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VORTEX MOTION FOR THE LANDAU-LIFSHITZ-GILBERT EQUATION WITH SPIN TRANSFER TORQUE

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Abstract. We study the Landau-Lifshitz-Gilbert equation for the dynamics of a magnetic vortex system. We include the spin-torque effects of an applied spin current, and rigorously derive an equation of motion ("Thiele equation") for vortices if the current is not too large. Our method of proof strongly utilizes the geometry of the problem in order to obtain the necessary energy estimates.

1. Introduction

1.1. **Physical background.** In the usual model of micromagnetics [23, 16], a ferromagnet is described by a domain $\Omega \subset \mathbb{R}^3$ representing the ferromagnetic sample, and its magnetization $\boldsymbol{m} : \Omega \to S^2$, a unit vector field. The time evolution of such a magnetization is described by the Landau-Lifshitz-Gilbert (LLG) equation [11]:

$$\frac{\partial \boldsymbol{m}}{\partial t} = \boldsymbol{m} \times \left(\alpha \frac{\partial \boldsymbol{m}}{\partial t} - \boldsymbol{h}_{\mathrm{eff}} \right). \tag{1}$$

Here $\times$ denotes the cross product in $\mathbb{R}^3$ and $\boldsymbol{h}_{\mathrm{eff}}$ the *effective field*, i.e., the negative L^2 gradient of the (free) energy of $\boldsymbol{m}$, and $\alpha > 0$ the Gilbert damping constant, a (small) dimensionless parameter.

In the presence of a spin-polarized current, (1) has to be modified by taking into account the so-called *spin-transfer torque*. This leads to the modified equation

$$\frac{\partial \boldsymbol{m}}{\partial t} + (v \cdot \nabla)\boldsymbol{m} = \boldsymbol{m} \times \left(\alpha \frac{\partial \boldsymbol{m}}{\partial t} - \boldsymbol{h}_{\mathrm{eff}} + \beta (v \cdot \nabla)\boldsymbol{m} \right), \tag{2}$$

where v is the direction of the current and β another dimensionless parameter of size comparable to α. This form of the LLG equation has been derived by Zhang-Li [43] and Thiaville et al. [37].

In certain thin-film regimes such as nanodots, the magnetization is mostly contained in the film plane. An interesting feature of such systems is the emergence of *vortices*, small regions where the magnetization turns out of plane and around which the in-plane part has a nonzero winding number. Such vortices carry two bits of information, the direction of winding and the polarity (i.e., the direction of the out-of plane component). They have been proposed as a possible means of magnetic data storage and have received much recent attention, especially as the polarity can be easily switched using magnetic fields [41] or applied currents [26]. As another field of possible applications for current-driven vortex motion, we mention the engineering of nanoscale microwave oscillators [30].

The motion of concentration phenomena under (1) has been described by Thiele [38] using a system of ODEs that was later adapted to vortex motion by Huber [15]. A spin-transfer term as in (2) can be easily addeded to these ODEs [37]. The resulting system for vortices with trajectories $t \mapsto a_j(t) \in \Omega$ $(j = 1, \ldots d)$ reads

$$F_j + G_j \times (\dot{a}_j - v) - \pi(\alpha_0 \dot{a}_j - \beta_0 v) = 0$$

with interaction forces $F_j = F_j(a_1, \ldots, a_d)$, gyro-vectors $G_j = 4\pi q_j \hat{\mathbf{e}}_3$, depending only on the topological index $q_j = \pm\frac{1}{2}$ of the vortex (which is half of the product of winding number and polarity), and with effective constants $\alpha_0, \beta_0 > 0$ (for $a \in \mathbb{R}^2 \equiv \mathbb{C}$, the notation $\hat{\mathbf{e}}_3 \times a$ means $-ia$). This system was analyzed in the case of periodic forcing in [18]. For a review of various theoretical and experimental approaches to vortex dynamics (the brief paragraphs above should not be seen as an attempt to do the extensive physical literature on this topic justice), we refer to [3].

In previous joint work with Spirn [21], we have rigorously derived a Thiele equation from (1) in the limit of small vortex size, for an exchange-dominated model energy. The aim of the present paper is to generalize this result to the LLG equation with spin transfer torque terms (2). Our results show that vortices can be manipulated using spin currents. In particular, spin currents allow us to move the vortices out of their equilibrium positions and to achieve nonequilibrium initial conditions for the current-free problem as studied in [21].

1.2. **Mathematical setting and results.** As an approximation of the physical micromagnetic (free) energy functional we will use the energy

$$E_\epsilon(\boldsymbol{m}) = \int_\Omega e_\epsilon(\boldsymbol{m})\, dx \tag{3}$$

under Dirichlet boundary conditions $\boldsymbol{m} = \boldsymbol{g}$ on $\partial\Omega$. Here Ω is a smooth and simply connected bounded domain in $\mathbb{R}^2$ and the energy density $e_\epsilon(\boldsymbol{m})$ is given for a map $\boldsymbol{m} = (m_1, m_2, m_3) \in H^1(\Omega; S^2)$ by

$$e_\epsilon(\boldsymbol{m}) = \frac{1}{2}|\nabla \boldsymbol{m}|^2 + \frac{1}{2\epsilon^2} m_3^2. \tag{4}$$

For the boundary condition, we assume that $\boldsymbol{g} \in C^\infty(\partial\Omega; S^1 \times \{0\})$ is a fixed map of degree $d \geq 1$. For the physical meaning of the functional and a justification the boundary condition, we refer to [21, Section 7].

We first sketch some static theory of this energy functional. As $\epsilon \searrow 0$, a sequence of $\boldsymbol{m}_\epsilon$ that satisfies the boundary condition $\boldsymbol{m}_\epsilon = \boldsymbol{g}$ will have divergent energy, since for topological reasons no continuous map with $m_3 \equiv 0$ can satisfy the boundary conditions. The same is true for maps of bounded energy; more precisely, one can show

$$E_\epsilon(\boldsymbol{m}_\epsilon) \geq d\pi \log \frac{1}{\epsilon} - C.$$

Given an upper bound matching this one up to a constant, for example for a sequence of minimizers, one obtains convergence of the rescaled energy density:

$$\frac{1}{\log \frac{1}{\epsilon}} e_\epsilon(\boldsymbol{m}_\epsilon) \to \pi \sum_{\ell=1}^{d} \delta_{a_\ell} \quad \text{as } \epsilon \searrow 0$$

in the sense of distributions, for some points $a = (a_1, \dots, a_d) \in \Omega^d$.

Formally, one expects that $\boldsymbol{m}_\epsilon$ will satisfy $m_{\epsilon 3} \approx 0$ outside a small region near the concentration points of the energy, and will cover one hemisphere of S^2 in such a small region. Therefore, one expects that concentration points of the energy are also concentration points of the magnetic vorticity $\omega_0(\boldsymbol{m})$, which is defined as

$$\omega_0(\boldsymbol{m}) = \left\langle \boldsymbol{m}, \frac{\partial \boldsymbol{m}}{\partial x_1} \times \frac{\partial \boldsymbol{m}}{\partial x_2} \right\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$. As $\omega_0(\boldsymbol{m})$ is the signed area element of $\boldsymbol{m}$ as a map into S^2, a cover of a hemisphere will contribute $\pm 2\pi$ to the vorticity. Additionally, $\boldsymbol{m}$ could cover the full sphere several times inside a small region in Ω, each covering yielding a contribution of $\pm 4\pi$. Under certain conditions on the energy, the latter can be shown not to happen, though, and we will mostly deal with $\boldsymbol{m}_\epsilon$ such that

$$\omega_0(\boldsymbol{m}_\epsilon) \to 4\pi \sum_{\ell=1}^{d} q_\ell \delta_{a_\ell}$$

in the sense of distributions, with weights $q_\ell \in \{\pm\frac{1}{2}\}$. The convention to use half-integers here represents the idea that a hemisphere corresponds to half a covering of S^2 and thus to an S^2 degree or skyrmion number of $\pm\frac{1}{2}$.

Although the energy $E_\epsilon(\boldsymbol{m}_\epsilon)$ diverges for any choice of $\boldsymbol{m}_\epsilon$, it is still possible to obtain a dependence of the energy on a configuration of points $a \in \Omega^d$ with $a_k \neq a_\ell$ for $k \neq \ell$. This is done by subtracting the core energy of the d vortices, each of which carries a typical energy of $\pi \log \frac{1}{\epsilon} + \gamma$, where γ is a universal constant related to the core profile. The limit of the optimal energy after subtracting the core energies is denoted as $W(a)$; this is the renormalized energy discussed in [4].

Related to the renormalized energy is the notion of energy excess of a map $\boldsymbol{m}$ relative to a vortex configuration a,

$$D_\epsilon(\boldsymbol{m}; a) = E_\epsilon(\boldsymbol{m}) - (\pi d \log \frac{1}{\epsilon} + d\gamma + W(a)).$$

It can be shown that $\liminf_{\epsilon\searrow 0} D_\epsilon(\boldsymbol{m}_\epsilon; a) \geq 0$ if $\frac{1}{\log\frac{1}{\epsilon}} e_\epsilon(\boldsymbol{m}_\epsilon) \to \pi \sum_{\ell=1}^{d} \delta_{a_\ell}$ as $\epsilon \searrow 0$, which is essentially the lower bound part of a Γ-convergence result. A matching upper bound also holds.

For a fully rigorous discussion of the results sketched above, we refer to [21, Section 2].

