# $\mathbf{C}^{\alpha}$-Regularity for a Class of Non-Diagonal Elliptic Systems with p-Growth 

Miroslav Bulíček, Jens Frehse

no. 483

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereichs 611 an der Universität Bonn entstanden und als Manuskript vervielfältigt worden.

# $\mathcal{C}^{\alpha}$-REGULARITY FOR A CLASS OF NON-DIAGONAL ELLIPTIC SYSTEMS WITH $p$-GROWTH 

MIROSLAV BULÍČEK AND JENS FREHSE


#### Abstract

We consider weak solutions to nonlinear elliptic systems in a $W^{1, p_{-}}$ setting which arise as Euler equations to certain variational problems. The solutions are assumed to be stationary in the sense that the differential of the variational integral vanishes with respect to variations of the dependent and independent variables. We impose new structure conditions on the coefficients which yield $\mathcal{C}^{\alpha}$-regularity and $\mathcal{C}^{\alpha}$-estimates for the solutions. These structure conditions cover variational integrals like $\int F(\nabla u) d x$ with potential $F(\nabla u):=$ $\tilde{F}\left(Q_{1}(\nabla u), \ldots, Q_{N}(\nabla u)\right)$ and positively definite quadratic forms in $\nabla u$ defined as $Q_{i}(\nabla u)=\sum_{\alpha \beta} a_{i}^{\alpha \beta} \nabla u^{\alpha} \cdot \nabla u^{\beta}$. A simple example consists in $\tilde{F}\left(\xi_{1}, \xi_{2}\right):=$ $\left|\xi_{1}\right|^{\frac{p}{2}}+\left|\xi_{2}\right|^{\frac{p}{2}}$ or $\tilde{F}\left(\xi_{1}, \xi_{2}\right):=\left|\xi_{1}\right|^{\frac{p}{4}}\left|\xi_{2}\right|^{\frac{p}{4}}$. Since the $Q_{i}$ need not to be linearly dependent our result covers a class of nondiagonal, possibly nonmonotone elliptic systems. The proof uses a new weighted norm technique with singular weights in an $L^{p}$-setting.


## 1. Introduction and statement of the result

This paper focuses on global $\mathcal{C}^{\alpha}$-estimates for a class of elliptic PDE's corresponding to the Euler operator of the following variational integral

$$
\begin{equation*}
J(u)=\int_{\Omega} F(x, \nabla u) d x \tag{1.1}
\end{equation*}
$$

with some coercive Caratheodory function $F: \Omega \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}_{+}$. Here, $\Omega$ denotes an open bounded domain in $\mathbb{R}^{d}$ with Lipschitz boundary and $u$ is an $\mathbb{R}^{N}$-valued function. The Euler operator of (1.1) has the form

$$
\begin{equation*}
L u=-\operatorname{div} F_{\eta}(x, \nabla u), \tag{1.2}
\end{equation*}
$$

where we set

$$
F_{\eta}(x, \eta):=\frac{\partial F(x, \eta)}{\partial \eta}: \Omega \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}^{d \times N}
$$

Here $\eta_{i}^{\alpha}$ corresponds to $\frac{\partial u^{\alpha}}{\partial x_{i}}$. We consider a class of $F$ 's that for certain $p \in(1, \infty)$ provides $p$-coercivity and $p$-growth estimates, more precisely we assume that there

[^0]exists $\delta_{0}, K \geq 0$ and $\alpha_{0}, \alpha_{1}>0$ such that
\[

$$
\begin{align*}
-K+\alpha_{0}\left(\delta_{0}+|\eta|^{2}\right)^{\frac{p}{2}} & \leq F(x, \eta) \leq \alpha_{1}\left(1+|\eta|^{2}\right)^{\frac{p}{2}} \\
& \left|F_{\eta}(x, \eta)\right| \leq \alpha_{1}\left(1+|\eta|^{2}\right)^{\frac{p-1}{2}} \tag{1.3}
\end{align*}
$$
\]

for all $\eta \in \mathbb{R}^{d \times N}$. We study the properties of a weak solution $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ of the following Euler equations corresponding to the variational problem (1.1)

$$
\begin{align*}
L u & =0 \text { in } \Omega, \\
u & =v \text { on } \partial \Omega \tag{1.4}
\end{align*}
$$

with $L$ given by (1.2) and with some given boundary data $v \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$. Note that under the growth conditions (1.3) the notion of a weak solution to (1.4) is meaningful. However, since (1.3) does not imply nor convexity neither quasiconvexity of $F$, the existence of a minima (or a solution to (1.4)) is not guaranteed. Furthermore, even if $F$ is a convex smooth function, the $\mathcal{C}^{\alpha}$ - or $L^{\infty}$-regularity of a minima to (1.1) or solutions to (1.4) may fail for $d \geq 3$, ( $d \geq 5$ respectively), see examples given in [14] and [22]. Such a regularity of a solution is known only for a particular class of $F$ 's, that is described below, and the main goal of this paper is to extend this class and to find new structural assumptions that imply the Hölder continuity ${ }^{1}$ of any weak solution to (1.4) and that do not rely on using any convexity assumptions on $F$ (although the existence of a weak solution to (1.4) might not be guaranteed and also the solution, if it exists, might not be a minimizer of the variational problem (1.1)).

