Improved Lp-Estimates for the Strain Velocities in Hardening Problems

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Abstract

Problems of elastic plastic deformation with kinematic or isotropic hardening and von Mises flowrule are considered. It is shown that the velocities of the stresses, strains and hardening parameters satisfy an improved L^p-property that is $\dot{\sigma}, \dot{\xi}, \nabla \dot{u} \in L^\infty(0,T;L^{2+2\delta}(\Omega))$ with some $\delta > 0$. For dimension $n = 2$ this implies continuity of $\dot{u}$ in spatial direction, furthermore it can be used as tool to prove boundary differentialbility $\sigma, \xi \in L^\infty(0,T;L^{1+\varepsilon})$ and $\sigma, \xi \in L^\infty(0,T;\mathcal{N}^{1/2+\delta',2})$, where $1/2+\delta'$ is the order of fractional Nichol'skii differentiability.

Keywords: elastic plastic deformation, isotropic & kinematic hardening, strain velocity
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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded and connected subset with Lipschitz boundary $\partial\Omega = \Gamma_D \dot{\cup} \Gamma_N$. We consider

$\sigma : \Omega \times [0,T] \to \mathbb{R}^{n\times n}_{\text{sym}}$ stress field
$\xi : \Omega \times [0,T] \to \mathbb{R}^m$ hardening parameter
$u : \Omega \times [0,T] \to \mathbb{R}^n$ displacement field

and $A \in \mathbb{R}^{n\times n\times n\times n}, H \in \mathbb{R}^{m\times m}$ denotes the elastic compliance respective hardening tensor. We furthermore assume A, H to be uniform elliptic, i.e.

$$\exists\, \alpha_A, \alpha_H > 0 : \; Am : m \geq \alpha_A |m|^2, Hv \cdot v \geq \alpha_H |v|^2 \tag{1.1}$$

for all vectors $m \in \mathbb{R}^{n\times n}, v \in \mathbb{R}^m$. We write $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\intercal)$ for the linearized strain tensor. The deviator $M_D = M - \frac{1}{n}\operatorname{tr}(M) Id$ of a matrix M, is the projection onto the subspace of matrices with zero trace. The variational inequality of elasto plasticity with isotropic or kinematic hardening and von Mises flowrule is

$$\int_\Omega (A\dot{\sigma} - \varepsilon(\dot{u})) : (\tau - \sigma) + H\dot{\xi} \cdot (\eta - \xi)\,\mathrm{d}x \geq 0\; \forall (\tau,\eta) : \mathscr{F}(\tau,\eta) \leq 0 \tag{1.2a}$$

$$\int_\Omega \sigma : \nabla w\,\mathrm{d}x = \int_\Omega f \cdot w\,\mathrm{d}x + \int_{\Gamma_N} p \cdot w\,\mathrm{d}\Gamma\; \forall w \in H^1_{\Gamma_D} \tag{1.2b}$$

$$\mathscr{F}(\sigma,\xi) = |\sigma_D| - (\xi + \kappa) \leq 0 \quad \text{isotropic case}, \xi \in \mathbb{R} \tag{1.2c}$$

$$\mathscr{F}(\sigma,\xi) = |\sigma_D - \xi_D| + \kappa \leq 0 \quad \text{kinematic case}, \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \tag{1.2d}$$

By $\kappa > 0$ we denote the yield limit. We may assume that the boundary integral $\int_{\Gamma_N} p \cdot \varphi\,\mathrm{d}\Gamma$ can be represented as

$$\int_{\Gamma_N} p \cdot \varphi\,\mathrm{d}\Gamma = \int_\Omega f_0 \cdot \nabla\varphi\,\mathrm{d}x \text{ for all } \varphi \in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n).$$

We assume (for simplicity)

$$f, \dot{f}, f_0, \dot{f}_0 \in L^\infty(0,T;L^\infty). \tag{1.3}$$

The standard existence theory to the hardening problem (1.2) leads to solutions $\sigma, \dot{\sigma}, \xi, \dot{\xi} \in L^\infty(0,T;L^2)$, $\varepsilon(\dot{u}) \in L^\infty(0,T;C^*(\overline{\Omega}))$.
We suppose the usual **safe load** condition (cf. Johnson [Joh78]). There exists an element $(\sigma^0,\xi^0) \in W^{1,\infty}\big(0,T;L^\infty(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times L^\infty(\Omega,\mathbb{R}^m)\big)$ and $\delta_0 > 0$ such that

$$\left\{\begin{array}{l} \mathscr{F}(\sigma^0,\xi^0) \leq -\delta_0 < 0 \\ \langle\sigma^0,\nabla w\rangle = \langle f,w\rangle + \int_{\Gamma_N} gw\,\mathrm{d}\Gamma \quad \forall w \in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n) \text{ a.e. with respect to } t \\ (\sigma^0,\xi^0)(0) = 0 \text{ in } \overline{\Omega} \times \{t=0\} \end{array}\right\} \tag{1.4}$$

and in addition

$$\ddot{\sigma}^0 \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))\,. \tag{1.5}$$

