Omega)$ uniformly as $p \to \infty$. Steinhauer (2003) showed that for $d = 2$ and all $p \in (2, \infty)$ fixed, the solution $\boldsymbol{\sigma}$ of the Norton-Hoff approximation of the Hencky problem belongs to $\mathcal{C}^{\alpha}(\Omega)^{2\times 2}$ for some $\alpha > 0$ (for similar result ses also Bildhauer and Fuchs (2007)). Recently, Blum and Frehse (2008) have considered a planar stress model where the three-dimensional situation is reduced to a two-dimensional Hencky type model where the von Mises condition (1.13) is replaced by the condition

$$\boldsymbol{\sigma} \in \mathbb{R}^{3\times 3}_{sym}; \sigma_{i3} = 0 \text{ for all } i = 1, 2, 3; |\boldsymbol{\sigma} - \frac{\operatorname{tr}\boldsymbol{\sigma}}{3}\mathbf{I}| \le \kappa.$$

The authors used the fact that tensors from this set have better properties than tensors satisfying $|\boldsymbol{\sigma}_D| = |\boldsymbol{\sigma} - \frac{\operatorname{tr}\boldsymbol{\sigma}}{2}\mathbf{I}| \le \kappa$, that is of course the relation corresponding to (1.13) and to the two-dimensional setting. and this fact allowed them to establish that the tangential derivatives of the stress $\boldsymbol{\sigma}$ belongs to the space $L^2$ and that the normal derivatives belongs to the space[5] $W^{-\frac{1}{2},2}$ provided that $\Omega$ is smooth enough ($\Omega$ is $\mathcal{C}^{1,1}$ domain). It remains to mention that for sufficiently smooth $\boldsymbol{f}$ it is an consequence of a priori estimates and the Nečas theorem that the solution of Hencky model $\boldsymbol{\sigma} \in L^q(\Omega)^{d\times d}$ for arbitrary $q \in [1, \infty)$ (see Temam (1985); Steinhauer (2003) for details).

[5]In fact the authors showed the estimate on derivatives in normal direction only for flat boundary, but it can be straight-forwardly generalized to arbitrary domains.

An observation made by Seregin (1996b) is of a completely different character. In the two-dimensional torus $\Omega := B(0,2) \setminus B(0,1)$ Seregin constructed a sequence of smooth approximations $(\sigma^p, \boldsymbol{u}^p)$ that converges to the solution of $(\mathcal{H})$ and in addition satisfying

$$\int_{\partial\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}^p) \cdot \nabla \boldsymbol{\sigma}^p \boldsymbol{n} \, dS \stackrel{p\to\infty}{\to} \infty \,. \tag{1.28}$$

Seregin's conjecture was that either his approximation is not suitable or $W^{1,2}$ regularity does not hold at least for non-convex domains. The fact that (1.28) strongly indicates irregular behavior of the solution to $(\mathcal{H})$ follows from the following observation. Applying $\nabla$ to (1.25) and multiplying the result by $\nabla\boldsymbol{\sigma}^p$ and finally integrating over $\Omega$ we obatin the identity where the integrals that give the desired estimates for $W^{1,2}$-regularity remain on the left hand side (we assume that $\psi^p_{pen}$ is a "good" penalization). However, on the right hand side one obtains (assuming $\boldsymbol{u} = \boldsymbol{0}$ on boundary; see Appendix for details) that

$$\int_\Omega \nabla\boldsymbol{\varepsilon}(\boldsymbol{u}^p) \cdot \nabla\boldsymbol{\sigma}^p \, dx = -\int_\Omega \boldsymbol{u}^p \cdot \triangle \boldsymbol{f} \, dx + \int_{\partial\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}^p) \cdot \nabla\boldsymbol{\sigma}^p \boldsymbol{n} \, dS.$$

Since the first integral on the right hand side can be usually bounded from apriori estimates if one puts a suitable assumption on $\boldsymbol{f}$, we indeed observe that (1.28) indicates some blow-up effects.

This paper shows how one can overcome such difficulties in some domains of special geometry (the domain from Seregin (1996b) is allowed) and how one can get the full estimates on tangential derivatives in arbitrary dimension. Moreover, if $\Omega$ is a $d$-dimensional ball, we establish the estimates on $L^2$-norm of $\boldsymbol{\sigma}$ on the boundary that are independent of approximation. Having in addition interior regularity and also regularity in tangential directions, this also indicates that at least $W^{\frac{1}{2}-\delta,2}$ regularity should hold that is in perfect coincidence of 2D or 3D results in Knees (2006); Blum and Frehse (2008). For simplicity we prove our results only for homogeneous Dirichlet boundary condition on $\partial\Omega_1$, i.e, we set

$$\boldsymbol{u}_0 \equiv \boldsymbol{0}.$$

To complete this overview of known results we recall the results concerning the regularity of the displacement $\boldsymbol{u}$. It is a consequence of embedding theorem that $\boldsymbol{u} \in L^{\frac{d}{d-1}}(\Omega)^d$ (see Temam (1985) for details). Moreover, Hardt and Kinderlehrer (1986) proved that there exists $\delta > 0$ such that $\boldsymbol{u} \in L^{\frac{d}{d-1}+\delta}(\Omega)^d$. Concerning the higher regularity of the displacement $\boldsymbol{u}$, in general dimension it is known that $\boldsymbol{\varepsilon}(\boldsymbol{u})$ belongs to the space of measures.

1.4. **Notation and auxiliary results.** By bold italic letters we always mean a vector or a vector-valued function on $\mathbb{R}^d$, i.e. $\boldsymbol{v}(x) := (v_1(x), \dots, v_d(x))$. Tensors of $d\times d$ order are written by bold Greek letters, e.g. $(\boldsymbol{\sigma})_{ij} := \sigma_{ij}$. Tensors of $d\times d\times d\times d$ order are written by bold capital letters. If $a \in L^p(\Omega)$ and $b \in L^{p'}(\Omega)$ ($p'$ denote the dual exponent, i.e., $p' := \frac{p}{p-1}$) we use the abbreviation $(a,b) := \int_\Omega ab \, dx$ and we use the same notation also for vector- or tensor-valued functions. We also use the standard notation for Lebesgue and Sobolev spaces. Moreover, for these spaces on unbounded domains $\Omega$ and for some $\alpha \in \mathbb{R}$ we define

$$L^p_\alpha(\Omega) := \{v; \int_\Omega (|v|(1+|x|)^\alpha)^p \, dx \le C\}$$

and similarly we define

$$W^{k,p}_{\alpha}(\Omega) := \{v; \forall i = 0,\dots,k \int_{\Omega}(|\nabla^i v|(1+|x|)^{\alpha})^p \, dx \le C\}.$$

Consequently, we define $\|u\|_{k,p,(\alpha)} := \sum_{i=0}^{k}(\int_{\Omega}(|\nabla^i u|(1+|x|)^{\alpha})^p \, dx)^{\frac{1}{p}}$. Since we consider only one type of unbounded domain we say that $\Omega$ is an infinite strip if $\Omega := \{(x_1,\dots,x_d); 0 < x_d < 1\}$.

For simplicity, we omit the dependence of $\mathsf{A}$ on $x$ and we assume that it is a constant symmetric tensor that for all $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{sym}$ satisfies

$$\nu_0|\boldsymbol{\sigma}|^2 \le \mathsf{A}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \le \nu_1|\boldsymbol{\sigma}|^2. \tag{1.29}$$

In order to be able to get some a priori uniform estimates we have to assume the existence of a “safe” load (introduced by Duvaut and Lions (1976), see also Johnson (1976)), i.e., the existence of some $\boldsymbol{\sigma}_0$ such that for some fixed $\delta_0 > 0$

$$|(\boldsymbol{\sigma}_0)_D| \le \kappa - \delta_0, \qquad -\operatorname{div}\boldsymbol{\sigma}_0 = \boldsymbol{f}, \qquad \boldsymbol{\sigma}_0\cdot\boldsymbol{n}|_{\partial\Omega_2} = \boldsymbol{f}_{\boldsymbol{n}}. \tag{1.30}$$

Next, we introduce a $\mu$-approximation of $(\mathcal{H})$ for which we get the final estimates on regularity near the boundary and consequently, after letting $\mu \to 0_+$ also for limit problem $(\mathcal{H})$. Such a procedure is somehow standard and from this reason we focus only on the estimates that are independent of $\mu$. Hence, we define (by using (1.25) and (1.27))

$$(\mathcal{H}_\mu) \qquad \begin{aligned} \mathsf{A}\boldsymbol{\sigma} + \mu^{-1}(|\boldsymbol{\sigma}_D| - \kappa)_+\frac{\boldsymbol{\sigma}_D}{|\boldsymbol{\sigma}_D|} &= \boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{in } \Omega,\\ -\operatorname{div}\boldsymbol{\sigma} &= \boldsymbol{f} && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega_1,\\ \boldsymbol{\sigma}\cdot\boldsymbol{n} &= \boldsymbol{f}_{\boldsymbol{n}} && \text{on } \partial\Omega_2. \end{aligned}$$

Note, that one can also introduce the corresponding approximation to $(\mathcal{PR})$ as

$$(\mathcal{PR}_\mu) \qquad \begin{aligned} \mathsf{A}\dot{\boldsymbol{\sigma}} + \mu^{-1}(|\boldsymbol{\sigma}_D| - \kappa)_+\frac{\boldsymbol{\sigma}_D}{|\boldsymbol{\sigma}_D|} &= \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) && \text{in } \Omega,\\ -\operatorname{div}\boldsymbol{\sigma} &= \boldsymbol{f} && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega_1,\\ \boldsymbol{\sigma}\cdot\boldsymbol{n} &= \boldsymbol{f}_{\boldsymbol{n}} && \text{on } \partial\Omega_2. \end{aligned}$$

We should mention that in Frehse and Málek (1999), that is somehow basis for this paper, the authors considered the Norton-Hoff approximation, for which they were able to establish their regularity results. The reason why we use slightly different structure is that we can simply use elliptic theory to obtain regularity of solution for our approximative problem and therefore at least on the level of approximation we have regularity up to the boundary. These results are formulated without proof in the following lemma where the constant $C$ is an universal constant depending only on the data but not on the order of approximation $\mu$. If there is any dependence on $\mu$ it is clearly denoted in the text.

