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Existence of Regular Solutions to a Class of Parabolic Systems in Two Space Dimensions with Critical Growth Behaviour

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Abstract

We consider parabolic systems

$$u_t - \operatorname{div}\big(a(t,x,u,\nabla u)\big) + a_0(t,x,u,\nabla u) = 0$$

in two space dimensions with initial and Dirichlet boundary conditions. The elliptic part including a_0 is derived from a potential with quadratic growth in ∇u and is coercive and monotone.

The term a_0 may grow quadratically in ∇u and satisfies a sign condition $a_0 \cdot u \geq -K$. We prove the existence of a regular long time solution verifying a regularity criterion of Arkhipova. No smallness is assumed on the data.

Keywords: Nonlinear parabolic systems, Hölder continuity, global solutions

AMS classification: 35K55, 35K50

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. We consider a parabolic system of the form

$$\begin{aligned} u_t - \sum_{i=1,2} D_i a_i(t,x,u,\nabla u) + a_0(t,x,u,\nabla u) = 0 \quad \text{in } [0,T^0]\times\Omega, \\ u = (u_1,\dots,u_N), \quad a_i = (a_i^1,\dots,a_i^N), \quad \nabla = \nabla_x \end{aligned} \tag{1}$$

with homogeneous Dirichlet boundary condition and initial value condition

$$u(0) = u_0, \quad u_0 \in H_0^1(\Omega), \tag{2}$$

where $H_0^1(\Omega)$ denotes the usual Sobolev spaces of L^2-functions such that $\nabla u \in L^2(\Omega)$, and the traces on the boundary vanish. We will show the existence of a regular long time solution under the assumptions that the coefficient functions are derived from a potential A with quadratic growth in ∇u together with coercivity and monotonicity conditions, while a_0 may grow quadratically in u. The precise

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assumptions will be specified in Section 2. Due to the critical growth behavior of the term a_0, the theory of monotone operators cannot be applied directly to obtain the existence of a long time weak solution to the system (1). However, with an additional coerciveness condition (see (9) below) for the lower order term a_0 we succeed to prove the existence of a solution

$$u \in L^\infty(0, T^0; H^1(\Omega)) \text{ such that } u_t \in L^2([0, T^0] \times \Omega) \tag{3}$$

which even is Hölder continuous. Thereby, the estimate of the modulus of continuity of u is crucial for our result. Once having that $u \in C^\alpha$ for some $\alpha \in (0, 1)$ regularity techniques are available to prove that the solution $u \in L^p(W^{1,p}) \cap L^2(H^2)$ with $\nabla u_t \in L^2(L^2)$, $u_t \in L^p(L^p) \bigcap L^\infty(L^{2+\delta})$, $p > 4$ and furthermore $\iint |\nabla u_t|^2 |u_t|^\delta dx\, dt < \infty$ which implies Hölder continuity of ∇u. This, in turn, implies higher order regularity using the linear theory of parabolic systems.

For scalar problems Hölder continuity of weak solutions of (1) and long time existence can be found in the famous book of Solonnikov [15] (together with Ladyzhenskaya-Uralzeva), where also a lot of basic facts about nonlinear parabolic systems can be found.

For diagonal systems with elliptic principal part coming from a variational integral

$$\sum_{i,k=1,2} \sum_{\nu,\mu=1}^{N} \int_\Omega b^{\nu\mu}(u) a_{ik}(x) D_i u^\mu D_k u^\nu, \tag{4}$$

and, more general, for harmonic flows, many results concerning long time existence of *weak* solutions are known, also for dimension $n \geq 2$, including results of partial regularity for the solution (see Hamilton's book [13] for a more extensive bibliography). Struwes result [21] implies for the two dimensional case the existence of long time solutions which may have isolated point singularities. With our condition (9) his methods would give full regularity of the global solution. The first global existence result for a regular harmonic flow is due to Eells-Sampson [6].

The case of *non-diagonal* parabolic systems with elliptic principal part coming from a variational integral

$$\sum_{i,k=1,2} \sum_{\nu,\mu=1}^{N} \int_\Omega b_{ik}^{\nu\mu}(t, x, u) D_i u^\mu D_k u^\nu, \text{ or } \int_\Omega A(t, x, u, \nabla u) dx$$

(with growth, coerciveness and monotonicity conditions) Arkhipova [4] proved that smooth solutions of (1) can be extended to a larger time interval provided that a uniform smallness property

$$\sup_{0<t\leq t_1} \sup_{x_0 \in \Omega} \int_{B_r(x_0)} |\nabla u|^2 dx \leq \varepsilon \text{ for } r \leq r_0 = r_0(\varepsilon) \tag{5}$$

for the solutions is available. From this condition she obtains further a priori estimates for smooth solutions in $L^2(W^{2,p})$, $p \geq 2$, with the additional regularity properties mentioned above; thus one can deduct the existence of smooth long time solutions. In the present paper we prove that the condition (5) is satisfied under the additional assumption $a_0 \cdot u \geq -K$ (see Section 2 for the complete assumptions).

The elliptic analogue of this paper is contained in [8]. We use ideas like inhomogeneous hole filling and logarithmic Morrey estimates from this paper, however the tools form the present paper would simplify the proofs of [8].

The plan of the paper is as follows: In Section 2 we recall some notations, formulate the precise assumptions for the coefficient a_i and the main result. In principle, it suffices to to prove Arhipova's smallness criterion (5). In this context only the Sections 3.2 and 3.4 are needed. However, we also present our approach obtaining uniform C^α-estimates for solutions which are sufficiently smooth. For this alternative proof, which is described in Section 3, we use a new hole filling argument [9] dedicated to evolutional problems. After all, our approach is based on the extension argument, too, since the global uniform C^α-estimate only works for smooth solutions (in fact, for solutions satisfying (3) and $u \in C^\alpha$.) An alternative way of proving the main theorem consists in using finite element approximations as in [10] and performing a discrete analogue of the proof. Thereby, long time C^α -solutions can be obtained directly without proving further regularity. However, the finite element technique used in this setting are as lengthy as the approach via extension and regularity arguments.

2 Assumptions for the coefficient functions and the main results

We start with a tabular list of frequently used common notations. We use

$W^{m,r}(\Omega)$ the Sobolev space of functions with derivatives of order $\leq m$ in L^r,

$H^m(\Omega) = W^{m,2}(\Omega)$,

$W_0^{m,r}(\Omega)$ the closure of smooth functions with compact support in $W^{m,r}(\Omega)$,

$C^{m+\alpha}(\Omega)$ the space of m-times continuously differentiable functions where the derivatives of order m are Hölder continuous with Hölder exponent α,

$L^r(V) = L^r(0,T;V)$ the L^r-space (in the sense of Bochner-Lebesgue) of V-valued functions on $[0,T]$, where V is any Banach-space,

$B_R(x_0)$ the open disc around x_0 of radius R,

$Q_R(t_1,x_0) = [\max(t_1 - R^2, 0), t_1] \times B_R(x_0)$ the (truncated) parabolic cylinder.

In general K stands for a constant which needs not to be specified and can change from line to line, we frequently use the symbol $\lesssim \ldots$ instead of $\leq K \ldots$.