Before we formulate the main results of the paper, we shortly recall known facts about smoothness of a solution to (1.4). First, the situation concerning $\mathcal{C}^{\alpha}{ }_{-}$ everywhere regularity of a solution to (1.4) does not change very much if the following additional standard $p$-ellipticity assumption is used:

$$
\begin{equation*}
\alpha_{0}\left(|\eta|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|\xi|^{2} \leq \frac{\partial^{2} F(x, \eta)}{\partial \eta^{2}} \cdot(\xi \otimes \xi) \leq \alpha_{1}\left(|\eta|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|\xi|^{2} \tag{1.5}
\end{equation*}
$$

for all $\eta, \xi \in \mathbb{R}^{d \times N}$. This assumption is sufficient for finding the unique weak solution to (1.4) that is also the minimizer to the variational problem (1.1). Moreover, it is known that for smooth data and for $\delta_{0}>0$ the solution obeys the property $|\nabla u|^{\frac{p}{2}} \in W^{1,2}(\Omega)$ and therefore for $p>d-2$ the solution $u$ is Hölder continuous which can be shown by using the embedding theorem. On the other hand in the opposite case $1<p \leq d-2$ the singular solution may appear even in the simplest case $p=2$, see eg. [14] for solution not being in $\mathcal{C}^{1}$, [22] for unbounded solution and [3] for discontinuous solutions for $F$ dependent not continuously on $x$. Hence, these results give an evidence that one has to find some new structural assumptions that are different/additional to those introduced in (1.5) in order to guarantee the Hölder continuity of the solution $u$. Up to our best knowledge, the only known possibility is to assume additionally to (1.5) that

$$
\begin{equation*}
F(x, \eta):=F(x,|\eta|) \quad \text { or } \quad F(x, \eta):=F_{0}(x,|\eta|)+(o(\eta))^{p} . \tag{1.6}
\end{equation*}
$$

[^1]In this setting, even $\mathcal{C}^{1, \alpha}$-regularity of the solution was proven by Uhlenbeck in [21], where the proof relies on the use of "scalar" techniques. As mentioned above the Uhlenbeck case (1.6) is the only one for which the global regularity of the solution is known. On the other hand there is huge amount of papers dealing with partial regularity of the solution only under the assumption (1.5). Since we deal with everywhere $\mathcal{C}^{\alpha}$ regularity and not with partial regularity of the solution we refer here the interested reader only to the detailed survey paper [12], where the problem of partial regularity of solution to (1.4) is described in a great detail, see also [6, 7].

In addition, the ellipticity condition (1.5) seems not be appropriate in many physical applications. Indeed, considering even the simplest elasticity problem, that means in our setting that $d=N=3$ and $p=2$, it has been shown that there can exist some non-dissipative processes and microstructures in the material such that the assumption (1.5) (and consequently convexity of $F$ ) is not valid anymore, see eg. $[16,18,17]$. Therefore, we see that the condition (1.5) is not optimal also from physical reasons and if such a microstructure occurs and (1.5) is not valid one has to look for additional methods and conditions on $F$ that lead to $\mathcal{C}^{\alpha}$-regularity of solution.

In the rest of the introduction, we present our results and describe the main novelty of the paper. The conditions on $F$, we deal with, are more general than the Uhlenbeck case (1.6), that however falls into this class, and lead to everywhere Hölder continuity of any weak solution to (1.4). Besides (1.3) which is very weak we require a kind of coercivity for the derivative of $F$, namely that ${ }^{2}$
(1.7) $\quad \alpha_{0}\left(\delta_{0}+|\eta|^{2}\right)^{\frac{p-2}{2}}|\eta|^{2} \leq F_{\eta}(x, \eta) \cdot \eta \leq C(F(x, \eta)+1) \quad$ for all $\eta \in \mathbb{R}^{d \times N}$.

Note that the second inequality is a consequence of (1.3). However, in this paper we will see that the method we use requires a stronger condition than the second inequality in (1.7) and has to reflect the $p$-setting of the problem. Thus, in the following we will assume that there is $\alpha_{2}>0$ and $0 \leq \theta<1$ such that (here $p>1$ is given by (1.3))

$$
\begin{equation*}
F_{\eta}(x, \eta) \cdot \eta \leq p F(x, \eta)+\alpha_{2}(1+F(x, \eta))^{\theta} \quad \text { for all } \eta \in \mathbb{R}^{d \times N} \tag{1.8}
\end{equation*}
$$

It seems that the presence of $p$ in (1.8) is a natural restriction since it "represents" the $p$-growth of $F$. For example, (1.8) holds for models of the Uhlenbeck type $F(x, \eta) \sim(1+|\eta|)^{p}$ as well as for more general models given eg. by $F(x, \eta) \sim$ $(1+|\eta|)^{p}+(Q(\eta, \eta))^{\frac{p}{2}}$, where $Q$ denotes an arbitrary positively-definit quadratic form, see also the examples given below that satisfy (1.8).

