Finite Element Based Second Moment Analysis for Elliptic Problems in Stochastic Domains

Helmut Harbrecht

no. 456

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereichs 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

Bonn, September 2009

Finite element based second moment analysis for elliptic problems in stochastic domains

Helmut Harbrecht

Abstract We present a finite element method for the numerical solution of elliptic boundary value problems on stochastic domains. The method computes, to leading order in the amplitude of the stochastic boundary perturbation relative to an unperturbed, nominal domain, the mean and the variance of the random solution. The variance is computed as the trace of the solution's two-point correlation which satisfies a deterministic boundary value problem on the tensor product of the nominal domain. This problem is discretized in the sparse tensor product space by a multilevel frame generated by standard finite elements. The computational complexity of the resulting approach stays essentially proportional to the number of finite elements required for the discretization of the nominal domain.

1 Introduction

Many problems in physics and engineering sciences lead to boundary value problems for an unknown function. In general, the numerical simulation is well understood provided that the input parameters are given exactly. Since, however, the input parameters are often not known exactly it is of growing interest to model such parameters stochastically.

A principal approach to solve boundary value problems with stochastic input parameters is the Monte Carlo Approach, see e.g. [16] and the references therein. However, it is hard and extremely expensive to generate a large number of suitable samples and to solve a deterministic boundary value problem on each sample. Thus, we aim here at a direct, deterministic method to compute the stochastic solution.

Deterministic approaches to solve stochastic partial differential equations have been proposed in e.g. [1, 7, 8, 9, 14, 15]. Therein, loadings and coefficients have

Helmut Harbrecht
Institute for Numerical Simulation, University of Bonn, Wegelerstr. 6, 53115 Bonn, Germany
e-mail: harbrecht@ins.uni-bonn.de

been considered as stochastic input parameter. Recently, in [6, 12, 17], also the underlying domain has been modeled as stochastic input parameter $D(\omega)$. For example, this enables the consideration of tolerances in the shape of products fabricated by line production. Other applications arise from blurred interfaces like cell membranes or molecular surfaces.

The present paper is dedicated to elliptic boundary value problems on stochastic domains. We assume small stochastic perturbations around a nominal domain with known statistics. Then, we can linearize to derive deterministic equations for the random solution's expectation

$$\mathrm{E}_u(\mathbf{x}) = \int_\Omega u(\mathbf{x},\omega)\,dP(\omega), \quad \mathbf{x}\in D,$$

and the two-point correlation

$$\mathrm{Cor}_u(\mathbf{x},\mathbf{y}) = \int_\Omega u(\mathbf{x},\omega)u(\mathbf{y},\omega)\,dP(\omega), \quad \mathbf{x},\mathbf{y}\in D.$$

From these quantities the variance is derived by $\mathrm{Var}_u(\mathbf{x}) = \mathrm{Cor}_u(\mathbf{x},\mathbf{x}) - \mathrm{E}_u^2(\mathbf{x})$. Thus, we are able to compute with leading order in the amplitude of the random boundary perturbation the solution's statistics.

2 Elliptic boundary value problems on stochastic domains

Let (Ω,Σ,P) be a suitable probability space. We consider the domain as the uncertain input parameter of an elliptic boundary value problem, i.e.,

$$\left.\begin{aligned} -\operatorname{div}\big[\mathbf{A}(\mathbf{x})\nabla u(\mathbf{x},\omega)\big] &= f(\mathbf{x}), \quad \mathbf{x}\in D(\omega)\\ u(\mathbf{x},\omega) &= g(\mathbf{x}), \quad \mathbf{x}\in \partial D(\omega)\end{aligned}\right\} \quad \omega\in\Omega. \tag{1}$$

To model the stochastic domain $D(\omega)$ let $\overline{D}$ denote a smooth reference domain and consider stochastic boundary variations in direction of the outer normal $\mathbf{U}(\mathbf{x},\omega) = \varepsilon\kappa(\mathbf{x},\omega)\mathbf{n}(\mathbf{x}) : \partial\overline{D}\to\mathbb{R}^n$ with $\kappa(\omega)\in L_P^2(\Omega, C^{2,1}(\partial\overline{D}))$ and $\|\kappa(\omega)\|_{C^{2,\alpha}(\partial\overline{D})}\le 1$ almost surely. Then, the stochastic domain $D(\omega)$ is described via perturbation of identity

$$\partial D(\omega) = \big\{\big(\mathbf{I}+\varepsilon\mathbf{U}(\omega)\big)(\mathbf{x}) = \mathbf{x}+\varepsilon\kappa(\mathbf{x},\omega)\mathbf{n}(\mathbf{x}) : \mathbf{x}\in\partial\overline{D}\big\}.$$

For what follows we assume that the expectation E_κ and the two-point correlation Cor_κ of κ are given. Without loss of generality (otherwise we redefine $\overline{D}$ correspondingly) we assume that the perturbation field κ is centered, i.e., that $\mathrm{E}_\kappa\equiv 0$.