We turn to dynamics. Our object of study is the equation

$$\frac{\partial \boldsymbol{m}}{\partial t} + \nabla_v \boldsymbol{m} = \boldsymbol{m} \times \left(\alpha_\epsilon \frac{\partial \boldsymbol{m}}{\partial t} + \beta_\epsilon \nabla_v \boldsymbol{m} - \boldsymbol{f}_\epsilon(\boldsymbol{m}) \right), \tag{5}$$

where $\boldsymbol{f}_\epsilon$ is the negative L^2 gradient of E_ϵ,

$$\boldsymbol{f}_\epsilon(\boldsymbol{m}) = \Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m} - \frac{1}{\epsilon^2}(m_3 \boldsymbol{e}_3 - m_3^2 \boldsymbol{m}),$$

and the coefficients α_ϵ and β_ϵ satisfy

$$\alpha_\epsilon \log \frac{1}{\epsilon} \to \alpha_0 > 0 \quad \text{and} \quad \beta_\epsilon \log \frac{1}{\epsilon} \to \beta_0 \in \mathbb{R} \quad \text{as } \epsilon \searrow 0.$$

The notation ∇_v denotes the operator $(v \cdot \nabla)$, and the vector field v is assumed to satisfy $v(t) = \lambda(t) w$ for some fixed $w \in S^1$ and a bounded function $\lambda \in C^\infty([0,\infty))$.

We also consider the system of ODEs

$$4\pi q_\ell i\,(\dot{a}_\ell - v) + \pi\,(\alpha_0 \dot{a}_\ell - \beta_0 v) + \frac{\partial W(a)}{\partial a_\ell} = 0 \quad (\ell = 1, \ldots, d), \tag{6}$$

which has a global solution satisfying $a_k(t) \neq a_\ell(t)$ for all $k \neq \ell$ and all $t > 0$ if this is true for $t = 0$.

To study solutions of (5) as $\epsilon \searrow 0$, we need initial data $\boldsymbol{m}_\epsilon^0$. We assume that $\boldsymbol{m}_\epsilon^0 \in C^\infty(\overline{\Omega}; S^2)$ with $\boldsymbol{m}_\epsilon^0 = \boldsymbol{g}$ on $\partial\Omega$. Furthermore, we assume that there exists an $a^0 \in \Omega^d$ with $a_k^0 \neq a_\ell^0$ for $k \neq \ell$ such that

$$\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon^0) \to \alpha_0 \pi \sum_{\ell=1}^{d} \delta_{a_\ell^0} \quad \text{and} \quad \omega_0(\boldsymbol{m}) \to 4\pi \sum_{\ell=1}^{d} q_\ell \delta_{a_\ell^0}$$

in the sense of distributions, and such that

$$\lim_{\epsilon \searrow 0} D_\epsilon(\boldsymbol{m}_\epsilon^0; a^0) = 0.$$

The latter condition means that the energy of the initial data is almost minimal given the vortex positions. The existence of such initial data can be inferred similarly as in [13, 17].

Now we can formulate our main result, which is that the motion of the concentration points of energy density and vorticity can be described by the ODE, and additionally, that the flow does not develop singularities.

Theorem 1. *There exists a number $L_0 > 0$ with the following property: For every $T_* > 0$, there exists an $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$ and every $\lambda \in C^\infty([0,\infty))$ with $\|\lambda\|_{L^\infty} < L_0$, there is a smooth solution $\boldsymbol{m}_\epsilon \in C^\infty([0,T_*) \times \overline{\Omega}; S^2)$ of (5) with $\boldsymbol{m}_\epsilon(0,\,\cdot\,) = \boldsymbol{m}_\epsilon^0$ and $\boldsymbol{m}_\epsilon(t,\,\cdot\,)|_{\partial\Omega} = \boldsymbol{g}$ for every $t \geq 0$. Let $a \in C^\infty([0,\infty); \Omega^d)$ be the solution of (6) with $a(0) = a^0$. Then for every $t \in [0, T_*)$,*

$$\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon(t,\,\cdot\,)) \to \pi\alpha_0 \sum_{\ell=1}^d \delta_{a_\ell(t)} \quad \textit{and} \quad \omega_0(\boldsymbol{m}_\epsilon(t,\,\cdot\,)) \to 4\pi \sum_{\ell=1}^d q_\ell \delta_{a_\ell(t)}$$

as $\epsilon \searrow 0$, in the sense of distributions.

For the proof of this result, many of the arguments are the same as for the $v = 0$ case treated in a previous work [21]. We generally do not repeat these arguments; some of the proofs in this paper are therefore not self-contained.

There are two new aspects, though. First, the spin current typically brings energy into the system, which needs to be estimated. The magnitude of this contribution is such that a simple estimate of the corresponding terms by their moduli is not sufficient. We use geometric observations here to achieve a better control, see Theorem 2. Second, we need additional information about the convergence of the quantity $\alpha_\epsilon \left\langle \frac{\partial \boldsymbol{m}_\epsilon}{\partial t}, \nabla \boldsymbol{m}_\epsilon \right\rangle$ as ϵ tends to 0. This was not available previously, and is obtained by testing the natural energy identity associated to the LLG equation with a time-dependent test function (see Theorem 3 below). Being able to control this term also allows us to simplify the derivation of the vortex motion law somewhat compared to the approach in [21]. This is done in Section 5, were we first show that the motion law holds locally in time, and then deduce the full statement of Theorem 1.

1.3. **Related mathematical work.** The energy functional (3)–(4) is closely related to the Ginzburg-Landau functional

$$E_{gl}(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2 \right) dx$$

for $u \in H^1(\Omega; \mathbb{C})$, which has been widely studied since Bethuel-Brezis-Hélein [4]. A few works also concern our energy functional: Static results for minimizers under certain condition have been obtained in [13], [2] and [33]. For dynamics, the undamped Schrödinger type problem was studied in [25]; the problem with damping was examined in [21]. (The result in [21] is stated under the assumption that $q_\ell = \frac{1}{2}$ for $\ell = 1, \ldots, d$, but this assumption is never used).

In the context of the Ginzburg-Landau functional, an analogous vortex motion law for hybrid Schrödinger and gradient flow type dynamics has been derived independently by Miot [28] for vortices in the whole space and by the authors and Spirn [20] in the setting of a bounded domain. The latter result was generalized to the gauged Ginzburg-Landau functional for superconductivity in small applied fields by Kurzke-Spirn [22]; the analogous result in large applied fields is due to Serfaty-Tice [35], who also add an applied current.

There are some similarities between our results here and those of Tice [39] and Serfaty-Tice [35] in the context of applied currents in superconductors. There, the applied current enters the equation as a boundary condition, but using a clever choice of gauge, it can be viewed instead as a term similar to the one studied here.

The crucial difference between the two problems lies in the properties of the vorticity for maps into $\mathbb{C}$ and S^2, respectively. In the classical Ginzburg-Landau theory used for superconductors, it is described by the Jacobian J. This is an ideal tool for encoding vortex degrees, and accordingly it has been studied in great detail in this context. In particular there are good compactness results for the Jacobian in space-time [34, 1].

For the S^2-valued problem, the vorticity is described by $\omega_0(\boldsymbol{m})$ in space-time. This is primarily a tool for measuring S^2-degrees, and even though it does give some information about the degree of vortices as studied here, it is difficult to separate the two. This fact is also reflected in the possibility of different skyrmion numbers even for vortices of the same degree.

A corresponding space-time compactness result is not available for the vorticity $\omega_0(\boldsymbol{m})$, nor indeed can it be expected. The powerful "product estimate" of Sandier-Serfaty [34], which is exploited to great effect in [39, 35], is not available, either. We use more geometric tools instead to control the vorticity. This method does not permit currents with arbitrary space dependence, and therefore we study only currents that are constant in space.

Another consequence of the different target geometry is the possibility of singularities for the LLG equation. Under the conditions studied here, we can rule out singularities by energy considerations, but this is only due to the well-prepared initial data.

Finally, for other mathematical works studying the motion of singularities in ferromagnets, we mention [6, 27] for the motion of Néel walls and [19, 29] for boundary vortices.

2. Mathematical tools

In this section we explain some of the notions used in the introduction in more detail and introduce other tools that are useful for the study of our problem.

2.1. **Notation.** We begin with some notation. We use the differential operators $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$. Recall that we have a fixed vector field $v = \lambda w$, where $w \in S^1$ is constant and $\lambda \in C^\infty([0,\infty))$. We also write $\nabla_v = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}$. Let

$$L = \sup_{0 \leq t < \infty} |\lambda(t)|.$$

We wish to prove results for small values of L only. Thus we can safely assume that $L \leq 1$ throughout the paper.

Suppose that we have d points $a_1, \ldots, a_d \in \Omega$ and let $a = (a_1, \ldots, a_d)$. We define

$$\rho(a) = \min \left\{ \frac{1}{2} \min_{k \neq \ell} |a_k - a_\ell|, \min_{\ell = 1, \ldots, d} \operatorname{dist}(a_\ell, \partial\Omega) \right\}.$$

If $x \in \mathbb{R}^2$ and $r > 0$, then $B_r(x)$ denotes the open unit disk in $\mathbb{R}^2$ with center x and radius r. Furthermore, we write

$$\Omega_r(a) = \Omega \backslash \bigcup_{\ell=1}^{d} \overline{B_r(a_\ell)} \quad \text{and} \quad \Omega_0(a) = \Omega \backslash \{a_1, \ldots, a_d\}.$$

We use the notation δ_x for the Dirac measure centered at x, and

$$\delta_a = \sum_{\ell=1}^{d} \delta_{a_\ell}.$$

We frequently identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, because this allows more convenient notation.

We work with scalar products in three different spaces, and we use different symbols in order to distinguish them. In $\mathbb{R}^2$, regarded as a tangent space of Ω, we use a dot. Tangent spaces of S^2 inherit a scalar product from $\mathbb{R}^3$, and this is denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$. Finally, we often identify $\mathbb{R}^2$ with the subspace $\mathbb{R}^2\times\{0\}$ of $\mathbb{R}^3$ and consider projections onto it. Then we write $(\,\cdot\,,\,\cdot\,)$ for the scalar product.

