

# **A Lower Bound for the Spectrum of the Linearized Allen-Cahn Operator Near a Singularity**

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# A LOWER BOUND FOR THE SPECTRUM OF THE LINEARIZED ALLEN-CAHN OPERATOR NEAR A SINGULARITY

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ABSTRACT. A lower bound for the principal eigenvalue of the linearized Allen-Cahn operator near a generic singularity is derived. The estimate leads to robust stability estimates past topological changes.

## 1. INTRODUCTION

Spectral estimates have recently been successfully employed to derive robust stability and error estimates for phase field models, cf. [FP03, KNS04, Bar05, BMO09]. In particular, the time-averaged principal eigenvalue of the linearized Allen-Cahn operator about the exact or approximate solution enters such estimates exponentially and thus logarithmic bounds for this quantity lead to useful estimates, cf. [BMO09]. For the smooth evolution of developed interfaces it is known that the eigenvalue remains uniformly bounded [AF93, Che94, dMS95] while for topological changes it attains the square of the inverse of the interface thickness. Numerical experiments in [BMO09] indicate that the principal eigenvalue grows like  $1/|t|$ ,  $t < 0$ , prior to a topological change at  $t = 0$ . Hence, an integration of it in time leads to a logarithmic bound.

For the mean curvature flow

$$V = -H$$

with a sphere of radius  $\sqrt{2}$  at  $t = -1$  as initial data, the evolution is defined through  $\dot{R} = -1/R$ , i.e.,  $R(t) = \sqrt{2}|t|^{1/2}$  for  $-1 \leq t < 0$ . The linearization of  $H$  in the class of circles is given by

$$H'(t) = -1/R(t)^2 = -(1/2)|t|^{-1}.$$

Since the Allen-Cahn problem approximates the mean curvature flow as the interface thickness tends to zero [Ilm93, BP95] we expect that a similar bound holds for the principal eigenvalue of the linearized Allen-Cahn operator. We adopt the techniques of [Che94] to give a proof of this statement under the following assumption.

**Assumption A.** *The solution  $\phi_\varepsilon$  of the Allen-Cahn problem in  $(-T, 0) \times B_2$  with  $B_2 := B_2(0) \subset \mathbb{R}^2$ , i.e., the function  $\phi_\varepsilon$  that satisfies*

$$\partial_t \phi_\varepsilon - \Delta \phi_\varepsilon = -\varepsilon^{-2} f(\phi_\varepsilon),$$

with  $f(u) := 2(u^2 - 1)u$  and  $0 < \varepsilon < 1$ , is for  $t \leq -\varepsilon^2 \log(\varepsilon^{-1})$  given by

$$(1.1) \quad \phi_\varepsilon(r, t) = \tanh((r - \sqrt{2}|t|^{1/2})/\varepsilon) + \varepsilon^2 q_\varepsilon(r, t)$$

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with a function  $q_\varepsilon$  that satisfies

$$(1.2) \quad |q_\varepsilon(r, t)| \leq c_0 |t|^{-1}.$$

This assumption is motivated by the expansion  $\phi_\varepsilon(x, t) = \tanh(d/\varepsilon) + \varepsilon^2 H^2 \xi(d/\varepsilon) + \mathcal{O}(\varepsilon^3)$  with  $d$  denoting the signed distance to the interface, cf. [PV92, BP95], is confirmed by numerical experiments reported in Appendix A below, and is expected to be justifiable rigorously by an appropriate construction of super- and subsolutions. The assumption enables us to prove the asserted result.

**Theorem 1.1.** *Suppose that Assumption A holds. Then the estimate*

$$\lambda_{AC}(t) := \inf_{0 \neq \psi \in H^1(B_2)} \frac{\int_{B_2} |\nabla \psi|^2 + \varepsilon^{-2} f'(\phi_\varepsilon(r, t)) \psi^2 dx}{\|\psi\|_{L^2(B_2)}^2} \geq -C_{AC} |t|^{-1}$$

holds for  $t \in [-T, -\varepsilon^2 \log(\varepsilon^{-1})]$  with an  $\varepsilon$ -independent constant  $C_{AC} \geq 0$ .

