The Ostrogradsky-Pierce Expansion: Probability Theory, Dynamical Systems and Fractal Geometry Points of View

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THE OSTROGRADSKY-PIERCE EXPANSION: PROBABILITY THEORY, DYNAMICAL SYSTEMS AND FRACTAL GEOMETRY POINTS OF VIEW.

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Abstract. We establish several new probabilistic, dynamical and number theoretical phenomena connected with the Ostrogradsky-Pierce expansion.

First of all we prove the singularity of the random version of this expansion.

Secondly we study properties of the symbolic dynamical system generated by the natural one-sided shift-transformation T on the difference-version of the Ostrogradsky-Pierce expansion. We show, in particular, that there are no probability measures which are simultaneously invariant and ergodic (w.r.t. T) and absolutely continuous (w.r.t. Lebesgue measure). This plays against the application of a direct ergodic approach to the development of a metric theory for the Ostrogradsky-Pierce expansion.

We develop instead the metric and dimensional theories for this expansion using probabilistic methods. In particular, it is shown that for Lebesgue almost all real numbers any digit i from the alphabet $A = \mathbb{N}$ appears only finitely many times in the difference-version of the Ostrogradsky-Pierce expansion, and the set of all reals with bounded digits of this expansion is of zero Hausdorff dimension.

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1. Introduction

It is by now well known that any real number $x \in (0,1)$ can be represented in the form

$$\sum_k \frac{(-1)^{k+1}}{q_1 q_2 \cdots q_k}, \quad \text{where } q_k \in \mathbb{N}, q_{k+1} > q_k, \quad k \in \mathbb{N}. \tag{1}$$

If x is irrational, then this expansion is unique. In the opposite case there are two different expansion of x into series of the above form ([21]).

In the western mathematical literature series of the above form are known as Pierce series, and in the eastern literature they are known as Ostrogradsky series of the first type. One can find notes on the history of the discovery and the development of such series in the paper [18]. Here we would like just to mention that such series can also be associated with the names of Lambert ([16]), Lagrange ([15]), Sierpiński ([24]). In what follows we shall use the notion "Ostrogradsky-Pierce expansion" for the above series. M.V. Ostrogradsky was probably the first (1860) who developed a few numerical properties of such an expansion (see, e.g., [21]), and T.A. Pierce was probably the first (1929) who used this expansion for a numerical estimation of algebraic roots of polynomials ([19]).

The Ostrogradsky-Pierce series converges rather quickly, giving a good approximation of irrational numbers by rationals, which are partial sums of the above series.

Let us recall ([1]) that the expression (1) can be rewritten in the form

$$\frac{1}{g_1} - \frac{1}{g_1(g_1+g_2)} + \cdots + \frac{(-1)^{n-1}}{g_1(g_1+g_2)\dots(g_1+g_2+\cdots+g_n)} + \cdots, \qquad (2)$$

where

$$g_1 = q_1 \quad \text{i} \quad g_{n+1} = q_{n+1} - q_n \quad \text{for all } n \in \mathbb{N}.$$

The expression (2) will be denoted by

$$\bar{\mathrm{O}}^1(g_1, g_2, \dots, g_n, \dots).$$

and is said to be $\bar{\mathrm{O}}^1$-expansion (or the Ostrogradsky-Pierce expansion with independent symbols), and coefficients $g_n = g_n(x)$ are called $\bar{\mathrm{O}}^1$-symbols (coefficients) of a real number $x \in (0,1)$. There are several papers on the metric theory of this expansion (see, e.g., [1, 4, 20, 23, 25] and references therein), but they should be considered only as first steps in the development of the general theory like the one existing for the continued fractions expansion. There are a lot of common features between these two expansions, but the Ostrogradsky-Pierce expansion generates essentially a more complicated "geometry of cylindrical intervals". It is known that the development of metric and ergodic theories of some expansion for reals can be essentially simplified if one can find a measure which is invariant and ergodic w.r.t. one-sided shift transformation on the corresponding expansion and absolutely continuous w.r.t. Lebesgue measure (see, e.g., [22]). For instance, having the Gauss measure (i.e., the probability measure with density $f(x) = \frac{1}{\ln 2}\frac{1}{1+x}$ on the unit interval) as invariant and ergodic measure w.r.t. the transformation $T(x) = \frac{1}{x}(mod 1)$, one can easily derive main metric and ergodic properties of continued fraction expansions (see, e.g., [5, 13, 22]).