We have we following known regularity properties:

Johnson [Joh78]	$\nabla\dot{u} \in L^\infty(0,T;L^2)$	
Seregin [Ser94]	$\sigma,\xi \in L^\infty(0,T;H^1_{loc})$	
	$\nabla\varepsilon(u) \in L^\infty(0,T;C^*_{loc}(\Omega))$	isotropic hardening
Alber & Nessenenko	$\sigma,\xi \in L^\infty(0,T;H^{\frac{1}{3}-\delta})$	kinematic hardening
Knees [Kne08]	$\sigma,\xi \in L^\infty(0,T;H^{\frac{1}{2}-\delta})$	kinematic hardening
Frehse & Löbach [FL09]	$\sigma,\xi \in L^\infty(0,T;H^{\frac{1}{2}-\delta}$	isotropic & kinematic
	$\sigma,\xi \in L^\infty(0,T;W^{1,1+\varepsilon})$	hardening
Löbach [Löb09]	$\sigma,\xi \in L^\infty(0,T;H^{\frac{1}{2}+\delta})$	isotropic & kinematic hardening

As one can see there is no improvement yet concerning the displacement velcioties $\dot{u}$. The purpose of the present paper is to present a slight L^p-improvement for $\nabla\dot{u}$ as well as for $\dot{\sigma}, \dot{\xi}$,

$$\nabla\dot{u} \in L^\infty(0,T;L^{2+2\delta}(\Omega)) \tag{1.6}$$

for some $\delta > 0$. The proof is based on a generalization of a Lemma of Gehring due to Giaquinta & Modica, establishing a reverse Hölder inequality. In case of two spatial dimensions, this implies continuity of $\dot{u}$ in spatial direction: $\dot{u} \in L^\infty(0,T;C^{2\delta}(\overline{\Omega}))$.
Futhermore the improved L^p-inclusion (1.6) implies an anisotropic Morrey condition, see (2.24) below, as it was shown in [FL09, Löb09]. This anisotropic condition yields

$$\nabla\sigma, \nabla\xi \in L^\infty(0,T;L^{1+\varepsilon}(\Omega)) \tag{1.7}$$

with some constant $\varepsilon = \varepsilon(\delta) > 0$ and it further yields fractional Nichol'skii-Space integrability of order $\alpha = \frac{1}{2} + \tilde{\delta}(\delta) > \frac{1}{2}$ for the stresses and hardening parameters at the boundary. Thus, the result of the present paper is also an alternative way to prove the anisotropic Morrey condition (2.24).
Furthermore, we present a slight improvement of (1.7) in theorem 2.4.
For Hencky plasticity the technique relying on reverse Hölder inequality has already been used by Hardt & Kinderlehrer [HK83] to obtain $u \in L^{\frac{n}{n-1}+\delta}$, $n \geq 2$.

2 Reverse Hölder inequality

We consider the following viscoplastic approximation to our initial hardening problem (1.2)

$$\varepsilon(\dot{u}_\mu) = A\dot{\sigma}_\mu + \frac{1}{\mu}\big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle\frac{\sigma_{\mu D}}{|\sigma_{\mu D}|} \tag{2.1a}$$

$$0 = H\dot{\xi}_\mu - \frac{1}{\mu}\big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle, \tag{2.1b}$$

where $\langle a\rangle = \max(a,0)$ denotes the Macauley bracket. The equation (2.1b) of the internal parameter ξ can be generalized to

$$0 = H\dot{\xi}_\mu - \frac{1}{\mu}\big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle + g_0(\sigma_\mu, \xi_\mu), \tag{2.2}$$

where g_0 is globally Lipschitz and monotone (in the sense of Minty) and all the results of the present paper also hold in this case.
One can prove that solutions $\big((\sigma_\mu, \xi_\mu), u_\mu\big)$ of (2.1a)&(2.1b) exist and that $\big((\sigma_\mu, \xi_\mu), u_\mu\big) \to \big((\sigma, \xi), u\big)$ as $\mu \to 0$, where $\big((\sigma, \xi), u\big)$ is the solution of the limiting problem (1.2). Furthermore one can show (for subsequences)

$$\frac{1}{\mu}\big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle \rightharpoonup \dot{\lambda} \qquad \text{weakly in } L^2$$

$$\frac{1}{\mu}\big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle^2 \to 0 \qquad \text{a.e. in } [0,T]\times\Omega,$$

as $\mu \to 0$. Here $\dot{\lambda} \geq 0$, $\dot{\lambda} \in L^2$ is the plastic multiplier giving the magnitude of the vector of plastic flow (for the details see Frehse & Löbach[FL08]).
For the boundary estimate we work with a mild condition for the boundaries Γ_D and Γ_N .