**Lemma 1.1.** *Let $\Omega$ be an open bounded smooth domain or infinite strip. Assume that $\boldsymbol{f} \in L^2(\Omega)$. In addition, if $\partial\Omega_1 = \emptyset$, assume that the compatibility condition*

$$-\int_{\Omega}\boldsymbol{f}\, dx = \int_{\partial\Omega_2}\boldsymbol{f}_{\boldsymbol{n}}\, dS \tag{1.31}$$

*holds. Then there exists a solution to* $(\mathcal{H}_\mu)$ *such that*

$$\|\boldsymbol{\sigma}\|_{1,2} + \|\boldsymbol{u}\|_{2,2} \le C(\mu^{-1}). \tag{1.32}$$

*Moreover, if* (1.30) *holds and* $\boldsymbol{f} \in W^{1,d}_{\alpha}(\Omega)^d$ *for some* $\alpha > \frac{d-1}{2}$ *then the following uniform estimate holds*

$$\begin{aligned} \|\boldsymbol{\sigma}\|_2 + \|\operatorname{div}\boldsymbol{u}\|_2 + \mu^{-1}\|(|\boldsymbol{\sigma}_D| - \kappa)_+(1 + |\boldsymbol{\sigma}_D|)\|_1 &\le C, \\ \|\boldsymbol{u}\|_{\frac{d}{d-1},(-\alpha)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{1,(-\alpha)} &\le C. \end{aligned} \tag{1.33}$$

*Moreover, assuming that* $\boldsymbol{f} \in W^{2,d}_{loc}(\Omega)^d$ *and that* $d \le 4$ *we have*

$$\|\nabla\boldsymbol{\sigma}\|_{2,loc} + \mu^{-1}\int_{\Omega_{loc}} \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}|\nabla\boldsymbol{\sigma}_D|^2 \, dx \le C. \tag{1.34}$$

Here we add only a few comments to the proof of Lemma 1.1. The existence and uniqueness of solution that satisfies (1.32) follows from standard elliptic theory. The first part of the estimate (1.33) can be shown by testing $(\mathcal{H}_\mu)_1$ by $\boldsymbol{\sigma} - \boldsymbol{\sigma}_0$ and by using (1.30) (Note that $\boldsymbol{\sigma}_0$ satisfies the safe load condition (1.30)). The second part then follows from the Korn-type inequality. For the proof of uniform ($\mu$-independent estimate) (1.34), see Lemma C.1 in Appendix.

For sake of completeness we recall the Korn-type inequality and the trace theorem for bounded domains. The proof can be found in Temam (1985). A proof for unbounded domains is given in Appendix of this paper.

**Lemma 1.2.** *Let* $\Omega \in \mathcal{C}^1$ *be an open bounded set. Define the space* $LD$ *as*

$$LD := \{\boldsymbol{u} \in L^1(\Omega)^d; \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^1(\Omega)^{d\times d}\} \tag{1.35}$$

*endowed with the norm* $\|\boldsymbol{u}\|_{LD} := \|\boldsymbol{u}\|_1 + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_1$. *Then the space* $LD$ *is continuously embedded into* $L^{d'}(\Omega)^d$. *Moreover, there exists the linear continuous surjective trace operator* $\operatorname{tr} : LD \to L^1(\partial\Omega)^d$ *such that for all* $\boldsymbol{u} \in \mathcal{C}^1(\overline{\Omega})^d$, $\operatorname{tr}\boldsymbol{u} = \boldsymbol{u}|_{\partial\Omega}$. *In addition, assume that either there exists* $\partial\Omega_3 \subset \partial\Omega$ *of non-zero measure such that* $\operatorname{tr}\boldsymbol{u} = \boldsymbol{0}$ *on* $\partial\Omega_3$ *or that* $\int_\Omega \boldsymbol{u} = \boldsymbol{0}$, *then there holds*

$$\|\boldsymbol{u}\|_{1,\partial\Omega} \le C\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_1, \tag{1.36}$$

$$\|\boldsymbol{u}\|_{d'} \le C\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_1. \tag{1.37}$$

## 2. Uniform estimates of tangential derivatives for $(\mathcal{H}_\mu)$

This section consists of the estimates of derivatives of $\boldsymbol{\sigma}$ in tangential directions for the approximative model $(\mathcal{H}_\mu)$. We deal with two types of domain $\Omega$. The first one consists of a ball or an annulus, the second one is an infinite band located between two parallel plates.

First, we assume the case for balls centered at zero with radii $R$, i.e., we assume that the normal vector $\boldsymbol{n}$ at each point of $\partial\Omega$ can be written as $\boldsymbol{n} = \pm|R|^{-1}(x_1,\dots,x_d)$. We also introduce the projection of the gradient of a scalar function $w$ to the tangent plane as (we denote $\boldsymbol{x} := (x_1,\dots,x_d)$)

$$\mathbf{T}w := |\boldsymbol{x}|^2\nabla w - (\nabla w\cdot\boldsymbol{x})\boldsymbol{x}. \tag{2.1}$$

In such a setting we prove the following uniform estimate for tangential derivatives of $\boldsymbol{\sigma}$, i.e., for $\mathbf{T}\boldsymbol{\sigma}$, where $(\mathbf{T}\boldsymbol{\sigma})_{ij} := \mathbf{T}\sigma_{ij}$.

**Theorem 2.1.** *Let $d$ be arbitrary, $R > r > 0$. Assume that $\Omega = B(\mathbf{0},R)\setminus B(\mathbf{0},r)$, $\partial\Omega_2 = \partial B(\mathbf{0},r)$ and $\partial\Omega_1 = \partial B(\mathbf{0},R)$. In addition assume that $\boldsymbol{f_n} \equiv \mathbf{0}$. If $\boldsymbol{\sigma}$ is a solution to $(\mathcal{H}_\mu)$ and $\boldsymbol{f} \in W^{2,d}(\Omega)^d$ then $\boldsymbol{\sigma}$ obeys the following uniform estimate*

$$\|\mathbf{T}\boldsymbol{\sigma}\|_2^2 + \mu^{-1}\int_\Omega \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}|\mathbf{T}\boldsymbol{\sigma}|^2 + \mu^{-1}\kappa\int_\Omega \chi_{\{|\boldsymbol{\sigma}_D|>\kappa\}}\frac{|\mathbf{T}|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3} \le C. \tag{2.2}$$

Theorem 2.1 extends the result established in Frehse and Málek (1999) and Blum and Frehse (2008) in two directions. First, it holds in arbitrary dimension while in Frehse and Málek (1999) and in Blum and Frehse (2008) the same[6] result was proved only in dimension two. Second, we permit also the Neumann boundary condition for $\boldsymbol{\sigma}$ which was also not included in Frehse and Málek (1999) and Blum and Frehse (2008). The result of Theorem 2.1 can be also easily extended to the case when $\partial\Omega_1 = \emptyset$ or $\partial\Omega_2 = \emptyset$, or onto the case when $\Omega = B(\mathbf{0},R)$, i.e., $\Omega$ is a ball. However, since this is in fact no generalization but just a simplification we do not deal with it.

In addition, if one considers only two-dimensional case than the result can be strengthen also for non-concentric balls (now circles) and we formulate the result in the following Corollary. Since the proof is based on using interior regularity result (see (1.34)) and on using the same procedure as in the proof of Theorem 2.1 with a proper cut-off function, we do not present it here for simplicity.