Next we specify our technical assumptions on the coefficients.
For simplicity we assume that the coefficients a_i are continuous differentiable with respect to all variables, moreover they must be derived from a potential with specific properties. In detail we have

1. There exists a C^1-function $A : [0,T_1] \times \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}$ such that $\frac{\partial}{\partial \eta_i^\nu} A(t,x,\mu,\eta) = a_i^\nu(t,x,\mu,\eta)$, $i = 1,2$, $\frac{\partial}{\partial \mu^\nu} A(t,x,\mu,\eta) = a_0^\nu(t,x,\mu,\eta)$.
2. The potential A and the coefficient functions a_i follow the growth conditions

$$\begin{aligned} &A(t,x,\eta) + |A_t(t,x,\mu,\eta)| + |a_0(t,x,\mu,\eta)| + \\ &\sum_{i=1,2} |a_i(t,x,\mu,\eta)|^2 \leq C_0|\eta|^2 + \kappa|\mu|^q + K \end{aligned} \tag{6}$$

for some $q < \infty$,

$$
\begin{aligned}
&\Big|\frac{\partial}{\partial\eta}a_i(\cdot,\cdot,\mu,\eta)\Big| \le \kappa|\mu|^q + K,\ i=1,2\\
&\Big|\frac{\partial}{\partial\eta}a_0(\cdot,\cdot,\mu,\eta)\Big| + \Big|\frac{\partial}{\partial\mu}a_i(\cdot,\cdot,\mu,\eta)\Big| \le K|\eta| + \kappa|\mu|^q + K, \qquad (7)\\
&\Big|\frac{\partial}{\partial\mu}a_0(\cdot,\cdot,\mu,\eta)\Big| \le K|\eta|^2 + \kappa|\mu|^q + K
\end{aligned}
$$

the coercivity conditions

$$\sum_{i=1,2}\sum_{\nu=1}^{N} a_i^\nu(\cdot,\cdot,\cdot,\eta)\eta_i^\nu \ge \alpha_0|\eta|^2 - K, \quad A(\cdot,\cdot,\mu,\eta) \ge \alpha_1|\eta|^2 - K\,, \qquad (8)$$

$$a_0(\cdot,\cdot,\mu,\eta)\cdot\mu \ge -K\,, \qquad (9)$$

the monotonicity condition

$$
\begin{aligned}
&\sum_{i=1,2}\sum_{\nu=1}^{N}\big(a^\nu(t,x,\mu,\eta) - a^\nu(t,x,\mu,\hat\eta)\big)\cdot(\eta^\nu - \hat\eta^\nu) \ge \alpha_2|\eta-\hat\eta|^2 \qquad (10)\\
&\eta,\hat\eta \in \mathbb{R}^{2N}\,,
\end{aligned}
$$

with positive constants C_0, α_0, α_1 and κ.

We note that some of the regularity assumptions may be weakened. Moreover, the assumption $\partial_\mu A = a_0$ can be weakened in such a way such that the elliptic part is "close to variational" (cf. [4] for more details.) In [4], the conditions (6) - (10) were required with $\kappa = 0$. For increasing the applicability, we added the term with $\kappa > 0$. Due to embedding arguments this term only leads to pollution terms and could be treated in [3] as well. The additional requirement here is the condition (9). This allows us to prove the uniform smallness condition which is sufficient for the extension of the local smooth solution.

Further we use the following assumptions on the data of the problem: The initial value

$$
\begin{aligned}
&u_0 = \hat u_0|_{t=0},\\
&\hat u_0 \in W^{1,p}([0,T^0]\times\Omega)\cap L^p(W^{2,p}(\Omega)), \quad \hat u_0(t,\cdot) = 0 \text{ on } \partial\Omega,
\end{aligned} \qquad (11)
$$

and the right hand side

$$f \in L^p([0,T^0]\times\Omega), (p > 4), \qquad (12)$$

in particular the data satisfy the Morrey conditions

$$R^{-2\gamma}\int_{B_R(x_0)} |\nabla u_0|^2\,dx \le K\,, \quad 0 < R \le R_0, \quad x_0 \in \Omega. \qquad (13)$$

$$R^{-2\gamma}\iint_{Q_R(t_1,x_0)} |f|^2\,dx\,dt \le K, \quad 0 < R \le R_0, t_1 \in (0,T^0), x_0 \in \Omega\,. \qquad (14)$$

Now we can formulate our main result.

Theorem 2.1 *Under the assumptions for the coefficients and the data listed above there exists a strong solution u to problem* (1) *with*

$$u \in L^\infty(0,T^0;H^1(\Omega)) \cap C^\alpha\big([0,T^0]\times\Omega\big), \quad u_t \in L^\infty(0,T^0;L^2(\Omega))\big)$$

for some $\alpha > 0$.

The additional C^α regularity implies the following

Corollary 2.2 *The solution is regular in the following sense:*

$$\begin{aligned} &u \in L^2(0,T^0;H^2(\Omega)) \cap L^2(0,T^0;W^{1,p}(\Omega)) \cap C^\alpha([0,T^0]\times\bar{\Omega})),\ p>4 \\ &u_t \in L^2(0,T^0,L^2) \cap L^\infty(\delta_0,T^0;L^{2+\delta}(\Omega)), \quad \nabla u_t \in L^2(\delta_0,T^0;L^2(\Omega)),\ \delta_0>0. \end{aligned} \tag{15}$$

These properties imply also $\nabla u \in C^\alpha([0,T^0]\times\bar{\Omega}))$.

Remark: Under appropriate smoothness conditions for $\partial\Omega$ and the data and eventually additionally compatibility conditions at $t=0$ a classical bootstrap argument leads higher regularity for u. The exponent α depends on the growth and coerciveness constants C_0, α_0 in (6) and (8), (9) and γ in (13), (14) and the Lipschitz constant of $\partial\Omega$. The C^α-semi-norm of u can be estimated uniformly with respect to the constants K in the data and the measure of Ω and the Lipschitz constant of $\partial\Omega$. However, the norms in $(15)_2$ need not have a bound uniform with respect to δ_0 as $\delta_0 \to 0$.

We understand a function $u \in L^2(0,T^0;(H^{1,2}_{\text{loc}})^N)$ with $u_t \in L^2([0,T^0]\times\Omega)$ as a weak solution u of (1) if u satisfies (2) and the equation

$$\int_0^{T^0}\!\!\int_\Omega \big[u_t\varphi + \sum_{i=1,2} a_i(\cdot,u,\nabla u)\cdot D_i\varphi + a_0(\cdot,u,\nabla u)\varphi\big]\,dx\,dt = \int_0^{T^0}\!\!\int_\Omega f\varphi\,dx\,dt \tag{16}$$

for all $\varphi \in C^1\big((0,T^0)\times\Omega\big)$ such that $\operatorname{supp}\varphi(t,\cdot\,) \subset\subset \Omega$ and $\varphi|_{T^0}=0$. One of the technical problems for the proof is the fact, that we have to work with the energy equality

$$\begin{aligned} &\int_{t_1}^{T}\!\!\int_\Omega u_t^2\varphi\,dx\,dt + \int_{t_1}^{T}\!\!\int_\Omega \big(-A\varphi_t - A_t\varphi + \sum_{i=1,2} a_i u_t D_i\varphi\big)\,dx\,dt \\ &\quad + \int_\Omega A\varphi\,dx\Big|_{t_1}^{T} = \int_{t_1}^{T}\!\!\int f u_t\varphi\,dx\,dt \text{ for a.a. } t_1, T. \end{aligned} \tag{17}$$