Neumann boundary:
For $x_0 \in U(\partial\Omega \setminus \Gamma_D)$ we assume

$$\operatorname{mes}\left(\Omega \cap T_R(x_0)\right) \geq aR^n,\ R \leq R_0 \tag{2.3}$$

with some $a > 0$, $T_R := B_{2R} \setminus B_R$, where $U(\partial\Omega \setminus \Gamma_D)$ is a neighbourhood of $\partial\Omega \setminus \Gamma_D$.

Dirichlet boundary:
For each $x_0 \in \Gamma_D$, there holds a "*Wiener type condition*" for the relative capacities (for the definition see for example [Fre82]).

$$\operatorname{cap}(T_R \cap \Gamma_D; B_{2R}) \geq c_0 R^{n-2}, \quad R \leq R_0 \tag{2.4}$$

For Lipschitz boundary the conditions (2.3) and (2.4) are satisfied. In the case of mixed boundary conditions, we simply assume, in addition, that the set which separates the Dirichlet and Neumann boundary is Lipschitz ($n \geq 3$) and that $\operatorname{cap}(\Gamma_D) > 0$.

Theorem 2.1 *Let $n \geq 2$, assume the safe load condition* (1.4) *&* (1.5)*, the ellipticity condition* (1.1) *and* (1.3)*. Then there exists a number $\delta > 0$ such that for the limiting problem* (1.2)*, i.e. $\mu = 0$, the inclusion*

$$\nabla\dot{u}, \dot{\sigma}, \dot{\xi}, \dot{\lambda} \in L^\infty(0,T; L^{2+2\delta}_{loc}(\Omega)) \tag{2.5}$$

holds. We have

$$\nabla\dot{u}, \dot{\sigma}, \dot{\xi}, \dot{\lambda} \in L^\infty(0,T; L^{2+2\delta}(\Omega))$$

if the additional assumptions (2.3) *&* (2.4) *at the boundary hold.*

Remark: The constant δ depends on the dimension n, on $\|H\|_\infty$ and on the values Λ, λ, where Λ ist largest eigenvalue and λ the minimal eigenvalue of the quadratic form associated to A.

The proof is based on a reverse Hölder inequality which is established in the following Lemma.

Lemma 2.2 *There exists a constant K such that for all balls $B_R = B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$, $R \leq R_0$ the limiting functions $(\sigma, \xi), u$ for $\mu = 0$ satisfy*

$$\frac{1}{R^n}\int_{B_R}\left(|\dot{\sigma}|^2 + |\dot{\xi}|^2 + |\nabla\dot{u}|^2\right)\mathrm{d}x \leq K\left(\frac{1}{R^n}\int_{B_{2R}}|\nabla\dot{u}|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}} + \frac{K}{R^n}\int_{B_{2R}}|\dot{f}|^2 + |\dot{f}_0|^2\mathrm{d}x. \tag{2.6}$$

proof (i) Let $\zeta = \zeta(x)$ be the usual Lipschitz continuous localization function, with $\operatorname{supp}\zeta \subset B_{2R} \subset \Omega$, $\zeta \equiv 1$ on B_R and $|\nabla\zeta| \leq R^{-1}$. In equations (2.1a) & (2.1b) we choose the pair of test functions $\zeta^2(\dot{\sigma}_\mu, \dot{\xi}_\mu)^\intercal$ and obtain a.e. with respect to t

$$\int_\Omega \zeta^2 \varepsilon(\dot{u}_\mu) : \dot{\sigma}_\mu \, \mathrm{d}x = \int_\Omega \zeta^2 \left[A\dot{\sigma}_\mu : \dot{\sigma}_\mu + H\dot{\xi}_\mu \cdot \dot{\xi}_\mu\right] \mathrm{d}x + \frac{1}{2\mu}\frac{d}{dt}\int_\Omega \big\langle |\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle^2 \, \mathrm{d}x. \quad (2.7)$$

We rewrite (2.7) using partial integration and the balance of forces (1.2b) and obtain in the case $f_0 = 0$

$$\begin{aligned}\int_\Omega \zeta^2 \varepsilon(\dot{u}_\mu) : \dot{\sigma}_\mu \, \mathrm{d}x &= -\int_\Omega \zeta^2 (\dot{u}_\mu - \overline{u}_{\mu R}) \cdot \dot{f} \, \mathrm{d}x - 2\int_\Omega (\dot{u}_\mu - \overline{u}_{\mu R}) \cdot \dot{\sigma}_\mu \zeta \nabla\zeta \, \mathrm{d}x \\ &\leq KR^2 \int_{B_{2R}} |\dot{f}|^2 \, \mathrm{d}x + \frac{1}{2}\lambda \int_{B_{2R}} |\zeta\dot{\sigma}_\mu|^2 \, \mathrm{d}x \\ &+ \frac{K}{R^2}\int_{B_{2R}} |\dot{u}_\mu - \overline{u}_{\mu R}|^2 \, \mathrm{d}x.\end{aligned} \quad (2.8)$$