The main aims of the present paper are:

1) to develop ergodic, metric and dimensional theories for the $\bar{\mathrm{O}}^1$ - expansion for real numbers (in particular, to find normal properties of real numbers, depending on asymptotic frequencies $\nu_i(x, \bar{\mathrm{O}}^1)$ of $\bar{\mathrm{O}}^1$-symbols ($i \in \mathbb{N}$), where $\nu_i(x, \bar{\mathrm{O}}^1) = \lim\limits_{n\to\infty} \frac{N_i(x,n)}{n}$, and $N_i(x,n)$ is the number of terms "i" among the first n $\bar{\mathrm{O}}^1$-coefficients of x);

2) to study properties of the symbolic dynamical system generated by the one-sided shift transformation on the $\bar{\text{O}}^1$-expansion:

$$\forall\ x = \bar{\text{O}}^1(g_1(x), g_2(x), \dots, g_n(x), \dots) \quad \in [0,1],$$

$$T(x) = T(\bar{\text{O}}^1(g_1(x), g_2(x), \dots, g_n(x), \dots)) = \bar{\text{O}}^1(g_2(x), g_3(x), \dots, g_n(x), \dots);$$

3) to study distributions of random variables

$$\eta = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\eta_1(\eta_1+\eta_2)\dots(\eta_1+\eta_2+\dots+\eta_k)} = \bar{\text{O}}^1(\eta_1, \eta_2, \dots, \eta_k, \dots),$$

whose $\bar{\text{O}}^1$-symbols η_k are independent random variables taking the values 1, 2, $\dots$, m, $\dots$ with probabilities p_{1k}, p_{2k}, $\dots$, p_{mk}, $\dots$ respectively, $p_{mk} \geq 0$, $\sum\limits_{m=1}^{\infty} p_{mk} = 1$, $\forall k \in \mathbb{N}$.

2. Sets $C[\bar{\text{O}}^1, \{V_n\}]$ and their metric and fractal properties.

Let $\{V_n\}$ be a given sequence of non-empty subsets of positive integers. Let us consider the set $C[\bar{\text{O}}^1, \{V_n\}]$, which is the closure of the set $C^*[\bar{\text{O}}^1, \{V_n\}]$ of all irrational numbers $x = \bar{\text{O}}^1(g_1(x), g_2(x), \dots, g_n(x), \dots)$ such that $g_n(x) \in V_n$ for all $n \in \mathbb{N}$.

It is clear that $C[\bar{\text{O}}^1, \{V_n\}]$ is a nowhere dense set if and only if the condition $V_n \neq \mathbb{N}$ holds for an infinite number of n's. The set of real numbers whose continued fraction expansion does not contain a given symbol $i \in \mathbb{N}$ is a Cantor-like set of zero Lebesgue measure. Indeed, almost all (in the sense of Lebesgue measure) real numbers contain a given digit i with non-zero asymptotic frequency $\nu_i^{c.f.} = \frac{1}{\ln 2} \ln \frac{(i+1)^2}{i(i+2)}$ (see, e.g., [5]). So, all points which can be written without using the symbol i belong to the exceptional zero-set. For the Ostrogradsky-Pierce expansion the metric properties of $C[\bar{\text{O}}^1, \{V_n\}]$ depend essentially on the sequence of sets V_k of admissible digits.

In [1, 4] some sufficient conditions for the set $C[\bar{\text{O}}^1, \{V_n\}]$ to be of zero resp. positive Lebesgue measure are found. We collect here some of these results without proof to be used later in the paper and to stress essential differences in metric theories of continued fractions and the Ostrogradsky-Pierce expansions.

Theorem 1. *Let* $V_k = \{1, 2, \dots, m_k\}$, $m_k \in \mathbb{N}$.

1) If $\sum\limits_{k=1}^{\infty} \frac{m_1+m_2+\dots+m_k}{m_{k+1}} < \infty$, *then* $\lambda(C[\bar{\text{O}}^1, \{V_n\}]) > 0$, *where* λ *denotes Lebesgue measure .*

2) If $\sum\limits_{k=1}^{\infty} \frac{k}{m_k} = \infty$, *then* $\lambda(C[\bar{\text{O}}^1, \{V_n\}]) = 0$.