Note that (17) is true for $t_1 = 0$ since in our setting the problem has a sufficiently regular solution on small time intervals (see the explanations below), hence $A(t,\cdot,u,\nabla u) \to A(t,\cdot,u_0,\nabla u_0)$ in $L^1(\Omega)$. For smooth solutions of (1) this identity follows by testing the equation with $u_t\varphi$, where $\varphi \geq 0$, φ Lipschitz. One can prove that (17) is satisfied if $u \in C^\alpha \cap L^\infty(H^1)$, $u_t \in L^2(L^2)$ (see Appendix). Furthermore, we need for the C^α-estimate of u that the quantity

$$\operatorname*{ess\,sup}_{t,x_0,R}\Big\{R^{-\alpha}\int_{B_R(x_0)} |\nabla u(t,\cdot)|^2\,dx\Big\}, \quad x_0\in\Omega,\ t\in[0,T],\ R\in(0,R_0],$$

is *bounded* (without using this bound in estimates). So we have to arrange a setting where we work with smooth solutions. For this reason we use the continuation method (similar as in [4]) to obtain global smooth solutions to parabolic systems. We give a scetch of the arguments below: By the local theory [1, 11, 4] there exists a small time interval $[0, t_1)$ such that (1) has a sufficiently regular solution u, i. e. u has the regularity properties (15) now on the time interval $[0, t_1]$ instead of $[0, T^0]$. For this local result, Arkhipova assumes still more regularity on the domain and the initial data than we did, however one can approximate the data and prove that the local solution is uniformly in C^α with respect to the approximations parameter, just by the methods exposed in this paper. Using this additional estimate the passage to the limit for subsequences of weakly convergent approximations is possible via the monotonicity argument. Since the C^α property is preserved, the regularity techniques described below can be applied and we obtain the properties (15). Hence we have, as a starting point, a local solution with the properties (15). Thus the set

$$\Xi = \left\{ t_1 \leq T^0 \middle| \text{ (1) has a solution } u \in [0, t_1] \text{ which enjoys (15) } \right\}$$

is not empty. Let $T^* = \sup\{t | t \in \Xi\}$. Suppose $T^* < T^0$. Then $T^* - \delta \in \Xi$ and we will show that u remains uniformly Hölder continuous as $\delta \to 0$ for some Hölder exponent α which does not depend on regularity properties of u other than those in (3). From the Hölder continuity one can derive (15) and further regularity via the following steps:

1. Estimation of the difference quotients in space direction together with boundary analysis gives $\nabla u \in L^2(L^2)$. Thereby on uses

$$\int |\nabla u|^4 dx \lesssim K[u]_\alpha^2 \int |\nabla^2 u|^2 dx + \text{ pollution terms} \tag{18}$$

with $[u]_\alpha = \sup_{x \neq y} |x - y|^{-\alpha} |u(x) - u(y)|$ ("interpolation between C^α and H^2"). This argument has been used in [20, 16, 17].

2. By an additional interpolation argument one achieves

$$\nabla u \in L^{4+\delta}(L^{4+\delta}).$$

3. Once having this, one uses difference quotient technique in time direction and arrives at

$$\nabla u_t \in L^2(L^2), \quad u_t \in L^\infty(L^2).$$

4. Finally, similar as in [2], one can apply the time derivative D_t to the equation and uses $u_t |u_t|^{\delta'}$ with small δ' as a test function. This yields

$$u_t \in L^\infty(L^{2+\delta'})$$

and one obtains $\nabla u \in L^\infty(C^\alpha)$ via the elliptic theory (considering u_t as right hand side. Since $\nabla u_t \in L^2(L^2)$, it follows then $\nabla u \in C^{\alpha/2}$ in space *and* time.

5. After having $\nabla u \in C^\alpha$, the linear theory of parabolic systems [15] can be applied.

The steps explained above correspond to the arguments in [4]. Now we may deal with a regular solution u of problem (1) which exists on the time interval $[0, T^* - \delta)$ for every $\delta > 0$. Since we will prove uniform C^α-estimates for u, the bounds for the norms in (15) are valid independent of δ as $\delta \to 0$, we obtain $\nabla u \in C^\alpha[(0, T^*] \times \Omega] \cap H^1$, $u_t \in L^\infty(L^2(0, T^*] \times \Omega)$ with the further regularity properties (15) in the interval $[0, T^*]$, and we can extend u into an interval $[T^*, T^* + \delta_1]$ applying the local theory of the first step again. This contradicts that T^* is maximal, thus u exists on $[0, T^0] \times \Omega$.

3 The uniform C^α-Estimate

3.1 Preliminary inequalities and the frame

Here and in the following sections we use the same name for u and its canonical extension (by zero) to $\mathbb{R}^2$. Using embedding theorems, the following estimates are simple consequences of a bound for u in $L^\infty(0, T^0; H_0^1(\Omega))$:

$$\operatorname*{ess\,sup}_{0 \le t \le T^0} \int_\Omega |u|^r dx \le K \text{ for all } r > 1, \tag{19}$$

$$\int_{B_R(x_0)} |u|^q dx \lesssim \left(\int_{B_R} |u|^r dx \right)^{r/q} R^{2-2q/r} \lesssim R^{2-\epsilon} \text{ for any } \epsilon > 0. \tag{20}$$

We recall an argument from [9], where we derived Hölder-continuity from a weighted Morrey type estimate. To this end we introduce the weight function $|W_t|$ as the modulus of the time derivative of

$$W(T, x_0, x, t) = \frac{T - t}{|x - x_0|^2 + T - t}, \text{ hence} \tag{21}$$

$$W_t(T, x_0, x, t) = \frac{-|x - x_0|^2}{(|x - x_0|^2 + T - t)^2}, \tag{22}$$

$$D_i W(T, x_0, x, t) = \frac{-2(x_i - x_{0,i})(T - t)}{(|x - x_0|^2 + T - t)^2}. \tag{23}$$

The properties (6), (8) lead to the following global bound independent x_0 and R

$$\sup_{R, x_0, T} \iint_{Q_R(x_0, T)} |\nabla u|^2 |W_t| dx\, dt \le K. \tag{24}$$

Indeed, we apply (17) with $\varphi = W_\epsilon = \frac{T-t}{|x-x_0|^2+T-t+\varepsilon}$, then

$$- \int_{t_1}^{T} \int_\Omega A W_{\epsilon,t} dx\, dt + \int_{t_1}^{T} \int_\Omega u_t^2 W dx\, dt \le \int_\Omega A(t_1, \cdot) W_\epsilon(t_1, \cdot) + dx$$

$$\int_{t_1}^{T} \int_\Omega \big(|A_t| W_\epsilon + \sum_{i=1,2} |a_i| |u_t| |D_i W_\epsilon| \big) dx\, dt + \int_{t_1}^{T} \int f u_t W_\epsilon \, dx\, dt.$$