*Remarks 1.1.* (i) Terms of order  $\varepsilon$  can be included if one restricts to  $t \in [-T, -\varepsilon^2 \log(\varepsilon^{-1})^2]$ . This however is not sufficient to prove robust stability estimates.

(ii) The arguments also apply to the linearized Cahn-Hilliard operator, cf. [Che94].

(iii) The same lower bound is expected to hold in three space dimensions.

(iv) Since interfaces become circular in two-dimensional Allen-Cahn evolutions, cf. [GH86], the considered situation is generic.

(v) An equivalent statement is to say that  $\lambda_{AC}(t) \geq -cH_m^2(t)$ , where  $H_m(t)$  is the maximal curvature of the interface.

## 2. ALLEN-CAHN PROFILE ON A BOUNDED INTERVAL

Given  $t \leq -\varepsilon^2 \log(\varepsilon^{-1})$  we consider the operator

$$\mathcal{L}_{\varepsilon, t}^0 := -d^2/dz^2 + f'(\theta_0) \quad \text{in } I_{\varepsilon, t} := (-|t|^{1/2}/(\sqrt{2}\varepsilon), 1/\varepsilon)$$

subject to homogeneous Neumann boundary conditions on  $\partial I_{\varepsilon, t}$  and define

$$L_{\varepsilon, t}^0(\Phi, \Psi) := \int_{I_{\varepsilon, t}} \Phi' \Psi' + f'(\theta_0) \Phi \Psi dz$$

for  $\Phi, \Psi \in H^1(I_{\varepsilon, t})$ . The function  $\theta_0(z) := \tanh(z) = (e^z - e^{-z})/(e^z + e^{-z})$  solves

$$-\theta'' + f(\theta) = 0, \quad \theta(0) = 0, \quad \lim_{z \rightarrow \pm\infty} \theta(z) = \pm 1.$$

Moreover,  $\theta'_0(z) = 1/\cosh^2(z)$  and  $\theta''_0(z) = -2 \sinh(z)/\cosh^3(z)$  satisfy

$$(2.1) \quad 0 < \theta'_0(z) \leq 4e^{-2|z|} \quad \text{and} \quad |\theta''_0(z)| \leq 8e^{-2|z|}.$$

**Lemma 2.1.** *The principal eigenvalue  $\lambda_1^0$  of  $\mathcal{L}_{\varepsilon, t}^0$  satisfies*

$$-c_4 e^{-(1+2\sqrt{2})|t|^{1/2}/\varepsilon} \leq \lambda_1^0 = \inf_{0 \neq \Psi \in H^1(I_{\varepsilon, t})} \frac{L_{\varepsilon, t}^0(\Psi, \Psi)}{\|\Psi\|_{L^2(I_{\varepsilon, t})}^2} \leq c_1 e^{-2\sqrt{2}|t|^{1/2}/\varepsilon},$$

where  $c_1, c_4 > 0$  are  $\varepsilon$ -independent constants.

*Proof.* The shifted operator  $\mathcal{L}_{\varepsilon,t}^0 + \max_{I_{\varepsilon,t}} |f'(\theta_0)|$  is self-adjoint and positive definite so that  $-\max_{I_{\varepsilon,t}} |f'(\theta_0)| \leq \lambda_1^0$ . Integration by parts,  $\mathcal{L}_{\varepsilon,t}^0 \theta'_0 = (-\theta''_0 + f(\theta_0))' = 0$ , and (2.1) show

$$(2.2) \quad \lambda_1^0 \leq \beta^2 L_{\varepsilon,t}^0(\theta'_0, \theta'_0) = \beta^2 \theta'_0 \theta''_0 \Big|_{-|t|^{1/2}/(\sqrt{2\varepsilon})}^{1/\varepsilon} \leq \beta^2 64 e^{-4|t|^{1/2}/(\sqrt{2\varepsilon})} =: c_1 e^{-2\sqrt{2}|t|^{1/2}/\varepsilon},$$