Here $\overline{u}_{\mu R} = ⨍_{B_{2R}} \dot{u}_\mu \, \mathrm{d}x$, and in case that $B_{2R} \cap \Gamma_D \neq \varnothing$ we choose $\overline{u}_{\mu R} = 0$.
For simplicity we set $f_0 = 0$. In the case of interior estimates one needs only to replace f by $f + \operatorname{div} f_0$ provided $\operatorname{div} \dot{f}_0 \in L^\infty(0,T;L^{2+2\delta_0})$, $\delta_0 > 0$. This would require more regularity for f_0. Alternatively. we may estimate the term arising from f_0 by

$$\begin{aligned}|(\dot{f}_0, \nabla((\dot{u} - \overline{u}_{\mu R})\zeta^2))| \leq K \int_{B_{2R}} |\dot{f}_0|^2 \, \mathrm{d}x + \varepsilon_0 \int_{B_{2R}} |\nabla\dot{u}|^2 \zeta^2 \, \mathrm{d}x \\ + K\int_{B_{2R}} |\dot{u} - \overline{u}_{\mu R}|^2 |\nabla\zeta|^2 \, \mathrm{d}x\end{aligned} \quad (2.9)$$

and the term $\varepsilon_0 \int |\nabla\dot{u}|^2 \zeta^2 \, \mathrm{d}x$ is absorbed choosing ε_0 small enough, while the third term in (2.9) already occurs.
Thus we obtain

$$\begin{aligned}\int_{B_{2R}} \left[\frac{\lambda}{2}|\dot{\sigma}_\mu|^2 + H|\dot{\xi}_\mu|^2\right] \zeta^2 \, \mathrm{d}x + \frac{1}{2\mu}\frac{d}{dt}\int_\Omega \big\langle |\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle^2 \zeta^2 \, \mathrm{d}x \\ \leq KR^2 \int_{B_{2R}} |\dot{f}|^2 + |\dot{f}_0|^2 \, \mathrm{d}x + \frac{K}{R^2}\int_{B_{2R}} |\dot{u}_\mu - \overline{u}_{\mu R}|^2 \, \mathrm{d}x.\end{aligned} \quad (2.10)$$

(ii) We express $\varepsilon(\dot{u}_\mu)$ by $\dot{\sigma}_\mu, \dot{\xi}_\mu$ eliminating the penalty term. From (2.1a) & (2.1b) we obtain pointwise a.e.

$$|\varepsilon(\dot{u}_\mu)| \leq |A\dot{\sigma}_\mu| + |H\dot{\xi}_\mu|$$

and

$$|\varepsilon(\dot{u}_\mu)|^2 \leq 2\Lambda^2 |\dot{\sigma}_\mu|^2 + 2H^2 |\dot{\xi}_\mu|^2. \quad (2.11)$$

We use Korns inequality for the function $(\dot{u}_\mu - \overline{u}_{\mu R})\zeta$ and obtain

$$\int_\Omega \left|\nabla\left[(\dot{u}_\mu - \overline{u}_{\mu R})\zeta\right]\right|^2 \mathrm{d}x \leq K\int_\Omega |\varepsilon\big((\dot{u}_\mu - \overline{u}_{\mu R})\zeta\big)^2|^2 \mathrm{d}x \tag{2.12}$$

and hence

$$\int_{B_{2R}} |\nabla \dot{u}_\mu|^2\zeta^2 \mathrm{d}x \leq K\int_{B_{2R}} |\varepsilon(\dot{u}_\mu)|^2\zeta^2 \mathrm{d}x + K\int_{B_{2R}} |\nabla\zeta|^2|\dot{u}_\mu - \overline{u}_{\mu R}|^2 \mathrm{d}x. \tag{2.13}$$

(cf. [FL08] for similar simple calculations). We insert (2.11) into (2.13), multiply with $\theta > 0$

$$\theta\int_{B_{2R}}|\nabla\dot{u}_\mu|^2\zeta^2 \mathrm{d}x \leq \theta\int_{B_{2R}}(2\Lambda|\dot{\sigma}_\mu|^2 + 2H^2|\dot{\xi}_\mu|^2)\zeta^2 \mathrm{d}x + \frac{\theta K}{R^2}\int_{B_{2R}}|\dot{u}_\mu - \overline{u}_{\mu R}|^2 \mathrm{d}x. \tag{2.14}$$