The left hand side can be estimated from below using $|W_{\epsilon,t}| = -W_{\epsilon,t}$, the coercivity condition (8) for A, the terms on the right hand side can be controlled by the growth

conditions, the assumptions on the data and using the a priori bounds for u (see [9, Sect. 3] for the details). Finally we may pass to the limit $\epsilon \to 0$ via the theorem on monotone convergence.
Further we frequently use the following localization function $\tau_R = \tau_R(x_0, T, \cdot, \cdot)$

$$\tau_R(t,x) = \psi(t)\zeta(|x-x_0|)\,, \tag{25}$$

where

$$\psi(t) = \begin{cases} 0, & t < T-2R^2, t > T\,, \\ R^{-2}\big(t-(T-2R^2)\big)\,, & T-2R^2 \le t < T-R^2\,, \\ 1, & T-R^2 \le t \le T\,, \end{cases}$$

and $\zeta \in C_0^1(\mathbb{R})$ such that $\zeta(r) = 1$ for $0 \le r \le R$, $\zeta(r) = 0$ for $|r| \ge 2R$, $|\zeta'(r)| \lesssim R^{-1}$. In particular we have

$$|D_i \tau_R| \lesssim R^{-1}, \quad |\partial_t \tau_R| \lesssim R^{-2}. \tag{26}$$

We need Morrey's Lemma in its qualitative assertion, therefore we recall the precise formulation for the reader's convenience.

Proposition 3.1 (Morrey [14, p.79]) *Suppose $u \in H^1(B_\rho(x_0))$, and*

$$\int\limits_{B_r(x')} |\nabla u|^2 dx \le M^2 \left(\frac{r}{\delta}\right)^{2\alpha}, \ \textit{for } 0 \le r \le \delta =: \operatorname{dist}(x', \partial B_\rho(x_0)) \tag{27}$$

for every $x \in B_\rho(x_0)$. Then $u \in C^\alpha(B_r(x_0))$ for each $r < \rho$ and

$$|u(x) - u(x')| \le C\, M \frac{|x-x'|^\alpha}{\delta^\alpha} \ \textit{for } |x-x'| \le \delta/2, \tag{28}$$

where $C > 0$ is an absolute constant independent of u, ρ and x'.

Remark. From Morrey's lemma it follows in particular: if

$$\int\limits_{B_r(x')} |\nabla u|^2 dx \le M^2 r^{2\alpha} \text{ independent of } x \in B_\rho(x_0), r \le \operatorname{dist}(x', \partial B_\rho(x_0)) \tag{29}$$

then for $x, x' \in B_{\rho/2}(x_0)$ with $|x-x'| \le \rho/4$,

$$|u(x) - u(x')| \le 2^\alpha C\, M |x-x'|^\alpha. \tag{30}$$

Indeed, from (29) it follows with $\delta = \operatorname{dist}(x', \partial B_\rho(x_0))$

$$\delta^{2\alpha} \int\limits_{B_r(x')} |\nabla u|^2 dx \le M^2 r^{2\alpha} \rho^{2\alpha}.$$

Since $\delta \ge \rho/2$ for $x' \in B_{\rho/2}(x_0)$, now (28) implies

$$|u(x) - u(x')| \le C\, M \rho^\alpha \frac{|x-x'|^\alpha}{\delta^\alpha} \le 2^\alpha C\, M |x-x'|^\alpha,$$

Proposition 3.2 *Let $u \in L^\infty(0, T^0; H_0^1(\Omega))$ be a function which has the properties* (15) *and fulfills the entropy equality* (17)*. Moreover, for $0 < R \leq R_0$, $t \in [0, T']$*

$$\iint\limits_{Q_R} |u_t|^2 + |\nabla u|^2 |W_t| \, dx \, dt \leq M^2 R^{2\alpha} \tag{31}$$

with a suitable constant $M > 0$ and a fixed $\alpha \in (0, \gamma]$, where γ is defined by the assumptions on the data. Then for $T \in [0, T']$, we have

$$\sup_{t \in [T-R^2, T]} \|u(t, \cdot)\|_{C^\alpha(\mathbb{R}^2)} \lesssim M + R^{1-\epsilon/2-\alpha} \tag{32}$$

where the constant depends on $\partial\Omega$, the constants in condition (6)-(10) *and the bounds for the norms in* (15)*.*

Proof. Let $(x_0, T) \in \Omega \times (0, T^0]$ be arbitrary. An elementary calculation shows $R^{-2} \lesssim |W_t(T, x_0, x, t)|$ as long as $|x_0 - x| \geq R/4$. From here we get

$$\iint\limits_{Q_{R/4}(x',t)} |u_t|^2 + \frac{|\nabla u|^2}{R^2} \, dx \, dt \lesssim \iint\limits_{Q_R(x_0,t)} |u_t|^2 + |\nabla u|^2 |W_t| \, dx \, dt \tag{33}$$

for $|x_0 - x'| \geq R/2$. Since x_0 can vary freely in Ω this implies (replace $R/4$ by R)

$$\iint\limits_{Q_R(x',t)} |u_t|^2 + \frac{|\nabla u|^2}{R^2} \, dx \, dt \leq \widetilde{M} R^{2\alpha} \quad \text{for } R \leq R_0/4. \tag{34}$$

Then we apply the coercivity condition (8), the entropy equality with $\varphi = \tau_R^2$, the decay properties (26) and the growth condition (6) and arrive at

$$\begin{aligned}
\int\limits_{B_R(x_0)} |\nabla u|^2 &\leq \frac{1}{\alpha_1} \int\limits_{B_R(x_0)} A dx + K R^2 \lesssim \int\limits_{B_{2R}(x_0)} A \tau_R^2 dx + R^2 \\
&\lesssim \iint\limits_{Q_{2R}} \left(u_t^2 + R^{-2} A + |A_t| + R^{-2} |a_i|^2 + |f|^2 \right) dx \, dt + R^2 \\
&\lesssim \iint\limits_{Q_{2R}} \left(u_t^2 + R^{-2} |\nabla u|^2 + R^{-2} |u|^q + |u|^2 + |u|^q + |f|^2 \right) dx \, dt + R^2 \\
&\lesssim R^{2\alpha} + R^{2-\epsilon} + R^2.
\end{aligned}$$

For the last inequality we also used the embedding (19) and the estimate (20). Now the assertion follows from Proposition 3.1. ■

Our goal is now to prove inequality (31) with a *global* hole filling argument depending on the following estimate

$$\iint\limits_{Q_R} u_t^2 + |\nabla u|^2 |W_t| \, dx \, dt \leq \Lambda \iint\limits_{Q_{2R} \setminus Q_R} u_t^2 + |\nabla u|^2 |W_t| \, dx \, dt + K R^{2\gamma} + J_{\text{crit}}. \tag{35}$$

In spite of the slightly different assumptions on the coefficients a_i, the entropy inequality used in [9] was exactly of the form (17). Together with the weak formulation of the boundary value problem this inequality was employed to derive the estimate (35) *without* the last term which comes from the term a_0 in (1), namely

$$J_{\text{crit}} = \int_{T-3R^2}^{T-R^2} \int_{B_{2R}} |\nabla u|^2 |u - \bar{u}_R| R^{-2}\, dx\, dt,$$

where

$$\bar{u}_R = \begin{cases} (4\pi R)^{-1} \int_{\partial B_{2R}} u(t,\cdot) ds & \text{if } B_{2R} \subset \Omega, \\ 0 & \text{if } B_{2R} \cap \complement\overline{\Omega} \neq 0. \end{cases}$$