Example.
1) If $m_k = 2^{k!}$, then $\lambda(C[\bar{\text{O}}^1, \{V_n\}]) > 0$.
2) If $m_k = k^2$, then $\lambda(C[\bar{\text{O}}^1, \{V_n\}]) = 0$.

Theorem 2. *Let* $V_k = \{v_k + 1, v_k + 2, \dots\}$, $v_k \in \mathbb{N}$.

If $\sum\limits_{k=1}^{\infty} \frac{v_k}{2^k} < +\infty$, *then* $\lambda(C[\bar{\text{O}}^1, \{V_n\}]) > 0$.

Corollary 1. *If* $V_k = V = \{v+1, v+2, ...\}$, *then* $\lambda(C[\bar{O}^1, \{V_n\}]) > 0$.

In the case of sets $C[\bar{O}^1, \{V_n\}]$ of zero Lebesgue measure, the next level of their study is the determination of their Hausdorff dimension $\dim_H(\cdot)$ (see, e.g., [8] for the definition and main properties of this main fractal dimension).

We shall study this problem for the case where $V_n = \{1, 2, ..., k_n\}$. A similar problem for the continued fraction expansion were studied by many authors during the last 60 years. Set

$$E_2 = \{x : x = \Delta^{c.f.}_{\alpha_1(x)...\alpha_k(x)...}, \alpha_k(x) \in \{1, 2\}\}.$$

In 1941 Good [9] shows that

$$0,5194 < \dim_H(E_2) < 0,5433.$$

In 1982 and 1985 Bumby [6, 7] improves these bounds:

$$0,5312 < \dim_H(E_2) < 0,5314.$$

In 1989 Hensley [10] shows that

$$0,53128049 < \dim_H(E_2) < 0,53128051.$$

In 1996 the same author ([11]) improves his estimate up to

$$0,5312805062772051416.$$

A new approach to the determination of the Hausdorff dimension of the set E_2 with a desired precision was developed by Jenkinson and Policott in 2001 [12].

Our nearest aim is to study fractal properties of sets which are $\bar{O}^1$-analogues of the above discussed set E_2, i.e., the set

$$\bar{O}^1_2 = \{x : x = \bar{O}^1(g_1(x)g_2(x)...g_k(x)...), g_k(x) \in \{1, 2\}\}$$

and its associate sets

$$\bar{O}^1_n = \{x : x = \bar{O}^1(g_1(x)g_2(x)...g_k(x)...), g_k(x) \in \{1, 2, ..., n\}\}, n \in N.$$

Firstly, let us mention, that from Theorem 1 it follows that all these sets are of zero Lebesgue measure, which is similar to the continued fractions case. But the following theorem shows that from the fractal geometry point of view the sets E_2 and $\bar{O}^1_2$ (as well as their generalizations $\bar{O}^1_n$) are cardinally different.

Theorem 3. *For any* $n \in \mathbb{N}$ *the Hausdorff dimension of the set* $\bar{O}^1_n$ *is equal to zero.*

Proof. Let $\bar{O}^1_{[c_1c_2...c_k]}$ be the cylindrical interval of the $\bar{O}^1$-expansion, i.e., the closure of all real numbers x from the unit interval such that $g_i(x) = c_i, i = 1, 2, ..., k$. It is known ([1]) that $\left|\bar{O}^1_{[c_1c_2...c_k]}\right| = \frac{1}{\sigma_1\sigma_2...\sigma_k(\sigma_k+1)}$, where $\sigma_j = c_1 + c_2 + ... + c_j$. Therefore, $\left|\bar{O}^1_{[c_1c_2...c_k]}\right| \leq \left|\bar{O}^1_{[11...1]}\right| = \frac{1}{k!(k+1)}$.