For small parameters $\varepsilon>0$ one can linearize (1) by means of shape optimization:

Theorem 1 ([11, 12]). *Assume that the compact set $K\Subset\overline{D}$ satisfies $K\subset D(\omega)$ almost surely. Then, it holds that*

$$E_u(\mathbf{x}) = \overline{u}(\mathbf{x}) + \mathscr{O}(\varepsilon^2), \quad \mathrm{Cov}_u(\mathbf{x},\mathbf{y}) = \varepsilon^2 \,\mathrm{Cor}_{du}(\mathbf{x},\mathbf{y}) + \mathscr{O}(\varepsilon^3), \quad \mathbf{x},\mathbf{y} \in K.$$

Herein, $\overline{u} \in H^1(\overline{D})$ *and* $\mathrm{Cor}_{du} \in H^{1,1}(\overline{D} \times \overline{D})$ *satisfy the deterministic boundary value problems*

$$\begin{aligned} -\operatorname{div}\left[\mathbf{A}(\mathbf{x})\nabla\overline{u}(\mathbf{x})\right] &= f(\mathbf{x}), \quad && \mathbf{x} \in \overline{D}, \\ \overline{u}(\mathbf{x}) &= g(\mathbf{x}), \quad && \mathbf{x} \in \partial\overline{D}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} (\mathrm{div}_{\mathbf{x}} \otimes \mathrm{div}_{\mathbf{y}})\left[\big(\mathbf{A}(\mathbf{x}) \otimes \mathbf{A}(\mathbf{y})\big)(\nabla_{\mathbf{x}} \otimes \nabla_{\mathbf{y}})\,\mathrm{Cor}_{du}(\mathbf{x},\mathbf{y})\right] &= 0, && \mathbf{x},\mathbf{y} \in \overline{D}, \\ \mathrm{div}_{\mathbf{x}}\left[\mathbf{A}(\mathbf{x})\nabla_{\mathbf{x}}\,\mathrm{Cor}_u(\mathbf{x},\mathbf{y})\right] &= 0, && \mathbf{x} \in \overline{D},\ \mathbf{y} \in \partial\overline{D}, \\ \mathrm{div}_{\mathbf{y}}\left[\mathbf{A}(\mathbf{y})\nabla_{\mathbf{y}}\,\mathrm{Cor}_u(\mathbf{x},\mathbf{y})\right] &= 0, && \mathbf{x} \in \partial D,\ \mathbf{y} \in \overline{D}, \\ \mathrm{Cor}_{du}(\mathbf{x},\mathbf{y}) = \mathrm{Cor}_{\kappa}(\mathbf{x},\mathbf{y})\left[\frac{\partial(\overline{u}-g)}{\partial\mathbf{n}}(\mathbf{x}) \otimes \frac{\partial(\overline{u}-g)}{\partial\mathbf{n}}(\mathbf{y})\right]&, && \mathbf{x},\mathbf{y} \in \partial\overline{D}. \end{aligned} \tag{3}$$

3 Finite element discretization

3.1 Parametric finite elements

Starting point of the definition of the sparse multilevel frame is a nested sequence of finite dimensional trial spaces

$$V_0 \subset V_1 \subset \cdots \subset V_j \cdots \subset H^1(\overline{D}). \tag{4}$$

In general, due to our smoothness assumptions on the domain, we have to deal with non-polygonal domains. To realize the multiresolution analysis (4) we will use parametric finite elements.

Let $\triangle$ denote the reference simplex in $\mathbb{R}^n$. We assume that the domain $\overline{D}$ is partitioned into a finite number of patches

$$\mathrm{clos}(\overline{D}) = \bigcup_k \tau_{0,k}, \quad \tau_{0,k} = \kappa_k(\triangle), \quad k = 1,2,\ldots,M,$$

where each $\kappa_k : \triangle \to \tau_{0,k}$ defines a diffeomorphism of $\triangle$ onto $\tau_{0,k}$. The intersection $\tau_{0,k} \cap \tau_{0,k'}$, $k \neq k'$, of the patches $\tau_{0,k}$ and $\tau_{0,k'}$ is either $\emptyset$, or a lower dimensional face. The parametric representation is supposed to be globally continuous which means that the diffeomorphisms κ_i and $\kappa_{i'}$ coincide at common patch interfaces except for orientation. A mesh of level j on $\overline{D}$ is then induced by regular subdivisions of depth j of $\triangle$ into 2^{jn} simplices. This generates the $2^{jn}M$ curved elements $\{\tau_{j,k}\}$.