Let us outline how (35) leads to (31). With

$$G = G(T, x_0, \cdot, \cdot) =: u_t^2 + |\nabla u|^2 |W_t(T, x_0, \cdot, \cdot)|$$

we conclude from (35)

$$\iint_{Q_R} G\, dx\, dt \leq \frac{\Lambda}{\Lambda + 1} \iint_{Q_{2R}} G\, dx\, dt + J_{\text{crit}} + CR^{2\gamma}\,. \tag{36}$$

We choose $\alpha = \alpha(\Lambda)$ small enough such that $4^\alpha \Lambda(\Lambda+1)^{-1} = \theta < 1$, $\alpha < \gamma$, and multiply (36) by $R^{-2\alpha}$. For fixed $\rho > 0$, we put

$$S_\rho = \sup_{R,T,x_0} R^{-2\alpha} \iint_{Q_R} G(T, x_0, \cdot, \cdot)\, dx\, dt,$$

where the supremum is taken over $0 < R \leq \rho$, $x_0 \in \Omega$ (or $x_0 \in \mathbb{R}^2$, respectively), $0 \leq T \leq T^* - \delta$. Now there are two possibilites: Either we we have

$$(2R)^{-\alpha} \iint_{Q_{2R}} G(T, x_0, \cdot, \cdot) dx\, dt \leq S_\rho \tag{37}$$

for all R, x_0 and T in question or

$$(2R)^{-\alpha} \iint_{Q_{2R}} G(T, x_0, \cdot, \cdot) dx\, dt > S_\rho$$

for some T, x_0, and R. The second case can only happen if $2R > \rho$ but then

$$S_\rho \leq \rho^{-2\alpha} \iint_{Q_{2R}} G(T, x_0, \cdot, \cdot) \leq K$$

due to (24), in which case we have already achieved the desired inequality (31) once ρ is fixed. Otherwise, we conclude from (37) and (36) that

$$S_\rho \leq \theta S_\rho + C + \sup\left\{ R^{-2\alpha} J_{\text{crit}} \,\middle|\, R \leq \rho, x_0 \in \Omega, T \leq T^* - \delta \right\} \tag{38}$$

To control the last term we observe that Hölder's inequality together with (33) gives

$$R^{-2\alpha} J_{\text{crit}} \leq K \left(\sup \iint_{Q_{2R}} G\, dx\, dt \right)^{1/2} S_\rho^{1/2} R^{-\alpha} \sup\left\{ \operatorname*{osc}_{B_{2R}(x_0)} u(t_1) \,\middle|\, t_1 \in [T - R^2, T] \right\}, \tag{39}$$

where $\operatorname{osc}_M u = \Big(\sum_{i=1,2} |\sup_M u_i - \inf_M u_i|^2\Big)^{1/2}$. We also have

$$\operatorname*{osc}_{B_{2R}(x_0)} u(t_1) \le K \sup_{x_1\in\Omega}\{ \operatorname*{osc}_{B_{R/2}(x_1)} u \} \text{ for } t_1 \in [T-R^2, T],$$

where the right hand side can be estimated using (32) by $K(R^\alpha S_\rho^{1/2} + R^{1-\epsilon})$. From (39) we thus obtain

$$R^{-2\alpha} J_{\text{crit}} \lesssim \Big[\iint\limits_{Q_{2R}} G\,dx\,dt\Big]^{1/2} S_\rho + R^{1-\epsilon-\alpha}.$$

The critical term in (38) can be absorbed, if we can show that for $\varepsilon_0 > 0$, there exists a ρ such that $\iint_{Q_{2R}} G\,dx\,dt \le \varepsilon_0$, in this case we can apply Proposition 3.2. The last smallness condition is the most delicate matter. In Section 3.2 and 3.4 we prove a uniform estimate

$$\iint\limits_{Q_{2R}} G\,dx\,dt \le \frac{K}{\sqrt{\ln|\ln R|}} \le \varepsilon_0 \text{ for } R \le \rho \le \rho(\varepsilon)\,. \tag{40}$$

Resume: Thus, the theorem will be proved if we obtain (35) and (40). Inequality (35) is derived in Section 3.2 and the first part of 3.4, inequality (40) in Section 3.3 and the second part of 3.4.

3.2 The Pre-Hole-Filling Inequality

If we use the energy-equality (17) with $\varphi = \tau_R^2$, and $\varphi = W\tau_R^2$, where $R > 0$ is fixed and τ_R is defined in (25) we may argue exactly in the same way estimate as in [9], since the potential A has similar growth and coerciveness properties. The combination of both inequalities ($\varphi = \tau_R^2$ and $\varphi = W\tau_R^2$ in (17)) gives the corresponding inequality stated in Proposition 3.1 of [9]. To this end, we introduce the following abbreviations:

$$\widetilde{I}_R := [T-2R^2, T-R^2], \quad \mathscr{H}_R := [T-2R^2, T] \times B_{2R} \backslash B_R. \tag{41}$$

Proposition 3.3 *Let f and u_0 fulfil the requirements* (13) *and* (14)*, respectively. Let $u \in L^\infty(0, T^0; H^1(\Omega))$ with $u_t \in L^2(L^2)$ satisfy the energy equality* (17)*; further let W be defined by* (21) *and τ_R by* (25)*. Then for all $x_0 \in \Omega$, $0 < T \le T^* - \delta$ and $0 < R < R_0$ the following estimate holds true:*

$$\begin{aligned}\iint\limits_{\operatorname{supp}\tau_R \bigcap (t>0)} (|\nabla u|^2 |W_t| + u_t^2)\tau_R^2\,dx\,dt \lesssim \iint\limits_{\widetilde{I}_R \bigcap\{t\ge 0\}\times B_{2R}} |\nabla u|^2 R^{-2}\,dx\,dt + \\ + \iint\limits_{\mathscr{H}_R \bigcap (t>0)} \Big(u_t^2 + |\nabla u|^2 |W_t|\Big)\,dx\,dt + KR^{2\gamma}\,.\end{aligned} \tag{42}$$

We do not repeat the proof from [9]; the proof is elementary, but lengthy, properties of τ_R and W are used, in particular the inequality $|W_t| \ge KR^{-2}$ on $\mathscr{H}_R$.

Inequality (42) is "almost" a hole filling inequality for the quantity $G = u_t^2 + |\nabla u|^2 |W_t|$, however, on the right hand side of (42) there arises the critical term

$$J^0_{\text{crit}} = \iint\limits_{\tilde{I}_R \bigcap\{t>0\}\times B_{2R}} |\nabla u|^2 R^{-2}\,dx\,dt\,.$$

Since the integration runs over B_{2R} and not over $B_{2R}\setminus B_R$, the factor R^{-2} cannot be estimated by $|W_t|$. Otherwise we would already have achieved a hole filling inequality

$$\iint_{Q_R} G\,dx\,dt \lesssim \iint_{Q_{2R}\setminus Q_R} G\,dx\,dt + KR^{\alpha}\,.$$

The term W_{crit} has to be estimated using Caccioppoli's inequality which is done in Section 3.4.