Let us fix a positive real number α. It is clear that the set $\bar{O}^1_n$ is contained in the union of the following cylinders:

$$\bar{O}^1_n \subset \bigcup_{i_1=1}^{n} \bigcup_{i_2=1}^{n} ... \bigcup_{i_k=1}^{n} \bar{O}^1_{[c_1c_2...c_k]}, \quad \forall k \in \mathbb{N},$$

which forms its $\varepsilon_k = \frac{1}{k!(k+1)}$-covering. The α-volume of this covering is equal to $n^k \cdot \left(\frac{1}{k!(k+1)}\right)^\alpha$. So, for the Hausdorff pre-measure $H^\alpha_{\varepsilon_k}(\bar{\text{O}}^1_n)$ we have:

$$H^\alpha_{\varepsilon_k}(\bar{\text{O}}^1_n) := \inf_{|E_i|\le\varepsilon_k} \sum_i |E_i|^\alpha \le n^k \cdot \left(\frac{1}{k!(k+1)}\right)^\alpha \to 0(k\to\infty), \quad \forall\alpha>0.$$

Therefore, $H^\alpha_{\varepsilon_k}(\bar{\text{O}}^1_n) = 0,\ \forall k\in\mathbb{N},\ \forall\alpha>0$.

So, $H^\alpha(\bar{\text{O}}^1_n) = \lim\limits_{k\to\infty} H^\alpha_{\varepsilon_k}(\bar{\text{O}}^1_n) = 0,\ \forall\alpha>0$ and, hence,

$$\dim_H(\bar{\text{O}}^1_n) := \inf\{\alpha : H^\alpha(\bar{\text{O}}^1_n) = 0\} = 0,$$

which proves the theorem. □

Let $B(\bar{\text{O}}^1)$ be the set of all real numbers from the unit interval with bounded $\bar{\text{O}}^1$-symbols (i.e., $x\in B(\bar{\text{O}}^1)$ iff there exists a positive integer K_x (depending on x) such that $g_k(x)\le K_x$ for all $k\in\mathbb{N}$).

Corollary 1. The set $B(\bar{\text{O}}^1)$ with bounded $\bar{\text{O}}^1$-symbols is an anomalously fractal set, i.e.,

$$\dim_H(B(\bar{\text{O}}^1)) = 0.$$

Corollary 2. The sequences $\{g_k(x)\}$ of $\bar{\text{O}}^1$-symbols are unbounded for all $x\in[0,1]$ except for a subset of zero Hausdorff dimension.

Remark. The set $B(c.f.)$ of real numbers with bounded continued fraction symbols is of full Hausdorff dimension ($\dim_H(B(c.f.)) = 1$), which stresses essential differences also in dimensional theories of the Ostrogradsky-Pierce and continued fraction expansions.

3. Properties of the symbolic dynamical system generated by the Ostrogradsky-Pierce expansion

Let us consider a dynamical system which is generated by the one-sided shift transformation T on the $\bar{\text{O}}^1$-expansion:

$$\forall\ x = \bar{\text{O}}^1(g_1(x), g_2(x), \dots, g_n(x), \dots) \quad \in [0,1],$$

$$T(x) = T(\bar{\text{O}}^1(g_1(x), g_2(x), \dots, g_n(x), \dots)) = \bar{\text{O}}^1(g_2(x), g_3(x), \dots, g_n(x), \dots).$$

Recall that a set A is said to be invariant w.r.t. a measurable transformation T, if $A = T^{-1}A$. A measure μ is said to be ergodic w.r.t. a transformation T, if any invariant set $A\in\mathfrak{B}$ is either of full or of zero measure μ. A measure μ is said to be invariant w.r.t. a transformation T, if for any set $E\in\mathfrak{B}$ one has $\mu(T^{-1}E) = \mu(E)$.

Let us remind that to develop metric and ergodic theories of any expansion it would very desirable to have a measure which is T-invariant, T-ergodic and absolutely continuous w.r.t. the Lebesgue measure (i.e., to find an analogue of the Gauss measure for the c.f.-expansion). Unfortunately, the following theorem shows that the above mentioned ergodic approach is not applicable for the Ostrogradsky-Pierce expansion.

Theorem 4. *There are no probability measures which are simultaneously invariant and ergodic w.r.t. the one-sided shift transformation T on the $\bar{\text{O}}^1$-expansion, and absolutely continuous w.r.t. the Lebesgue measure.*

Proof. Firstly we prove the lemma characterizing generic properties of asymptotic frequencies of digits (from the alphabet) in the Ostrogradsky-Pierce expansion of real numbers.