The second and the most restrictive assumption, we assume here, is that $F$ satisfies a splitting condition ${ }^{3}$ up to a lower order term. It means we assume that there are symmetric matric-valued function $A: \Omega \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}^{N \times N}$, a symmetric matrix valued $b: \Omega \rightarrow \mathbb{R}^{d \times d}$ and a matrix-valued function $G: \Omega \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}^{d \times N}$ such that

$$
\begin{equation*}
F_{\eta_{i}^{\alpha}}(x, \eta)=\sum_{\beta=1}^{N} \sum_{j=1}^{d} A^{\alpha \beta}(x, \eta) b_{i j}(x) \eta_{j}^{\beta}+G_{i}^{\alpha}(x, \eta) \tag{1.9}
\end{equation*}
$$

[^2]for all $\eta \in \mathbb{R}^{d \times N}$, all $i=1, \ldots, d$, all $\alpha=1, \ldots, N$ and a.a. $x \in \Omega$. Observe that (1.9) is an additional structure condition which does not reflect the fact, that the $F_{\eta_{i}^{\alpha}}$ come from a potential. To fix ideas, the reader may switch to examples (1.11)-(1.13) where the examples of quantities $A^{\alpha \beta}$ are calculated. Moreover, we require that for all $\mu \in \mathbb{R}^{N}$, all $\eta \in \mathbb{R}^{d \times N}$, all $v \in \mathbb{R}^{d}$ and a.a. $x \in \Omega$ there holds
\[

$$
\begin{align*}
& \alpha_{0}\left(|\eta|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|\mu|^{2} \leq \sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(x, \eta) \mu^{\alpha} \mu^{\beta} \leq \alpha_{1}\left(|\eta|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|\mu|^{2} \\
& \alpha_{0}|v|^{2} \leq \sum_{i, j=1}^{d} b_{i j}(x) v_{i} v_{j} \leq \alpha_{1}|v|^{2}  \tag{1.10}\\
&|G(\eta)| \leq \alpha_{2}\left(1+|\eta|^{2}\right)^{\frac{\theta(p-1)}{2}}, \quad \text { for some } \theta \in[0,1)
\end{align*}
$$
\]

Despite the splitting condition of $F$, we are still able to treat rather general systems with really non-diagonal coupling. To show this, we give several examples of potentials $F$ that satisfy (1.8)-(1.10) and that are far from being of Uhlenbeck type (1.6) and that admit also non-diagonal coupling. Let us define

$$
Q_{\ell}(\xi, \mu):=\sum_{\alpha, \beta=1}^{N} \sum_{i, j=1}^{d} a_{\ell}^{\alpha \beta} H_{\ell}^{i j} \xi_{i}^{\alpha} \mu_{j}^{\beta}
$$

where $H_{\ell}$ and $a_{\ell}$ are symmetric positively definite constant matrices and consider for some $p \in(1, \infty)$ potentials of the form

$$
\begin{array}{ll}
F(\eta):=Q_{1}(\eta, \eta)|\eta|^{p-2}, & \text { with } H_{1}^{i j}:=h \delta^{i j} \text { and } h>0 \\
F(\eta):=\left(Q_{1}(\eta, \eta)\right)^{\frac{p}{2}}, & \\
F(\eta):=\left(Q_{1}(\eta, \eta)\right)^{\frac{p}{2}}+\left(Q_{2}(\eta, \eta)\right)^{\frac{p}{2}}, & \text { with } H_{1}=h H_{2} \text { with } h \geq 0
\end{array}
$$

or in general of the form

$$
\begin{equation*}
F(\eta):=\tilde{F}\left(Q_{1}(\eta, \eta), \ldots, Q_{M}(\eta, \eta)\right), \quad \text { with } H_{i}=h_{i} H_{1} \text { with } h_{i} \geq 0 \tag{1.14}
\end{equation*}
$$

for all $i=2, \ldots, M$. Furthermore, for $\tilde{F}$, we prescribe the following conditions: There exists $p \in(1, \infty)$ and $\left\{p_{i}\right\}_{i=1}^{M} \in \mathbb{R}$ such that $\sum_{i=1}^{M} p_{i}=p$ and

$$
\begin{align*}
& \alpha_{0} \prod_{i=1}^{M} \xi_{i}^{\frac{p_{i}}{2}} \leq \tilde{F}\left(\xi_{1}, \ldots, \xi_{M}\right) \leq \alpha_{1}\left(1+\prod_{i=1}^{M} \xi_{i}^{\frac{p_{i}}{2}}\right),  \tag{1.15}\\
&\left|\frac{\partial \tilde{F}\left(\xi_{1}, \ldots, \xi_{M}\right)}{\partial \xi_{k}}\right| \leq \alpha_{1} \xi_{k}^{-1} \tilde{F}\left(\xi_{1}, \ldots, \xi_{M}\right) \text { for all } k=1, \ldots, M  \tag{1.16}\\
& \alpha_{0} \prod_{i=1}^{M} \xi_{i}^{\frac{p_{i}}{2}} \leq 2 \sum_{k=1}^{M} \frac{\partial \tilde{F}\left(\xi_{1}, \ldots, \xi_{M}\right)}{\partial \xi_{k}} \xi_{k} \leq p \tilde{F}\left(\xi_{1}, \ldots, \xi_{M}\right) . \tag{1.17}
\end{align*}
$$