The ansatz functions $\Phi_j = \{\varphi_{j,k} : k \in \mathscr{I}_j\}$ are defined via parameterization, lifting continuous piecewise linear Lagrangian finite elements from $\triangle$ to the domain $\overline{D}$ by using the mappings κ_i and gluing across patch boundaries. Setting $V_j = \operatorname{span}\Phi_j$ yields (4), where $\dim V_j \sim 2^{jn}$.

To treat the non-homogeneous Dirichlet data in (2) and (3), we shall further distinguish between interior basis functions $\Phi_j^{\overline{D}} = \{\varphi_{j,k} : k \in \mathscr{I}_j^{\overline{D}}\}$ with $\varphi_{j,k}\big|_{\partial\overline{D}} \equiv 0$ and boundary functions $\Phi_j^{\partial\overline{D}} = \{\varphi_{j,k} : k \in \mathscr{I}_j^{\partial\overline{D}}\}$ with $\varphi_{j,k}\big|_{\partial\overline{D}} \not\equiv 0$.

The solution of the mean field equation (2) by multigrid accelerated finite element methods is straightforward and along the lines of standard literature, see e.g. [2, 3]. Therefore we will skip all the details here.

3.2 Multilevel frames for sparse tensor product spaces

We will discretize (3) in the *sparse* tensor product space $\widehat{V}_J = \sum_{j+j' \le J} V_j \otimes V_{j'}$. Abbreviating $N_J := \dim V_J$ there holds $\widehat{N}_J := \dim \widehat{V}_J \sim N_J \log N_J$ which is substantially smaller than the dimension N_J^2 of the full tensor product space $V_J \otimes V_J = \sum_{j,j' \le J} V_j \otimes V_{j'}$. Nevertheless, the approximation power in $\widehat{V}_J$ is essentially the same as in the full tensor product space provided that there is extra regularity in terms of the anisotropic Sobolev spaces $H^{s,s}(\overline{D} \times \overline{D})$, see [5, 15].

To discretize functions in $\widehat{V}_J$ one traditionally uses hierarchical bases like wavelet or multilevel bases, see for example [5]. In the present paper we use instead multilevel frames as proposed in [13], i.e., we represent functions by the redundant but stable collection $\widehat{\Phi}_J := \{\varphi_{j,k} \otimes \varphi_{j',k'} : k \in \mathscr{I}_j,\ k' \in \mathscr{I}_{j'},\ j + j' \le J\}$. Thus, the structural and computational advantages of finite element methods are combined with the efficiency of sparse grid approximations.

It has been shown in [13] that $\operatorname{card}(\widehat{\Phi}_J) \sim \widehat{N}_J \sim N_J \log N_J$, i.e., this frame has still optimal cardinality. Notice that the frame $\widehat{\Phi}_J$ is the restriction to $\widehat{V}_J$ of the two-fold tensor product of the frame that underlies the BPX-preconditioner [4].

3.3 Galerkin discretization

We shall be concerned with Galerkin's method for solving the boundary value problem (3) in the sparse tensor product space. We abbreviate the mean's Neumann data by $\sigma := \partial(\overline{u} - g)/\partial\mathbf{n}$ and their approximate version by $\sigma_J := \langle \nabla(\overline{u}_J - g), \mathbf{n} \rangle$, with $\overline{u}_J \in V_J$ being the finite element solution of (2). Instead of the Dirichlet data of (3),

$$f := (\sigma \otimes \sigma) \operatorname{Cor}_\kappa \in H^{1/2,1/2}(\partial\overline{D} \times \partial\overline{D}), \tag{5}$$

we have only access to the approximation $f_J := (\sigma_J \otimes \sigma_J) \operatorname{Cor}_\kappa$ which lives on the full tensor product grid. Thus, we follow [12] and insert the L^2-orthoprojector $\widehat{\Pi}_J$ onto the sparse tensor product space $\widehat{V}_J|_{\partial\overline{D} \times \partial\overline{D}}$ according to

$$\widehat{f}_J := (\sigma_J \otimes \sigma_J) \widehat{\Pi}_J \operatorname{Cor}_\kappa. \tag{6}$$

We shall fix notation. Define for all $0 \le j, j' \le J$ the univariate stiffness matrices, and with respect to the traces of the ansatz functions, the mass matrices and the multiplication operators,

$$\begin{gathered}\mathbf{A}^{\Theta}_{j,j'} := (\mathbf{A}\nabla\Phi^{\Theta}_{j'}, \nabla\Phi^{\overline{D}}_{j})_{L^2(\overline{D})}, \quad \Theta \in \{\overline{D}, \partial\overline{D}\},\\ \mathbf{G}_{j,j'} := (\Phi^{\partial D}_{j'}, \Phi^{\partial\overline{D}}_{j})_{L^2(\partial\overline{D})}, \quad \mathbf{M}_{j,j'} := (\sigma_J\Phi^{\partial\overline{D}}_{j'}, \Phi^{\partial\overline{D}}_{j})_{L^2(\partial\overline{D})}.\end{gathered} \tag{7}$$