2.2. **Energy density and vorticity.** Suppose that we have a map $\boldsymbol{m}:\Omega\to S^2$. We now have a closer look at the energy density $e_\epsilon(\boldsymbol{m})$ and the vorticity $\omega_0(\boldsymbol{m})$. Note that ω_0 is a Jacobian for S^2-valued maps, and it replaces the Jacobian

$$J(u)=\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2}$$

for a function $u:\Omega\to\mathbb{C}$ that plays such an important role in the usual theory of Ginzburg-Landau vortices. In some cases, it is convenient to consider the projection m of $\boldsymbol{m}$ onto $\mathbb{R}^2$, and then we also use $J(m)$. Note also that J has another representation involving the quantity

$$j(u)=(iu,\nabla u),$$

namely $J(u)=\frac{1}{2}\operatorname{curl} j(u)$.

When we have a map $\boldsymbol{m}:(0,T)\times\Omega\to S^2$ (where the variable $t\in(0,T)$ is typically interpreted as time), then we also have a space-time vorticity, which is most conveniently represented as a differential form. We define

$$\omega_1(\boldsymbol{m})=\left\langle \boldsymbol{m}\times\frac{\partial\boldsymbol{m}}{\partial x_2},\frac{\partial\boldsymbol{m}}{\partial t}\right\rangle \quad\text{and}\quad \omega_2(\boldsymbol{m})=\left\langle \boldsymbol{m}\times\frac{\partial\boldsymbol{m}}{\partial t},\frac{\partial\boldsymbol{m}}{\partial x_1}\right\rangle,$$

and then we set

$$\omega(\boldsymbol{m})=\omega_0(\boldsymbol{m})dx_1\wedge dx_2+\omega_1(\boldsymbol{m})dx_2\wedge dt+\omega_2(\boldsymbol{m})dt\wedge dx_1.$$

It is readily checked that $\omega(\boldsymbol{m})$ has a vanishing exterior derivative if $\boldsymbol{m}$ has continuous second derivatives. That is, we have $d\omega(\boldsymbol{m})=0$. Essentially the same tool has been introduced previously by Brezis, Coron, and Lieb [5] in connection with singular harmonic maps and minimal connections of defects in three spatial dimensions.

It will be important for our arguments to keep track of how the energy density and the (spatial) vorticity evolve in time. If $\boldsymbol{m}$ is sufficiently smooth, then we compute

$$\frac{\partial}{\partial t}e_\epsilon(\boldsymbol{m})=\operatorname{div}\left\langle\frac{\partial\boldsymbol{m}}{\partial t},\nabla\boldsymbol{m}\right\rangle-\left\langle\frac{\partial m}{\partial t},f_\epsilon(\boldsymbol{m})\right\rangle. \tag{7}$$

The equation $d\omega(\boldsymbol{m})=0$ can also be expressed as

$$\frac{\partial}{\partial t}\omega_0(\boldsymbol{m})=\operatorname{curl}\left\langle\boldsymbol{m}\times\frac{\partial\boldsymbol{m}}{\partial t},\nabla\boldsymbol{m}\right\rangle. \tag{8}$$

The right hand sides can of course be rewritten once we use the LLG equation to substitute the appropriate expressions for $\frac{\partial\boldsymbol{m}}{\partial t}$.

The following formulas will also be useful.

Lemma 1. *For $\boldsymbol{m}\in C^2(\Omega;S^2)$,*

$$\omega_0(\boldsymbol{m})=3m_3J(m)+\operatorname{curl}(m_2m_3\nabla m_1-m_1m_3\nabla m_2)$$

and

$$J(m)=m_3\omega_0(\boldsymbol{m}).$$

Proof. The first identity is verified by a direct calculation. For the second, we observe that

$$J(m) = (m_1^2 + m_2^2 + m_3^2)J(m).$$

Furthermore,

$$\begin{aligned} m_1^2 J(m) &= \frac{1}{2} m_1 \frac{\partial}{\partial x_1} m_1^2 \frac{\partial m_2}{\partial x_2} - \frac{1}{2} m_1 \frac{\partial m_2}{\partial x_1} \frac{\partial}{\partial x_2} m_1^2 \\ &= -\frac{1}{2} m_1 \frac{\partial}{\partial x_1}(m_2^2 + m_3^2)\frac{\partial m_2}{\partial x_2} + \frac{1}{2} m_1 \frac{\partial m_2}{\partial x_1}\frac{\partial}{\partial x_2}(m_2^2 + m_3^2) \\ &= -m_1 m_3 \frac{\partial m_3}{\partial x_1}\frac{\partial m_2}{\partial x_2} + m_1 m_3 \frac{\partial m_2}{\partial x_1}\frac{\partial m_3}{\partial x_2}. \end{aligned}$$

Similarly,

$$m_2^2 J(m) = -m_2 m_3 \frac{\partial m_1}{\partial x_1}\frac{\partial m_3}{\partial x_2} + m_2 m_3 \frac{\partial m_3}{\partial x_1}\frac{\partial m_1}{\partial x_2}.$$

Combining these formulas, the second identity follows as well. □

2.3. **Renormalized energy.** Next, we give a precise definition of the renormalized energy W and the energy excess D_ϵ. If we have an $a \in \Omega^d$ comprising pairwise distinct points and we consider a limiting configuration with vortices of degree 1 at these points, then we can represent it as a map $m : \Omega_0(a) \to S^1$ of the form

$$m(z) = e^{i\theta(z)} \prod_{\ell=1}^{d} \frac{z - a_\ell}{|z - a_\ell|}.$$

The energetically most favorable limiting map satisfies the equation $\Delta\theta = 0$ in Ω. This equation is complemented by Dirichlet boundary conditions for θ such that $(m, 0) = \boldsymbol{g}$ on $\partial\Omega$. We write $m_*(\,\cdot\,; a)$ for the configuration with these properties, and $\boldsymbol{m}_*(\,\cdot\,; a) = (m_*(\,\cdot\,; a), 0)$. When there is no danger of confusion, we often use the shorthand notation $m_* = m_*(\,\cdot\,; a)$ and $\boldsymbol{m}_* = \boldsymbol{m}_*(\,\cdot\,; a)$. Note that m_* can also be characterized as the unique map $\Omega_0(a) \to S^1$ satisfying the boundary conditions and

$$\begin{aligned} \operatorname{div} j(m_*) &= 0, \\ \operatorname{curl} j(m_*) &= 2\pi\delta_a \end{aligned}$$

in Ω. The Dirichlet energy of $\boldsymbol{m}_*$ is infinite, but since the asymptotic behavior near the singularities is independent of the positions of these points, we can discard the corresponding (infinite) energy contribution and thus calculate a renormalized energy. This is

$$W(a) = \lim_{r \searrow 0} \left(\frac{1}{2} \int_{\Omega_r(a)} |\nabla m_*(x; a)|^2 \, dx - d\pi \log \frac{1}{r} \right).$$

When we have a family of maps $\boldsymbol{m}_\epsilon : \Omega \to S^2$, with $\boldsymbol{m}_\epsilon = \boldsymbol{g}$ on $\partial\Omega$, converging to $\boldsymbol{m}_*$, then we can give an asymptotic lower bound for the energy $E_\epsilon(\boldsymbol{m}_\epsilon)$. This consists of the renormalized energy $W(a)$ and an additional contribution for each vortex of the amount

$$\pi \log \frac{1}{\epsilon} + \gamma,$$

where γ is a constant and can be interpreted as the energy contained in each vortex core. In order to calculate γ, we define

$$I_\epsilon = \inf \left\{ \int_{B_1(0)} e_\epsilon(\boldsymbol{m}) \, dx : \ \boldsymbol{m} \in H^1(B_1(0); S^2) \text{ with } \boldsymbol{m}(x) = (x, 0) \text{ on } \partial B_1(0) \right\}.$$

Then

$$\gamma = \lim_{\epsilon \searrow 0} \left(I_\epsilon - \pi \log \frac{1}{\epsilon} \right).$$

For $\boldsymbol{m} \in H^1(\Omega; S^2)$, we set

$$W_\epsilon(a) = W(a) + d\pi \log \frac{1}{\epsilon} + d\gamma$$

and

$$D_\epsilon(\boldsymbol{m}; a) = E_\epsilon(\boldsymbol{m}) - W_\epsilon(a).$$

It is shown in [21] that the development of vortices at the points $a_1, \dots, a_d$ with boundary data $\boldsymbol{g}$ is only possible if

$$\liminf_{\epsilon \searrow 0} D_\epsilon(\boldsymbol{m}_\epsilon; a) \geq 0.$$

With a standard construction going back to [4], it can also be shown that equality is possible here. Thus D_ϵ asymptotically measures the energy excess for a given set of vortices.