For simplicity, we do not include the possible dependence on $\delta_{0}$ into the potential given by (1.11)-(1.14), but it is evident that such a minor generalization is also possible. It is also clear that we can add to each potential in (1.11)-(1.14) some lower order term (having $q$-growth with $q<p$ ) and that this lower order term does not change our results. It can be shown (we refer the interested reader to the Appendix for precise computations) that all potentials given in (1.11)-(1.12) satisfy (1.3), and (1.7)-(1.8). Moreover, the potentials given by (1.12)-(1.13) are convex
and satisfy the ellipticity condition (1.5). The same is not true for (1.11) and (1.14) in general, it depends on the choice of matrices $a$ and $H$ and also on their relation to $p$ and/or to $\tilde{F}$, see again Appendix for more details. Concerning the splitting condition (1.9) on $F$, the situation is more delicate. The potential given by (1.12) evidently satisfies the splitting condition. On the other hand for (1.11) and (1.13) we have to assume a special structure on $H$ (also given in (1.11) and (1.13)). To illustrate validity of the splitting condition (1.9), we get for $F$ given by (1.11) that

$$
F_{\eta_{i}^{\alpha}}(\eta)=\sum_{\beta=1}^{N} \sum_{j=1}^{d} 2 a_{1}^{\alpha \beta} H_{1}^{i j} \eta_{j}^{\beta}|\eta|^{p-2}+(p-2) Q_{1}(\eta, \eta)|\eta|^{p-4} \delta^{\alpha \beta} \delta^{i j} \eta_{j}^{\beta} .
$$

Therefore, defining

$$
A^{\alpha \beta}(x, \eta):=2 h a_{1}^{\alpha \beta}|\eta|^{p-2}+(p-2) Q_{1}(\eta, \eta)|\eta|^{p-4} \delta^{\alpha \beta}, \quad b^{i j}(x):=\delta^{i j}
$$

we see that $F$ satisfies (1.9) with $A^{\alpha \beta}$ and $b^{i j}$ and consequently satisfies the splitting condition. Similarly, one can proceed also with $F$ given by (1.13), see Appendix for more details. For the most general example (1.14) we only note here, that the validity of the splitting conditions follows again from the choice of $Q_{i}$, the conditions (1.15)-(1.16) represents the assumption (1.3) and finally that (1.17) implies (1.7)(1.8). It is also evident that potentials (1.11)-(1.14) can generate non-diagonal systems, and in addition they do not need to be convex.

Above, we presented all structural assumptions on $F$ that are used in the paper. However, since we do not use the ellipticity condition (1.5), we must add an additional hypothesis on the qualitative properties of the weak solution $u$ to (1.4). In order to simplify the presentation, we omit the dependence of $F$ on $x$ here and formulate the remaining assumptions and the main results only for $F$ being $x$-independent. The full setting with $x$-dependence of $F$ is discussed in Section 6. We say in what follows that a weak solution $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ to (1.4) satisfies the Noether equation if

$$
\begin{equation*}
\sum_{i=1}^{d} \sum_{\alpha=1}^{N} D_{i}\left(F_{\eta_{i}^{\alpha}}(\nabla u) \cdot D_{k} u^{\alpha}\right)=D_{k} F(\nabla u) \text { in } \Omega, \quad \text { for all } k=1, \ldots, d \tag{1.18}
\end{equation*}
$$

in the sense of distribution, where we denote $D_{i}:=\frac{\partial}{\partial x_{i}}$. Note that by using the assumption (1.3) it is evident that (1.18) is meaningful for any weak solution. Moreover, the identity (1.18) can be formally derived from (1.4) $)_{1}$ by multiplying the $\alpha$-th equation in (1.4) by $D_{k} u^{\alpha}$, summing the results over $\alpha=1, \ldots, N$ and by using the potential structure of the equations. An alternative way, how to derive (1.18) from the variational problem (1.1) is to minimize w.r.t. internal variables, see $[1,5]$ for details. Such an formal procedure can be justified if one assumes in addition that $|\nabla u|^{\frac{p}{2}} \in W^{1,2}$. Such regularity of the solution usually holds for convex potentials satisfying (1.5). We refer to Section 2 where the validity of (1.18) is proved rigorously for potentials satisfying (1.5). Solutions to Euler equation (1.4), which in addition satisfy (1.18) are also called stationary solutions in literature, c.f. [5]. Since the notion stationary is often used in many context we prefer to follow Giaquinta and Hildebrandt [8] who call the above equation (1.18) Noether equation in honor of Emmy Noether [15], who used this equation to study invariance properties of variational problems, see also [20] for other properties of the Noether equation. An alternative naming was used by Klaus Steffen in [19] who calls equation (1.18) Euler equation of second order. It is worth noticing that the
same equation was used for proving partial regularity results for harmonic mapping (see [5]). To point out the difference between the harmonic mapping setting and this paper, we should mention that while for harmonic mapping one is usually not able to prove the Caccioppoli inequality, it is not the case in our setting and this inequality is the second key tool for proving the Hölder continuity of the solution. The importance of the Noether equation also arises in elasticity theory. In this setting the term on the left hand side of (1.18) is sometimes called the Eshelby tensor (see [4]), and the Noether equation was also used for proving the uniqueness of minima to (1.1) even for non-convex functions $F$ in elasticity theory, see [10].