where we used that  $|t|^{1/2}/(\sqrt{2\varepsilon}) \leq 1/\varepsilon$  and defined  $\beta^2 := \|\theta'_0\|_{L^2(I_{\varepsilon,t})}^{-2} \leq 1$ . Set  $m := \max\{f'(-1), f'(1)\} = 4$  and let  $a_0 > 0$  be such that  $f'(\theta_0(z)) \geq 3m/4$  for all  $|z| \geq a_0$ . Since  $a_0$  is independent of  $\varepsilon$  we may assume that  $\pm(a_0 + 1) \in I_{\varepsilon,t}$ . Owing to (2.2) we may assume that  $\lambda_1^0 \leq m/4$ . Then, the positive eigenfunction  $\Psi_1^0$  satisfies

$$-(\Psi_1^0)'' + (f'(\theta_0) - \lambda_1^0)\Psi_1^0 = 0,$$

with  $f'(\theta_0) - \lambda_1^0 \geq m/2$  in  $I_{\varepsilon,t} \setminus [-a_0, a_0]$ . Since  $\|\Psi_1^0\|_{L^2(I_{\varepsilon,t})} = 1$  we may choose  $a'_0 \in [a_0, a_0 + 1]$  such that  $\Psi_1^0(\pm a'_0) \leq 1$ . A comparison argument with the functions

$$\Phi_+(z) := \Psi(a'_0) \frac{\cosh(c(1/\varepsilon - z))}{\cosh(c(1/\varepsilon - a'_0))}, \quad \Phi_-(z) := \Psi(-a'_0) \frac{\cosh(c(|t|^{1/2}/(\sqrt{2\varepsilon}) + z))}{\cosh(c(|t|^{1/2}/(\sqrt{2\varepsilon}) - a'_0))},$$

where  $c = \sqrt{m/2}$ , shows that  $\Psi_1^0(z) \leq \Phi_+(z)$  for  $z \geq a'_0$  and  $\Psi_1^0(z) \leq \Phi_-(z)$  for  $z \leq -a'_0$ , cf. Lemma B.1 for details. We thus deduce that for  $z \in I_{\varepsilon,t} \setminus [-(a_0 + 1), a_0 + 1]$  we have

$$\Psi_1^0(z) \leq \Phi_{\pm}(z) \leq 2e^{-c|z|} e^{ca'_0} \leq 2e^{-c|z|} e^{c(a_0+1)} =: c_2 e^{-\sqrt{2}|z|}.$$

This, integration by parts,  $\mathcal{L}_{\varepsilon,t}^0 \theta'_0 = 0$ , and (2.1) imply

$$\lambda_1^0 \int_{I_{\varepsilon,t}} \Psi_1^0 \theta'_0 dz = \int_{I_{\varepsilon,t}} (\mathcal{L}_{\varepsilon,t}^0 \Psi_1^0) \theta'_0 dz = \theta''_0 \Psi_1^0 \Big|_{-|t|^{1/2}/(\sqrt{2\varepsilon})}^{1/\varepsilon} \geq -16c_2 e^{-(\sqrt{2}+4)|t|^{1/2}/(\sqrt{2\varepsilon})}.$$

It remains to prove a lower bound for  $\int_{I_{\varepsilon,t}} \Psi_1^0 \theta'_0 dz$ . Since  $\theta'_0 > 0$  it suffices to show that  $\Psi_1^0$  is uniformly bounded from below in  $(-a^*, a^*)$  for some  $a^*$  independent of  $\varepsilon$ . Since  $\|\Psi_1^0\|_{L^2(I_{\varepsilon,t})} = 1$  and  $\Psi_1^0(z) \leq c_2 e^{-\sqrt{2}|z|}$ ,  $|z| \geq a_0 + 1$ , there exists an  $\varepsilon$ -independent number  $a^* > 0$  such that

$$\int_{-a^*}^{a^*} |\Psi_1^0(z)|^2 dz \geq 1/2.$$

The coefficients of  $\mathcal{L}_{\varepsilon,t}^0 - \lambda_1^0$  are uniformly bounded so that an application of Harnack's inequality, cf. , e.g., [GT01], to the identity