Two-fold tensor products of these finite element matrices lead to the necessary matrices on the sparse tensor product space:

$$\widehat{\mathbf{A}}^{\Theta,\Xi}_J = \big[\mathbf{A}^{\Theta}_{j_1,j_2} \otimes \mathbf{A}^{\Xi}_{j'_1,j'_2}\big]_{j_1+j_2,j'_1+j'_2 \le J}, \quad \Theta,\Xi \in \{\overline{D}, \partial\overline{D}\},$$

$$\widehat{\mathbf{G}}_J = \big[\mathbf{G}_{j_1,j_2} \otimes \mathbf{G}_{j'_1,j'_2}\big]_{j_1+j_2,j'_1+j'_2 \le J}, \quad \widehat{\mathbf{M}}_J = \big[\mathbf{M}_{j_1,j_2} \otimes \mathbf{M}_{j'_1,j'_2}\big]_{j_1+j_2,j'_1+j'_2 \le J},$$

$$\left.\begin{aligned}\widehat{\mathbf{B}}^{\Theta}_J &= \big[\mathbf{A}^{\Theta}_{j_1,j_2} \otimes \mathbf{G}_{j'_1,j'_2}\big]_{j_1+j_2,j'_1+j'_2 \le J},\\ \widehat{\mathbf{C}}^{\Theta}_J &= \big[\mathbf{G}_{j_1,j_2} \otimes \mathbf{A}^{\Theta}_{j'_1,j'_2}\big]_{j_1+j_2,j'_1+j'_2 \le J},\end{aligned}\right\} \quad \Theta \in \{\overline{D}, \partial\overline{D}\}.$$

Finally, we need the data vector $\widehat{\mathbf{c}}_J = \big[(\mathrm{Cor}_\kappa, \Phi^{\partial\overline{D}}_j \otimes \Phi^{\partial\overline{D}}_{j'})_{L^2(\partial\overline{D}\times\partial\overline{D})}\big]_{j+j'\le J}$. Notice that (6) reads in the discrete form as $\widehat{\mathbf{f}}_J = \widehat{\mathbf{M}}_J \widehat{\mathbf{G}}_J^{-1} \widehat{\mathbf{c}}_J$.

In what follows we abbreviate Cor_{du} by v. To determine the approximate counterpart $\widehat{v}_J \in \widehat{V}_J$ we shall separate the degrees of freedom in order to solve the boundary value problem (3) successively: $\widehat{v}_J = \widehat{v}^{\overline{D},\overline{D}}_J + \widehat{v}^{\overline{D},\partial\overline{D}}_J + \widehat{v}^{\partial\overline{D},\overline{D}}_J + \widehat{v}^{\partial\overline{D},\partial\overline{D}}_J$, where

$$\widehat{v}^{\Theta,\Xi}_J := \sum_{j+j'\le J} (\Phi^{\Theta}_j \otimes \Phi^{\Xi}_{j'}) \widehat{\mathbf{v}}^{\Theta,\Xi}_{j,j'}, \quad \Theta,\Xi \in \{\overline{D}, \partial\overline{D}\}.$$

Then we proceed as follows (see [11] for the details).

1. Determine $\widehat{u}^{\partial\overline{D},\partial\overline{D}}_J$ as the L^2-orthoprojection of the approximate Dirichlet data $\widehat{f}_J$ (6) onto the discrete trace space $\widehat{V}_J\big|_{\partial\overline{D}\times\partial\overline{D}}$ according to

$$\widehat{\mathbf{G}}_J \widehat{\mathbf{v}}^{\partial\overline{D},\partial\overline{D}}_J = \widehat{\mathbf{M}}_J \widehat{\mathbf{G}}_J^{-1} \widehat{\mathbf{c}}_J. \tag{8}$$

2. Compute $\widehat{v}^{\overline{D},\partial\overline{D}}_J$ such that $(\widehat{v}^{\partial\overline{D},\partial\overline{D}}_J + \widehat{v}^{\overline{D},\partial\overline{D}}_J)\big|_{\overline{D}\times\partial\overline{D}} \in H^1(\overline{D}) \otimes H^{1/2}(\partial\overline{D})$ satisfies the homogeneous boundary condition on $\overline{D}\times\partial\overline{D}$. In complete analogy determine $\widehat{v}^{\partial\overline{D},\overline{D}}_J$, which gives raise to

$$\widehat{\mathbf{B}}^{\overline{D}}_J \widehat{\mathbf{v}}^{\overline{D},\partial\overline{D}}_J = -\widehat{\mathbf{B}}^{\partial\overline{D}}_J \widehat{\mathbf{v}}^{\partial\overline{D},\partial\overline{D}}_J, \qquad \widehat{\mathbf{C}}^{\overline{D}}_J \widehat{\mathbf{v}}^{\partial\overline{D},\overline{D}}_J = -\widehat{\mathbf{C}}^{\partial\overline{D}}_J \widehat{\mathbf{v}}^{\partial\overline{D},\partial\overline{D}}_J. \tag{9}$$