3.3 Auxiliary lemmata concerning logarithmic Dirichlet growth

In order to continue the proof of the main theorem we consider the following situation: Let

$$G : \big[[0,T_1]\times\Omega\big]^2 \to \mathbb{R} \text{ be measurable, } G \geq 0 \text{ a.e. }\,, \tag{43}$$

$$\operatorname{ess\,sup}\left\{\int_0^{T_1}\int_\Omega G(t_1,x_0;\ldots)\,dx\,dt \big| (t_1,x_0)\in[0,T_1]\times\Omega\right\} \leq K\,. \tag{44}$$

Then we have the following elementary auxiliary result.

Lemma 3.4 *Let $R = 2^{-N}$, $N\in\mathbb{N}$, $N\geq 2$. For every $t_1\in(0,T1]$, and $x_0\in\Omega$ there exists*

$$r' = r'(t_1,x_0) \in \big[2^{-N^e}, 2^{-N}\big] = \big[2^{-(|\ln(R)|/\ln 2)^e}, R\big], e = \exp(1)\,,$$

such that

$$\iint_{Q_{2r'}(t_1,x_0)\setminus Q_{r'}(t_1,x_0)} G(t_1,x_0;\ldots)\,dx\,dt \leq \frac{\ln 2}{|\ln r'|\ln|\ln r'|}\iint_{Q_{2R}(t_1,x_0)} G(t_1,x_0;\ldots)\,dx\,dt$$

Proof: We fix $t_1>0$, $x_0\in\Omega$ and put $S(j) =: Q(t_1,x_0;2^{-j+1})\setminus Q(t_1,x_0;2^{-j})$, then the sets $S(j)$ are mutually disjoint and for $R=2^{-N}$, we have

$$\bigcup_{j=N}^{\infty} S(j) = Q(t_1,x_0;2R)\,.$$

If $M>N$, assertion (44) implies

$$\sum_{j=N}^{M} := \sum_{j=N}^{M}\iint_{S(j)} G(t_1,x_0;\cdot)\,dx\,dt \leq \iint_{Q_{2R}(t_1,x_0)} G(t_1,x_0;\cdot)\,dx\,dt\,. \tag{45}$$

Suppose now that

$$\iint_{S(j)} G(t_1,x_0;\cdot)\,dx\,dt > \frac{1}{j\ln j}\iint_{Q_{2R}} G(t_1,x_0;\cdot)\,dx\,dt \tag{46}$$

for all integers $j \in [N, M]$. Then

$$\sum_{j=N}^{M} \geq \sum_{j=N}^{M-1} \geq \int\limits_{N}^{M} \frac{d\xi}{\xi \ln \xi} \iint\limits_{Q_{2R}} G(t_1, x_0; \cdot)\, dx\, dt =$$
$$= \big[\ln \ln M - \ln \ln N\big] \iint\limits_{Q_{2R}} G(t_1, x_0; \cdot)\, dx\, dt\,.$$

We have

$$\big[\ln \ln M - \ln \ln N\big] > 1 \tag{47}$$

if $\ln M > e \ln N$ which is equivalent to $M > N^e$. However, (47) contradicts (45) and we conclude: If $M \geq (N+1)^e$ there exists an $N_0 \in [N, M]$ such that

$$\iint\limits_{S(N_0)} G(t_1, x_0; \cdot)\, dx\, dt \leq \frac{1}{N_0 \ln N_0} \iint\limits_{Q_{2R}} G(t_1, x_0; \cdot)\, dx\, dt\,. \tag{48}$$

For $r' = 2^{-N_0}$ we have

$$\frac{1}{N_0 \ln N_0} = \frac{\ln 2}{|\ln r'| \ln(|\ln r'|/\ln 2)} \leq \frac{\ln 2}{|\ln r'| \ln |\ln r'|}\,.$$

The Lemma is proved. ■

Due to estimate (24) and $u_t \in L^2(L^2)$, Lemma 3.4 leads to the inequality

$$\begin{aligned}\iint\limits_{Q_{2r'}(t_1,x_0)\setminus Q_{r'}(t_1,x_0)} & |\nabla u|^2 |W_t(t_1, x_0; \cdot)|\, dx\, dt \\ & \leq \frac{\ln 2}{|\ln r'| \ln |\ln r'|} \iint |\nabla u|^2 |W_t(t_1, x_0; \cdot)|\, dx\, dt\end{aligned} \tag{49}$$

for some $r' \in [2^{-|\ln R/\ln 2|^e}, R]$, $r' = r'(t_1, x_0)$, where $R \in (0, R_0]$ is arbitrarily given.

The second Lemma uses Trudinger's inequality [12, Thm.7.15] which implies that the quantity

$$\operatorname*{ess\,sup}_t \int\limits_\Omega e^{\alpha |u|^2} \leq K \tag{50}$$

for some constant α depending only on $\operatorname*{ess\,sup}_t \int\limits_\Omega |\nabla u|^2\, dx$ (in two space dimensions).

Lemma 3.5 *Let u satisfy* (50). *Then*

$$\int\limits_{B_R} |u|\, dx \leq K_0 \sqrt{|\ln R|} R^2$$

for all balls B_R with $R < 1/2$, K_0 depending on α, K in (50).

Proof: For any $\alpha' < \alpha$, we have

$$|u|e^{\alpha'|u|^2} \le K_{\alpha,\alpha'} e^{\alpha|u|^2} . \tag{51}$$

Let $M = \{x \big| |u(x)| \ge \big[(2/\alpha')\ln(1/R)\big]^{1/2}$. Then $e^{\alpha'|u|^2} \ge R^{-2}$ on M, and from (50) and (51) we conclude

$$K \gtrsim \int_M e^{\alpha|u|^2}\,dx \gtrsim \int_M |u|e^{\alpha'|u|^2}\,dx \gtrsim \int_M |u|\,dx R^{-2} . \tag{52}$$

On $\complement M \cap B_R$ we have

$$\int_{\complement M \cap B_R} |u|\,dx \le \big[(2/\alpha')\ln(1/R)\big]^{1/2} R^2 ,$$

which together with (52) prove Lemma 3.5. ∎

3.4 Cacciopoli's Inequality and Uniformly Small Dirichlet Growth

In this chapter we derive uniform smallness of the quantities

$$\iint_{Q_r} \big[|u_t|^2 + |\nabla u|^2\big]\big|W_t(t_1 x_0, ., \cdot)\big|\,dx\,dt$$

for r sufficiently small. To this end we use the function $(u-c)\tilde\tau_R^2$ as a test function in the weak formulation (16) of the parabolic system where $c = c(t)$ will be defined later on. The localization function $\tilde\tau_R$ has similar properties as the function τ_R in (25), but the support is shifted, this means

$$\tilde\tau_R = \zeta(x - x_0)\cdot\psi\big(t + (T - R^2)\big), \text{ hence } \operatorname{supp}\tilde\tau_R \subset Q_{2R}\setminus Q_R . \tag{53}$$

Then $\tilde\tau_R$ is Lipschitz continuous for $t \le T - R^2$. With this test function we obtain (here we omit the integration domain support $\tilde\tau_R$ for simplicity)

$$\begin{aligned}
&\iint u_t(u-c)\tilde\tau_R^2\,dx\,dt + \sum_{i=1,2}\iint a_i(\cdot,\cdot,u,\nabla u)D_i u\tilde\tau_R^2\,dx\,dt \\
&\quad + \iint a_0(\cdot,\cdot,u,\nabla u)(u-c)\tilde\tau_R^2\,dx\,dt + \\
&\quad + 2\sum_{i=1,2}\iint a_i(\cdot,\cdot,\nabla u)(u-c)(D_i\tilde\tau_R)\tilde\tau_R\,dx\,dt \\
&\quad = \iint f(u-c)\tilde\tau_R^2\,dx\,dt .
\end{aligned} \tag{54}$$