**Corollary 2.1.** *Assume that $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary $\partial\Omega$. In addition, assume that there are $\boldsymbol{x}_0 \in \Omega$ and $r > 0$ such that the one-dimensional Hausdorf measure of[7] $\Gamma := \partial\Omega_1 \cap \partial B(\boldsymbol{x}_0, r)$ is not zero. In addition assume that $\boldsymbol{f_n} \equiv \mathbf{0}$. If $\boldsymbol{\sigma}$ is a solution to $(\mathcal{H}_\mu)$ and $\boldsymbol{f} \in W^{2,2}(\Omega)^2$ then $\boldsymbol{\sigma}$ obeys the following uniform estimate*

$$\|\mathbf{T}\boldsymbol{\sigma}\xi\|_2^2 + \mu^{-1}\int_\Omega \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}|\mathbf{T}\boldsymbol{\sigma}|^2\xi + \mu^{-1}\kappa\int_\Omega \chi_{\{|\boldsymbol{\sigma}_D|>\kappa\}}\frac{|\mathbf{T}|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3}\xi \le C(h^{-1}), \tag{2.3}$$

*where $\xi \in \mathcal{D}(\mathbb{R}^2)$ such that $0 \le \xi \le 1$, and for sufficiently small $h$ there holds $\xi(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \Omega$ such that dist $(\boldsymbol{x},\Gamma) \le h$ and dist $(\boldsymbol{x},\partial\Gamma) \ge 2h$, and $\xi(\boldsymbol{x}) = 0$ if dist $(\boldsymbol{x},\Gamma) \ge 2h$ or dist $(\boldsymbol{x},\partial\Gamma) \le h$.*

The second setting we want to deal with is the case when $\partial\Omega$ consists of two infinite hyperplanes, i.e., we assume that $\Omega := \{\boldsymbol{x} \in \mathbb{R}^d; 0 < x_d < 1\}$. For this model we establish the following result:

**Theorem 2.2.** *Let $d$ be arbitrary, $\partial\Omega_1 = \{\boldsymbol{x};\ x_d = 0\}$ and $\partial\Omega_2 = \{\boldsymbol{x};\ x_d = 1\}$. Assume that $\boldsymbol{f} \in W^{2,d}_\alpha(\Omega)^d$ for some $\alpha > \frac{d-1}{2}$. Setting $D_s = \frac{\partial}{\partial x_s}$ for all $s = 1,\dots,d-1$, then the following estimate holds for the same $s$:*

$$\|D_s\boldsymbol{\sigma}\|_2^2 + \mu^{-1}\int_\Omega \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}|D_s\boldsymbol{\sigma}|^2 + \mu^{-1}\kappa\int_\Omega \chi_{\{|\boldsymbol{\sigma}_D|>\kappa\}}\frac{|D_s|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3} \le C. \tag{2.4}$$

We would like to emphasize that the right-hand side of (2.4) is $\mu$-independent and that we are able to include also the Neumann boundary condition. It is also clear from the proof that as in Theorem 2.1 we can extend our results to case with only Neumann boundary conditions or Dirichlet boundary conditions. For simplicity we omit it here.

[6] In fact, in Blum and Frehse (2008) the result holds for slightly different yield condition.

[7] The same result holds if one replaces $\partial\Omega_1$ by $\partial\Omega_2$.

*Proof of Theorem 2.1.* For simplicity, we assume that $R = 1$ and $r = \frac{1}{2}$. We also introduce the notation for tangential derivatives in one tangential direction as[8]

$$T_{a_1\dots a_{d-2}} := \varepsilon_{ija_1\dots a_{d-2}} x_i D_j \tag{2.5}$$

that fully describes all tangential derivatives on $\partial\Omega$. Note that the symbol $\varepsilon_{a_1\dots a_d}$ (with $a_k \in \{1,\dots,d\}$) represents the $d$-generalization of the classical Levi-Civita symbol, i.e., we assume that it is fully antisymmetric $d-$order tensor, or in other words it can be defined as $\varepsilon_{a_1\dots a_d} :=$ sign $\{a_1,\dots,a_d\}$. To be more specific, in three dimensional setting, the definition (2.5) includes just three tangential derivatives

$$T_1 := x_2D_3 - x_3D_2, \quad T_2 := x_3D_1 - x_1D_3, \quad \text{and} \quad T_3 := x_1D_2 - x_2D_1.$$

Finally, it is easy to observe that the operator $\mathbf{T}$ is generated by operators $T_{a_1\dots a_{d-2}}$, namely we have that $|\mathbf{T}u(x)|^2 = C(d)|\boldsymbol{x}|^2 \sum |T_{a_1\dots a_{d-2}}u(x)|^2$. Having (2.5), one can easily observe the following commutator relation

$$D_l T_{a_1\dots a_{d-2}} - T_{a_1\dots a_{d-2}} D_l = \varepsilon_{lma_1\dots a_{d-2}} D_m. \tag{2.6}$$

Next, we apply the operator $T_{a_1\dots a_{d-2}}$ to $(\mathcal{H}_\mu)$, take the scalar product with $T_{a_1\dots a_{d-2}}\boldsymbol{\sigma}$, sum over all indices $a_1,\dots,a_{d-2}$ and integrate the result over[9] $\Omega$, we simply get (using Einstein summation convention)

$$\begin{aligned}(\mathbf{A}T_{a_1\dots a_{d-2}}\boldsymbol{\sigma}, T_{a_1\dots a_{d-2}}\boldsymbol{\sigma}) &+ \mu^{-1}\int_\Omega \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}|T_{a_1\dots a_{d-2}}\boldsymbol{\sigma}_D|^2 \\ &+ 4^{-1}\mu^{-1}\kappa \int_\Omega \chi_{\{|\boldsymbol{\sigma}_D|>\kappa\}} \frac{|T_{a_1\dots a_{d-2}}|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3} = (T_{a_1\dots a_{d-2}}\boldsymbol{\varepsilon}(\boldsymbol{u}), T_{a_1\dots a_{d-2}}\boldsymbol{\sigma}).\end{aligned} \tag{2.7}$$

The left hand side of (2.7) can be estimated as follows

$$\begin{aligned}(\text{LHS}) \geq C(d)\Big(\nu_0\|\mathbf{T}\boldsymbol{\sigma}\|^2 &+ \mu^{-1}\int_\Omega \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}|\mathbf{T}\boldsymbol{\sigma}_D|^2 \\ &+ \mu^{-1}\kappa \int_\Omega \chi_{\{|\boldsymbol{\sigma}_D|>\kappa\}} \frac{|\mathbf{T}|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3}\Big).\end{aligned} \tag{2.8}$$

The essential step is to estimate the right hand side of (2.7). In order to do it rigorously, we define for arbitrary $\delta > 0$ the cut-off function $V_\delta \in \mathcal{D}(\Omega)$ as $V_\delta(\boldsymbol{x}) := V_\delta(|\boldsymbol{x}|^2)$ such that $V_\delta(s) = 0$ for $s \in [1-\delta, 1]$, $V(s) = 1$ for $s \in [0, 1-2\delta]$ and such that $|V_\delta'| \leq C\delta^{-1}$ uniformly for all $\delta$. Then the RHS of (2.7) can be rewritten as

$$\begin{aligned}(T_{a_1\dots a_{d-2}}\boldsymbol{\varepsilon}(\boldsymbol{u}), T_{a_1\dots a_{d-2}}\boldsymbol{\sigma}) &= (T_{a_1\dots a_{d-2}}D_ju_i, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta) \\ &\quad + (T_{a_1\dots a_{d-2}}D_ju_i, T_{a_1\dots a_{d-2}}\sigma_{ij}(1-V_\delta)).\end{aligned} \tag{2.9}$$

Clearly, the second term in (2.9) tends to 0 as $\delta \to 0$. So it remains to discuss the first integral in (2.9). Next, for fixed $\delta$, we define for arbitrary $0 < \eta < \delta$ the standard mollification $\boldsymbol{u}^\eta := \boldsymbol{u} * \nu^\eta$ where $\nu^\eta$ is standard mollification kernel with

[8]Note that some of them are taken several times. Also, for $d = 2$ we set $T := x_1D_2 - x_2D_1$.

[9]Note that due to the even nonuniform estimates (1.32) such procedure is rigorous at the level of $\mu$-approximations

support in a ball of radii $\eta$. Hence, for fixed $\delta$, we can decompose the first integral in (2.9)

$$(T_{a_1\dots a_{d-2}}D_ju_i, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta) = (T_{a_1\dots a_{d-2}}D_j(u_i - u_i^\eta), T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta) + (T_{a_1\dots a_{d-2}}D_ju_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta) \tag{2.10}$$

such that the first integral tends to zero (for fixed $\delta$) as $\eta \to 0$. Finally, we start to estimate the remaining integral from (2.10). Hence,

$$\begin{aligned}
I_\eta^\delta &:= (T_{a_1\dots a_{d-2}}D_ju_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta)\\
&\overset{(2.6)}{=} (D_jT_{a_1\dots a_{d-2}}u_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta) - \varepsilon_{jma_1\dots a_{d-2}}(D_mu_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta)\\
&\overset{\text{by parts}}{=} -(T_{a_1\dots a_{d-2}}D_jT_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}V_\delta) - (D_jT_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}T_{a_1\dots a_{d-2}}V_\delta)\\
&\quad + \varepsilon_{kla_1\dots a_{d-2}}(D_jT_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}V_\delta x_kn_l)_{\partial\Omega} - \varepsilon_{jma_1\dots a_{d-2}}(D_mu_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta).
\end{aligned}$$

Since $\varepsilon_{kla_1\dots a_{d-2}}$ is antisymmetric w.r.t. $k, l$ and $x_kn_l$ is symmetric on $\partial\Omega$ the third integral vanishes (it is a consequence of the fact that $T_{a_1\dots a_{d-2}}$ is a derivative in a tangential direction). Also since $V_\delta$ depends only on $|\boldsymbol{x}|$ and consequently $T_{a_1\dots a_{d-2}}V_\delta = 0$, the second integral vanishes as well. Therefore

$$\begin{aligned}
I_\eta^\delta &= -(T_{a_1\dots a_{d-2}}D_jT_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}V_\delta) - \varepsilon_{jma_1\dots a_{d-2}}(D_mu_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta)\\
&\overset{(2.6)}{=} -(D_jT_{a_1\dots a_{d-2}}T_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}V_\delta) + \varepsilon_{jma_1\dots a_{d-2}}(D_mT_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}V_\delta)\\
&\quad - \varepsilon_{jma_1\dots a_{d-2}}(D_mu_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta).
\end{aligned}$$