3. Compute the function $\widehat{v}^{\overline{D},\overline{D}}_J \in H^{1,1}_0(\overline{D}\times\overline{D})$ inside the tensor product domain $\overline{D}\times\overline{D}$ according to

$$\widehat{\mathbf{A}}^{\overline{D},\overline{D}}_J \widehat{\mathbf{v}}^{\overline{D},\overline{D}}_J = -\widehat{\mathbf{A}}^{\partial\overline{D},\partial\overline{D}}_J \widehat{\mathbf{v}}^{\partial\overline{D},\partial\overline{D}}_j - \widehat{\mathbf{A}}^{\overline{D},\partial\overline{D}}_J \widehat{\mathbf{v}}^{\overline{D},\partial\overline{D}}_j - \widehat{\mathbf{A}}^{\partial\overline{D},\overline{D}}_J \widehat{\mathbf{v}}^{\partial\overline{D},\overline{D}}_j. \tag{10}$$

3.4 Error estimates

Let $h_J := 2^{-J} \sim \max_k\{\operatorname{diam}\tau_{J,k}\}$ denote the mesh size associated with the subspace V_J on $\overline{D}$. Then, from standard finite element theory for elliptic operators (e.g. [2, 3]), we derive the following facts with respect to the approximate mean.

Proposition 1. *Equation* (2) *can be solved in linear complexity. The approximate mean $\overline{u}_J$ satisfies the error estimate $\|\overline{u}-\overline{u}_J\|_{L^2(\overline{D})} \lesssim h_J^2\|\overline{u}\|_{H^2(\overline{D})}$ provided that the given data are sufficiently smooth.*

In the Galerkin scheme we have to employ the perturbed Dirichlet data $\widehat{f}_J$ (6) instead of the original Dirichlet data f (5) to compute the approximate solution $\widehat{v}_J$ of (3). Therefore, we obtain only a reduced rate of convergence.

Theorem 2. *Assume that $\overline{u} \in W^{2,\infty}(\overline{D})$ and $\operatorname{Cor}_\kappa \in H^{1,1}(\partial\overline{D}\times\partial\overline{D})$. Then, the approximate solution $\widehat{v}_J \in \widehat{V}_J$ to* (3) *satisfies the error estimate*

$$\|v-\widehat{v}_J\|_{L^2(\overline{D}\times\overline{D})} \lesssim h_J \|\operatorname{Cor}_\kappa\|_{H^{1,1}(\partial\overline{D}\times\partial\overline{D})}\|\overline{u}\|^2_{W^{2,\infty}(\overline{D})}.$$

Proof. The assertion follows immediately from [11] if we show that the consistency error of the right hand side satisfies

$$\|f-f_J\|_{L^2(\partial\overline{D}\times\partial\overline{D})} \lesssim h_J \|\operatorname{Cor}_\kappa\|_{H^{1,1}(\overline{D}\times\overline{D})}\|\overline{u}\|^2_{W^{2,\infty}(\overline{D})}. \tag{11}$$

To show this estimate we proceed as follows:

$$\begin{aligned}
\|f-f_J\|_{L^2(\partial\overline{D}\times\partial\overline{D})} &= \big\|(\sigma\otimes\sigma)\operatorname{Cor}_\kappa-(\sigma_J\otimes\sigma_J)\widehat{\Pi}_J\operatorname{Cor}_\kappa\big\|_{L^2(\partial\overline{D}\times\partial\overline{D})}\\
&\le \big\|(\sigma\otimes\sigma-\sigma_J\otimes\sigma_J)\operatorname{Cor}_\kappa\big\|_{L^2(\partial\overline{D}\times\partial\overline{D})}+\big\|(\sigma_J\otimes\sigma_J)(I-\widehat{\Pi}_J)\operatorname{Cor}_\kappa\big\|_{L^2(\partial\overline{D}\times\partial\overline{D})}\\
&\le \|\sigma\otimes\sigma-\sigma_J\otimes\sigma_J\|_{L^\infty(\partial\overline{D}\times\partial\overline{D})}\|\operatorname{Cor}_\kappa\|_{L^2(\partial\overline{D}\times\partial\overline{D})}\\
&\qquad+\|\sigma_J\|^2_{L^\infty(\partial\overline{D})}\big\|(I-\widehat{\Pi}_J)\operatorname{Cor}_\kappa\big\|_{L^2(\partial\overline{D}\times\partial\overline{D})}.
\end{aligned} \tag{12}$$