For the first version of Cacciopoli's inequality we choose $c = 0$ and use the hypothesis $a_0 \cdot u \ge -K$ and the coerciveness property for $\sum_{i=1,2} a_i(\cdot,u,\nabla u)D_i u$. This yields an estimate

$$\begin{aligned}
\iint |\nabla u|^2\tilde\tau_R^2\,dx\,dt &\lesssim \iint |u_t||u|\tilde\tau_R^2\,dx\,dt + \\
&+ R^{-1}\iint_{Q_{2R}\setminus Q_R} |\nabla u||u|\,dx\,dt + \iint f|r|\tilde\tau_R dx\,dt
\end{aligned} \tag{55}$$

We define

$$\mathscr{A}_R := B_{2R} \setminus B_R, \qquad \tilde{u}_R(t) = |\mathscr{A}_R|^{-1} \int\limits_{\mathscr{A}_R} u\, dx$$

(using the natural extension of u). From (55) we conclude

$$\begin{aligned}
\iint |\nabla u|^2 \tilde{\tau}_R^2\, dx\, dt &\lesssim \iint\limits_{\operatorname{supp} \tilde{\tau}_R} |u_t||u - \tilde{u}_R|\, dx\, dt + \iint\limits_{\operatorname{supp} \nabla\tilde{\tau}_R} \frac{|\nabla u|}{R}|u - \tilde{u}_R|\, dx\, dt \\
&\quad + \iint\limits_{\operatorname{supp} \tilde{\tau}_R} |u_t||\tilde{u}_R| dx\, dt + \iint\limits_{\operatorname{supp} \nabla\tilde{\tau}_R} \frac{|\nabla u|}{R}|\tilde{u}_R| dx\, dt \\
&\quad + \iint\limits_{\operatorname{supp} \tilde{\tau}_R} |f||u - \tilde{u}_R| + |f||\tilde{u}_R| dx\, dt \\
&\lesssim \iint\limits_{\operatorname{supp} \tilde{\tau}_R} \big(R^2|u_t|^2 + \frac{|u - \tilde{u}_R|^2}{R^2}\big) dx\, dt + \iint\limits_{\operatorname{supp} \nabla\tilde{\tau}_R} |\nabla u|^2 dx\, dt \\
&\quad + R^{2+2\gamma} + \varepsilon_2 \iint\limits_{\operatorname{supp} \tilde{\tau}_R} \frac{|\tilde{u}_R|^2}{R^2} dx\, dt \\
&\quad + \frac{1}{\varepsilon_2} \left(\iint\limits_{\operatorname{supp} \tilde{\tau}_R} \big(R^2|u_t|^2 + \frac{|u - \tilde{u}_R|^2}{R^2}\big) dx\, dt + \iint\limits_{\operatorname{supp} \nabla\tilde{\tau}_R} |\nabla u|^2 dx\, dt \right) \qquad (56)
\end{aligned}$$

For the second inequality we used applied Young's inequality to all products and assumption (14) for f Now we apply the following variant of Poincaré's inequality (cf [9, Prop.6.1])

$$\int\limits_{B_{2R}(x_0)} |u - \tilde{u}_R(t)|^2\, dx \lesssim \int\limits_{B_{2R}} |x - x_0|^2 |\nabla u|^2\, dx \text{ independent of } x_0, R,$$

divide the resulting inequality by R^2, und recall the the following elementary inequalities, which are valid independent on x_0 and R:

$$R^{-2} \lesssim |W_t| \text{ on } \operatorname{supp} \nabla\tilde{\tau}_R, \quad R^{-4}|x - x_0|^2 \lesssim |W_t| \text{ on } Q_{2R}. \qquad (57)$$

Finally with $\operatorname{supp} \nabla\tilde{\tau}_R \subset \operatorname{supp} \tilde{\tau}_R \subset Q_{2R} \setminus Q_R$ and $G(T, x_0, \cdot, \cdot) = |u_t|^2 + |\nabla u|^2 |W_t|$ we arrive at

$$R^{-2} \iint |\nabla u|^2 \tilde{\tau}_R^2\, dx\, dt \lesssim \iint\limits_{Q_{2R}\setminus Q_R} G\, dx\, dt + \varepsilon_2 \iint\limits_{\operatorname{supp} \tilde{\tau}_R} \frac{|\tilde{u}_R|^2}{R^2} + \frac{1}{\varepsilon_2} \iint\limits_{Q_{2R}\setminus Q_R} G\, dx\, dt + R^{2\gamma} \qquad (58)$$

Let us fix R for a moment. From Lemma 3.4 we conclude that there exists

$$r = r(T, x_0) \in \big[2^{-(|\ln R|/\ln 2)^e}, 2R\big]$$

such that

$$\iint\limits_{Q_{2r}\setminus Q_r} G\, dx\, dt \lesssim \big(\ln|\ln r|\big)^{-1} |\ln r|^{-1},$$

and from Lemma 3.5

$$\varepsilon_2 \iint\limits_{\operatorname{supp}\tilde{\tau}_r} \frac{|\tilde{u}_r|^2}{r^2} \lesssim \varepsilon_2 |\ln r| \,.$$

We choose

$$\varepsilon_2 = |\ln r|^{-1} (\ln|\ln r|)^{-1/2} \,,$$

and we conclude

$$r^{-2} \iint |\nabla u|^2 \tilde{\tau}_r^2 \, dx \, dt \lesssim (\ln|\ln r|)^{-1/2}$$

Thus we have shown: Let $\varepsilon > 0$. Then for any $R \le R_0$, there exists a radius

$$r \in \left[2^{-|\ln(R/2)|^e}, 2R\right] \tag{59}$$

such that

$$r^{-2} \int\limits_{T-3r^2}^{T-r^2} \int\limits_{B_{2r}} |\nabla u|^2 \tilde{\tau}_r^2 \, dx \, dt \le \varepsilon, \qquad r = r(T, x_0) \,.$$

Thus we have estimated the critical term J^0_{crit} in the pre-hole-filling inequality (42); i. e.

$$J^0_{\text{crit}} < \varepsilon$$

for the above choice of r.
Inspecting the other terms on the right hand side of (42) we see: For sufficiently small R and for r as in (59) we may apply Lemma 3.4, to all terms on the right hand side hence we obtain from (42)

$$\iint \left(|\nabla u|^2 |W_t| + u_t^2\right) \tau_r^2 \, dx \, dt \le 4\varepsilon \tag{60}$$

for $r = r(T, x_0, R)$.

We diminish the domain of integration in (60) choosing the minimal

$$r = 2^{-(|\ln R|/\ln 2)^e} \tag{61}$$

and conclude

Lemma 3.6 *For $\varepsilon > 0$ there exists $r_* > 0$ such that*

$$\iint\limits_{Q_{r_*}} \left(|\nabla u|^2 |W_t| + u_t^2\right) dx \, dt \le \varepsilon. \tag{62}$$

This smallness property holds uniformly with respect to δ, x_0, $T = T^* - \delta$.