Lemma 1. *Let $\nu_i(x, \bar{\mathrm{O}}^1)$ be the asymptotic frequency of a symbol i in the $\bar{\mathrm{O}}^1$-expansion of x (if the limit $\lim\limits_{k\to\infty} \frac{N_i(x,k)}{k}$ exists). Then for Lebesgue almost all real numbers $x \in [0,1]$ and for any symbol $i \in \mathbb{N}$ the asymptotic frequency $\nu_i(x, \bar{\mathrm{O}}^1)$ is equal to zero.*

Proof. Let x be a random variable which is uniformly distributed on the unit interval, i.e., the Lebesgue measure coincides with the probability measure μ_x. Let i be a given positive integer, and let us consider the following sequence of random variables:

$$\xi_k = \xi_k(x) = 0, \quad \text{if} \quad g_k(x) \neq i;$$

$$\xi_k = \xi_k(x) = 1, \quad \text{if} \quad g_k(x) = i.$$

It is clear that $N_i(x,k) = \xi_1(x) + \xi_2(x) + ... + \xi_k(x)$. Let

$$G_i = \{x : \lim_{k\to\infty} \frac{N_i(x,k)}{k} = 0\}.$$

The event $x \in G_i$ does not depend on any finite number of $\bar{\mathrm{O}}^1$-symbols of x. Therefore, either $\mu_x(G_i) = 0$ or $\mu_x(G_i) = 1$.

Fix $V_n = V = \{i+1, i+2, ...\}$. If $x \in C[\bar{\mathrm{O}}^1, \{V_n\}]$, then $N_i(x,k) = 0, \forall k \in \mathbb{N}$. Therefore, $C[\bar{\mathrm{O}}^1, \{V_n\}] \subset G_i$. From the corollary of Theorem 2 it follows directly that $\lambda(C[\bar{\mathrm{O}}^1, \{V_n\}]) > 0$. So, $\mu_x(G_1) = \lambda(G_1) = 1$. □

To prove the theorem ad absurdum, let us assume that there exists an absolutely continuous probability measure ν, which is invariant and ergodic w.r.t. the above defined transformation T. Then, by Birkhoff ergodic theorem, for ν-almost all $x \in [0,1]$ and for any function $\varphi \in L^1([0,1], \nu)$ we get:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(x)) = \int_0^1 \varphi(x) d(\nu(x)) = \int_0^1 \varphi(x) f_\nu(x) dx,$$

where $f_\nu(x)$ is the density of ν.

Choose $\varphi_i(x) = 1$, if $x \in \bar{\mathrm{O}}^1_{[i]}$, and $\varphi_i(x) = 0$ otherwise. Then

$$\int_0^1 \varphi_i(x) f_\nu(x) dx = \int_{\bar{\mathrm{O}}^1_{[i]}} f_\nu(x) dx > 0 \ \text{ for at least one } \ i \in \mathbb{N}.$$

Let the latter condition hold for the index i_0.

On the other hand, from the above Lemma it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_{i_0}(T^j(x)) = \lim_{n\to\infty} \frac{N_{i_0}(x,n)}{n} = 0$$

for λ-almost all $x \in [0,1]$.

Hence,

$$\lim_{n\to\infty} \frac{N_{i_0}(x,n)}{n} = 0$$

for λ-almost all $x \in [0,1]$, and simultaneously

$$\lim_{n\to\infty} \frac{N_{i_0}(x,n)}{n} > 0$$

for a set of positive Lebesgue measure. This contradiction proves the theorem. $\square$

Remark. From the proof given above it follows that there are no probability measures which are simultaneously invariant and ergodic w.r.t. the one-sided shift transformation T acting on the $\bar{\text{O}}^1$-expansion, and which contains an absolutely continuous component in its Lebesgue decomposition.