We now estimate the two terms on the right hand side of this inequality separately. The L^2-orthoprojection onto the sparse grid space satisfies (cf. [5, 15])

$$\big\|(I-\widehat{\Pi}_J)\operatorname{Cor}_\kappa\big\|_{L^2(\partial\overline{D}\times\partial\overline{D})} \lesssim h_J\|\operatorname{Cor}_\kappa\|_{H^{1,1}(\partial\overline{D}\times\partial\overline{D})}. \tag{13}$$

Pointwise error estimates for piecewise linear finite elements (see e.g. [3]) imply

$$\|\sigma-\sigma_J\|_{L^\infty(\partial\overline{D})} = \|\langle\nabla(\overline{u}-\overline{u}_J),\mathbf{n}\rangle\|_{L^\infty(\partial\overline{D})} \le \|\nabla\overline{u}-\nabla\overline{u}_J\|_{L^\infty(\overline{D})} \lesssim h_J\|\overline{u}\|_{W^{2,\infty}(\overline{D})}.$$

This induces by standard tensor product arguments

$$\|\sigma\otimes\sigma-\sigma_J\otimes\sigma_J\|_{L^\infty(\partial\overline{D}\times\partial\overline{D})} \lesssim h_J\big(\|\sigma\|_{L^\infty(\partial\overline{D})}+\|\sigma_J\|_{L^\infty(\partial\overline{D})}\big)\|\overline{u}\|_{W^{2,\infty}(D)}. \tag{14}$$

Inserting (13), (14) and $\|\sigma_J\|_{L^\infty(\partial\overline{D})} \le \|\sigma-\sigma_J\|_{L^\infty(\partial\overline{D})}+\|\sigma\|_{L^\infty(\partial\overline{D})}$ into the estimate (12) yields the desired consistency result (11). □

3.5 Fast second moment computation

The linear systems of equations arising from the sparse multilevel discretization can be assembled and solved in essentially linear complexity when using the following ingredients, developed in the papers [11, 12, 13].

(i) Due to the non-uniqueness of the representation of functions in frame coordinates, all system matrices have a large kernel. Since the associated right hand side vectors lie in the related images, Krylov subspace methods converge without further modifications (see, e.g. [10, 13]). In practice, we apply the conjugate gradient method to solve (8)–(10).

(ii) The diagonally scaled system matrices are essentially well conditioned in the sense that all nonzero eigenvalues behave essentially like a fixed constant. Therefore, the conjugate gradient method converges with a rate that is essentially independent of the discretization level J (e.g. [10]).

(iii) Iterative solvers involve only matrix-vector multiplications. The fast matrix-vector multiplication developed in [11, 13] is of essentially linear complexity. Besides standard prolongations and restrictions, it involves only system matrices (7) with $0 \le j = j' \le J$, i.e., standard finite element matrices. By using prolongations and restrictions, all coarse level matrices are successively derived from the finest grid matrices in linear time.

(iv) Numerical quadrature in the sparse tensor product space is performed as follows. We expand the two-point correlation into the hierarchical basis $\widehat{\Psi}_J := \{\varphi_{j,k} \otimes \varphi_{j',k'} : k \in \mathscr{I}_j \setminus \mathscr{I}_{j-1},\ k' \in \mathscr{I}_{j'} \setminus \mathscr{I}_{j'-1},\ j + j' \le J\} \subset \widehat{\Phi}_J$ of the sparse tensor product space $\widehat{V}_J$. For $\mathrm{Cor}_\kappa \in H^{s,s}(\partial\overline{D} \times \partial\overline{D})$ with $0 \le s < 2$ an approximation $\widehat{\mathrm{Cor}}_{\kappa,J}$ is obtained such that $\| \mathrm{Cor}_\kappa - \widehat{\mathrm{Cor}}_{\kappa,J} \|_{L^2(\partial\overline{D}\times\partial\overline{D})} = h_J^s \| \mathrm{Cor}_\kappa \|_{H^{s,s}(\partial\overline{D}\times\partial\overline{D})}$. In the case $s = 2$ there appears an additional $\sqrt{|\log h_J|}$ factor, see [5] for the details.

Proposition 2. *Combining the ingredients* (i)–(iv) *one arrives at an algorithm that computes the solution's second moment in a complexity that stays essentially proportional to the number of unknowns used to discretize the mean field equation* (2).