$$(\mathcal{L}_{\varepsilon,t}^0 - \lambda_1^0)\Psi_1^0 = 0 \quad \text{in } (-a^* - 1, a^* + 1)$$

implies the existence of a constant  $c_3 > 0$  such that

$$\inf_{z \in (-a^*, a^*)} \Psi_1^0(z) \geq c_3 \sup_{z \in (-a^*, a^*)} \Psi_1^0(z) \geq c_3 \left( \frac{1}{2a^*} \int_{-a^*}^{a^*} (\Psi_1^0)^2 dz \right)^{1/2} \geq c_3 \frac{1}{(4a^*)^{1/2}}.$$

This proves  $\lambda_1^0 \geq -c_4 e^{-(1+2\sqrt{2})|t|^{1/2}/\varepsilon}$  and finishes the proof.  $\square$

### 3. REDUCTION TO THE ONE-DIMENSIONAL SITUATION

Under the assumptions on  $\phi_\varepsilon$  stated in Assumption A, the estimation of  $\lambda_{AC}$  reduces to a one-dimensional problem. For  $\psi \in C^1(B_2)$  we have

$$\int_{B_2} \varepsilon |\nabla \psi|^2 + \varepsilon^{-1} f'(\phi_\varepsilon) \psi^2 dx \geq 2\pi \int_{|t|^{1/2}/\sqrt{2}}^{1+\sqrt{2}|t|^{1/2}} (\varepsilon |\psi_r|^2 + \varepsilon^{-1} f'(\phi_\varepsilon) \psi^2) r dr.$$

The transformation  $z = (r - \sqrt{2}|t|^{1/2})/\varepsilon$  and the rescaling  $\Psi(z) := \varepsilon^{1/2} \psi(r)$  lead to

$$(3.1) \quad \int_{B_2} \varepsilon |\nabla \psi|^2 + \varepsilon^{-1} f'(\phi_\varepsilon) \psi^2 dx \geq \frac{2\pi}{\varepsilon} \int_{I_{\varepsilon,t}} (|\Psi_z|^2 + f'(\tilde{\phi}_\varepsilon) \Psi^2) \tilde{J}(z) dz =: \frac{2\pi}{\varepsilon} L_{\varepsilon,t}(\Psi, \Psi),$$

where  $I_{\varepsilon,t} = (-|t|^{1/2}/(\sqrt{2}\varepsilon), 1/\varepsilon)$ ,  $\tilde{J}(z) = \varepsilon z + \sqrt{2}|t|^{1/2}$ , and  $\tilde{\phi}_\varepsilon(z, t) = \phi_\varepsilon(\varepsilon z + \sqrt{2}|t|^{1/2}) = \theta_0(z) + \varepsilon^2 \tilde{q}_\varepsilon(z, t)$  with  $\tilde{q}_\varepsilon(z, t) = q_\varepsilon(\varepsilon z + \sqrt{2}|t|^{1/2}, t)$ . Since

$$\|\psi\|_{L^2(B_2)}^2 \geq 2\pi \int_{|t|^{1/2}/\sqrt{2}}^{1+\sqrt{2}|t|^{1/2}} \psi^2 r dr = 2\pi \int_{I_{\varepsilon,t}} \Psi^2 \tilde{J}(z) dz$$

Theorem 1.1 follows from the next lemma.

**Lemma 3.1.** *For  $t \in [-T, -\varepsilon^2 \log(\varepsilon^{-1})]$  the principal eigenvalue  $\lambda_1$  of  $L_{\varepsilon,t}$  defined in (3.1) satisfies*

$$-c_8 \varepsilon^2 |t|^{-1} \leq \lambda_1 = \inf_{0 \neq \Psi \in H^1(I_{\varepsilon,t})} \frac{L_{\varepsilon,t}(\Psi, \Psi)}{\|\Psi \tilde{J}^{1/2}\|_{L^2(I_{\varepsilon,t})}^2} \leq c_6 \varepsilon |t|^{-1},$$

where  $c_6, c_8 > 0$  are  $\varepsilon$ -independent constants.