The first theorem, we present here, still deals with the ellipticity condition (1.5). Having this condition we do not need to assume the validity of the Noether equation and the result holds for the unique weak solution to (1.4) that surely exits by using the monotone operator theory.
Theorem 1.1. Let $p \in(1, \infty)$ and $F$ satisfy (1.5)-(1.10) and be $x$-independent. Then there exists $\alpha>0$ such that the unique weak solution $u$ to (1.4) belongs to $\mathcal{C}_{\text {loc }}^{\alpha}\left(\Omega ; \mathbb{R}^{N}\right)$. Moreover, for all $\Omega^{\prime} \subset \subset \Omega$ we have the estimate

$$
\|u\|_{\mathcal{C}^{\alpha}\left(\Omega^{\prime}\right)} \leq C\left(\Omega^{\prime}\right)\|v\|_{1, p} .
$$

In addition, there exists $C>0$ such that for all $x_{0} \in \Omega$ and all $\varepsilon>0$ the solution $u$ satisfies the following potential inequality

$$
\begin{align*}
& \int_{B_{r}\left(x_{0}\right)} \frac{\varepsilon|\nabla u|^{p}}{r^{\varepsilon}\left|x-x_{0}\right|^{d-p-\varepsilon}}+\frac{\left|\nabla u \cdot\left(x-x_{0}\right)\right|^{2}}{\left|x-x_{0}\right|^{d-p+2}} d x \\
& \quad \leq C\left(\frac{r}{R}\right)^{\alpha p}\left(R^{p}+\int_{B_{R}\left(x_{0}\right)} \frac{|\nabla u|^{p}}{R^{d-p}} d x\right) \tag{1.19}
\end{align*}
$$

for all $0<2 r<R<\operatorname{dist}\left(x_{0}, \partial \Omega\right)$.
The second result that does not require the ellipticity condition (1.5), and for which the solution might not be a minimizer of any variational problem, is the following.

Theorem 1.2. Let $p \in(1, \infty)$ and $F$ be $x$-independent and satisfy (1.3) and (1.7)(1.10). Then there exists $\alpha>0$ such that any weak solution $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ to (1.4) satisfying Noether's equation (1.18) belongs to $\mathcal{C}_{\text {loc }}^{\alpha}\left(\Omega ; \mathbb{R}^{N}\right)$ and satisfies the estimate (1.19). Moreover, for all $\Omega^{\prime} \subset \subset \Omega$ we have

$$
\|u\|_{\mathcal{C}^{\alpha}\left(\Omega^{\prime}\right)} \leq C\left(\Omega^{\prime}\right)\|v\|_{1, p} .
$$

The novelty of the present paper concerning the analytical techniques is that we establish a weighted hole-filling inequality in a setting where a Green function is not available (cf. [9]) and where a sub-optimal weight can be used (" $\varepsilon$-hole filling").

The rest of the paper is devoted to the proofs of Theorem 1.1 and Theorem 1.2. In Section 2 there is shown that the assumption on ellipticity (1.5) is sufficient for proving that the unique weak solution of (1.4) satisfies the Noether equation (1.18). Consequently, Theorem 1.2 then implies Theorem 1.1. Next, in Section 3 we derive a hole-filling like inequality for a gradient of $u$ with a certain weight. In Section 4 the Caccioppoli-like inequality is established and it again takes the form of hole-filling like inequality again with a certain weight. Finally, in Section 5, we combine both results. The $\varepsilon$-hole-filling like method is used to derive the estimate (1.19) and then it is a routine to show the Hölder continuity of a solution. For
sake of simplicity in the proofs presented in Sections 3-5 we drop the pollution term in (1.8) and (1.10), i.e., we set $\theta=0$. We also consider only the case for which $b$, that is introduced in (1.9), is the identity matrix. The (formal) proof for general $\theta \in(0,1)$ and $b$ satisfying (1.10) is given in the last section, where also some possible extensions, as estimates up to the boundary and the dependence of $b$ on the spatial variable $x$, are shortly discussed.

## 2. The Noether equation

In this section we recall the uniform a priori estimates for the unique solution $u$ to (1.4) based on the ellipticity condition (1.5). Moreover we recall the proof that the solution is in fact the strong one and that it also satisfies the Noether equation (1.18).