Next, we apply the integration by parts to the first term and use $(\mathcal{H}_\mu)_2$. We obtain

$$\begin{aligned}
I_\eta^\delta &= -(T_{a_1\dots a_{d-2}}T_{a_1\dots a_{d-2}}u_i^\eta, f_iV_\delta) + (T_{a_1\dots a_{d-2}}T_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}D_jV_\delta)\\
&\quad - (T_{a_1\dots a_{d-2}}T_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}n_jV_\delta)_{\partial\Omega} + \varepsilon_{jma_1\dots a_{d-2}}(D_mT_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}V_\delta)\\
&\quad - \varepsilon_{jma_1\dots a_{d-2}}(D_mu_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta).
\end{aligned}$$

The third integral again vanishes because $\sigma_{ij}n_j = 0$ on $\partial\Omega_2$ and $V_\delta = 0$ on $\partial\Omega_1$. Next, integration by parts applied to the second integral leads to (boundary integral again vanishes since $T_{a_1\dots a_{d-2}}$ is tangential derivative)

$$\begin{aligned}
I_\eta^\delta &= -(T_{a_1\dots a_{d-2}}T_{a_1\dots a_{d-2}}u_i^\eta, f_iV_\delta) - (T_{a_1\dots a_{d-2}}u_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}D_jV_\delta)\\
&\quad - (T_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}T_{a_1\dots a_{d-2}}(D_jV_\delta)) + \varepsilon_{jma_1\dots a_{d-2}}(D_mT_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}V_\delta)\\
&\quad - \varepsilon_{jma_1\dots a_{d-2}}(D_mu_i^\eta, T_{a_1\dots a_{d-2}}\sigma_{ij}V_\delta).
\end{aligned}$$

Next, we let $\eta \to 0$ and then $\delta \to 0$ and investigate behavior of involved terms separately. This procedure is simple with the first, fourth and fifth term on the right hand side since we have to our disposal the estimate (non-uniform in $\mu$) (1.32). It is also easy to take the limit $\eta \to 0$ in the second and third term and observe that

$$\begin{aligned}
&\lim_{\eta\to 0}(T_{a_1\dots a_{d-2}}u_i, T_{a_1\dots a_{d-2}}\sigma_{ij}^\eta D_jV_\delta) = (T_{a_1\dots a_{d-2}}u_i, T_{a_1\dots a_{d-2}}\sigma_{ij}D_jV_\delta),\\
&\lim_{\eta\to 0}(T_{a_1\dots a_{d-2}}u_i^\eta, \sigma_{ij}T_{a_1\dots a_{d-2}}(D_jV_\delta)) = (T_{a_1\dots a_{d-2}}u_i, \sigma_{ij}T_{a_1\dots a_{d-2}}(D_jV_\delta)).
\end{aligned}$$

Moreover, since $T_{a_1\dots a_{d-2}}V_\delta$=0 and the commutator relation (2.6) holds we obtain

$$(T_{a_1\dots a_{d-2}}u_i, \sigma_{ij}T_{a_1\dots a_{d-2}}(D_jV_\delta)) = \varepsilon_{jma_1\dots a_{d-2}}(T_{a_1\dots a_{d-2}}u_i, \sigma_{ij}D_mV_\delta)).$$

The next step is to let $\delta \to 0$ in remaining integrals. Since $\nabla^2 \boldsymbol{u} \in L^2(\Omega)^{d\times d\times d}$ and $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega_1$, we see (since $T_{a_1\dots a_{d-2}}$ is derivative in tangential direction) that $T_{a_1\dots a_{d-2}} u_i = 0$ on $\partial\Omega_1$ and therefore $\frac{T_{a_1\dots a_{d-2}} u_i(\boldsymbol{x})}{\text{dist }(\boldsymbol{x},\partial\Omega_1)} \in L^2(\Omega)$. Hence,

$$(T_{a_1\dots a_{d-2}} u_i, T_{a_1\dots a_{d-2}} \sigma_{ij} D_j V_\delta) \le C \left\| \frac{T_{a_1\dots a_{d-2}} u_i(\boldsymbol{x})}{\text{dist }(\boldsymbol{x},\partial\Omega)} \right\|_2 \|\nabla \boldsymbol{\sigma}\|_{2,B(\boldsymbol{0},1)\setminus B(\boldsymbol{0},1-2\delta)} \overset{\delta\to 0}{\to} 0.$$

Using the same procedure we also obtain that

$$(T_{a_1\dots a_{d-2}} u_i, \sigma_{ij} D_m V_\delta)) \overset{\delta\to 0}{\to} 0.$$

Thus, we can conclude that

$$\begin{aligned}(T_{a_1\dots a_{d-2}}\boldsymbol{\varepsilon}(\boldsymbol{u}), T_{a_1\dots a_{d-2}}\boldsymbol{\sigma}) &= \lim_{\delta\to 0}\lim_{\eta\to 0} I_\eta^\delta \\ &= -(T_{a_1\dots a_{d-2}} T_{a_1\dots a_{d-2}} u_i, f_i) + \varepsilon_{jma_1\dots a_{d-2}}(D_m T_{a_1\dots a_{d-2}} u_i, \sigma_{ij}) \\ &\quad - \varepsilon_{jma_1\dots a_{d-2}}(D_m u_i, T_{a_1\dots a_{d-2}}\sigma_{ij}) =: I_1 + I_2 + I_3.\end{aligned} \tag{2.11}$$

Regarding $I_1$ we first integrate by parts (note that boundary term vanishes due to the fact that $T_{a_1\dots a_{d-2}}$ represents a derivative in tangential direction) and then conlcude that

$$I_1 = -(u_i, T_{a_1\dots a_{d-2}} T_{a_1\dots a_{d-2}} f_i) \le C\|\boldsymbol{u}\|_{\frac{d}{d-1}} \|\boldsymbol{f}\|_{2,d} \overset{(1.33)}{\le} C \tag{2.12}$$

Regarding $I_2$ we use the commutator estimate (2.6) and then integrate by parts w.r.t. $T_{a_1\dots a_{d-2}}$ (boundary integral again vanishes) and obtain

(2.13)

$$\begin{aligned} I_2 &\overset{(2.6)}{=} \varepsilon_{jma_1\dots a_{d-2}}(T_{a_1\dots a_{d-2}} D_m u_i, \sigma_{ij}) + \varepsilon_{jma_1\dots a_{d-2}}\varepsilon_{mna_1\dots a_{d-2}}(D_n u_i, \sigma_{ij}) \\ &= -\varepsilon_{jma_1\dots a_{d-2}}(D_m u_i, T_{a_1\dots a_{d-2}}\sigma_{ij}) + \varepsilon_{jma_1\dots a_{d-2}}\varepsilon_{mna_1\dots a_{d-2}}(D_n u_i, \sigma_{ij}).\end{aligned}$$

Consequently, combining (2.11) and (2.13), we deduce that

$$\begin{aligned} I_2 + I_3 &= -2\varepsilon_{jma_1\dots a_{d-2}}(D_m u_i, T_{a_1\dots a_{d-2}}\sigma_{ij}) + \varepsilon_{jma_1\dots a_{d-2}}\varepsilon_{mna_1\dots a_{d-2}}(D_n u_i, \sigma_{ij}) \\ &=: I_4 + I_5.\end{aligned}$$

Next, recalling the definition of $\varepsilon_{jma_1\dots a_{d-2}}$ and we observe that it is enough to take the sum in for $I_5$ only over indices $j = n$. Moreover, using the antisymmetry of $\varepsilon_{jma_1\dots a_{d-2}}$ and integrating by parts (boundary integral disappears because $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega_1$ and $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega_2$) we are led to the following observation:

$$\begin{aligned} I_5 &= -\varepsilon_{mna_1\dots a_{d-2}}\varepsilon_{mna_1\dots a_{d-2}}(D_n u_i, \sigma_{in}) = -(d-1)!(D_n u_i, \sigma_{in}) \\ &= (d-1)!(u_i, D_n\sigma_{in}) \overset{(\mathcal{H}_\mu)_2}{=} -(d-1)!(\boldsymbol{u}, \boldsymbol{f}) \le C\|\boldsymbol{u}\|_{\frac{d}{d-1}}\|\boldsymbol{f}\|_d \overset{(1.33)}{\le} C.\end{aligned}$$

Regarding $I_4$, we have

$$\begin{aligned} I_4 &= -2\varepsilon_{jma_1\dots a_{d-2}}(D_m u_i + D_i u_m, T_{a_1\dots a_{d-2}}\sigma_{ij}) + 2\varepsilon_{jma_1\dots a_{d-2}}(D_i u_m, T_{a_1\dots a_{d-2}}\sigma_{ij}) \\ &= -2\varepsilon_{jma_1\dots a_{d-2}}(D_m u_i + D_i u_m, T_{a_1\dots a_{d-2}}(\sigma_D)_{ij}) \\ &\quad - 2\varepsilon_{jma_1\dots a_{d-2}}(D_m u_i + D_i u_m, T_{a_1\dots a_{d-2}}(\sigma_{ij} - (\sigma_D)_{ij})) \\ &\quad\quad + 2\varepsilon_{jma_1\dots a_{d-2}}(D_i u_m, T_{a_1\dots a_{d-2}}\sigma_{ij}) =: I_6 + I_7 + I_8.\end{aligned}$$