Now add the resulting equation(2.14) to (2.10). This yields

$$\begin{aligned}\int_\Omega \left[\frac{\lambda}{2}|\dot{\sigma}_\mu|^2 + H|\dot{\xi}_\mu|^2 + \frac{1}{2\mu}\frac{d}{dt}\big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle^2\right]\zeta^2 \mathrm{d}x + \theta\int_{B_{2R}}|\nabla\dot{u}_\mu|^2\zeta^2 \mathrm{d}x \\ \leq KR^2\int_{B_{2R}}|\dot{f}|^2 + |\dot{f}_0|^2 \mathrm{d}x + K\frac{1}{R^2}\int_{B_{2R}}|\dot{u}_\mu - \overline{u}_{\mu R}|^2 \mathrm{d}x \\ + 2K\theta\int_{B_{2R}}\left[\Lambda^2|\dot{\sigma}_\mu|^2 + H^2|\dot{\xi}_\mu|^2\right] \mathrm{d}x.\end{aligned} \tag{2.15}$$

Choosing $\theta = \theta(\Lambda, H) > 0$ small enough, we obtain

$$\begin{aligned}\int_\Omega \left[\frac{\lambda}{4}|\dot{\sigma}_\mu|^2 + \frac{1}{2}H|\dot{\xi}_\mu|^2\right]\zeta^2 \mathrm{d}x + \theta\int_{B_{2R}}|\nabla\dot{u}_\mu|^2\zeta^2 \mathrm{d}x + \frac{d}{dt}\frac{1}{2\mu}\int_\Omega\big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle^2\zeta^2 \mathrm{d}x \\ \leq KR^2\int_{B_{2R}}|\dot{f}|^2 + |\dot{f}_0|^2 \mathrm{d}x + \frac{K}{R^2}\int_{B_{2R}}|\dot{u}_\mu - \overline{u}_{\mu R}|^2 \mathrm{d}x.\end{aligned} \tag{2.16}$$

We integrate inequality (2.16) in time from t_1 to t_2, the integrated penalty term reads

$$\frac{1}{2\mu}\int_\Omega \big\langle|\sigma_{\mu D}| - (\xi_\mu + \kappa)\big\rangle^2 \mathrm{d}x\Big|_{t_1}^{t_2} \tag{2.17}$$

however this quantity tends to 0 a.e. as $\mu \to 0$ for a subsequence.

The terms on the left hand-side of (2.16) are treated via lower semicontinuity with respect to weak convergence $\nabla\dot{u}_\mu \rightharpoonup \nabla\dot{u}$, $(\dot{\sigma}_\mu, \dot{\xi}_\mu) \rightharpoonup (\dot{\sigma}, \dot{\xi})$ for subsequences.
The term $\int|\dot{u}_\mu - \overline{u}_{\mu R}|^2 \mathrm{d}x$ converges due to Lemma 3.1 to the limit term $\int|\dot{u} - \overline{u}_R|^2 \mathrm{d}x$. Thus we obtain a.e. with respect to t_1, t_2

$$c_0\int_{t_1}^{t_2}\int_\Omega\Big(|\dot{\sigma}|^2 + |\dot{\xi}|^2 + |\nabla\dot{u}|^2\Big)\zeta^2 \mathrm{d}x \leq \frac{K}{R^2}\int_{t_1}^{t_2}\int_{B_{2R}}|\dot{u} - \overline{u}_R|^2 \mathrm{d}x + KR^2\int_{t_1}^{t_2}\int_{B_{2R}}|\dot{f}|^2 + |\dot{f}_0|^2 \mathrm{d}x. \tag{2.18}$$

We divide by $t_2 - t_1 > 0$ and pass to the limit $t_1 \to t_2 = t$. This yields, a.e. with respect to t,

$$c \int_{B_{2R}} \left(|\dot{\sigma}|^2 + |\dot{\xi}|^2 + |\nabla \dot{u}|^2 \right) \zeta^2 \,\mathrm{d}x \leq KR^2 \int_{B_{2R}} |\dot{f}|^2 + |\dot{f}_0|^2 \,\mathrm{d}x + \frac{K}{R^2} \int_{B_{2R}} |\dot{u} - \overline{\dot{u}}_R|^2 \,\mathrm{d}x. \tag{2.19}$$

We emphasize, that (2.19) has been proven only for the limit $\mu = 0$. We estimate via Sobolevs inequality

$$\frac{1}{R^2} \int_{B_{2R}} |\dot{u} - \overline{\dot{u}}_R|^2 \,\mathrm{d}x \leq KR^n \left(R^{-n} \int_{B_{2R}} |\nabla \dot{u}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}}. \tag{2.20}$$