Lemma 2.1. Let $\Omega$ be an open bounded set with Lipschitz boundary and $p \in(1, \infty)$. Assume that $F$ satisfies (1.5) and $v \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$. Then there exists a unique weak solution $u$ to (1.4) such that

$$
\begin{align*}
\|u\|_{1, p} & \leq C\left(1+\|v\|_{1, p}\right),  \tag{2.1}\\
\left\||\nabla u|^{\frac{p}{2}}\right\|_{W^{1,2}\left(\Omega^{\prime}\right)} & \leq C\left(\Omega^{\prime}, v\right) . \tag{2.2}
\end{align*}
$$

Moreover, the Noether equation (1.18) is satisfied in the following sense

$$
\begin{equation*}
\int_{\Omega}\left(\sum_{i, j=1}^{d} \sum_{\alpha=1}^{N} F_{\eta_{i}^{\alpha}}(\nabla u) D_{i} \psi_{j} D_{j} u^{\alpha}\right)=\int_{\Omega} F(\nabla u) \operatorname{div} \psi d x \tag{2.3}
\end{equation*}
$$

for all $\psi \in \mathcal{C}_{0}^{0,1}\left(\Omega ; \mathbb{R}^{d}\right)$.
Having this Lemma, it is easy by using Theorem 1.2 to prove Theorem 1.1. Indeed, let us assume that all assumptions of Theorem 1.1 are satisfied. Then by using Lemma 2.1, we see that the weak solution also satisfy the Noether equation (1.18) and therefore we can apply Theorem 1.2 to prove the statement of Theorem 1.1.

Proof of Lemma 2.1. The existence of a unique weak solution $u$ to (1.4) that satisfies (2.1) follows from the standard Monotone operator theory (due to the uniform $p$-convexity of $F$ ). Thus, to prove (2.2), we introduce a mollified problem for which we have existence of "smooth" solution, then we derive (2.2) for the approximation where the bounded will not depend on the order of approximation and therefore in the limit still satisfies (2.2). Hence we introduce the operator $L_{\varepsilon}$ as

$$
L_{\varepsilon} u:=\varepsilon \triangle^{2 m} u+L u
$$

for some sufficiently large $m \in \mathbb{N}$. We also find a regularization of $v$ such that

$$
\begin{aligned}
& v^{\varepsilon} \rightarrow v \\
& \varepsilon v^{\varepsilon} \rightarrow 0 \text { strongly in } W^{1, p}\left(\mathbb{R}^{d} ; \mathbb{R}^{N}\right), \\
& \text { strongly in } W^{4 m, 2}\left(\mathbb{R}^{d} ; \mathbb{R}^{N}\right) .
\end{aligned}
$$

Here $v$ denotes the extension of the boundary value onto the whole space. Finally, we study the problem

$$
\begin{align*}
& L_{\varepsilon} u^{\varepsilon}=0 \text { in } \Omega \\
& \nabla^{l} u^{\varepsilon}=\nabla^{l} v^{\varepsilon} \text { on } \partial \Omega \quad \text { for all } l=0, \ldots, m \tag{2.4}
\end{align*}
$$

The solvability, uniqueness and existence of strong solution to (2.4) is standard if one takes sufficiently large $m$. Moreover, we get an uniform estimate

$$
\begin{equation*}
\varepsilon \int_{\Omega}\left|\nabla^{m} u\right|^{2} d x \leq K \tag{2.5}
\end{equation*}
$$

Therefore we can test (1.4) by $-\operatorname{div}\left(\nabla u^{\varepsilon} \tau^{2}\right)$, where $\tau$ is an nonnegative smooth function with compact support in $\Omega$. Doing so, and using integration by parts and the estimate (2.5), we find that (dropping all terms with correct sign)

$$
\begin{equation*}
-\int_{\Omega} L u^{\varepsilon} \operatorname{div}\left(\nabla u \tau^{2}\right) d x \leq C(\tau) \tag{2.6}
\end{equation*}
$$

Hence, we integrate twice by parts to observe that (we omit writing $\varepsilon$ in what follows)

$$
\begin{aligned}
-\int_{\Omega} L u \operatorname{div} & \left(\nabla u \tau^{2}\right) d x=\sum_{i, j=1}^{d} \sum_{\alpha=1}^{N} \int_{\Omega} D_{i}\left(F_{\eta_{i}^{\alpha}}(\nabla u)\right) D_{j}\left(\tau^{2} D_{j} u^{\alpha}\right) d x \\
& =\sum_{i, j=1}^{d} \sum_{\alpha=1}^{N} \int_{\Omega} D_{j}\left(F_{\eta_{i}^{\alpha}}(\nabla u)\right) D_{i}\left(\tau D_{j} u^{\alpha}\right) d x \\
& =\sum_{i, j, k=1}^{d} \sum_{\alpha, \beta=1}^{N} \int_{\Omega} F_{\eta_{i}^{\alpha} \eta_{k}^{\beta}}(\nabla u) D_{j k} u^{\beta}\left(D_{i} \tau^{2} D_{j} u^{\alpha}+\tau^{2} D_{j i} u^{\alpha}\right) d x
\end{aligned}
$$