To estimate $I_6$ we use the equation $(\mathcal{H}_\mu)_1$ and obtain

$$\frac{1}{2}(D_m u_i + D_i u_m, T_{a_1\dots a_{d-2}}\sigma_{ij}) \le \|\mathsf{A}\boldsymbol{\sigma}\|_2\|\mathsf{T}\boldsymbol{\sigma}_D\|_2 + \mu^{-1}((|\boldsymbol{\sigma}_D| - \kappa)_+|\mathsf{T}\boldsymbol{\sigma}_D|) \tag{2.14}$$

Consequently, using the assumption on $\mathbf{A}$ (1.29), the estimates that are uniform w.r.t. $\mu$ (1.33), the relation (2.8), and the Young inequality, we observe that

$$I_6 \leq Cd!\nu_1\|\boldsymbol{\sigma}\|_2\|\mathbf{T}\boldsymbol{\sigma}\|_2 + Cd!\mu^{-1}((|\boldsymbol{\sigma}_D|-\kappa)_+|\mathbf{T}\boldsymbol{\sigma}_D|) \leq C + \frac{1}{2}(\text{LHS}) \text{ of } (2.7).$$

Further, since $\sigma_{ij} - (\sigma_D)_{ij} = d^{-1}\delta_{ij}tr\boldsymbol{\sigma}$, we can simplify the equation for $I_7$ and obtain

$$I_7 = -2d^{-1}\varepsilon_{jma_1\dots a_{d-2}}(D_m u_j + D_j u_m, T_{a_1\dots a_{d-2}}\operatorname{tr}\boldsymbol{\sigma}).$$

Since $\varepsilon_{jma_1\dots a_{d-2}}$ is antisymmetric w.r.t. $j,m$ and $D_m u_j + D_j u_m$ is symmetric, we can easily conclude that $I_7 = 0$. Finally, to estimate $I_8$ we integrate twice by parts, first[10] w.r.t. $T_{a_1\dots a_{d-2}}$ (boundary integral vanishes, since $T_{a_1\dots a_{d-2}}$ is derivative in tangential direction on the boundary), secondly we use the commutator relation (2.6) and finally we integrate by parts w.r.t. $D_i$ (the boundary integral again vanishes since $T_{a_1\dots a_{d-2}}\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega_1$ and $\boldsymbol{\sigma n} = \mathbf{0}$ on $\partial\Omega_2$), more precisely

$$\begin{aligned}
I_8 &= -2\varepsilon_{jma_1\dots a_{d-2}}(T_{a_1\dots a_{d-2}}D_i u_m, \sigma_{ij})\\
&= -2\varepsilon_{jma_1\dots a_{d-2}}(D_i T_{a_1\dots a_{d-2}}u_m, \sigma_{ij}) + 2\varepsilon_{jma_1\dots a_{d-2}}\varepsilon_{ina_1\dots a_{d-2}}(D_n u_m, \sigma_{ij})\\
&= -2\varepsilon_{jma_1\dots a_{d-2}}(T_{a_1\dots a_{d-2}}u_m, f_j) + 2\varepsilon_{jma_1\dots a_{d-2}}\varepsilon_{ina_1\dots a_{d-2}}(D_n u_m, \sigma_{ij})\\
&=: I_9 + I_{10}.
\end{aligned}$$

In $I_9$ we integrate again by parts to obtain

$$I_9 = 2\varepsilon_{jma_1\dots a_{d-2}}(u_m, T_{a_1\dots a_{d-2}}f_j) \leq C\|\boldsymbol{u}\|_{\frac{d}{d-1}}\|\boldsymbol{f}\|_{1,d} \overset{(1.33)}{\leq} C.$$

Finally, we need to estimate $I_{10}$. To do it, we first observe that it is enough to take into account the case when $j = i$ and $m = n$, or the case when $j = n$ and $i = m$. (Another choice of indices leads to $\varepsilon_{jma_1\dots a_{d-2}}\varepsilon_{ina_1\dots a_{d-2}} = 0$.) Hence we have

$$\begin{aligned}
I_{10} &= -2\varepsilon_{mja_1\dots a_{d-2}}\varepsilon_{ina_1\dots a_{d-2}}(D_n u_m, \sigma_{ij})\\
&= -2\varepsilon^2_{mna_1\dots a_{d-2}}(D_n u_m, \sigma_{mn}) + 2\varepsilon^2_{mia_1\dots a_{d-2}}(D_m u_m, \sigma_{ii})\\
&= 2(d-2)!(\delta_{mn}-1)\left((D_n u_m, \sigma_{mn}) - (D_m u_m, \sigma_{nn})\right)\\
&= 2(d-2)!\left(-(D_n u_m, \sigma_{mn}) + (D_n u_n, \sigma_{nn}) - (D_n u_n, \sigma_{nn}) + (D_m u_m, \sigma_{nn})\right)\\
&= -2(d-2)!(\nabla\boldsymbol{u}, \boldsymbol{\sigma}) + 2(d-2)!(\operatorname{div}\boldsymbol{u}, \operatorname{tr}\boldsymbol{\sigma}).
\end{aligned}$$

Consequently, integrating by parts in the first term and using (1.33) we conclude

$$I_{10} = -2(d-2)!(\boldsymbol{u}, \boldsymbol{f}) + 2(d-2)!(\operatorname{div}\boldsymbol{u}, \operatorname{tr}\boldsymbol{\sigma}) \leq C\,.$$

Combining all the above estimates for $I_1$–$I_{10}$ and inserting them into (2.7), we observe that the estimate (2.2) holds. The proof of Theorem 2.1 is complete. □

*Proof of Theorem 2.2.* First, we define the cut-off function $V_k$ as

$$V_k(\boldsymbol{x}) := V_k(x_1,\dots,x_{d-1}) = \begin{cases} 1 & \text{if } |x_i| \leq k \quad \forall i \in \{1,\dots,d-1\},\\ 0 & \text{if } \exists i \in \{1,\dots,d-1\}, |x_i| > 2k, \end{cases} \tag{2.15}$$

such that

$$|D_s V_k| \leq \frac{C}{k}. \tag{2.16}$$

[10]This procedure is possible because for the approximate problem we have enough regularity, namely (1.32)

Then, for arbitrary fixed $s \in \{1, \dots, d-1\}$, we apply $D_s$ to the equation $(\mathcal{H}_\mu)$, take the scalar product of the result with $D_s\boldsymbol{\sigma} V_k$ and integrate the result over $\Omega$. This results to the equality

$$\begin{aligned}\int_\Omega \mathsf{A} D_s\boldsymbol{\sigma} \cdot D_s\boldsymbol{\sigma} V_k + \mu^{-1} \int_\Omega \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|} |D_s\boldsymbol{\sigma}_D|^2 V_k \\ + 4^{-1}\mu^{-1}\kappa \int_\Omega \chi_{\{|\boldsymbol{\sigma}_D|>\kappa\}} \frac{|D_s|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3} V_k = \int_\Omega D_s\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot D_s\boldsymbol{\sigma} V_k := I_1.\end{aligned} \tag{2.17}$$

It follows from the assumption (1.29) that LHS of (2.17) leads precisely (after letting $k \to \infty$ and using e.g. the Fatou lemma) to the LHS of (2.4). So, it remains to estimate $I_1$. To do it, we use integration by parts[11] and since $D_s u_i|_{\partial\Omega_1} = 0$ and $D_s\sigma_{id}|_{\partial\Omega_2} = 0$ all boundary integrals are zero. Hence, using also the fact that $-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}$, we observe that

$$\begin{aligned}I_1 &= (D_sD_ju_i, D_s\sigma_{ij}V_k) = (D_su_i, D_sf_iV_k) - (D_su_i, D_s\sigma_{ij}D_jV_k) \\ &= -(u_i, D_s^2f_iV_k) - (u_i, D_sf_iD_sV_k) - (D_su_i, D_s\sigma_{ij}D_jV_k) := I_2 + I_3 + I_4.\end{aligned}$$

Using the a priori estimates (1.33) and the assumption on $\boldsymbol{f}$, one can easily conclude that

$$I_2 \le \|\boldsymbol{u}\|_{\frac{d}{d-1},(-\alpha)} \|\boldsymbol{f}\|_{2,d,(\alpha)} \le C.$$

Using also the property of $V_k$ (2.16), we immediately obtain that

$$I_3 \overset{k\to\infty}{\to} 0.$$

Finally, to estimate $I_4$, we use (1.32) and (2.16) and observe that

$$I_4 \le C(\mu^{-1})\frac{1}{k} \overset{k\to\infty}{\to} 0\,,$$

which completes the proof of Theorem 2.2. □

## 3. $L^2$-ESTIMATE OF $\boldsymbol{\sigma}$ ON THE BOUNDARY

In this section we establish $L^2$ estimates on boundary for $\boldsymbol{\sigma}$ provided that $\Omega$ is a ball. Even if the proof is very straightforward it seems that it is a new result.