4 Numerical results

We consider the boundary value problem (1) with $\mathbf{A} \equiv \mathbf{I}$, $f \equiv 1/4$, $g(\mathbf{x}) = x \cdot y$, and $\overline{D}$ being the unit circle (i.e. $n = 2$). If we prescribe Gaussian correlation $\mathrm{Cor}_\kappa(\mathbf{x},\mathbf{y}) = e^{-\|\mathbf{x}-\mathbf{y}\|^2}$ we get the solution's approximate mean and variance shown in Figure 1. It turns out that the variance increases when approaching the boundary of the domain, i.e., the solution's sensitivity with respect to boundary perturbations is the larger the nearer the boundary. This effect is stronger in regions where the modulus of the Dirichlet data g is large. The non-symmetry is induced by the present inhomogeneity $f \equiv 1/4$. Notice that the variance scales linearly in the perturbation parameter ε and thus decreases correspondingly as $\varepsilon \to 0$. Further numerical results, especially a comparison with an MC simulation, can be found in [11].

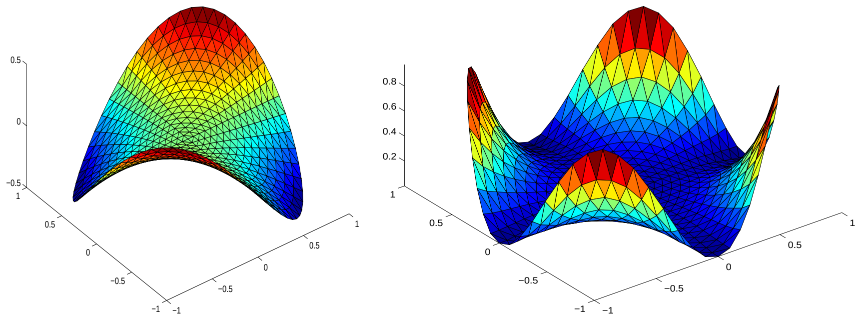

Fig. 1 Approximate mean and two-point correlation of u.

References

1. I. Babuška, F. Nobile, and R. Tempone. Worst case scenario analysis for elliptic problems with uncertainty. *Numer. Math.* **101** (2005), 185–219.
2. D. Braess. *Finite elements. Theory, fast solvers, and applications in solid mechanics*. Second edition. Cambridge University Press, Cambridge, 2001.
3. S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*. Texts in Applied Mathematics, 15. Springer-Verlag, New York, 1994.
4. J. Bramble, J. Pasciak, and J. Xu. Parallel multilevel preconditioners. *Math. Comput.* **55** (1990) 1–22.
5. H.J. Bungartz and M. Griebel. Sparse Grids. *Acta Numerica* **13** (2004) 147–269.
6. C. Canuto and T. Kozubek. A fictitious domain approach to the numerical solution of PDEs in stochastic domains. *Numer. Math.* **107** (2007), 257–293.
7. M.K. Deb, I. Babuška, and J.T. Oden. Solution of stochastic partial differential equations using Galerkin finite element techniques. *Comput. Methods Appl. Mech. Eng.* **190** (2001), 6359–6372.
8. P. Frauenfelder, C. Schwab, and R.A. Todor. Finite elements for elliptic problems with stochastic coefficients. *Comput. Meth. Appl. Mech. Engrg.* **194** (2004), 205–228.
9. R.G. Ghanem and P.D. Spanos. *Stochastic finite elements: a spectral approach*. Springer-Verlag, 1991.
10. M. Griebel. *Multilevelmethoden als Iterationsverfahren über Erzeugendensystemen.* Teubner Skripten zur Numerik. B.G. Teubner, Stuttgart, 1994.
11. H. Harbrecht. A finite element method for elliptic problems with stochastic input data. *Preprint No. 408*, Berichtsreihe des SFB 611, Universität Bonn, 2008, submitted.
12. H. Harbrecht, R. Schneider, and C. Schwab. Sparse Second Moment Analysis for Elliptic Problems in Stochastic Domains. *Numer. Math.* **109** (2008), 167–188.
13. H. Harbrecht, R. Schneider, and C. Schwab. Multilevel frames for sparse tensor product spaces. *Numer. Math.* **110** (2008), 199–220.
14. H.G. Matthies and A. Keese. Galerkin Methods for Linear and Nonlinear Elliptic Stochastic Partial Differential Equations. *Comput. Methods Appl. Mech. Engrg.* **194** (2005), 1295–1331.
15. T. von Petersdorff and C. Schwab. Sparse wavelet methods for operator equations with stochastic data. *Applications of Mathematics* **51** (2006) 145–180.
16. P. Protter. *Stochastic integration and differential equations: a new approach*. 3rd edition, Springer-Verlag, 1995.
17. D.M. Tartakovsky and D. Xiu. Numerical methods for differential equations in random domains. *SIAM J. Sci. Comput.* **28** (2006), 1167–1185.