Consequently, using (1.5) we get that

$$
\begin{aligned}
-\int_{\Omega} L u \operatorname{div}\left(\nabla u \tau^{2}\right) d x \geq & \alpha_{0} \int_{\Omega}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}\left|\nabla^{2} u\right|^{2} \tau^{2} d x \\
& -C \int_{\Omega}\left(\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}\left|\nabla^{2} u\right|^{2} \tau^{2}\right)^{\frac{1}{2}}\left(1+|\nabla u|^{\frac{p}{2}}\right) d x
\end{aligned}
$$

Thus, using the Young inequality, (2.1) and (2.6) we find that for any $\Omega^{\prime} \subset \subset \Omega$

$$
\begin{equation*}
\int_{\Omega^{\prime}}\left(\delta_{0}+\left|\nabla u^{\varepsilon}\right|^{2}\right)^{\frac{p-2}{2}}\left|\nabla^{2} u^{\varepsilon}\right|^{2} d x \leq C\left(\Omega^{\prime}, v\right) \tag{2.7}
\end{equation*}
$$

and we see that (2.2) follows for $u^{\varepsilon}$ and having uniformity of such an estimate we get (2.2) and (2.7) also for the unique weak solution to (1.4). Moreover, by using the standard monotone operator theory, it is simple to deduce the strong convergence of $u^{\varepsilon}$ to $u$ in $W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$.

It remains to prove (2.3). Since $u^{\varepsilon}$ is sufficiently smooth, we can test (2.4) by $\varphi:=\sum_{i=1}^{d} \psi_{i} D_{i} u^{\varepsilon}$, where $\psi \in \mathcal{D}\left(\Omega ; \mathbb{R}^{d}\right)$. Therefore, after integration by parts we find that

$$
\begin{equation*}
0=\int_{\Omega} \sum_{\alpha=1}^{N}\left(L_{\varepsilon} u^{\varepsilon}\right)^{\alpha} \varphi^{\alpha} d x=\int_{\Omega} \sum_{i, j=1}^{d} \sum_{\alpha=1}^{N} F_{\eta_{i}^{\alpha}}\left(\nabla u^{\varepsilon}\right) D_{i}\left(\psi_{j} D_{j}\left(u^{\varepsilon}\right)^{\alpha}\right)+O(\varepsilon) \tag{2.8}
\end{equation*}
$$

where "pollution" term on the right hand side of (2.8) tends to 0 as $\varepsilon \rightarrow 0_{+}$. By using the potential structure of the equation, we see that the term on the right
hand side of (2.8) can be rewritten as

$$
\begin{aligned}
& \sum_{i, j=1}^{d} \sum_{\alpha=1}^{N} F_{\eta_{i}^{\alpha}}\left(\nabla u^{\varepsilon}\right) D_{i}\left(\psi_{j} D_{j}\left(u^{\varepsilon}\right)^{\alpha}\right) \\
& \quad=\psi \cdot \nabla F\left(\nabla u^{\varepsilon}\right)+\sum_{i, j=1}^{d} \sum_{\alpha=1}^{N} F_{\eta_{i}^{\alpha}}\left(\nabla u^{\varepsilon}\right) D_{i} \psi_{j} D_{j}\left(u^{\varepsilon}\right)^{\alpha} .
\end{aligned}
$$

Next, we substitute this into (2.8), integrate by parts and let $\varepsilon \rightarrow 0_{+}$. Hence, using strong convergence of $u^{\varepsilon}$ we derive (2.3).

## 3. Estimates based on $p$-Structure and the Noether equation

The next step is to deduce a weighted local estimate for any $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ that satisfies (2.3). In order to get the local estimates, we introduce the standard notation

$$
\begin{aligned}
& B_{R}\left(x_{0}\right):=\left\{x \in \mathbb{R}^{d} ;\left|x-x_{0}\right| \leq R\right\}, \\
& A_{R}\left(x_{0}\right):=B_{2 R}\left(x_{0}\right) \backslash B_{R}\left(x_{0}\right)
\end{aligned}
$$

and we define the cut-off function $\tau_{R}$ as

$$
\tau_{R}(s):=\tau(s / R)
$$

for some $\tau \in \mathcal{D}([0,2])$ being nonnegative non-increasing function such that $\tau \equiv 1$ on $[0,1]$.

Lemma 3.1. Let $p \in(1, d)$ and $F$ satisfy (1.7)-(1.10). Then there is $C>0$ such that for any $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ satisfying the Noether equation (2.3), any $x_{0} \in \Omega$ and any $R>0$ such that $B_{2 R}\left(x_{0}\right) \subset \Omega$ the following estimate holds

$$
\begin{gather*}
\int_{B_{R}\left(x_{0}\right)} \frac{\left(|\nabla u|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}\left|\left(x-x_{0}\right) \cdot \nabla u\right|^{2}}{\left|x-x_{0}\right|^{d-p+2}} d x  \tag{3.1}\\
\leq C \int_{A_{R}\left(x_{0}\right)} \frac{\left(|\nabla u|^{2}+\delta_{0}\right)^{\frac{p}{2}}}{\left|x-x_{0}\right|^{d-p}} d x+C R^{p} .
\end{gather*}
$$