**Theorem 3.1.** *Let $d$ be arbitrary, $\Omega = B(\mathbf{0}, R)$ be a ball and $\boldsymbol{f} \in W^{1,d}(\Omega)^d$. Assume that either $\partial\Omega_1 = \emptyset$ and $\boldsymbol{f_n} \in L^\infty(\partial\Omega)^d$, or $\partial\Omega_2 = \emptyset$. Then there holds*

$$\int_{\partial\Omega} |\boldsymbol{\sigma}|^2 + \mu^{-1}((|\boldsymbol{\sigma}_D| - \kappa)_+)^2 \, dS \le C. \tag{3.1}$$

*Proof of Theorem 3.1.* First we denote by symbol $D_N$ the normal derivative, i.e., we define

$$D_N := x_iD_i.$$

Note that we immediately obtain the commutator relation

$$D_jD_N - D_ND_j = D_j. \tag{3.2}$$

First we consider the case $\partial\Omega_2 = \emptyset$, which means that $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$. We multiply $(\mathcal{H}_\mu)$ by $D_N\boldsymbol{\sigma}$ and integrate the result over $\Omega$ to get

$$I_1 + I_2 + I_3 := (\mathsf{A}\boldsymbol{\sigma}, D_N\boldsymbol{\sigma}) + \mu^{-1}(\frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}\boldsymbol{\sigma}_D, D_N\boldsymbol{\sigma}) - (\boldsymbol{\varepsilon}(\boldsymbol{u}), D_N\boldsymbol{\sigma}) = 0. \tag{3.3}$$

[11] We prove it only formally, since we do not know what is the meaning of $D_sD_j\sigma_{ij}$. However the whole procedure can be made rigorously, for details see the proof of Theorem 2.1.

Using the symmetry of $\mathbf{A}$ we observe that

(3.4)
$$I_1 = \frac{1}{2}\int_\Omega D_N\left(\mathbf{A}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\right) = \frac{R}{2}\int_{\partial\Omega} \mathbf{A}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} - \frac{d}{2}(\mathbf{A}\boldsymbol{\sigma},\boldsymbol{\sigma}) \overset{(1.33),(1.29)}{\geq} \frac{\nu_0}{2}\int_{\partial\Omega}|\boldsymbol{\sigma}|^2\, dS - C.$$

For $I_2$ we use similar procedure to get

$$\begin{aligned} I_2 &= \frac{1}{2\mu}\int_\Omega D_N((|\boldsymbol{\sigma}_D| - \kappa)_+)^2 \\ &= \frac{R}{2\mu}\int_{\partial\Omega}((|\boldsymbol{\sigma}_D| - \kappa)_+)^2\, dS - \frac{d}{2\mu}\int_\Omega((|\boldsymbol{\sigma}_D| - \kappa)_+)^2 \\ &\overset{(1.33)}{\geq} \frac{R}{2\mu}\int_{\partial\Omega}((|\boldsymbol{\sigma}_D| - \kappa)_+)^2\, dS - C. \end{aligned} \tag{3.5}$$

Finally, considering $I_3$, we integrate by parts (boundary integral disappears because $\boldsymbol{u} = \mathbf{0}$ on boundary), use the commutator relation (3.2) and the equation $-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}$. Then we conclude that

$$-I_3 = (D_j u_i, D_N\sigma_{ij}) = (\boldsymbol{u}, D_N\boldsymbol{f}) + (\boldsymbol{u},\boldsymbol{f}) \overset{(1.33)}{\leq} C\|\boldsymbol{f}\|_{1,d} \leq C. \tag{3.6}$$

Combining (3.3)–(3.6) we easily get (3.1).

For second case, i.e., if $\partial\Omega_1 = \emptyset$ we slightly change the procedure as follows. We apply $D_N$ to $(\mathcal{H}_\mu)_1$ and then multiply the result by $\boldsymbol{\sigma}$ and integrate over $\Omega$ to get

$$I_1 + I_2 + I_3 := (\mathbf{A}D_N\boldsymbol{\sigma},\boldsymbol{\sigma}) + \mu^{-1}(D_N(\frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|}\boldsymbol{\sigma}_D),\boldsymbol{\sigma}) - (D_N\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\sigma}) = 0. \tag{3.7}$$

The reason for this change follows from the way how $I_3$ is treated. Now we can again integrate by parts in $I_3$ (all boundary integrals vanish because $\boldsymbol{\sigma}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$) to obtain

(3.8)
$$\begin{aligned} -I_3 &= (D_N D_j u_i, \sigma_{ij}) = -(D_j D_N u_i, \sigma_{ij}) - (D_j u_i, \sigma_{ij}) = -(D_N u_i, f_i) - (u_i, f_i) \\ &= (d-1)(u,\boldsymbol{f}) + (\boldsymbol{u}, D_N\boldsymbol{f}) - R^2\int_{\partial\Omega} u_i f_i\, dS \\ &\overset{(1.33)(1.36)}{\leq} C(\|\boldsymbol{f}\|_{1,d} + \|\boldsymbol{f}\|_{\infty,\partial\Omega}) \leq C. \end{aligned}$$

For $I_1$ we have exactly same relation as (3.4). So it remains to estimate $I_2$.

$$\begin{aligned} I_2 &= \mu^{-1}\int_\Omega \left((|\boldsymbol{\sigma}_D| - \kappa)_+ + \kappa\xi_{\{|\boldsymbol{\sigma}_D|>\kappa\}}\right) D_N|\boldsymbol{\sigma}_D| \\ &= \mu^{-1}\int_\Omega D_N\left(((|\boldsymbol{\sigma}_D| - \kappa)_+)^2 + \kappa(|\boldsymbol{\sigma}_D| - \kappa)_+\right) \\ &= R\mu^{-1}\int_{\partial\Omega}\left(((|\boldsymbol{\sigma}_D| - \kappa)_+)^2 + \kappa(|\boldsymbol{\sigma}_D| - \kappa)_+\right)\, dS \\ &\quad - d\mu^{-1}\int_\Omega((|\boldsymbol{\sigma}_D| - \kappa)_+)^2 + \kappa(|\boldsymbol{\sigma}_D| - \kappa)_+ \\ &\overset{(1.33)}{\geq} R\mu^{-1}\int_{\partial\Omega}\left(((|\boldsymbol{\sigma}_D| - \kappa)_+)^2 + \kappa(|\boldsymbol{\sigma}_D| - \kappa)_+\right)\, dS - C. \end{aligned} \tag{3.9}$$

Consequently, having (3.7)–(3.9), (3.1) easily follows. □

## Appendix A. Why the proof of Theorem 2.1 fails for ellipse

This part is devoted to a short discussion why the proof of Theorem 2.1 cannot be simply adapted to other domains, e.g., ellipse. Hence we assume that $\Omega \subset \mathbb{R}^2$ is an ellipse with center at $\mathbf{0}$ and axis of the length $a$, $b$. In this setting, the tangential derivative (now we denote it by $T$) is given by

$$T := ax_1 D_2 - bx_2 D_1$$

and the corresponding commutator relation has the form

$$D_1 T - T D_1 = aD_2, \qquad D_2 T - T D_2 = -bD_1. \tag{A.1}$$

Next, we will follow step by step the proof of Theorem 2.1 (we just omit the regularization procedure and we will make all computation formally). Hence, applying $T$ to $(\mathcal{H}_\mu)$, taking the scalar product with $T\boldsymbol{\sigma}$ and integrating it over $\Omega$, we get

$$\begin{aligned} \nu_0 \|T\boldsymbol{\sigma}\|_2 + \mu^{-1} \int_\Omega \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|} |T\boldsymbol{\sigma}_D|^2 \\ + 4^{-1}\mu^{-1}\kappa \int_\Omega \chi_{\{|\boldsymbol{\sigma}_D|>\kappa\}} \frac{|T|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3} \le C(TD_j u_i, T\sigma_{ij}). \end{aligned} \tag{A.2}$$

Following the proof of Theorem 2.1 (we also use the same notation for $I_n$ for corresponding terms), we integrate by parts (all boundary integrals again vanish)

$$\begin{aligned} I &:= (TD_j u_i, T\sigma_{ij}) \overset{(A.1)}{=} (D_j T u_i, T\sigma_{ij}) - (aD_2 u_i, T\sigma_{i1}) + (bD_1 u_i, T\sigma_{i2}) \\ &= -(Tu_i, D_j T\sigma_{ij}) - (aD_2 u_i, T\sigma_{i1}) + (bD_1 u_i, T\sigma_{i2}) \\ &= -\underbrace{-(Tu_i, Tf_i)}_{I_1} - \underbrace{(Tu_i, aD_2\sigma_{i1}) + (Tu_i, bD_1\sigma_{i2})}_{I_2} - \underbrace{(aD_2 u_i, T\sigma_{i1}) + (bD_1 u_i, T\sigma_{i2})}_{I_3}. \end{aligned}$$

Similarly as before, we deduce that

$$I_1 \le C\|\boldsymbol{u}\|_{\frac{d}{d-1}} \|\boldsymbol{f}\|_{2,d}. \tag{A.3}$$

Next,

$$\begin{aligned} I_2 + I_3 &= -a(D_2 u_i, T\sigma_{i1}) - ab(D_1 u_i, \sigma_{i1}) + b(D_1 u_i, T\sigma_{i2}) - ab(D_2 u_i, \sigma_{i2}) \\ &\quad - (aD_2 u_i, T\sigma_{i1}) + (bD_1 u_i, T\sigma_{i2}) \\ &= \underbrace{-ab(\boldsymbol{u}, \boldsymbol{f})}_{I_5} - \underbrace{2a(D_2 u_i, T\sigma_{i1}) + 2b(D_1 u_i, T\sigma_{i2})}_{I_4}. \end{aligned}$$

For $I_4$ we derive

$$\begin{aligned} I_4 &= -2a(D_2 u_i + D_i u_2, T(\sigma_D)_{i1}) + 2b(D_1 u_i + D_i u_1, T(\sigma_D)_{i2}) \\ &\quad - 2a(D_2 u_i + D_i u_2, T(\sigma - \sigma_D)_{i1}) + 2b(D_1 u_i + D_i u_1, T(\sigma - \sigma_D)_{i2}) \\ &\quad + 2a(D_i u_2, T\sigma_{i1}) - 2b(D_i u_1, T\sigma_{i2}) =: I_6 + I_7 + I_8. \end{aligned}$$