Note that W coincides with the renormalized energy for the Ginzburg-Landau theory (whereas W_ϵ differs by a constant from the corresponding expression). Thus we can use known results when we study this function. In particular, we have a well-known expression for its gradient in terms of ∇m_*. If $\phi \in C_0^\infty(\Omega)$ such that $\nabla^\perp \nabla \phi$ vanishes near the vortices, then we have [4, 8]

$$\pi \sum_{\ell=1}^{d} \nabla^\perp \phi(a_\ell) \cdot \frac{\partial}{\partial a_\ell} W(a) = - \int_\Omega \nabla^\perp \nabla \phi : (\nabla m_* \otimes \nabla m_*)\, dx. \tag{9}$$

3. The Landau-Lifshitz-Gilbert equation

Recall that we study the LLG equation

$$\frac{\partial \boldsymbol{m}}{\partial t} + \nabla_v \boldsymbol{m} = \boldsymbol{m} \times \left(\alpha_\epsilon \frac{\partial \boldsymbol{m}}{\partial t} + \beta_\epsilon \nabla_v \boldsymbol{m} - \boldsymbol{f}_\epsilon(\boldsymbol{m}) \right). \tag{10}$$

This is equivalent to

$$\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial t} + \boldsymbol{m} \times \nabla_v \boldsymbol{m} + \alpha_\epsilon \frac{\partial \boldsymbol{m}}{\partial t} + \beta_\epsilon \nabla_v \boldsymbol{m} = \boldsymbol{f}_\epsilon(\boldsymbol{m}). \tag{11}$$

In this section we study the equation for a fixed $\epsilon \in (0, \frac{1}{2}]$. As usual, we use Dirichlet boundary data given by a smooth map $\boldsymbol{g} : \partial\Omega \to S^1 \times \{0\}$ with degree d and initial data $\boldsymbol{m}^0 \in H^1(\Omega; S^2)$ with

$$E_\epsilon(\boldsymbol{m}^0) \leq d\pi \log \frac{1}{\epsilon} + C_0. \tag{12}$$

3.1. **Weak solutions and bubbling.** The LLG equation is well understood, and as long as we do not require estimates that are uniform in ϵ (which we eventually will), we can use known results to describe the behavior of its solutions. Typically, the equation is studied in a simplified form, and thus some of the arguments from the literature need to be modified somewhat, but this is not difficult.

Consider first a smooth solution. Using the form (11) of the equation and taking the scalar product with $\frac{\partial \boldsymbol{m}}{\partial t}$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} e_\epsilon(\boldsymbol{m}) + \alpha_\epsilon \left| \frac{\partial m}{\partial t} \right|^2 - \operatorname{div} \left\langle \frac{\partial \boldsymbol{m}}{\partial t}, \nabla \boldsymbol{m} \right\rangle \\ + \left\langle \boldsymbol{m} \times \nabla_v \boldsymbol{m}, \frac{\partial m}{\partial t} \right\rangle + \beta_\epsilon \left\langle \nabla_v \boldsymbol{m}, \frac{\partial \boldsymbol{m}}{\partial t} \right\rangle = 0. \end{aligned} \tag{13}$$

Integrating over Ω, we find

$$\begin{aligned}(14)\quad & \frac{d}{dt}E_\epsilon(\boldsymbol{m}(t,\,\cdot\,)) \\ & = -\int_{\{t\}\times\Omega}\left(\alpha_\epsilon\left|\frac{\partial \boldsymbol{m}}{\partial t}\right|^2 + \left\langle \boldsymbol{m}\times\nabla_v\boldsymbol{m},\frac{\partial \boldsymbol{m}}{\partial t}\right\rangle + \beta_\epsilon\left\langle \nabla_v\boldsymbol{m},\frac{\partial \boldsymbol{m}}{\partial t}\right\rangle\right)dx.\end{aligned}$$

Using Young's inequality, we derive an estimate of the form

$$\frac{d}{dt}E_\epsilon(\boldsymbol{m}(t,\,\cdot\,)) + \frac{\alpha_\epsilon}{2}\int_{\{t\}\times\Omega}\left|\frac{\partial \boldsymbol{m}}{\partial t}\right|^2 dx\,dt \le c_\epsilon E_\epsilon(\boldsymbol{m}(t,\,\cdot\,))$$

for a constant c_ϵ that depends on α_ϵ, β_ϵ, and L. Hence we obtain an estimate for the growth of the energy, and in particular $E_\epsilon(\boldsymbol{m}(t,\,\cdot\,))$ cannot tend to infinity in finite time as long as $\boldsymbol{m}$ remains smooth.

On the other hand, it must be expected that solutions blow up in finite time in general. But with known arguments, we can construct weak solutions with very good properties. These arguments were first used for the harmonic map heat flow, which is the gradient flow for the Dirichlet functional and thus closely related to the Landau-Lifshitz-Gilbert equation. In particular, Struwe [36] studied the heat flow on surfaces without boundary and showed that there exist solutions with only finitely many singularities. These results were generalized to domains with boundary by Chang [7] and to a version of the Landau-Lifshitz-Gilbert equation by Guo and Hong [12].

The essential observation in these papers is that the solutions remain smooth as long as concentration of a certain amount of energy does not occur at a single point in Ω. The proof uses above all two ingredients: a local version of the above energy identity and estimates that imply regularity for solutions with small energy. The latter is dependent on the fact that the nonlinearities in the equation are critical with respect to the Sobolev space $H^1(\Omega)$ (that is naturally associated to the energy). For equation (10), the same arguments still work. We can obtain a local energy identity from (13), and the additional nonlinearities are subcritical, which is even better.

A consequence is that a certain amount of energy is lost at each singularity. Since we have only a finite amount of energy available after a finite time, this means that there can only be isolated singular points. That is, there exists a weak solution $\boldsymbol{m}\in L^\infty_{\mathrm{loc}}([0,\infty);H^1(\Omega;S^2))$ of (10) which is smooth away from isolated singular points (t^i,x^i), $i=1,2,\ldots$, in space-time.

The precise structure of the singularities has been examined in great detail by a number of authors [9, 31, 42, 32, 24, 40] for the harmonic map heat flow; in particular, it has been shown by Qing [31] and by Ding and Tian [9] that all the lost energy at a singularity goes into the development of so-called harmonic "bubbles". These are critical points of the Dirichlet energy (called harmonic maps) in $\mathbb{R}^2$, obtained by rescaling a solution of the flow near a singularity and passing to the limit. The relevant inequalities in this theory do not depend on the exact structure of the equation, but mostly on L^2-bounds for the L^2-gradient of the corresponding energy. Thus they can be used also in the case of equation (10). This gives very precise information about the behavior of the energy density $e_\epsilon(\boldsymbol{m})$ and the vorticity $\omega(\boldsymbol{m})$ near the singular points in terms of harmonic maps. Furthermore, harmonic maps between $\mathbb{R}^2$ and S^2 with finite energy are completely classified [10]. Combining all this information, we can describe the behavior of our weak solution $\boldsymbol{m}$ at the singularities as follows.

For every i there exists an integer q_i such that for every sufficiently small $r > 0$,

$$\int_{\{t^i\}\times B_r(x^i)} e_\epsilon(\boldsymbol{m})\,dx + 4\pi|q_i| \le \liminf_{t\nearrow t^i} \int_{\{t\}\times B_r(x^i)} e_\epsilon(\boldsymbol{m})\,dx \tag{15}$$

and

$$\int_{\{t^i\}\times B_r(x^i)} \omega_0(\boldsymbol{m})\,dx + 4\pi q_i = \lim_{t\nearrow t^i} \int_{\{t\}\times B_r(x^i)} \omega_0(\boldsymbol{m})\,dx. \tag{16}$$

Here the number q_i has a geometrical interpretation of the combined degree of all bubbles at the given singularity. If there is no cancellation, then we even have equality in the first formula.

When we study the solution in a time interval $(t_1, t_2]$, then we only have to consider the singularities with $t_1 < t^i \le t_2$, of course. For this purpose, we use the notation $I(t_1, t_2)$ for the set of all these indices from now on.

Note that we do not have uniqueness of weak solutions of the LLG equation. But among all weak solutions satisfying

$$\limsup_{t\searrow t_0} E_\epsilon(\boldsymbol{m}(t,\,\cdot\,)) \le E(\boldsymbol{m}(t_0,\,\cdot\,))$$

for every $t_0 \ge 0$, the weak solution examined above is unique [14]. Thus we call it the *energy decreasing solution* (although, strictly speaking, this is a misnomer, as energy may be brought into the system by the spin current).

3.2. **A uniform energy estimate.** The aim of this section is to prove the following energy estimate.

Theorem 2. *Let $C_1 > C_0$. Then there exist $L_0 > 0$, $\epsilon_0 > 0$, and $\tau > 0$ with the following property. Suppose that $\boldsymbol{m}^0 \in H^1(\Omega; S^2)$ satisfies $\boldsymbol{m}^0 = \boldsymbol{g}$ on $\partial\Omega$ and inequality* (12)*. If $L \le L_0$ and $\epsilon \le \epsilon_0$, then the energy decreasing solution $\boldsymbol{m}$ of* (10) *with initial data $\boldsymbol{m}^0$ and boundary data $\boldsymbol{g}$ satisfies*

$$E_\epsilon(\boldsymbol{m}(t_0,\,\cdot\,)) + \frac{\alpha_\epsilon}{2}\int_0^{t_0}\int_\Omega \left|\frac{\partial \boldsymbol{m}}{\partial t}\right|^2 dx\,dt \le d\pi \log\frac{1}{\epsilon} + C_1$$

for every $t_0 \in [0, \tau)$.

The proof of this result is based on an integration of (13) in space-time. The subsequent estimate of one of the resulting expression requires the following formula.