*Proof.* Let  $\Psi \in H^1(I_{\varepsilon,t})$  and define  $\widehat{\Psi} := \tilde{J}^{1/2} \Psi$ . Noting  $\Psi_z \tilde{J}^{1/2} = \widehat{\Psi}_z - \varepsilon \tilde{J}^{-1} \widehat{\Psi}/2$ , where  $\tilde{J}^{-1} := 1/\tilde{J}$ , we deduce

$$\begin{aligned} L_{\varepsilon,t}(\Psi, \Psi) &= \int_{I_{\varepsilon,t}} \widehat{\Psi}_z^2 + f'(\tilde{\phi}_\varepsilon(z, t)) \widehat{\Psi}^2 dz + \frac{\varepsilon^2}{4} \int_{I_{\varepsilon,t}} \tilde{J}^{-2} \widehat{\Psi}^2 dz - \varepsilon \int_{I_{\varepsilon,t}} \tilde{J}^{-1} \widehat{\Psi} \widehat{\Psi}_z dz \\ &= L_{\varepsilon,t}^0(\widehat{\Psi}, \widehat{\Psi}) + \int_{I_{\varepsilon,t}} (f'(\tilde{\phi}_\varepsilon(z, t)) - f'(\theta_0(z))) \widehat{\Psi}^2 dz + \frac{\varepsilon^2}{4} \int_{I_{\varepsilon,t}} \tilde{J}^{-2} \widehat{\Psi}^2 dz - \frac{\varepsilon}{2} \int_{I_{\varepsilon,t}} \tilde{J}^{-1} (\widehat{\Psi}^2)_z dz. \end{aligned}$$

A Taylor expansion of the quadratic function  $f'$  about  $\theta_0$  shows

$$(3.2) \quad f'(\tilde{\phi}_\varepsilon(z, t)) - f'(\theta_0(z)) = \varepsilon^2 f''(\theta_0) \tilde{q}_\varepsilon + f'''(\theta_0) (\varepsilon^2 \tilde{q}_\varepsilon)^2 / 2 =: \varepsilon^2 r_\varepsilon$$

with  $|r_\varepsilon| \leq c_5 |t|^{-1}$  owing to (1.2). An integration by parts and  $(\tilde{J}^{-1})_z = -\tilde{J}^{-2} \varepsilon$  lead to

$$- \int_{I_{\varepsilon,t}} \tilde{J}^{-1} (\widehat{\Psi}^2)_z dz = -\varepsilon \int_{I_{\varepsilon,t}} \tilde{J}^{-2} \widehat{\Psi}^2 dz - \tilde{J}^{-1} \widehat{\Psi}^2 \Big|_{-|t|^{1/2}/(\sqrt{2}\varepsilon)}^{1/\varepsilon}.$$

This implies

$$(3.3) \quad L_{\varepsilon,t}(\Psi, \Psi) = L_{\varepsilon,t}^0(\widehat{\Psi}, \widehat{\Psi}) + \varepsilon^2 \int_{I_{\varepsilon,t}} r_\varepsilon \widehat{\Psi}^2 dz - \frac{\varepsilon^2}{4} \int_{I_{\varepsilon,t}} \tilde{J}^{-2} \widehat{\Psi}^2 dz - \frac{\varepsilon}{2} \tilde{J}^{-1} \widehat{\Psi}^2 \Big|_{-|t|^{1/2}/(\sqrt{2}\varepsilon)}^{1/\varepsilon}.$$