The integral $I_6$ can be estimated in the exactly same way as in the proof of Theorem 2.1. Note that the fact that we test the equation by tensor with zero trace is essential. For $I_8$, we can compute

$$\begin{aligned} \frac{1}{2} I_8 &= -a(u_2, D_i T\sigma_{i1}) + b(u_1, D_i T\sigma_{i2}) = a(u_2, Tf_1) - b(u_1, Tf_2) \\ &\quad - a^2(u_2, D_2\sigma_{11}) + ab(u_2, D_1\sigma_{21}) + ab(u_1, D_2\sigma_{12}) - b^2(u_1, D_1\sigma_{22}) =: I_9 + I_{10} \end{aligned}$$

and we see that $I_9$ can be again simply bounded. Thus, it remains to estimate $I_7$ and $I_{10}$. Since $\sigma_{ij} - (\sigma_D)_{ij} = 2^{-1}\delta_{ij}tr\boldsymbol{\sigma}$, we can simplify the relation for $I_7$ as

$$I_7 = (b-a)(D_1u_2 + D_2u_1, T\,\mathrm{tr}\,\boldsymbol{\sigma})$$

and we see that $I_7 = 0$ only if $a = b$, i.e., for ball. From a similar reason the computation that has been successfully used in the proof of Theorem 2.1 to bound $I_{10}$ cannot be used here and we are not able to handle $I_{10}$.

## Appendix B. Trace theorem and Korn-type inequality in unbounded domains

This part of the appendix is devoted to estimates in unbounded domain that corresponds to

$$\|\cdot\|_{d'} \le C\|\boldsymbol{\varepsilon}(\cdot)\|_1, \quad \|\cdot\|_{1,\partial\Omega} \le C(\|\cdot\|_1 + \|\boldsymbol{\varepsilon}(\cdot)\|_1)$$

that are valid in bounded open sufficiently smooth domains (for proof see for example Temam (1985)). For simplicity[12], we assume that $\Omega = \{(x_1,\dots,x_d); 0 < x_d < 1\}$. For such domain we can proof the following

**Lemma B.1.** *Let $d \ge 2$ and $\boldsymbol{u} \in W^{1,2}(\Omega)^d$. Then for all $\alpha < -\frac{(2-p)(d-1)}{2p}$ and all[13] $p \in [1,2)$, there holds*

$$\|\boldsymbol{u}\|_{\frac{dp}{d-p},\alpha} \le C(\alpha)\|\boldsymbol{\varepsilon}(u)\|_{p,\alpha} \tag{B.1}$$

*whenever the right hand side of* (B.1) *is finite.*

*Proof.* From the definition $W^{1,2}(\Omega)$ we can find a sequence $\{\boldsymbol{u}^n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathbb{R})^d$ such that $\nabla\boldsymbol{u}^n \to \nabla\boldsymbol{u}$ strongly in $L^2$. Defining $V(\boldsymbol{x}) := (K + |\boldsymbol{x}|)^{-\alpha}$ and using the standard Korn inequality (for the case $p = 1$ see (Temam, 1985, the proof of Theorem 1.2, page 125)) we get that

$$\|\boldsymbol{u}^n V\|_{\frac{dp}{d-p}} \le C\|\boldsymbol{\varepsilon}(\boldsymbol{u}^n V)\|_p, \tag{B.2}$$

where the constant $C$ depends only on the dimension. Next, we can compute

$$\begin{aligned}\|\boldsymbol{\varepsilon}(\boldsymbol{u}^n V)\|_p &\le \|\boldsymbol{\varepsilon}(\boldsymbol{u}^n)V\|_p + \|\boldsymbol{u}^n|\nabla V|\|_p \\ &\le \|\boldsymbol{\varepsilon}(\boldsymbol{u}^n)V\|_p + \|\boldsymbol{u}^n V\|_{\frac{dp}{d-p}}\|V^{-1}|\nabla V|\|_d\end{aligned} \tag{B.3}$$

Next, since $|\nabla V|V^{-1} \le C(d)(K+|\boldsymbol{x}|)^{-1}$, we see that (using the fact the our domain is "infinite" only in $(d-1)$ variables) we have that $\|V^{-1}|\nabla V|\|_d \le C(K)$, where $C(K) \to 0$ as $K \to \infty$. Hence, we can find such $K \in (0,\infty)$ that

$$\|V^{-1}|\nabla V|\|_d \le \frac{1}{2}.$$

Substituting this into (B.3) and using (B.2), we finally get

$$\|\boldsymbol{u}^n V\|_{\frac{dp}{d-p}} \le C\|\boldsymbol{\varepsilon}(\boldsymbol{u}^n)V\|_p. \tag{B.4}$$

[12] In fact, the result can be extended in many directions. First one is that we may assume more general domains for which we get estimates with slightly different weights. Second possible extension is that we can prove the result under weaker hypothesis on $\boldsymbol{u}$, namely we could assume that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathsf{F}_1 + \mathsf{F}_2$$

where $\mathsf{F} \in L^q(\Omega)$ and $\mathsf{F}_2$ being a measure.

[13] If $d \ge 3$ we may consider also the case $p = 2$.

In order to pass to the limit in (B.4), we use the Fatou lemma to get the lower limit on LHS of (B.4). Such a procedure is possible, because of locally compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^2_{loc}$. To pass to the limit on the right hand side it is enough to show (since $\boldsymbol{\varepsilon}(\boldsymbol{u}^n)$ is compact in $L^2$) that $V \in L^{\frac{2p}{2-p}}$. This is however equivalent to

$$\alpha > \frac{(2-p)(d-1)}{2p},$$

which is exactly the assumption put on $\alpha$. Thus, the proof is complete. □

As a simple consequence of Lemma B.1 is the following

**Corollary B.1.** *Let $\boldsymbol{u} \in W^{1,2}(\Omega)$ be such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathsf{F}_1 + \mathsf{F}_2,$$

*where $\mathsf{F}_1 \in L^2(\Omega)^{d\times d}$ and $\mathsf{F}_2 \in L^1(\Omega)^{d\times d}$. Then for all $\alpha < -\frac{d-1}{2}$ there holds*

$$\|\boldsymbol{u}\|_{d',(\alpha)} \le C(\alpha)(\|\mathsf{F}_2\|_2 + \|\mathsf{F}_1\|_1). \tag{B.5}$$

*Proof.* Using (B.1) we obtain that

$$\|\boldsymbol{u}\|_{d',(\alpha)} \le C(\|\mathsf{F}_1\|_{1,(\alpha)} + \|\mathsf{F}_2\|_{1,(\alpha)}) \le C(\|\mathsf{F}_2\|_2\|(1+|\boldsymbol{x}|)^\alpha\|_2 + \|\mathsf{F}_2\|_1)$$

and (B.5) easily follows provided that $(1+|\boldsymbol{x}|)^\alpha \in L^2 \iff \alpha < -\frac{d-1}{2}$. □

We also establish the estimates concerning the behavior of the trace.

**Lemma B.2.** *Let $d \ge 2$ and $\boldsymbol{u} \in W^{1,2}(\Omega)^d$. Then for all $\alpha < -\frac{d-1}{2}$, there holds*

$$\int_{\partial\Omega} |\boldsymbol{u}|(1+|x|)^\alpha \, dS \le C(\alpha)(\|\boldsymbol{u}\|_{1,(\alpha)} + \|\boldsymbol{\varepsilon}(u)\|_{1,\alpha}) \tag{B.6}$$

*whenever the right hand side of* (B.6) *is finite.*

*Proof.* Here, we give only a formal proof. For details we refer to the procedure described in (Temam, 1985, Proof of Theorem 1.1, page 121). For simplicity we analyye the behavior of $\boldsymbol{u}$ only on the part of the boundary consisting from $\Gamma := \{x; x_d = 1\}$. First, we estimate the behavior of $u_d$. For a.a. $x \in \Gamma$, we have

$$u_d(x) = u_d(x_1, \dots, x_{d-1}, s) + \int_s^1 \frac{\partial}{\partial x_d} u_d(x_1, \dots, x_{d-1}, s) \, ds.$$

Multiplying it by $(1+|x|)^\alpha$, taking absolute value and integrating over $\Gamma$ w.r.t. $x_1, \dots, x_{d-1}$ we obtain

$$\begin{aligned}\int_\Gamma |u_d|(1+|x|)^\alpha \, dS \le \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |u_d(x_1, \dots, x_{d-1}, s)|(1+|x|)^\alpha \, dx_1 \dots dx_{d-1} \\ + \|\frac{\partial u_d}{\partial x_d}\|_{1,(\alpha)}.\end{aligned}$$

Consequently, integrating the result over $s \in (0,1)$ and using the fact that $|\frac{\partial u_d}{\partial x_d}| \le |\boldsymbol{\varepsilon}(\boldsymbol{u})|$ we obtain

$$\int_\Gamma |u_d|(1+|x|)^\alpha \, dS \le \|\boldsymbol{u}\|_{1,(\alpha)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{1,(\alpha)}.$$