Lemma 2. *Consider the linear function $\phi(x) = w_1x_2 - w_2x_1$ on $\mathbb{R}^2$. Suppose that $\Omega \subset \{x \in \mathbb{R}^2 :\ a < \phi(x) < b\}$. Then*

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega \left\langle \boldsymbol{m}\times\nabla_v\boldsymbol{m}, \frac{\partial\boldsymbol{m}}{\partial t}\right\rangle dx\,dt \\
&\qquad = \int_\Omega (b-\phi)\left(\lambda(t_2)\omega_0(\boldsymbol{m}(t_2,\,\cdot\,)) - \lambda(t_1)\omega_0(\boldsymbol{m}(t_1,\,\cdot\,))\right) dx \\
&\qquad\quad - \int_{t_1}^{t_2}\int_\Omega (b-\phi)\dot\lambda\omega_0(\boldsymbol{m})\,dx\,dt + 4\pi\sum_{i\in I(t_1,t_2)} (b-\phi(x^i))\lambda(t^i)q_i.
\end{aligned}$$

Proof. For $a < s < b$, let

$$\Omega_s = \{x \in \Omega :\ \phi(x) < s\}, \quad P_s = (t_1, t_2)\times\Omega_s,$$

and

$$Q_s = \{(t, x) \in (t_1, t_2)\times\Omega :\ \phi(x) = s\}.$$

Furthermore, let I_s be the set of all indices $i \in I(t_1, t_2)$ such that $x^i \in \Omega_s$. We set $\sigma(t, x) = \lambda(t)\omega(\boldsymbol{m}(t, x))$ and we compute

$$d\sigma = \dot\lambda\omega_0(\boldsymbol{m})dt\wedge dx_1\wedge dx_2$$

away from the singularities of $\boldsymbol{m}$. Because of (16), we have

$$\int_{\partial P_s} \sigma = \int_{P_s} \dot{\lambda}\omega_0(\boldsymbol{m})dt \wedge dx_1 \wedge dx_2 - 4\pi \sum_{i\in I_s} \lambda(t^i) q_i.$$

Note that $\omega_1(\boldsymbol{m}) = \omega_2(\boldsymbol{m}) = 0$ on $(t_1, t_2) \times \partial\Omega$. Decomposing ∂P_s into several parts, we therefore obtain

$$\begin{aligned}\int_{Q_s} \left\langle \boldsymbol{m} \times \nabla_v \boldsymbol{m}, \frac{\partial \boldsymbol{m}}{\partial t} \right\rangle dt \wedge (w_1 dx_1 + w_2 dx_2) = -\int_{Q_s} \sigma \\ = \int_{\Omega_s} (\lambda(t_2)\omega_0(\boldsymbol{m}(t_2, \cdot\,)) - \lambda(t_1)\omega_0(\boldsymbol{m}(t_1, \cdot\,)))\, dx \\ - \int_{P_s} \dot{\lambda}\omega_0(\boldsymbol{m})dt \wedge dx_1 \wedge dx_2 + 4\pi \sum_{i\in I_s} \lambda(t^i) q_i.\end{aligned}$$

Now we integrate over s. This yields the required identity. □

Proof of Theorem 2. We may assume that $C_1 \le C_0 + 1$. Let

$$T_0 = \sup \left\{ t_0 > 0 : \ E_\epsilon(\boldsymbol{m}(t, \cdot\,)) \le d\pi \log \frac{1}{\epsilon} + C_1 \text{ for all } t \in [0, t_0) \right\}.$$

Consider the formula (14) away from the singular times. In the interval $[0, T_0)$, using the inequality

$$\int_{t_1}^{t_2} \int_\Omega \left| \frac{\partial \boldsymbol{m}}{\partial t} \right| |\nabla \boldsymbol{m}|\, dx\, dt \le \delta \int_{t_1}^{t_2} \int_\Omega \left| \frac{\partial \boldsymbol{m}}{\partial t} \right|^2 dx\, dt + \frac{C}{\delta}(t_2 - t_1) \log \frac{1}{\epsilon},$$

and taking the loss of energy (15) at possible singularities into account, we prove the estimates

$$\begin{aligned}(17)\quad E_\epsilon(\boldsymbol{m}(t_2, \cdot\,)) + \frac{\alpha_\epsilon}{2} \int_{t_1}^{t_2} \int_\Omega \left| \frac{\partial \boldsymbol{m}}{\partial t} \right|^2 dx\, dt + 4\pi \sum_{i \in I(t_1, t_2)} |q_i| \\ \le E_\epsilon(\boldsymbol{m}(t_1, \cdot\,)) - \int_{t_1}^{t_2} \int_\Omega \left\langle \boldsymbol{m} \times \nabla_v \boldsymbol{m}, \frac{\partial \boldsymbol{m}}{\partial t} \right\rangle dx\, dt + \frac{C\beta_\epsilon^2}{\alpha_\epsilon}(t_2 - t_1) \log \frac{1}{\epsilon}\end{aligned}$$

and

$$\begin{aligned}(18)\quad E_\epsilon(\boldsymbol{m}(t_2, \cdot\,)) + \frac{\alpha_\epsilon}{2} \int_{t_1}^{t_2} \int_\Omega \left| \frac{\partial \boldsymbol{m}}{\partial t} \right|^2 dx\, dt \\ \le E_\epsilon(\boldsymbol{m}(t_1, \cdot\,)) + \frac{C(1 + \beta_\epsilon^2)}{\alpha_\epsilon}(t_2 - t_1) \log \frac{1}{\epsilon}.\end{aligned}$$

(Here and subsequently C denotes various constants that depend only on $\boldsymbol{g}$, C_0, α_0, β_0, and the geometry of Ω.)

We want to show that there exists a number $\tau > 0$ such that $T_0 \ge \tau$ whenever L and ϵ are sufficiently small. To this end, fix $\tau \in (0, 1]$ and suppose $T_0 < \tau$. Because of (18), there exists a set $\Theta \subset (0, T_0)$ with measure

$$|\Theta| \le \frac{C\epsilon}{\alpha_\epsilon^2}(1 + \beta_\epsilon^2) \log \frac{1}{\epsilon},$$

such that for every $t \in (0, T_0)\backslash\Theta$,

$$\int_{\{t\}\times\Omega} |\boldsymbol{f}_\epsilon(\boldsymbol{m})|^2\, dx \le \frac{1}{\epsilon},$$

and Θ contains none of the singular times. It is shown in [21, Theorem 2.1] that this inequality, together with the energy bound

$$E_\epsilon(\boldsymbol{m}(t,\,\cdot\,)) \le d\pi \log\frac{1}{\epsilon} + C_1,$$

implies

$$\int_{\{t\}\times\Omega} m_3^2 |\nabla \boldsymbol{m}|^2\,dx \le C$$

outside of Θ. Choose $T_1, T_2 \in (0, T_0)\backslash\Theta$ with

$$\max\{T_1, T_0 - T_2\} \le \frac{C\epsilon}{\alpha_\epsilon^2}(1+\beta_\epsilon^2)\log\frac{1}{\epsilon}.$$

We apply Lemma 2 in the interval (T_1, T_2). According to Lemma 1, some of the terms in the resulting identity can be estimated as follows (for a suitable choice of a and b):

$$\begin{aligned}
&-\int_{\{T_2\}\times\Omega}(b-\phi)\lambda\omega_0(\boldsymbol{m})\,dx\\
&= 3\int_{\{T_2\}\times\Omega}(\phi-b)\lambda m_3^2\omega_0(\boldsymbol{m})\,dx - \int_{\{T_2\}\times\Omega}\lambda m_3(m_2\nabla_v m_1 - m_1\nabla_v m_2)\,dx\\
&\le 3L(b-a)\int_{\{T_2\}\times\Omega} m_3^2|\nabla\boldsymbol{m}|^2\,dx + CL\epsilon\log\frac{1}{\epsilon} \le CL.
\end{aligned}$$

We have a similar estimate for the term involving T_1 instead of T_2. Furthermore,

$$-\int_{(T_1,T_2)\backslash\Theta}\int_\Omega (b-\phi)\dot{\lambda}\omega_0(\boldsymbol{m})\,dx\,dt \le C\tau\|\dot{\lambda}\|_{L^\infty(0,T_0)}.$$

We also have

$$\begin{aligned}
-\int_{(T_1,T_2)\cap\Theta}\int_\Omega (b-\phi)\dot{\lambda}\omega_0(\boldsymbol{m})\,dx\,dt &\le \frac{C\epsilon}{\alpha_\epsilon^2}(1+\beta_\epsilon^2)\left(\log\frac{1}{\epsilon}\right)^2\|\dot{\lambda}\|_{L^\infty(0,T_0)}\\
&\le C\epsilon\left(\log\frac{1}{\epsilon}\right)^4\|\dot{\lambda}\|_{L^\infty(0,T_0)}.
\end{aligned}$$

If $(b-a)L \le 1$, then we can combine Lemma 2 with (17) and we obtain

$$\begin{aligned}
&E_\epsilon(\boldsymbol{m}(T_2,\,\cdot\,)) + \frac{\alpha_\epsilon}{2}\int_{T_1}^{T_2}\int_\Omega \left|\frac{\partial\boldsymbol{m}}{\partial t}\right|^2 dx\,dt\\
&\qquad\le E_\epsilon(\boldsymbol{m}(T_1,\,\cdot\,)) + CL + C\tau + C\left(\tau + \epsilon\left(\log\frac{1}{\epsilon}\right)^4\right)\|\dot{\lambda}\|_{L^\infty(0,T_0)}.
\end{aligned}$$

Furthermore, if we use (18) between the times 0 and T_1 and between T_2 and T_0, we obtain

$$\begin{aligned}
&E_\epsilon(\boldsymbol{m}(T_0,\,\cdot\,)) + \frac{\alpha_\epsilon}{2}\int_0^{T_0}\int_\Omega \left|\frac{\partial\boldsymbol{m}}{\partial t}\right|^2 dx\,dt\\
&\le d\pi\log\frac{1}{\epsilon} + C_0 + CL + C\tau + C\epsilon\left(\log\frac{1}{\epsilon}\right)^5 + C\left(\tau + \epsilon\left(\log\frac{1}{\epsilon}\right)^4\right)\|\dot{\lambda}\|_{L^\infty(0,T_0)}.
\end{aligned}$$

If τ, L, and ϵ are sufficiently small, then this contradicts the definition of T_0. Thus $T_0 \ge \tau$. The required estimate for $\frac{\partial\boldsymbol{m}}{\partial t}$ then also follows from the last inequality. $\square$

4. A convergence result

Now we consider a sequence of initial data $\boldsymbol{m}_\epsilon^0 \in H^1(\Omega; S^2)$ with boundary values $\boldsymbol{g}$ and with

$$\alpha_\epsilon e_\epsilon(\boldsymbol{m}_\epsilon^0) \to \alpha_0 \pi \delta_{a^0} \quad \text{and} \quad \omega_0(\boldsymbol{m}_\epsilon^0) \to 4\pi \sum_{\ell=1}^d q_\ell \delta_{a_\ell^0},$$

for a certain $a = (a_1, \ldots, a_d) \in \Omega^d$ and $q_1, \ldots, q_d = \pm\frac{1}{2}$, in the sense of distributions. Furthermore, we assume that

$$\lim_{\epsilon \searrow 0} D_\epsilon(\boldsymbol{m}_\epsilon^0; a^0) = 0.$$

Let $\boldsymbol{m}_\epsilon$ denote the energy decreasing solutions of (10) belonging to $\boldsymbol{m}_\epsilon^0$. The last inequality implies in particular that

$$E_\epsilon(\boldsymbol{m}_\epsilon^0) \leq d\pi \log \frac{1}{\epsilon} + C_0$$

for a constant C_0 that is independent of ϵ. Therefore, Theorem 2 gives an estimate for the energy that is uniform in ϵ.