Inserting this into (2.19) we have shown a reverse Hölder inequality for the function

$$\Upsilon = \sqrt{|\dot{\sigma}|^2 + |\dot{\xi}|^2 + |\nabla \dot{u}|^2}.$$

From (2.20) we arrive at an inequality for the local maximal function of Υ

$$M_{R_0}(\Upsilon)(x) = \sup_{R<R_0} \sup \fint_{B_R(x)} |\Upsilon| dy \tag{2.21}$$

which reads

$$M_{\frac{1}{2}d(x)}(|\Upsilon|^2)(x) \leq KM_{d(x)}(|\Upsilon|^{\frac{n+2}{n}})(x) + KM_{d(x)}(|\dot{f}|^2 + |\dot{f}_0|^2)(x), \tag{2.22}$$

where $d(x) := \mathrm{dist}(x, \partial\Omega)$.□

The local version of theorem 2.1 now follows from a generalization of Gehring's Lemma due to Giaquinta & Modica(see e.g. [Gia83] Chapter V, Theorem 1.2). The statement up to the boundary is very similar, however, if $B_{2R} \cap \Gamma_D \neq \varnothing$ in the case of Dirichlet boundary, or mixed boundary conditions, one has to replace the constant $c = \overline{\dot{u}_R}$ by 0. Furthermore one proceeds as in the paper [FL08], but one does not use Poincaré's inequality to estimate

$$\int_{B_{2R}} |\dot{u} - c|^2 \,\mathrm{d}x \leq \int_{B_{m \cdot R}} |\dot{u} - c|^2 \,\mathrm{d}x \leq KR^2 \int_{B_{m \cdot R}} |\nabla \dot{u}|^2 \,\mathrm{d}x$$

but

$$\leq K \left(\int_{B_{2R}} |\nabla \dot{u}|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}}$$

by the Sobolev-Poincaré inequality.□

Corallary 2.3 *Let $n = 2$, then*

$$\dot{u} \in L^\infty(0, T; C^{2\delta}(\overline{\Omega})). \tag{2.23}$$

This follows immediately from the Sobolev inequalities.

Remark: It is an open problem to find a condition which implies continuity of $\dot{u}$ in t-direction.

Another application of theorem 2.1 yields an alternative approach to refined regularity properties of σ, ξ near the boundary:
The improved L^p-property of $\nabla \dot{u}$ implies, via Hölder's, inequality an anisotropic Morrey condition, which reads near the boundary

$$\int_{B'_r} \int_{\varphi(x')}^{\varphi(x')+r} |\dot{\sigma}|^2 + |\dot{\xi}|^2 + |\nabla \dot{u}|^2 \,\mathrm{d}x' \,\mathrm{d}x_n \leq K r_0^{\delta}, \quad 0 < r \leq r_0. \tag{2.24}$$

Here $x' = (x_1, \dots, x_{n-1})$, $B'_r = \{y' \in \mathbb{R}^{n-1} \big| |y' - x'_0| \leq r\}$ where we have performed an orthogonal transformation such that the boundary $\partial\Omega$ can be localy represented by the set

$$\partial\Omega \cap B_{R_0}(x_0) = \{(x', \varphi(x') + \beta) \in \mathbb{R}^n \big| x' \in B'_{R_0}\},\ x_0 \in \partial\Omega.$$

This means that the boundary can be represented localy as the graph of a Lipschitz continuous function φ. As it was shown in [FL09, Löb09] (2.24) is an important tool to prove the regularity property

$$D_n\sigma, D_n\xi \in L^\infty(0,T; L^{1+\varepsilon})$$

and

$$\sigma, \xi \in L^\infty(0,T; H^{\frac{1}{2}+\delta})$$

of the normal derivatives of the stresses and hardening parameters up to the boundary. In [FL09, Löb09] the authours proved (2.24) with the plate filling technique, the approach via the improved L^p-property is another possibility. We can also derive the tube filling inequality form [FL08], which leads to C^α-estimates for u in the case of two spatial dimensions. From [FL09, Löb09], we know that the anisotropic Morrey condition (2.24) implies the weighted estimates

$$\int_{B'_{R_0}} \int_{\varphi(x')}^{\varphi(x')+r} \left(|D_n\sigma|^2 + |D_n\xi|^2\right) x_n^{1-\delta'} \,\mathrm{d}x \leq K \tag{2.25}$$

with $\delta' = \delta'(\delta) > 0$ and

$$\int_{B'_{r_0}} \int_{\varphi(x')}^{\varphi(x')+r} \left(|\sigma|^2 + |\xi|^2\right) x_n^{\delta_1 - 1} \,\mathrm{d}x \leq K \tag{2.26}$$

for all $\delta_1 > 0$.
We observe that (2.25),(2.26) and theorem 2.1 imply

Theorem 2.4 *Under the assumptions of theorem 2.1 there holds*

$$\int_{B'_{r_0}} \int_{\varphi(x')}^{\varphi(x')+r} \big(|D_n\sigma| + |D_n\xi|\big)^{1+\varepsilon'} (1+|\sigma|+|\xi|)^{1+\varepsilon'} \,\mathrm{d}x_n\,\mathrm{d}x' \le K$$

with some $\varepsilon' = \varepsilon'(\delta_0) > 0$.