Next, we prove the same result but for different component of $\boldsymbol{u}$. For simplicity we prove it only for $u_1$, for other $u_i$'s the proof is same. First, we deduce that

$$
\begin{aligned}
u_1(x) + u_d(x) &= u_1(x_1+s-1, x_2, \dots, x_{d-1}, s) + u_d(x_1+s-1, x_2, \dots, x_{d-1}, s) \\
&\quad + \int_s^1 \frac{d}{dt}\left(u_1(x_1+t-1, x_2, \dots, x_{d-1}, t)\right)\, dt \\
&\quad + \int_s^1 \frac{d}{dt}\left(u_d(x_1+t-1, x_2, \dots, x_{d-1}, t)\right)\, dt.
\end{aligned} \tag{B.7}
$$

Since

$$
\begin{aligned}
&\left| \frac{d}{dt}\left(u_1(x_1+t-1, x_2, \dots, x_{d-1}, t) + u_d(x_1+t-1, x_2, \dots, x_{d-1}, t)\right)\right| \\
&\qquad\qquad \le 2|\boldsymbol{\varepsilon}(\boldsymbol{u}(x_1+t-1, x_2, \dots, x_{d-1}, t))|
\end{aligned}
$$

we can take absolute value in (B.7) multiply the result by $(1+|x|)^\alpha$ and integrate w.r.t. $x_1, \dots, x_{d-1}$ and $s$ to get

$$
\int_\Gamma |u_1 - u_d|(1+|x|)^\alpha \, dS \le C\|\boldsymbol{u}\|_{1,(\alpha)} + C\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{1,(\alpha)}.
$$

Using the same procedure for arbitrary $u_i$ we finally conclude (B.6). □

## Appendix C. Interior regularity estimates

For the sake of completeness we end this paper with the proof of interior regularity result for our approximative problem and consequently (as the estimates are $\mu$ independent) also for limit problem. To be more concrete under non-uniform assumption (1.32) we prove uniform estimate (1.34). This result can be found also in Löbach (2007) by using the same approximation or in Bensoussan and Frehse (1993), via the Norton-Hoff approximation that has the advantage that it works in arbitrary dimension. The approach here works also for our different approximation but from very essential reasons we are able to prove local regularity result only for $d \le 4$ (see also Löbach (2007)).

**Lemma C.1.** *Let $d \le 4$. Assume that $(\boldsymbol{\sigma}, \boldsymbol{u})$ is a solution to $(\mathcal{H}_\mu)$ satisfying* (1.33) *and* (1.32). *If $\boldsymbol{f} \in W^{2,d}_{loc}(\Omega)$ then* (1.34) *holds.*

*Proof.* Let $\boldsymbol{x} \in \Omega$ and let $\varepsilon > 0$ be such that $B(\boldsymbol{x}, 2\varepsilon) \subset \Omega$. We find $\xi \in \mathcal{D}(B(\boldsymbol{x}, 2\varepsilon))$ such that $\xi \equiv 1$ in $B(\boldsymbol{x}, \varepsilon)$. Next, we apply the operator $\nabla$ to equation $(\mathcal{H}_\mu)_1$, take the scalar product with $\nabla\boldsymbol{\sigma}\xi^{2m}$ for some $m \in \mathbb{N}$ and integrate the result over $\Omega$. Note that having (1.32), such procedure is rigorous. Hence, we get the equation

$$
(\mathbf{A}\nabla\boldsymbol{\sigma}, \boldsymbol{\sigma}\xi^{2m}) + \mu^{-1}(\nabla \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+ \boldsymbol{\sigma}_D}{|\boldsymbol{\sigma}_D|}, \nabla\boldsymbol{\sigma}\xi^{2m}) = (\nabla\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\boldsymbol{\sigma}\xi^{2m}). \tag{C.1}
$$

Using (1.29), we immediately deduce that

$$
\begin{aligned}
\text{LHS of (C.1)} &\ge c\mu^{-1} \int_{B(x,2\varepsilon)} \frac{(|\boldsymbol{\sigma}_D| - \kappa)_+}{|\boldsymbol{\sigma}_D|} |\nabla\boldsymbol{\sigma}_D|^2 \xi^{2m} \\
&\quad + c\mu^{-1}\kappa \int_{B(x,2\varepsilon)} \chi_{\{|\boldsymbol{\sigma}_D| > \kappa\}} \frac{|\nabla|\boldsymbol{\sigma}_D|^2|^2}{|\boldsymbol{\sigma}_D|^3} \xi^{2m} + \nu_0 \int_{B(x,2\varepsilon)} \nabla\boldsymbol{\sigma} \cdot \nabla\boldsymbol{\sigma}\xi^{2m}\, dx
\end{aligned} \tag{C.2}
$$

that is in fact the LHS of the estimate (1.34). We follow by estimating RHS of (C.2). Hence, we use integration by parts and the identity $(\mathcal{H}_\mu)_2$ to get

$$
\begin{aligned}
&\text{RHS of (C.1)} = (D_k D_j u_i, D_k \sigma_{ij} \xi^{2m}) \\
&= (D_k u_i, D_k f_i \xi^{2m}) - (D_k u_i, D_k \sigma_{ij} D_j \xi^{2m}) \\
&= -(\boldsymbol{u}, \triangle \boldsymbol{f} \xi^{2m}) - (u_i, D_k f_i D_k \xi^{2m}) - (D_k u_i, D_k \sigma_{ij} D_j \xi^{2m}) \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{C.3}
$$

Using (1.33) and the fact that $\boldsymbol{f} \in W^{2,d}_{loc}(\Omega)$ we immediately deduce that

$$J_1 + J_2 \le C(\varepsilon, m).$$

The remaining integral is estimated as follows

$$
\begin{aligned}
J_3 &= -(D_k u_i + D_i u_k, D_k \sigma_{ij} D_j \xi^{2m}) + (D_i u_k, D_k \sigma_{ij} D_j \xi^{2m}) \\
&= -(D_k u_i + D_i u_k, D_k \sigma_{ij} D_j \xi^{2m}) + (u_k, D_k f_j D_j \xi^{2m}) - (u_k, D_k \sigma_{ij} D_i D_j \xi^{2m}) \\
&=: J_4 + J_5 + J_6.
\end{aligned}
$$

Next, for $J_4$ we use the equation $(\mathcal{H}_\mu)_1$ to obtain that (using that $|\nabla \xi^{2m}| \le C(\varepsilon)\xi^m$)

$$J_4 \le C(\varepsilon) \|\boldsymbol{\sigma}\|_2 \|\nabla \boldsymbol{\sigma} \xi^m\|_2 + C(\varepsilon)\mu^{-1} \int_{B(x,2\varepsilon)} (|\boldsymbol{\sigma}_D| - \kappa)_+ |\nabla \boldsymbol{\sigma}| \xi^m \, dx.$$

Consequently, having the point-wise estimate $|\nabla \boldsymbol{\sigma}| \le C|\nabla \boldsymbol{\sigma}_D| + C|\operatorname{div} \boldsymbol{\sigma}|$ and using the Young inequality, a priori estimate (1.33) and the assumption on $\boldsymbol{f}$ we conclude

$$
\begin{aligned}
J_4 &\le \frac{\nu_0}{2} \|\nabla \boldsymbol{\sigma} \xi^m\|_2^2 + C(\varepsilon, \nu_0) \|\boldsymbol{\sigma}\|_2^2 \\
&\quad + C(\varepsilon)\mu^{-1} \int_{B(x,2\varepsilon)} (|\boldsymbol{\sigma}_D| - \kappa)_+ \left( C|\boldsymbol{f}| + \frac{1}{2} \frac{|\nabla \boldsymbol{\sigma}_D \xi^m|^2}{|\boldsymbol{\sigma}_D|} + C|\boldsymbol{\sigma}_D| \right) dx \\
&\le C(\varepsilon, \nu_0) + \frac{1}{2} (\text{LHS of (C.1)}).
\end{aligned}
$$

The term $J_5$ can be simply estimated as

$$J_5 \le \|\boldsymbol{u}\|_{d'} \|\boldsymbol{f}\|_{1,d} \overset{(1.33)}{\le} C.$$

For the remaining integral, we have

$$J_6 = (\operatorname{div} \boldsymbol{u}, \boldsymbol{\sigma} \cdot \nabla^2 \xi^{2m}) + (u_k, \sigma_{ij} D_i D_j D_k \xi^{2m}) \overset{(1.33)}{\le} C + \int_\Omega |\boldsymbol{u}||\boldsymbol{\sigma}||\nabla^3 \xi^{2m}| \, dx.$$

The last term is simply bounded if dimension $d = 2$ (using (1.33)). For $d = 3, 4$ we use Hölder inequality to conclude that

$$\int_\Omega |\boldsymbol{u}||\boldsymbol{\sigma}||\nabla^3 \xi^{2m}| \, dx \le \|\boldsymbol{u} \xi^{-m} |\nabla^3 \xi^{2m}|\|_{d'} \|\boldsymbol{\sigma} \xi\|_d =: J_7.$$

Next, setting $m = 3$ we see that $\xi^{-m}|\nabla \xi^{2m}| \le C$. By means of the embedding theorem, we obtain

$$J_7 \le C\|\boldsymbol{u}\|_{d'} \|\nabla(\boldsymbol{\sigma} \xi^m)\|_2 \overset{(1.33)}{\le} C + C\||\nabla \boldsymbol{\sigma}| \xi^m\| \le C + \frac{1}{8} (\text{LHS of (C.1)}).$$

Finally, combining all above estimates, we come to the conclusion that (1.34) holds. □

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