Theorem 3. *There exist a number $T > 0$, a sequence $\epsilon_k \searrow 0$, and a curve $a \in H^1(0, T; \Omega^d)$ with $a(0) = a^0$, such that for every $t \in (0, T)$,*

$$\alpha_{\epsilon_k} e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(t, \cdot)) \to \alpha_0 \pi \delta_{a(t)}$$

weakly in $(C_0^0(\Omega))^*$ and*

$$J(m_{\epsilon_k}(t, \cdot)) \to \pi \delta_{a(t)}, \quad \omega_0(\boldsymbol{m}_{\epsilon_k}(t, \cdot)) \to 4\pi \sum_{\ell=1}^d q_\ell \delta_{a_\ell(t)}$$

in $W^{-1,1}(\Omega)$. Moreover,

$$\inf_{t \in (0,T)} \rho(a(t)) > 0.$$

For all $t_1, t_2 \in (0, T)$ with $t_1 \leq t_2$ and for all $\eta \in C^1(\overline{\Omega})$,

(19)
$$\pi \sum_{\ell=1}^d (\eta(a_\ell(t_1)) - \eta(a_\ell(t_2))) = \lim_{k \to \infty} \left(\frac{\alpha_{\epsilon_k}}{\alpha_0} \int_{t_1}^{t_2} \int_\Omega \nabla \eta \cdot \left\langle \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t}, \nabla \boldsymbol{m}_{\epsilon_k} \right\rangle dx\, dt \right),$$

$$\pi \int_{t_1}^{t_2} |\dot{a}|^2 \, dt \leq \liminf_{k \to \infty} \left(\frac{\alpha_{\epsilon_k}}{\alpha_0} \int_{t_1}^{t_2} \int_\Omega \left| \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right| dx\, dt \right), \tag{20}$$

$$\pi \mathrm{id} \sum_{\ell=1}^d \int_{t_1}^{t_2} \eta(a_\ell(t))\, dt = \lim_{k \to \infty} \left(\frac{\alpha_{\epsilon_k}}{\alpha_0} \int_{t_1}^{t_2} \int_\Omega \eta \nabla \boldsymbol{m}_{\epsilon_k} \otimes \nabla \boldsymbol{m}_{\epsilon_k} \, dx\, dt \right), \tag{21}$$

$$4\pi \sum_{\ell=1}^d q_\ell \int_{t_1}^{t_2} v^\perp \cdot \dot{a}_\ell \, dt = -\lim_{k \to \infty} \int_{t_1}^{t_2} \int_\Omega \left\langle \boldsymbol{m}_{\epsilon_k} \times \nabla_v \boldsymbol{m}_{\epsilon_k}, \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right\rangle dx\, dt. \tag{22}$$

Proof. Most of these statements are proved in another paper [21, Theorem 4.1] for $v = 0$ and $q_\ell = \frac{1}{2}$ (note that (21) corresponds to equation (31) in that work). It is readily checked that most of the arguments in this proof make little use of the exact structure of the equation, and therefore they still work in the situation studied here. The main ingredients for the proof are the identities (7) and (8), both of which hold for every smooth $\boldsymbol{m}$, and the inequalities

$$D_\epsilon(\boldsymbol{m}_\epsilon(t, \cdot); a^0) \leq \kappa, \quad 0 \leq t < T,$$

for some $\kappa < 4\pi$ and

$$\int_0^T \int_\Omega \left| \frac{\partial \boldsymbol{m}_\epsilon}{\partial t} \right|^2 \, dx \, dt \le C \log \frac{1}{\epsilon}$$

for a constant C independent of ϵ. It is also used that (7) gives rise to a nice local energy identity.

In the situation of this theorem, the inequalities follow from Theorem 2, provided that T is chosen small enough. The local energy identity (given below) for equation (10) has a few extra terms relative to the identity from [21], but they are quite easy to handle and do not invalidate the arguments, except for the proof of (20).

It is also shown in this proof that $\boldsymbol{m}_{\epsilon_k}$ remains smooth in $(0,T)$ under the conditions of the theorem.

Formulas (19) and (22), on the other hand, are not proved in [21], and (20) needs another derivation for equation (10). Note, however, that we do obtain the statement $a \in H^1(0,T;\Omega^d)$ from the previous work.

We now use (13) again. Testing it with a function $\xi \in C^1([0,T] \times \overline{\Omega})$, we obtain

$$\begin{aligned}
&\alpha_\epsilon \int_{t_1}^{t_2} \int_\Omega \xi \left| \frac{\partial \boldsymbol{m}_\epsilon}{\partial t} \right|^2 \, dx \, dt + \int_{\{t_2\} \times \Omega} \xi e_\epsilon(\boldsymbol{m}_\epsilon) \, dx \\
&= \int_{\{t_1\} \times \Omega} \xi e_\epsilon(\boldsymbol{m}_\epsilon) \, dx - \int_{t_1}^{t_2} \int_\Omega \nabla \xi \cdot \left\langle \frac{\partial \boldsymbol{m}_\epsilon}{\partial t}, \nabla \boldsymbol{m}_\epsilon \right\rangle \, dx \, dt \\
&\quad - \int_{t_1}^{t_2} \int_\Omega \xi \left(\left\langle \boldsymbol{m}_\epsilon \times \nabla_v \boldsymbol{m}_\epsilon, \frac{\partial \boldsymbol{m}_\epsilon}{\partial t} \right\rangle + \beta_\epsilon \xi \left\langle \nabla_v \boldsymbol{m}_\epsilon, \frac{\partial \boldsymbol{m}_\epsilon}{\partial t} \right\rangle \right) \, dx \, dt \\
&\quad + \int_{t_1}^{t_2} \int_\Omega \frac{\partial \xi}{\partial t} e_\epsilon(\boldsymbol{m}_\epsilon) \, dx \, dt.
\end{aligned}$$

In fact, an approximation argument shows that the identity is true for every $\xi \in L^\infty(0,T;C^1(\overline{\Omega}))$ with $\dot{\xi} \in L^1(0,T;C^0(\overline{\Omega}))$. Choose $r > 0$ such that $r < \frac{1}{2}\rho(a(t))$ for all $t \in [t_1,t_2]$. Choose a cut-off function $\tilde{\chi} \in C_0^\infty(B_r(0);[0,\infty))$ with $\chi \equiv 1$ in $B_{r/2}(0)$ and a function $b \in H^1(0,T;\mathbb{R}^2)$. Now define

$$\chi(t,x) = \tilde{\chi}(x - a_1(t))$$

and consider the function

$$\xi(t,x) = \chi(t,x) b(t) \cdot (x - a_1(t)).$$

This has the regularity required for the above identity. Now we multiply the resulting terms by α_ϵ, restrict our attention to the sequence ϵ_k, and use the convergence

$$\alpha_{\epsilon_k} e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(t,\,\cdot\,)) \to \alpha_0 \pi \delta_{a(t)}.$$

In the limit $k \to \infty$, we obtain

$$\pi \alpha_0 \int_{t_1}^{t_2} \int_\Omega b \cdot \dot{a}_1 \, dx \, dt = -\lim_{k \to \infty} \left(\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \chi b \cdot \left\langle \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t}, \nabla \boldsymbol{m}_{\epsilon_k} \right\rangle \, dx \, dt \right).$$

Of course this is true for $a_2, \ldots, a_d$ as well. If we choose $b(t) = \nabla \eta(a_1(t))$, then (19) follows.

Inequality (20) is a consequence of (19) and (21), because for χ and b as above, it follows that

$$\alpha_0 \pi \int_{t_1}^{t_2} b \cdot \dot{a}_1 \, dt \le \lim_{k \to \infty} A_k \left(\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \chi \left| \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right|^2 \, dx \, dt \right)^{1/2},$$

where

$$A_k = \left(\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \chi (b \otimes b) : (\nabla \boldsymbol{m}_{\epsilon_k} \otimes \nabla \boldsymbol{m}_{\epsilon_k}) \, dx \, dt \right)^{1/2} \to \left(\alpha_0 \pi \int_{t_1}^{t_2} |b|^2 \, dt \right)^{1/2}.$$

By approximation, we obtain the same inequality for $b \in L^2(0,T;\mathbb{R}^2)$. Inserting $b = \dot{a}_1$ and observing that the corresponding inequality holds for the other vortices as well, we obtain (20).