Bestellungen nimmt entgegen:

Sonderforschungsbereich 611
der Universität Bonn
Poppelsdorfer Allee 82
D - 53115 Bonn

Telefon: 0228/73 4882
Telefax: 0228/73 7864
E-Mail: astrid.link@ins.uni-bonn.de http://www.sfb611.iam.uni-bonn.de/

Verzeichnis der erschienenen Preprints ab No. 430

430. Frehse, Jens; Málek, Josef; Ružička, Michael: Large Data Existence Result for Unsteady Flows of Inhomogeneous Heat-Conducting Incompressible Fluids

431. Croce, Roberto; Griebel, Michael; Schweitzer, Marc Alexander: Numerical Simulation of Bubble and Droplet Deformation by a Level Set Approach with Surface Tension in Three Dimensions

432. Frehse, Jens; Löbach, Dominique: Regularity Results for Three Dimensional Isotropic and Kinematic Hardening Including Boundary Differentiability

433. Arguin, Louis-Pierre; Kistler, Nicola: Small Perturbations of a Spin Glass System

434. Bolthausen, Erwin; Kistler, Nicola: On a Nonhierarchical Version of the Generalized Random Energy Model. II. Ultrametricity

435. Blum, Heribert; Frehse, Jens: Boundary Differentiability for the Solution to Hencky's Law of Elastic Plastic Plane Stress

436. Albeverio, Sergio; Ayupov, Shavkat A.; Kudaybergenov, Karim K.; Nurjanov, Berdach O.: Local Derivations on Algebras of Measurable Operators

437. Bartels, Sören; Dolzmann, Georg; Nochetto, Ricardo H.: A Finite Element Scheme for the Evolution of Orientational Order in Fluid Membranes

438. Bartels, Sören: Numerical Analysis of a Finite Element Scheme for the Approximation of Harmonic Maps into Surfaces

439. Bartels, Sören; Müller, Rüdiger: Error Controlled Local Resolution of Evolving Interfaces for Generalized Cahn-Hilliard Equations

440. Bock, Martin; Tyagi, Amit Kumar; Kreft, Jan-Ulrich; Alt, Wolfgang: Generalized Voronoi Tessellation as a Model of Two-dimensional Cell Tissue Dynamics

441. Frehse, Jens; Specovius-Neugebauer, Maria: Existence of Hölder Continuous Young Measure Solutions to Coercive Non-Monotone Parabolic Systems in Two Space Dimensions

442. Kurzke, Matthias; Spirn, Daniel: Quantitative Equipartition of the Ginzburg-Landau Energy with Applications

443. Bulíček, Miroslav; Frehse, Jens; Málek, Josef: On Boundary Regularity for the Stress in Problems of Linearized Elasto-Plasticity

444. Otto, Felix; Ramos, Fabio: Universal Bounds for the Littlewood-Paley First-Order Moments of the 3D Navier-Stokes Equations

445. Frehse, Jens; Specovius-Neugebauer, Maria: Existence of Regular Solutions to a Class of Parabolic Systems in Two Space Dimensions with Critical Growth Behaviour

446. Bartels, Sören; Müller, Rüdiger: Optimal and Robust A Posteriori Error Estimates in $L^{\infty}(L^2)$ for the Approximation of Allen-Cahn Equations Past Singularities

447. Bartels, Sören; Müller, Rüdiger; Ortner, Christoph: Robust A Priori and A Posteriori Error Analysis for the Approximation of Allen-Cahn and Ginzburg-Landau Equations Past Topological Changes

448. Gloria, Antoine; Otto, Felix: An Optimal Variance Estimate in Stochastic Homogenization of Discrete Elliptic Equations

449. Kurzke, Matthias; Melcher, Christof; Moser, Roger; Spirn, Daniel: Ginzburg-Landau Vortices Driven by the Landau-Lifshitz-Gilbert Equation

450. Kurzke, Matthias; Spirn, Daniel: Gamma-Stability and Vortex Motion in Type II Superconductors

451. Conti, Sergio; Dolzmann, Georg; Müller, Stefan: The Div–Curl Lemma for Sequences whose Divergence and Curl are Compact in $W^{-1,1}$

452. Barret, Florent; Bovier, Anton; Méléard, Sylvie: Uniform Estimates for Metastable Transition Times in a Coupled Bistable System

453. Bebendorf, Mario: Adaptive Cross Approximation of Multivariate Functions

454. Albeverio, Sergio; Hryniv, Rostyslav; Mykytyuk, Yaroslav: Scattering Theory for Schrödinger Operators with Bessel-Type Potentials

455. Weber, Hendrik: Sharp Interface Limit for Invariant Measures of a Stochastic Allen-Cahn Equation

456. Harbrecht, Helmut: Finite Element Based Second Moment Analysis for Elliptic Problems in Stochastic Domains