Remark: Compared to (1.7) Theorem 2.4 allows the additional weight $\big(1+|\sigma|+|\xi|\big)^{1+\varepsilon'}$ in the estimate for $\nabla\sigma, \nabla\xi$. We have $\varepsilon' < \varepsilon$.

3 Strong convergence

The final section is devoted to prove the strong convergence of $\dot\sigma_\mu, \dot\xi_\mu$.

In [Löb08] it was shown, that for solutions $\sigma_\mu, \xi_\mu, \nabla u_\mu$ of the viscoplastic approximate problem (2.1) holds

$$\sigma_\mu \to \sigma,\ \xi_\mu \to \xi \text{ strongly in } L^2(0,T;L^2) \tag{3.1a}$$

$$\dot\sigma_\mu \rightharpoonup \dot\sigma,\ \dot\xi_\mu \rightharpoonup \dot\xi \tag{3.1b}$$

$$\nabla \dot u_\mu \rightharpoonup \nabla \dot u \text{ weakly in } L^2(0,T;L^2) \tag{3.1c}$$

$$\frac{1}{\mu}\int_0^T\int_\Omega \big\langle |\sigma_{\mu D}| - (\xi_\mu+\kappa)\big\rangle^2 \,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{3.1d}$$

as $\mu \to 0$. The above functions are bounded uniformly in $L^\infty(0,T;L^2)$ as the viscosity parameter μ tends to zero.
From (3.1d) we derive

$$\begin{aligned} T_1 := \frac{1}{\mu}\int_{t_1}^{t_2}\int_\Omega \big\langle |\sigma_{\mu D}| - (\xi_\mu+\kappa)\big\rangle \left(\frac{\sigma_{\mu D}}{|\sigma_{\mu D}|} : \dot\sigma_\mu - \dot\xi_\mu\right) \mathrm{d}x\,\mathrm{d}t \\ = \frac{1}{2\mu}\int_\Omega \big\langle |\sigma_{\mu D}| - (\xi_\mu+\kappa)\big\rangle^2 \,\mathrm{d}x\Big|_{t_1}^{t_2} \to 0 \end{aligned} \tag{3.2}$$

a.e. with respect to t_1, t_2. In [FL08] it was proven that

$$\frac{1}{\mu}\big\langle |\sigma_{\mu D}| - (\xi_\mu+\kappa)\big\rangle \frac{\sigma_{\mu D}}{|\sigma_{\mu D}|} \rightharpoonup \dot\lambda_{isotropic}\frac{\sigma_D}{|\sigma_D|}$$

and $\dot\lambda = 0$ if $|\sigma_D| < \xi + \kappa$. In the kinematic case a similar reasoning leads to

$$\frac{1}{\mu}\big\langle |\sigma_{\mu D} - \xi_{\mu D}| - \kappa\big\rangle \frac{\sigma_{\mu D} - \xi_{\mu D}}{|\sigma_{\mu D} - \xi_{\mu D}|} \rightharpoonup \dot\lambda_{kinematic}\frac{\sigma_D - \xi_D}{|\sigma_D - \xi_D|}.$$

Lemma 3.1 *Under the assumption of the safe load condition* (1.4) & (1.5)*, the ellipticity condition* (1.1) *and* (1.3) *we have the strong convergence* $\dot\sigma_\mu \to \dot\sigma$ *and* $\dot\xi_\mu \to \dot\xi$ *in* $L^2(0,T;L^2)$.

proof We confine ourselves to the study of the isotropic case. The kinematic case can be treated in a similar manner. We test the penalized equation (2.1) with $(\dot{\sigma}_\mu - \dot{\sigma}^0, \dot{\xi}_\mu - \dot{\xi}^0)$, where (σ^0, ξ^0) satisfies the safe load condition (1.4). The term $(\varepsilon(\dot{u}_\mu), \dot{\sigma}_\mu - \dot{\sigma}^0)$ vanishes and we obtain

$$\begin{aligned} 0 = &\int_{t_1}^{t_2}\int_\Omega A\dot{\sigma}_\mu : (\dot{\sigma}_\mu - \dot{\sigma}^0) + H\dot{\xi}_\mu \cdot (\dot{\xi}_\mu - \dot{\xi}^0)\,\mathrm{d}x\,\mathrm{d}t \\ &+ T_1 - \frac{1}{\mu}\int_{t_1}^{t_2}\int_\Omega \langle |\sigma_{\mu D}| - (\xi_\mu + \kappa)\rangle \left(\frac{\sigma_{\mu D}}{|\sigma_{\mu D}|} : \dot{\sigma}^0 - \dot{\xi}^0\right)\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.3}$$