Finally, we use the formula from Lemma 2 (without the contributions of the bubbles, as we know that $\boldsymbol{m}_{\epsilon_k}$ is smooth). Letting $k \to \infty$, we first obtain

$$\begin{aligned}
&\lim_{k\to\infty} \int_{t_1}^{t_2} \int_\Omega \left\langle \boldsymbol{m}_{\epsilon_k} \times \nabla_v \boldsymbol{m}_{\epsilon_k}, \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right\rangle dx\, dt \\
&= 4\pi b \sum_{\ell=1}^d q_\ell \left(\lambda(t_2) - \lambda(t_1) - \int_{t_1}^{t_2} \dot{\lambda}\, dt \right) \\
&\quad + 4\pi \sum_{\ell=1}^d q_\ell \left(v^\perp(t_1)\cdot a_\ell(t_1) - v^\perp(t_2)\cdot a_\ell(t_2) + \int_{t_1}^{t_2} \dot{\lambda} w^\perp \cdot a_\ell\, dt \right).
\end{aligned}$$

An integration by parts then yields the desired formula. □

5. The motion law

In this section we prove Theorem 1. Let $\hat{a} \in C^\infty([0,\infty);\Omega^d)$ be the unique solution of the initial value problem for the corresponding Thiele equation in complex form

$$4\pi q_\ell i\,(\dot{a}_\ell - v) + \pi\,(\alpha_0 \dot{a}_\ell - \beta_0 v) + \frac{\partial W(a)}{\partial a_\ell} = 0 \quad (\ell = 1,\dots,d)$$

with initial values

$$\hat{a}(0) = a^0 \in \Omega^d.$$

We suppose $v(t) = \lambda(t) w$ for some fixed $w \in S^1$ and the constants $\alpha_0 > 0$ and $\beta_0 \in \mathbb{R}$ are given by the following limits

$$\alpha_0 = \lim_{\epsilon \searrow 0} \alpha_\epsilon \log\frac{1}{\epsilon} \quad \text{and} \quad \beta_0 = \lim_{\epsilon\searrow 0} \beta_\epsilon \log\frac{1}{\epsilon}.$$

We choose $T > 0$ and a sequence $\epsilon_k \searrow 0$ that satisfy the conclusions of Theorem 3, and let a be the corresponding curve in Ω^d. From the proof of Theorem 3 we recall that solutions remain smooth in $(0,T)$ for small ϵ as shown in [21, Theorem 4.1], so we can concentrate on the verification of the motion law.

We fix a radius $r \in (0, \rho(a^0)/2]$ and choose $T_0 \in (0,T)$ to be small enough such that the trajectories of a_ℓ and $\hat{a}_\ell$ do not exit $B_{r/2}(a_\ell^0)$ before time T_0 for all $\ell = 1,\dots,d$. As in [21] we choose $\phi, \psi \in C_0^\infty(\Omega)$ such that for every ℓ, both ϕ and ψ are affine with $\nabla\psi = \nabla^\perp\phi$ in $B_r(a_\ell^0)$. We define

$$\begin{aligned}
\xi_k(t) = \int_{\{t\}\times\Omega} &\left(\alpha_{\epsilon_k} \psi\, e_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}) + \phi\, \omega_0(\boldsymbol{m}_{\epsilon_k})\right) dx \\
&- \pi \sum_{\ell=1}^d \left(\alpha_0 \psi(\hat{a}_\ell(t)) + 4 q_\ell \phi(\hat{a}_\ell(t))\right),
\end{aligned}$$

converging, for every $t \in [0,T)$, to

$$\xi(t) = \pi \sum_{\ell=1}^d \Big(\alpha_0 \Big(\psi(a_\ell(t)) - \psi(\hat{a}_\ell(t)) \Big) + 4 q_\ell \Big(\phi(a_\ell(t)) - \phi(\hat{a}_\ell(t)) \Big) \Big).$$

In estimating ξ we follow the strategy from [21]. Thanks to our new convergence result (20) for the kinetic term $\alpha_\epsilon \left\langle \nabla \boldsymbol{m}, \frac{\partial \boldsymbol{m}}{\partial t} \right\rangle$ the argument can be slightly simplified and relies at this point only on the dynamic identity for the vorticity. With the notation

$$\tilde{e}_\epsilon(\boldsymbol{m}) = \frac{1}{2}\left(|\nabla |m||^2 + |\nabla m_3|^2 + \frac{m_3^2}{\epsilon^2} \right)$$

and the norm

$$\|\psi\|_{W^{-1,1}(\Omega)} = \sup\left\{\psi(u) : \ u \in W^{1,\infty}(\Omega) \text{ with } \|u\|_{W^{1,\infty}(\Omega)} \leq 1\right\},$$

the result carries over literally.

Lemma 3. *There exist a constant C and a sequence $\lambda_k \to 0$ such that for all $t_1, t_2 \in [0, T_0]$ with $t_1 \leq t_2$ and every $k \in \mathbb{N}$,*

$$\xi_k(t_2) - \xi_k(t_1) \leq C \int_{t_1}^{t_2} \int_{\Omega_r(a^0)} \left(\tilde{e}_\epsilon(\boldsymbol{m}_{\epsilon_k}) + \left| \frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*(\,\cdot\,;\hat{a}(t))) \right|^2 \right) dx\, dt \\ + C \int_{t_1}^{t_2} \|J(m_{\epsilon_k}) - \pi\delta_{\hat{a}}\|_{W^{-1,1}(\Omega)}\, dt + \lambda_k.$$

Proof. First we calculate from the differential equation for $\hat{a}_\ell$ and

$$\frac{d}{dt}\psi(\hat{a}_\ell(t)) = \nabla\psi(\hat{a}_\ell(t)) \cdot \dot{\hat{a}}_\ell(t) = \nabla^\perp\phi(\hat{a}_\ell(t)) \cdot \dot{\hat{a}}_\ell(t)$$

the identity

$$\begin{aligned} \pi \sum_{\ell=1}^{d} \frac{d}{dt} \left(\alpha_0\psi(\hat{a}_\ell(t)) + 4q_\ell\phi(\hat{a}_\ell(t))\right) \\ = \sum_{\ell=1}^{d} 4\pi q_\ell \nabla_v\phi(\hat{a}_\ell(t)) + \pi\beta_0\nabla_v\psi(\hat{a}_\ell(t)) - \frac{\partial W(\hat{a})}{\partial a_\ell} \cdot \nabla\psi(\hat{a}_\ell(t)) \end{aligned} \tag{23}$$

and recall from (9) with $\boldsymbol{m}_* = \boldsymbol{m}_*(\,\cdot\,;\hat{a})$:

$$-\sum_{\ell=1}^{d} \nabla\psi(\hat{a}_\ell(t)) \cdot \frac{\partial W(\hat{a})}{\partial a_\ell} = \int_{\{t\}\times\Omega} \nabla^\perp\nabla\phi : (\nabla m_* \otimes \nabla m_*)\, dx.$$

Using conservation of vorticity

$$\left(\frac{\partial}{\partial t} + \nabla_v\right)\omega_0(\boldsymbol{m}) = \operatorname{curl}\left(\operatorname{div}(\nabla\boldsymbol{m}\otimes\nabla\boldsymbol{m}) - \left\langle \left(\alpha_\epsilon \frac{\partial}{\partial t} + \beta_\epsilon\nabla_v\right)\boldsymbol{m}, \nabla\boldsymbol{m}\right\rangle\right)$$

(that can be found by multiplying the equation by $\nabla\boldsymbol{m}$ and taking the curl thereafter) we find after integration by parts

$$\begin{aligned} \frac{d}{dt}\int_{\{t\}\times\Omega} \phi\,\omega_0(\boldsymbol{m}_{\epsilon_k})\, dx = & \int_{\{t\}\times\Omega} \nabla_v\phi\ \omega_0(\boldsymbol{m}_{\epsilon_k})\, dx \\ & + \alpha_{\epsilon_k}\int_{\{t\}\times\Omega} \nabla\psi \cdot \left\langle \frac{\partial\boldsymbol{m}_{\epsilon_k}}{\partial t}, \nabla\boldsymbol{m}_{\epsilon_k}\right\rangle dx \\ & + \beta_{\epsilon_k}\int_{\{t\}\times\Omega} \nabla\psi \cdot \langle\nabla_v\boldsymbol{m}_{\epsilon_k}, \nabla\boldsymbol{m}_{\epsilon_k}\rangle\, dx \\ & + \int_{\{t\}\times\Omega} \nabla^\perp\nabla\phi : (\nabla\boldsymbol{m}_{\epsilon_k}\otimes\nabla\boldsymbol{m}_{\epsilon_k})\, dx. \end{aligned}$$

Integrating this identity in time and passing to the limit $\epsilon_k \searrow 0$, the terms stemming from the current converge, in view of Theorem 3 and the fact that $\nabla\phi$ and $\nabla\psi$ are constant in $B_r(a_\ell^0)$, to

$$4\pi\sum_{\ell=1}^{d} q_\ell\nabla_w\phi(a_\ell^0)\int_{t_1}^{t_2}\lambda(t)\, dt \quad \text{and} \quad \pi\beta_0\sum_{\ell=1}^{d}\nabla_w\psi(a_\ell^0)\int_{t_1}^{t_2}\lambda(t)\, dt,$$

respectively, and agree with the corresponding terms from (23). Moreover, by Theorem 3, we know

$$\alpha_{\epsilon_k} \int_{t_1}^{t_2} \int_\Omega \nabla\psi \cdot \left\langle \nabla \boldsymbol{m}_{\epsilon_k}, \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right\rangle dx\, dt \to -\pi\alpha_0 \sum_{\ell=1}^{d} \Big(\psi(a_\ell(t_2)) - \psi(a_\ell(t_1)) \Big).$$

Therefore, it suffices to estimate the integrals

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla\phi : (\nabla \boldsymbol{m}_{\epsilon_k} \otimes \nabla \boldsymbol{m}_{\epsilon_k} - \nabla m_* \otimes \nabla m_*)\, dx\, dt.$$