We conclude with (2.1) and (2.2) that

$$\begin{aligned}\lambda_1 &= \inf_{\|\Psi \tilde{J}^{1/2}\|_{L^2(I_{\varepsilon,t})}=1} L_{\varepsilon,t}(\Psi, \Psi) \leq L_{\varepsilon,t}(\beta \tilde{J}^{-1/2} \theta'_0, \beta \tilde{J}^{-1/2} \theta'_0) \\ &\leq \beta^2 L_{\varepsilon,t}^0(\theta'_0, \theta'_0) + c_5 \varepsilon^2 |t|^{-1} + \beta^2 \frac{\varepsilon}{2} (\theta'_0(-\sqrt{2}|t|^{1/2}/(2\varepsilon)))^2 \leq c_6 \varepsilon |t|^{-1},\end{aligned}$$

where we used that  $e^{-2\sqrt{2}|t|^{1/2}/\varepsilon} \leq 1$  for  $\varepsilon$  sufficiently small. Let  $\Psi_1$  be the positive eigenfunction corresponding to  $\lambda_1$  with  $\|\Psi_1 \tilde{J}^{1/2}\|_{L^2(I_{\varepsilon,t})} = 1$  and note that  $\Psi_1$  satisfies

$$-\tilde{J}^{-1} \frac{d}{dz} \left( \tilde{J} \frac{d}{dz} \Psi_1 \right) + f'(\tilde{\phi}_\varepsilon(z, t)) \Psi_1 = \lambda_1 \Psi_1$$

in  $I_{\varepsilon,t}$ . We may assume that  $\lambda_1 \leq m/4$ , and  $f'(\tilde{\phi}_\varepsilon(z, t)) \geq 3m/4$  for  $z \geq a_0$  with an  $\varepsilon$ -independent number  $a_0 > 0$  such that  $a_0 + 1 \leq 1/\varepsilon$ . Let  $a'_0 \in [a_0, a_0 + 1]$  such that  $\Psi_1(a'_0) \tilde{J}^{1/2}(a'_0) \leq 1$ . Employing the comparison function

$$\Phi(z) = \Psi_1(a'_0) \frac{\cosh(c(1/\varepsilon - z))}{\cosh(c(1/\varepsilon - a'_0))},$$

where  $c = \sqrt{m/2}$ , we deduce that, cf. Lemma B.2 for details,

$$\Psi_1(z) \leq c_7 e^{-\sqrt{2}z} \tilde{J}^{-1/2}(a'_0).$$

With  $\hat{\Psi}_1 := \tilde{J}^{1/2} \Psi_1$  we deduce from (3.3) and Lemma 2.1 that

$$\begin{aligned}\lambda_1 &= L_{\varepsilon,t}(\Psi_1, \Psi_1) \geq L_{\varepsilon,t}^0(\hat{\Psi}_1, \hat{\Psi}_1) - c_5 \varepsilon^2 |t|^{-1} - \frac{\varepsilon^2}{4} \sup_{z \in I_{\varepsilon,t}} \tilde{J}^{-2}(z) - \frac{\varepsilon}{2} (\Psi_1(1/\varepsilon))^2 \\ &\geq \lambda_1^0 - c_5 \varepsilon^2 |t|^{-1} - \varepsilon^2 |t|^{-1} - c_7 \varepsilon e^{-2\sqrt{2}/\varepsilon} \tilde{J}^{-1}(a'_0) \geq -c_8 \varepsilon^2 |t|^{-1},\end{aligned}$$

provided that  $\varepsilon$  is sufficiently small so that  $e^{-2\sqrt{2}/\varepsilon} \tilde{J}^{-1}(a'_0) \leq \varepsilon$ .  $\square$

*Remark 3.1.* For  $t \leq -\varepsilon^2 \log(\varepsilon^{-1})^2$  the upper bound  $\lambda_1 \leq c_6 \varepsilon^2 |t|^{-1}$ .

## APPENDIX A. EXPERIMENTAL VERIFICATION OF ASSUMPTION A

For a triangulation  $\mathcal{T}$  of  $B_2$  with mesh-size  $h \sim 2^{-8}$  we approximated the Allen-Cahn problem with a semi-implicit time-stepping scheme with step-size  $\tau = h/10$  for the initial data  $u_0(r) = \tanh((r - \sqrt{2}|t_0|^{1/2})/\varepsilon)$  at  $t_0 = -1/4$ . In Figure 1 we plotted for  $\varepsilon = 2^{-\ell}$ ,  $\ell = 2, 3, 4, 5$  the quantity

$$\eta_\varepsilon(t) := \varepsilon^{-2} \max_{z \in \mathcal{N}} |u_h(z, t) - \tanh((|z| - \sqrt{2}|t|^{1/2})/\varepsilon)|,$$

where  $\mathcal{N}$  denotes the set of nodes in the triangulation  $\mathcal{T}$ . The results show that  $\eta_\varepsilon(t) \leq c|t|^{-1}$  and thus justify Assumption A.