Passing to the limit $\mu = 0$ and using (3.2) we obtain

$$\begin{aligned} &\lim_{\mu\to 0}\int_{t_1}^{t_2}\int_\Omega A\dot{\sigma}_\mu : \dot{\sigma}_\mu + H\dot{\xi}_\mu \cdot \dot{\xi}_\mu\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\mu\to 0}\left(\int_{t_1}^{t_2}\int_\Omega A\dot{\sigma}_\mu : \dot{\sigma}^0 + H\dot{\xi}_\mu \cdot \dot{\xi}^0 - T_1 + \frac{1}{\mu}\int_{t_1}^{t_2}\int_\Omega \langle |\sigma_{\mu D}| - (\xi_\mu + \kappa)\rangle\Big(\frac{\sigma_{\mu D}}{|\sigma_{\mu D}|} : \dot{\sigma}^0 - \dot{\xi}^0\Big)\right)\mathrm{d}x\,\mathrm{d}t \\ &= \underbrace{\int_{t_1}^{t_2}\int_\Omega A\dot{\sigma} : \dot{\sigma}^0 + H\dot{\xi} \cdot \dot{\xi}^0 + \dot{\lambda}\left(\frac{\sigma_D}{|\sigma_D|} : \dot{\sigma}^0 - \dot{\xi}^0\right)\mathrm{d}x\,\mathrm{d}t.}_{=:T_0} \end{aligned} \tag{3.4}$$

We now test the limiting equation

$$\begin{aligned} \varepsilon(\dot{u}) &= A\dot{\sigma} + \dot{\lambda}\frac{\sigma_D}{|\sigma_D|} \\ 0 &= H\dot{\xi} - \dot{\lambda} \end{aligned} \tag{3.5}$$

with $\left(\dot{\sigma} - \dot{\sigma}^0, \dot{\xi} - \dot{\xi}^0\right)$. In the general case we have $0 = H\dot{\xi} + g_0(\sigma, \xi)$ in equation (3.5), the additional term g_0 is not difficult to treat.

Again, we have $0 = \left(\varepsilon(\dot{u}), \dot{\sigma} - \dot{\sigma}^0\right)$ and obtain

$$\int_{t_1}^{t_2}\int_\Omega A\dot{\sigma} : \dot{\sigma} + H\dot{\xi}\cdot\dot{\xi} + \dot{\lambda}\big(\frac{\sigma_D}{|\sigma_D|} : \dot{\sigma} - \dot{\xi}\big)\,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega A\dot{\sigma} : \dot{\sigma}^0 + H\dot{\xi}\cdot\dot{\xi}^0 + \dot{\lambda}\big(\frac{\sigma_D}{|\sigma_D|} : \dot{\sigma}^0 - \dot{\xi}^0\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.6}$$

Now we use $\frac{\sigma_D}{|\sigma_D|} : \dot{\sigma} - \dot{\xi} = \frac{d}{dt}\big(|\sigma_D| - \xi\big)$ and on the set $S_- := \{|\sigma_D| - (\xi + \kappa) < 0\}$ we have $\dot{\lambda} = 0$, hence

$$\dot{\lambda}\Big(\frac{\sigma_D}{|\sigma_D|} : \dot{\sigma} - \dot{\xi}\Big) = 0 \text{ on } S_-.$$

On the set $S_= := \{|\sigma_D| - (\xi + \kappa) = 0\}$ we have $\frac{d}{dt}\big(|\sigma_D| - \xi\big) = 0$ a.e. and hence

$$\dot{\lambda}\Big(\frac{\sigma_D}{|\sigma_D|} : \dot{\sigma} - \dot{\xi}\Big) = 0 \text{ a.e. on } S_=.$$

The set $S_+ = \{|\sigma_D| - (\xi + \kappa) > 0\}$ is empty and thus

$$\dot\lambda\Big(\frac{\sigma_D}{|\sigma_D|} : \dot\sigma - \xi\Big) = 0 \text{ a.e.} \tag{3.7}$$

Combining (3.6) and (3.7), we conclude

$$\int_{t_1}^{t_2}\int_\Omega A\dot\sigma : \dot\sigma + H\dot\xi \cdot \dot\xi \,\mathrm{d}x\,\mathrm{d}t = T_0 \tag{3.8}$$

where T_0 was introduced in (3.4). Hence we proved

$$\lim_{\mu\to 0}\int_{t_1}^{t_2}\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu \cdot \dot\xi_\mu \,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega A\dot\sigma : \dot\sigma + H\dot\xi \cdot \dot\xi \,\mathrm{d}x\,\mathrm{d}t$$

a.e. with respect to $t_1, t_2 \in [0,T]$ and the strong L^2-convergence follows for $L^2(\varepsilon, T; L^2)$ for every $0 < \varepsilon < T$. Finally the strong convergence of $\dot\sigma_\mu, \dot\xi_\mu$ in $L^2(0,T;L^2)$ follows since, in addition $\dot\sigma_\mu, \dot\xi_\mu$ are uniformly bounded in $L^\infty(0,T;L^2)$.□

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