This result can be naturally applied to study the Lebesgue structure of the random Ostrogradsky-Pierce expansion, i.e., the random variable

$$\eta = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\eta_1(\eta_1+\eta_2)\dots(\eta_1+\eta_2+\dots+\eta_k)} = \bar{\text{O}}^1(\eta_1,\eta_2,\dots,\eta_k,\dots), \quad (3)$$

whose $\bar{\text{O}}^1$-symbols η_k are independent identically distributed random variables taking the values $1, 2, \dots, m, \dots$ with probabilities $p_1, p_2, \dots, p_m, \dots$ respectively, i.e.,

$$P\{\eta_k = m\} = p_m \quad \text{with} \quad p_m \geq 0, \quad \sum_{m=1}^{\infty} p_m = 1 \quad \forall k \in \mathbb{N}.$$

Theorem 5. *Let $\{\eta_k\}$ be a sequence of independent identically distributed random variables taking the values $1, 2, \dots, m, \dots$ with probabilities $p_1, p_2, \dots, p_m, \dots$ respectively. Then the random variable η defined by (3) has either:*

1) degenerate distribution (if $p_i = 1$ for some $i \in \mathbb{N}$);

2) or pure singularly continuous distribution (in all other cases).

Proof. 1) The correctness of the first assertion follows directly from the necessary and sufficient condition for discreteness of η in the general independent case (see, e.g., [1]): the random variable η is purely discretely distributed if and only if $\prod_{k=1}^{\infty} \max_i p_{ik} > 0$.

2) Let us prove that in the case of continuity the distribution of η does not contain any absolutely continuous component. To this end we need an auxiliary lemma.

Lemma 2. *If $\{\eta_k\}$ are independent and identically distributed random variables, then the measure μ_η is invariant and ergodic w.r.t. the one-sided shift transformation T.*

Proof. 1) Let A be an invariant set w.r.t. T. Then $T(T^{-1}A) = T(A)$ and, so, $A = TA$. Therefore $A = T^{-1}A = T^{-1}(TA)$.

If $x = \bar{\text{O}}^1(g_1(x)g_2(x)...g_k(x)...)$ and $x \in A$, then

$$T^{-1}(Tx) = \{x : x = \bar{\text{O}}^1(c_1 g_2(x)...g_k(x)...), c_1 \in N\} \subset A.$$

Therefore the event $\{x \in A\}$ does not depend on the first $\bar{\text{O}}^1$-symbol of the point x. Similarly one can show that this event does not depend on the initial n $\bar{\text{O}}^1$-symbols of x. Then, from Kolmogorov's "zero and one" law it follows that either $\mu_\eta(A) = 0$ or $\mu_\eta(A) = 1$. So, μ_η is ergodic w.r.t. T.

2) Since the Borel σ-algebra $\mathcal{B}$ is generated by the family of $\bar{\mathrm{O}}^1$-cylinders, i.e., sets of the form $\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_n]}$, it is sufficient to show that the measure μ_η is invariant on these cylinders ([5]). It is clear that $\mu_\eta(\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_n]}) = p_{c_1} \cdot p_{c_2} \cdot \ldots \cdot p_{c_n}$. Since $T^{-1}(\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_n]}) = \bar{\mathrm{O}}^1_{[ic_1c_2\ldots c_n]}, i \in \mathbb{N}$, we have

$$\mu_\eta(T^{-1}(\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_n]})) = \sum_{i=1}^{\infty} \mu_\eta(\bar{\mathrm{O}}^1_{[ic_1c_2\ldots c_n]}) =$$

$$= p_{c_1} \cdot p_{c_2} \cdot \ldots \cdot p_{c_n} \sum_{i=1}^{\infty} p_i = p_{c_1} \cdot p_{c_2} \cdot \ldots \cdot p_{c_n} = \mu_\eta(\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_n]}),$$

which proves the lemma. □

Let us choose a positive integer i_0 such that $p_{i_0} > 0$ and consider the set $M_{i_0} = \{x : x \in [0,1], \nu_i(x, \bar{\mathrm{O}}^1) = p_{i_0} > 0\}$. Since the symbols of $\bar{\mathrm{O}}^1$-expansion are independent w.r.t. the measure μ_η, from the strong law of large number it follows that this set is of full μ_η-measure.

Let us now consider the set $L^*_{i_0} = \{x : x \in [0,1],\ \nu_{i_0}(x, \bar{\mathrm{O}}^1) = 0\}$. From Lemma 1 it follows directly that $\lambda(L^*_{i_0}) = 1$. The sets M_{i_0} and $L^*_{i_0}$ have no mutual intersection. The first one is a support of the probability measure μ_η, and the second one is a support of the Lebesgue measure on the unit interval. So, $\mu_\eta \perp \lambda$, which completes the proof of the theorem. □

Corollary. The random variable η with independent identically distributed increments of the Ostrogradsky-Pierce expansion has a pure distribution, and it is can not be absolutely continuous.