3.5 Cacciopoli's Inequality II

In equation (54) we choose

$$\bar{u}_R(t) = \begin{cases} |\partial B_{2R}|^{-1} \int\limits_{\partial B_{2R}} u \, do & \text{if } B_{2R} \subset \Omega \\ 0 & \text{else} \end{cases}$$

and proceed estimating as in [9], Section 4, Proposition 4.1, and Theorem 6.1. Repeating the arguments of the proofs we arrive at

$$\iint_{\tilde I_R\times B_{2R}} |\nabla u|^2 R^{-2}\,dx \lesssim \iint_{Q_{2R}\setminus Q_R} \left[|\nabla u|^2|W_t| + |u_t|^2\right] dx\,dt + J_{\text{crit}} + R^{2\gamma}\,. \tag{63}$$

We have to comment on one technical detail here. If the ball $B_{2R}\subset\Omega$ we arrive at the estimate (63) with $\iint_{Q_{2R}\setminus Q_R}\dots$ on the right hand side. In the case $B_{2R}\cap\complement\Omega\neq\emptyset$ we have to enlarge the domain of integration from B_{2R} to B_{MR} in order to apply the inequality

$$\int_{B_{2R}} |u|^2 dx \le C_p \int_{B_{MR}} |x-x_0|^2|\nabla u|^2 dx \tag{64}$$

(see [9, Section 6]), basically because the measure of the points where u vanishes must be large enough. If the boundary $\partial\Omega$ is Lipschitz then $M=4$ is sufficient. In order to keep the setting in Section 3 we may argue in the following way: If even $B_R\cap\complement\Omega\neq\emptyset$, then (64) is´valid with $M=2$, and no change is necessary at all. If only $B_{2R}\cap\complement\Omega\neq\emptyset$´we change the localization function ζ in (25) in such a way that still $\zeta=1$ on $B_R(x_0)$, $\nabla\zeta\lesssim R^{-1}$, $\operatorname{supp}\zeta\subset B_{3R/2}$. If $B_{3R/2}\subset\Omega$ we proceed as in the "interior" case, if $B_{3R/2}\cap\complement\Omega\neq\emptyset$, the preimage of zero in B_{2R} is large enough, and we may use (64) with $M=2$.

The difference to the arguments used in [9] is that we have the additional term (coming from a_0)

$$J_{\text{crit}} = \iint_{Q_{2R}} |\nabla u|^2|u-\bar u_R|(t)|\,dx\,.$$

Obviously, (63) is also an estimate for the critical term J^0_{crit} occuring in (42). So, we combine (63) and (42) and arrive at the desired inequality (35) which finishes the proof of the theorem 2.1.

Appendix

In this appendix we show: The properties (3) and $u\in C^\alpha$ suffice to prove the entropy inequality (17).

Lemma A.1 *Let u be a weak solution of* (16) *satisfying the standard conditions* (3) *and $u\in L^\infty(0,T^0;C^\alpha(\bar\Omega))$. Then u satisfies the entropy inequality* (17).

Proof: By well known theorems, due to the additional properties $u\in C^\alpha$, we have $u\in L^2(0,T^0,H^2(\Omega))\cap H^1([0,T^0]\times\Omega)\cap L^\infty(0,T^0;H^1(\Omega))$, (difference quotient technique!) and the system of differential equations (1) is satisfied point-wise almost everywhere. With

$$a^{\nu\mu}_{ik}(t,x,z,\eta) = \partial_{\eta^\mu_k} a^\nu_i(t,x,z,\eta),\ a^{\nu\mu}_{i0}(t,x,z,\eta) = \partial_{z^\mu} a^\nu_i(t,x,z,\eta),$$
$$a^\nu_{i,x_i}(t,x,z,\eta) = \partial_{x_i} a^\nu_i(t,x,z,\eta)$$

this means

$$\begin{aligned} L^\nu u^\nu :=& u^\nu_t - \sum_{\mu=1}^N \sum_{\substack{i=1,2\\k=0,1,2}} a^{\nu\mu}_{ik}(t,x,u,\nabla u) D_i D_k u^\mu \\ &+ \sum_{i=1,2} a^\nu_{i,x_i}(t,x,u,\nabla u) + a^\nu_0(t,x,u,\nabla u) = f^\nu \end{aligned}$$

By simple interpolation arguments we deduct from $u \in L^2(H^2) \cap L^\infty$ that $\nabla u \in L^4(L^4)$. Now we choose smooth approximations $\tilde{u}_m$ such that

$$u_m \to u \text{ in } L^2(H^2) \cap L^4(W^{1,4}), u_{m,t} \to u_t \text{ in } L^2([0,T^0 \times \Omega), u_m|_{t=0} \to u_0 \in H^1(\Omega).$$

and

$$\operatorname*{ess\,sup}_t \int |\nabla u_m| \le K \quad \text{uniformly in } m.$$

Then we have $L^\nu u_m^\nu = f_m$, and from (7) we obtain $f_m \to f$ in L^2. Clearly u_m fulfils (16) with u replaced by u_m and f by f_m, respectively. Since u_m is smooth now, we may use $u_{m,t}\varphi$ as a test function where again $\varphi \in C^1\big((0,T^0) \times \Omega\big)$ such that $\operatorname{supp}\varphi(t,\,\cdot\,) \subset\subset \Omega$ and $\varphi|_{T^0} = 0$. Then we arrive at

$$\begin{aligned} \int_0^{T^0}\!\!\int_\Omega \big[u_{m,t}^2\varphi + \sum_{i=1,2} a_i(\cdot, u_m, \nabla u_m) \cdot D_i(u_{m,t}\varphi) \\ + a_0(\cdot, u_m, \nabla u_m)u_{m,t}\varphi\big]\,dx\,dt = \int_0^{T^0}\!\!\int_\Omega f_m u_{m,t}\varphi\,dx\,dt \end{aligned} \tag{65}$$

We rewrite

$$\begin{aligned} \sum_{i=1,2} a_i(\cdot, u_m, \nabla u_m) \cdot D_i u_{m,t}\,\varphi + a_0(\cdot, u_m, \nabla u_m)u_{m,t}\,\varphi \\ = \frac{d}{dt}A(\cdot, u_m, \nabla u_m)\varphi - A_t(\cdot, u_m, \nabla u_m), \end{aligned}$$

which leads to the approximated entropy inequality (with $A^m = A(t, x, u_m, \nabla u_m)$)

$$\begin{aligned} \int_{t_1}^{T}\!\!\int_\Omega u_{m,t}^2\varphi\,dx\,dt + \int_{t_1}^{T}\!\!\int_\Omega \big(-A^m\varphi_t - A_t^m\varphi + \sum_{i=1,2} a_i(\cdot, u_m\nabla u_m)u_{m,t}D_i\varphi\big)\,dx\,dt \\ + \int_\Omega A^m\varphi\,dx\Big|_{t_1}^{T} = \int_{t_1}^{T}\!\!\int f_m u_{m,t}\varphi\,dx\,dt. \end{aligned}$$

Here we may pass to the limit in all space-time integrals, and for almost all t_1 and T in the boundary integrals, which leads to (17) for u.

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