In view of the decomposition

$$\nabla \boldsymbol{m} \otimes \nabla \boldsymbol{m} = \nabla|m| \otimes \nabla|m| + \nabla m_3 \otimes \nabla m_3 + \frac{j(m)}{|m|} \otimes \frac{j(m)}{|m|}$$

valid for $\boldsymbol{m} = (m, m_3) \in C^\infty(\Omega; S^2)$, we estimate the contributions to the tensor $\nabla \boldsymbol{m}_{\epsilon_k} \otimes \nabla \boldsymbol{m}_{\epsilon_k}$ including $\nabla|m_{\epsilon_k}|$ and $\nabla m_{\epsilon_k 3}$ in terms of $\tilde{e}_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k})$, and we proceed with

$$\begin{aligned}
&\frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} \otimes \frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*) \otimes j(m_*) \\
&\quad = \left(\frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*) \right) \otimes \left(\frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*) \right) \\
&\qquad + (1 - |m_{\epsilon_k}|) \left(\frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} \otimes j(m_*) + j(m_*) \otimes \frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} \right) \\
&\qquad + (j(m_{\epsilon_k}) - j(m_*)) \otimes j(m_*) + j(m_*) \otimes (j(m_{\epsilon_k}) - j(m_*)).
\end{aligned}$$

As in [21] the integral coming from the second term can be estimated in terms of the energy $\epsilon_k E_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k})$, so we can concentrate on the estimation of

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla\phi : ((j(m_{\epsilon_k}) - j(m_*)) \otimes j(m_*))\, dx\, dt$$

and

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla\phi : (j(m_*) \otimes (j(m_{\epsilon_k}) - j(m_*)))\, dx\, dt.$$

Taking into account that both integrands can be considered as products of the form

$$\sigma \cdot (j(m_{\epsilon_k}) - j(m_*))$$

for smooth vector fields $\sigma \in C^\infty([0, T_0] \times \overline{\Omega}; \mathbb{R}^2)$ independent of k, we argue by the same Hodge decomposition argument used in [21, Lemma 6.1]. Writing

$$-\sigma = \nabla u + \nabla^\perp h,$$

where $u, h \in C^\infty([0, T_0] \times \overline{\Omega})$ with $u = 0$ on $[0, T_0] \times \partial\Omega$ we infer

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_\Omega \sigma \cdot (j(m_{\epsilon_k}) - j(m_*))\, dx\, dt \\
&\quad = \int_{t_1}^{t_2} \int_\Omega u \operatorname{div} j(m_{\epsilon_k})\, dx\, dt + 2 \int_{t_1}^{t_2} \left(\int_\Omega h\, J(m_{\epsilon_k})\, dx - \pi \sum_{\ell=1}^{d} h(\hat{a}_\ell) \right) dt.
\end{aligned}$$

We recall the following identity

$$\left(\frac{\partial}{\partial t} + \nabla_v \right) m_3 + (\nabla - \beta_\epsilon v) \cdot j(m) = \alpha_\epsilon \left(im, \frac{\partial m}{\partial t} \right)$$

which is nothing but the third component of the equation. Note that in view of our energy bounds

$$\alpha_\epsilon \left(im_{\epsilon_k}, \frac{\partial m_{\epsilon_k}}{\partial t} \right) \to 0 \quad \text{and} \quad \beta_{\epsilon_k} j(m_{\epsilon_k}) \to 0$$

in $L^1([0, T_0] \times \Omega)$, respectively, as $\epsilon_k \searrow 0$. Moreover,

$$\int_{t_1}^{t_2} \int_\Omega u \left(\frac{\partial}{\partial t} + \nabla_v \right) m_{\epsilon_k 3} \, dx \, dt = \int_\Omega u m_{\epsilon_k 3} \, dx \Big|_{t=t_1}^{t_2} - \int_{t_1}^{t_2} \int_\Omega m_{\epsilon_k 3} \left(\frac{\partial}{\partial t} + \nabla_v \right) u \, dx \, dt,$$

and we infer that

$$\int_{t_1}^{t_2} \int_\Omega u \operatorname{div} j(m_{\epsilon_k}) \, dx \, dt \to 0.$$

Thus we have a constant c, independent of k, such that

$$\int_{t_1}^{t_2} \int_\Omega \nabla^\perp \nabla \phi \cdot (j(m_{\epsilon_k}) - j(m_*)) \otimes j(m_*) \, dx \, dt$$
$$\leq c \int_{t_1}^{t_2} \| J(m_{\epsilon_k}) - \pi \delta_{\hat{a}} \|_{W^{-1,1}(\Omega)} \, dt + c\sqrt{\alpha_{\epsilon_k}}.$$

The same conclusion can also be drawn for $j(m_*) \otimes (j(m_{\epsilon_k}) - j(m_*))$. □

We will also need the following complementary estimate which is idenpendent of the dynamic equation and is proven in [21, Lemma 6.2].

Lemma 4. *For $t \in [0, T_0]$, let*

$$\hat{h}_k(t) = \int_{\{t\} \times \Omega_r(a^0)} \left(\tilde{e}_\epsilon(\boldsymbol{m}_{\epsilon_k}) + \frac{1}{8} \left| \frac{j(m_{\epsilon_k})}{|m_{\epsilon_k}|} - j(m_*(\,\cdot\,; \hat{a}(t))) \right|^2 \right) dx$$
$$- D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(t, \,\cdot\,); \hat{a}(t)).$$

Then there exists a constant C such that for almost all $t_1, t_2 \in [0, T_0]$ with $t_1 \leq t_2$,

$$\limsup_{k \to \infty} \int_{t_1}^{t_2} \hat{h}_k(t) \, dt \leq C \int_{t_1}^{t_2} |\hat{a}(t) - a(t)| \, dt.$$

Proof of Theorem 1. The proof follows by the usual Gronwall argument. As in [21] we consider, for $t \in [0, T_0]$, the functions

$$\zeta_k(t) = D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(t, \,\cdot\,); \hat{a}(t))$$

and

$$\chi_k(t) = \left\| J(m_{\epsilon_k}(t, \,\cdot\,)) - \pi \delta_{\hat{a}(t)} \right\|_{W^{-1,1}(\Omega)}.$$

From the corresponding energy identities we obtain

$$E_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(t_2, \,\cdot\,)) - E_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(t_1, \,\cdot\,))$$
$$= - \int_{t_1}^{t_2} \int_\Omega \left(\alpha_{\epsilon_k} \left| \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right|^2 + \beta_{\epsilon_k} \left\langle \nabla_v \boldsymbol{m}_{\epsilon_k}, \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right\rangle + \left\langle \boldsymbol{m}_{\epsilon_k} \times \nabla_v \boldsymbol{m}_{\epsilon_k}, \frac{\partial \boldsymbol{m}_{\epsilon_k}}{\partial t} \right\rangle \right) dx \, dt$$

and

$$W(\hat{a}(t_1)) - W(\hat{a}(t_2)) = \pi \int_{t_1}^{t_2} \left(\alpha_0 |\dot{\hat{a}}|^2 - \sum_{\ell=1}^d \left(\beta_0 v + 4 q_\ell v^\perp \right) \cdot \dot{\hat{a}}_\ell \right) dt$$

for $0 \le t_1 \le t_2 \le T_0$. In view of Theorem 3 we can select a subsequence such that $\zeta_k(t) \to \zeta(t)$ almost everywhere for a bounded function $\zeta : [0, T_0] \to \mathbb{R}$ with

$$\begin{aligned}\zeta(t_2) - \zeta(t_1) &\le \pi \int_{t_1}^{t_2} \alpha_0 \left(|\dot{\hat{a}}|^2 - |\dot{a}|^2\right) + (\beta_0 + 2)|v||\dot{\hat{a}} - \dot{a}|\, dt \\ &\le C_1 \int_{t_1}^{t_2} |\dot{\hat{a}} - \dot{a}|\, dt\end{aligned}$$

for almost all $t_1 \le t_2$ and some constant C_1. We infer that ζ has bounded variation in $[0, T_0]$ with a distributional estimate

$$\dot{\zeta} \le C_1 |\dot{\hat{a}} - \dot{a}|.$$

From this point the argument is the same as in the case $v = 0$. In fact,

$$\chi_k(t) \to \chi(t) = \sum_{\ell=1}^{d} |\hat{a}_\ell(t) - a_\ell(t)| \quad \text{pointwise and in } L^1(0, T_0).$$

Lemma 3 and Lemma 4 now imply

$$\xi(t_2) - \xi(t_1) \le C_2 \int_{t_1}^{t_2} (\zeta(t) + \chi(t))\, dt$$

for a constant C_2, and with an appropriate choice of ϕ and ψ we obtain the desired integral inequality

$$|\dot{\hat{a}}(t) - \dot{a}(t)| \le C_3 \int_0^t |\dot{\hat{a}}(\tau) - \dot{a}(\tau)|\, d\tau.$$

As $\hat{a}(0) = a(0)$, Gronwall's lemma implies $\hat{a} = a$ in $[0, T_0]$. Moreover,

$$\limsup_{k\to\infty} D_{\epsilon_k}(\boldsymbol{m}_{\epsilon_k}(T_0, \cdot\,); a(T_0)) \le 0.$$

which enables us to iterate the argument for new initial times T_0, and we eventually obtain the motion law for all times before T.
To prove the full statement of Theorem 1, we note that Theorem 3 can be applied (choosing further subsequences) with T as the new initial time. We can thus iterate the argument again and obtain the statement until a chosen terminal time T_*. Note that by uniqueness of energy decreasing solutions, solutions $\boldsymbol{m}_\epsilon$ extend, for small ϵ, smoothly to $(0, T_*)$. Finally, thanks to the unique solvability of the limiting ODE, the convergence result for energy density and vorticity can be seen to hold without taking subsequences, as any subsequence of $\epsilon \searrow 0$ will have a further subsequence converging to the same limit. □

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