Remark. Based on the latter theorem it is easy to construct all possible types of singularly continuous measures (see, e.g., [2] for the classification).

If $p_{2m-1} = 0$, and $p_{2m} > 0$, then the distribution of η is of GC-type, because the topological support of the measure μ_η is of zero Lebesgue measure.

If $p_m > 0, \forall m \in N$, then the distribution of η is of GS-type, because the topological support of the measure μ_η coincides with the whole unit interval.

Usually the construction of the singularly continuous measures of the pure GP-type is a more complicated problem, but the latter theorem and theorem 2 show us a very easy way for such a construction: if $p_1 = 0$ and $p_{m+1} > 0, \forall m \in N$, then the distribution of η is of GP-type, because the topological support of the measure μ_η is a nowhere dense set of positive Lebesgue measure.

4. On normal properties of reals in the $\bar{\mathrm{O}}^1$ - expansion and singularity of random Ostrogradsky-Pierce expansions in the general independent case

A property "Υ" of real numbers is said to be normal if it holds for almost all (in the sense of the Lebesgue measure) real numbers. Typical normal properties are "to be irrational", "to be transcendental". These properties do not depend on a chosen system of numeration (expansion). Having a fixed expansion, it is convenient to formulate normal properties via properties of symbols (digits) of this expansion. For instance, for the classical decimal expansion the following properties are normal: "to have infinitely many zeroes (in the expansion)", " does not contain any period", "to contain any digit from the alphabet with the asymptotic frequency $\frac{1}{10}$". For the continued fractions expansion as an example of typical normal property one may consider "to contain a symbol i from the alphabet with the asymptotic frequency $\frac{1}{\ln 2}\ln\frac{(i+1)^2}{i(i+2)}$" (see, e.g., [5] for details and other examples). The investigation of normal properties of real numbers written in some expansion is an important part in the development of the metric theory of the corresponding expansion, because to determinate the Lebesgue measure (or any other equivalent measures) of a given subset, one may ignore real numbers loosing normal properties. They are also extremely helpful for the study of properties of the probability distributions connected to the corresponding expansion.

In the initial sections of our paper we already derived two normal properties of real numbers written via $\bar{\mathrm{O}}^1$ - expansion:

1) for Lebesgue almost all real numbers $x \in [0,1]$ the sequences $\{g_k(x)\}$ of their $\bar{\mathrm{O}}^1$-symbols are unbounded;

2) for Lebesgue almost all real numbers $x \in [0,1]$ and for any symbol $i \in \mathbb{N}$ the asymptotic frequency $\nu_i(x, \bar{\mathrm{O}}^1)$ is equal to zero.

The following theorem gives us a rather unusual property of the $\bar{\mathrm{O}}^1$-expansion and it can be considered as an essential strengthening of the latter property.

Theorem 6. *For Lebesgue almost all real numbers $x \in [0,1]$ and for any symbol $i \in \mathbb{N}$ one has:*

$$\limsup_{n\to\infty} N_i(x,n) < +\infty,$$

i.e., in the $\bar{\mathrm{O}}^1$- expansion of almost all real numbers any digit i from the alphabet $A = \mathbb{N}$ appears only finitely many times!

Proof. Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}$, and let $P = \lambda$ be the Lebesgue measure on the unit interval. For any $i \in \mathbb{N}$ and $k \in \mathbb{N}$ set

$$A_k^i := \{x : x = \bar{\mathrm{O}}^1(g_1(x), g_2(x), \dots, g_n(x), \dots); g_k(x) = i\} =$$
$$= \bigcup_{c_1=1}^{\infty} \dots \bigcup_{c_{k-1}=1}^{\infty} \bar{\mathrm{O}}^1[c_1 c_2 \dots c_{k-1} i],$$

where $\bar{\mathrm{O}}^1[c_1c_2...c_{k-1}i]$ is the $\bar{\mathrm{O}}^1$-cylinder. Since $\left|\bar{\mathrm{O}}^1_{[c_1c_2...c_k]}\right| = \frac{1}{\sigma_1\sigma_2...\sigma_k(\sigma_k+1)}$, where $\sigma_j = c_1+c_2+...+c_j$, we have $\lambda(\bar{\mathrm{O}}^1[c_1c_2...c_{k-1}i]) \le \lambda(\bar{\mathrm{O}}^1[c_1c_2...c_{k-1}1])$.