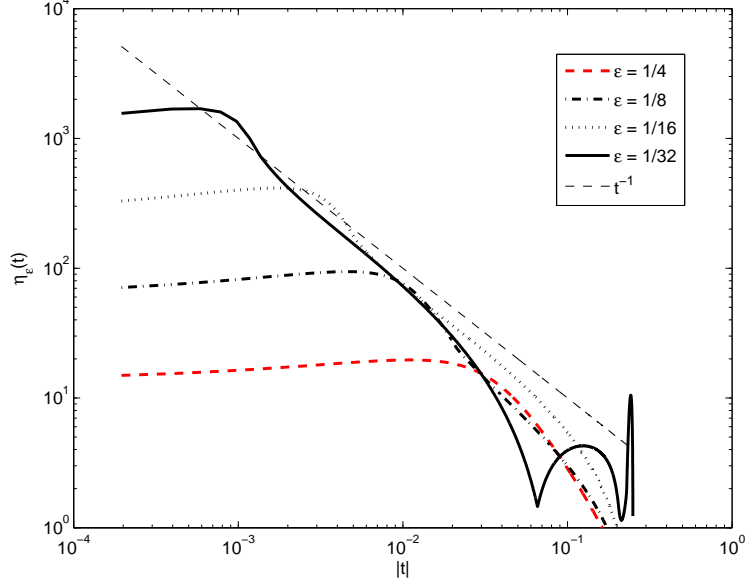


FIGURE 1. Experimental bounds on the second order term in the asymptotic expansion.

## APPENDIX B. COMPARISON PRINCIPLES

**Lemma B.1.** *a) Let  $a < b$ , set  $I := (a, b) \subseteq \mathbb{R}$ , let  $p, q \in C(I)$ , and suppose  $p \geq q \geq 0$ . Assume  $\Psi, \Phi \in C^2(\mathbb{R})$ , satisfy  $\Psi \geq 0$ ,  $\Psi(a) = \Phi(a)$ ,  $\Psi'(b) = \Phi'(b) = 0$ , and*

$$-\Psi'' + p\Psi = 0, \quad -\Phi'' + q\Phi = 0 \quad \text{in } I.$$

*Then  $\Psi \leq \Phi$ . The same conclusion holds if  $\Psi(b) = \Phi(b)$  and  $\Psi'(a) = \Phi'(a) = 0$ .*

*b) Let  $a < b$  set  $I_+ := (a, b)$  and  $I_- := (-b, -a)$ . For  $\psi \in C(\mathbb{R})$  and  $c \geq 0$  the functions  $\Phi_\pm : I_\pm \rightarrow \mathbb{R}$ , defined by*

$$\Phi_\pm : z \mapsto \Psi(\pm a) \frac{\cosh(c(b \mp z))}{\cosh(c(b-a))}$$

*satisfy  $\Phi_\pm(\pm a) = \Psi(\pm a)$ ,  $\Phi'_\pm(\pm b) = 0$ , and  $-\Phi''_\pm + c^2\Phi_\pm = 0$  in  $I_\pm$ . For  $z \in I_\pm$  we have*

$$(B.1) \quad |\Phi_\pm(z)| \leq 2|\Psi(\pm a)|e^{-c|z|}e^{ca}.$$

*Proof.* a) The function  $E := \Psi - \Phi$  satisfies  $E(a) = 0$ ,  $E'(b) = 0$ , and  $-E'' + qE \leq 0$  in  $I$ . Suppose there exist  $a \leq \alpha < \beta \leq b$  such that  $E|_{(\alpha, \beta)} > 0$  and  $E(\alpha) = E(\beta) = 0$ . Then  $E'(\alpha) > 0$  and  $E'(\beta) \leq 0$  contradict  $E'' \geq qE \geq 0$  in  $(\alpha, \beta)$ , i.e., the fact that  $E'$  is monotonically increasing in  $(\alpha, \beta)$ . Hence,  $E \leq 0$ , i.e.,  $\Psi \leq \Phi$ . The second case is analogous. b) The identities follow from  $\cosh'' = \cosh$  and  $\sinh(0) = 0$ . The estimates are consequences of the bounds

$$(B.2) \quad \frac{\cosh(c(b \mp z))}{\cosh(c(b-a))} = \frac{e^{c(b \mp z)} + e^{-c(b \mp z)}}{e^{c(b-a)} + e^{-c(b-a)}} = \frac{e^{\mp cz} e^{cb}}{e^{-ca} e^{cb}} \left( \frac{1 + e^{-2c(b \mp z)}}{1 + e^{-2c(b-a)}} \right) \leq 2e^{\mp cz} e^{ca},$$

where we used  $e^{-2c(b \mp z)} \leq 1$  for  $z < b$  and  $-b < z$ , respectively.  $\square$

**Lemma B.2.** *Let  $a < b$  such that  $I = (a, b) \subseteq I_{\varepsilon,t} := (-|t|^{1/2}/(\sqrt{2\varepsilon}), 1/\varepsilon)$ ,  $p \in C(I)$ , and  $c \geq 0$  such that  $p \geq c^2$ . Let  $\Psi \in C^2(I)$  be non-negative with  $\Psi'(b) = 0$  and*

$$-\tilde{J}^{-1} \frac{d}{dz} \left( \tilde{J} \frac{d}{dz} \right) \Psi + p\Psi = 0$$

*in  $I$ , where  $\tilde{J}^{-1} = 1/\tilde{J}$  with  $\tilde{J}(z) = \varepsilon z + \sqrt{2}|t|^{1/2}$ . Then,  $\Psi(z) \leq 2\Psi(a)e^{-cz}e^{ca}$ .*

*Proof.* Defining

$$\Phi : I \rightarrow \mathbb{R}, \quad z \mapsto \Psi(a) \frac{\cosh(c(b-z))}{\cosh(c(b-a))}.$$

we have  $-\Phi'' + c^2\Phi = 0$ ,  $\Phi(a) = \Psi(a)$ , and  $\Phi'(b) = 0$ . With  $\tilde{J}_z = \varepsilon$  and  $\Phi' \leq 0$ ,  $\tilde{J} > 0$  in  $I$  we deduce

$$-\tilde{J}^{-1} \frac{d}{dz} \left( \tilde{J} \frac{d}{dz} \right) \Phi + c^2\Phi = -\varepsilon\tilde{J}^{-1}\Phi' - \tilde{J}^{-1}\tilde{J}\Phi'' + c^2\Phi = -\varepsilon\tilde{J}^{-1}\Phi' \geq 0.$$

Since  $p \geq c^2$  and  $\Psi \geq 0$  the function  $E := \Psi - \Phi$  satisfies

$$-\tilde{J}^{-1} \frac{d}{dz} \left( \tilde{J} \frac{d}{dz} \right) E + c^2E = \varepsilon\tilde{J}^{-1}\Phi' \leq 0$$

and  $E(a) = 0$ ,  $E'(b) = 0$ . Suppose that  $(\alpha, \beta) \subseteq I$  is maximal with  $E|_{(\alpha,\beta)} > 0$ . Then, since  $\tilde{J} > 0$  we have  $\tilde{J}(\alpha)E'(\alpha) > 0$  and  $\tilde{J}(\beta)E'(\beta) \leq 0$ . This contradicts

$$\frac{d}{dz}(\tilde{J}E') = \frac{d}{dz} \left( \tilde{J} \frac{d}{dz} \right) E \geq \tilde{J}c^2E \geq 0$$

and shows that  $E \leq 0$ , i.e.,  $\Psi \leq \Phi$ . The proof of the estimate follows from (B.2).  $\square$

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