So,

$$\lambda(A_k^i) \le \lambda(A_k^1) = \sum_{c_1=1}^{\infty} \dots \sum_{c_{k-1}=1}^{\infty} \lambda(\bar{\mathrm{O}}^1[c_1c_2...c_{k-1}1]) =$$
$$= \sum_{c_1=1}^{\infty} \dots \sum_{c_{k-1}=1}^{\infty} \frac{1}{\sigma_1\sigma_2...\sigma_{k-1}(\sigma_{k-1}+1)(\sigma_{k-1}+2)} = \frac{1}{2^k}.$$

Let $A_\infty^i = \limsup_{k\to\infty} A_k^i$. It is evident that $\sum_{k=1}^{\infty} \lambda(A_k^i) \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, and, therefore, applying the Borel-Cantelli lemma to the sequence of events $\{A_k^i\} (k \in \mathbb{N})$, which are mutually depending w.r.t. the Lebesgue measure, we get

$\lambda(A^i) = 0$. Thus, for any symbol $i \in \mathbb{N}$ and for λ-almost all $x \in [0,1]$ the $\bar{\text{O}}^1$-expansion of x contains only finitely many symbols "i". $\square$

In the previous section, based on the ergodic approach, we were studying the structure of probability distributions of random variables with independent *identically distributed* $\bar{\text{O}}^1$-symbols. In the present Section, we shall study properties of the distribution of the random variable η in the general independent case, i.e., in the case where η_k are independent but, generally speaking, not identically distributed.

Theorem 7. *Let $\{\eta_k\}$ be a sequence of independent random variables taking values $1, 2, 3, \ldots$ with probabilities $p_{1k}, p_{2k}, p_{3k}, \ldots$ correspondingly, $(\sum\limits_{i=1}^{\infty} p_{ik} = 1,\ \forall k \in \mathbb{N})$.*

If there exists a symbol "i_0" such that

$$\sum_{k=1}^{\infty} p_{i_0 k} = +\infty, \tag{4}$$

then the random variable

$$\eta = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\eta_1(\eta_1+\eta_2)\ldots(\eta_1+\eta_2+\cdots+\eta_k)} = \bar{\text{O}}^1(\eta_1, \eta_2, \ldots, \eta_k, \ldots),$$

is singularly distributed (w.r.t. λ).

Proof. Let

$$A_k^{i_0} := \{x : x = \bar{\text{O}}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots); g_k(x) = i_0\},$$

and let

$$A^i_\infty = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A^i_k = \limsup_{k\to\infty} A^i_k.$$

The events $A_k^{i_0}, k \in \mathbb{N}$ are independent w.r.t. the probability measure μ_η and $\mu_\eta(A_k^{i_0}) = p_{i_0k}$. So, by the inverse Borel-Cantelli lemma for independent events, the condition

$$\sum_{k=1}^{\infty} \lambda(A_k^{i_0}) = \sum_{k=1}^{\infty} p_{i_0k} = +\infty$$

implies the equality $\mu_\eta(A^i_\infty) = 1$. On the other hand, from Theorem 6 it follows directly that $\lambda(A^i_\infty) = 0$, which proves a mutual singularity of the measure μ_η and the Lebesgue measure. $\square$

Corollary. If there exists a symbol "i_0" such that $\sum\limits_{k=1}^{\infty} p_{i_0k} = +\infty$, then the random variable η with independent increments of the Ostrogradsky-Pierce expansion has:

1) a pure discrete distribution if and only if $\prod\limits_{k=1}^{\infty} \max\limits_i p_{ik} > 0$;

2) a singularly continuous distribution in all other cases.

Remark. Condition (4) plays an important role in our proof of the singularity of μ_η, but we strongly believe that the distribution of η is orthogonal with respect to the Lebesgue measure without any additional restrictions.

Conjecture. For any choice of the stochastic matrix $\|p_{ik}\|$ the random variable η with independent increments of the Ostrogradsky-Pierce expansion is singular w.r.t. Lebesgue measure.

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