Proof. To simplify the paper, here we give the rigorous proof only for the case when $\theta=0, b_{i j}=\delta_{i j}$ and $G \equiv 0$, the parameters $\theta, b$ and $G$ appear in the assumptions (1.7)-(1.10). The proof in the full generality is described in Section 6. We assume that $x_{0}=0$ and that $B_{2 R}(0) \subset \Omega$. For other $x_{0}$ the proof is similar. The proof of (3.1) is based on using $x|x|^{p-d} \tau_{R}^{p}(|x|)$ as a test function in (2.3). Since such setting is not possible (due to low regularity), we first regularize the test function and then we pass to the limit. Thus, for some fixed $\varepsilon>0$ we use in (2.3) the test function being of the form

$$
\psi(x):=\frac{x \tau_{R}^{p}(|x|)}{|x|^{d-p}+\varepsilon}, \quad \varepsilon>0
$$

Such setting is possible since $\psi$ is a Lipschitz function with compact support. Simple computation gives

$$
\begin{align*}
D_{j} \psi_{i} & =\frac{\delta_{i j} \tau_{R}^{p}(|x|)}{|x|^{d-p}+\varepsilon}-(d-p) \frac{x_{i} x_{j}|x|^{d-p-2} \tau_{R}^{p}(|x|)}{\left(|x|^{d-p}+\varepsilon\right)^{2}}+p \frac{x_{i} x_{j} \tau_{R}^{p-1}(|x|) \tau_{R}^{\prime}(|x|)}{|x|\left(|x|^{d-p}+\varepsilon\right)}  \tag{3.2}\\
\operatorname{div} \psi & =\frac{p \tau_{R}^{p}(|x|)|x|^{d-p}+\varepsilon d \tau_{R}^{p}(|x|)}{\left(|x|^{d-p}+\varepsilon\right)^{2}}+p \frac{|x| \tau_{R}^{p-1}(|x|) \tau_{R}^{\prime}(|x|)}{|x|^{d-p}+\varepsilon}
\end{align*}
$$

Next, we evaluate all terms in (2.3) with our $\psi$. Therefore using (3.2) we get that

$$
\begin{align*}
& \int_{\Omega} F(\nabla u) \operatorname{div} \psi d x \\
& \quad=\int_{\Omega} F(\nabla u)\left(\frac{p \tau_{R}^{p}(|x|)|x|^{d-p}+\varepsilon d \tau_{R}^{p}(|x|)}{\left(|x|^{d-p}+\varepsilon\right)^{2}}+p \frac{|x| \tau_{R}^{p-1}(|x|) \tau_{R}^{\prime}(|x|)}{|x|^{d-p}+\varepsilon}\right) d x \tag{3.3}
\end{align*}
$$

Similarly, using (3.2) again and using the assumption (1.9) we obtain that (note that here is the point where the proof is simplified due to the fact that $b_{i j}=\delta_{i j}$ and $G \equiv 0$ )

$$
\begin{aligned}
& \int_{\Omega}\left(\sum_{i, j=1}^{d} \sum_{\alpha=1}^{N} F_{\eta_{i}^{\alpha}}(\nabla u) D_{i} \psi_{j} D_{j} u^{\alpha}\right) d x \\
& =\int_{\Omega}\left(\sum _ { i , j = 1 } ^ { d } \sum _ { \alpha , \beta = 1 } ^ { N } A ^ { \alpha \beta } ( \nabla u ) D _ { i } u ^ { \beta } D _ { j } u ^ { \alpha } \left(\frac{\delta_{i j} \tau_{R}^{p}(|x|)}{|x|^{d-p}+\varepsilon}-(d-p) \frac{x_{i} x_{j}|x|^{d-p-2} \tau_{R}^{p}(|x|)}{\left(|x|^{d-p}+\varepsilon\right)^{2}}\right.\right. \\
& \left.\left.\quad+p \frac{x_{i} x_{j} \tau_{R}^{p-1}(|x|) \tau_{R}^{\prime}(|x|)}{|x|\left(|x|^{d-p}+\varepsilon\right)}\right)\right) d x \\
& =\int_{\Omega} \sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(\nabla u)\left(\frac{\nabla u^{\alpha} \cdot \nabla u^{\beta} \tau_{R}^{p}(|x|)}{|x|^{d-p}+\varepsilon}-(d-p) \frac{\left(x \cdot \nabla u^{\alpha}\right)\left(x \cdot \nabla u^{\beta}\right)|x|^{d-p-2} \tau_{R}^{p}(|x|)}{\left(|x|^{d-p}+\varepsilon\right)^{2}}\right. \\
& \left.\quad+p \frac{\left(x \cdot \nabla u^{\alpha}\right)\left(x \cdot \nabla u^{\beta}\right) \tau_{R}^{p-1}(|x|) \tau_{R}^{\prime}(|x|)}{|x|\left(|x|^{d-p}+\varepsilon\right)}\right) d x .
\end{aligned}
$$

Thus, using these identities in (2.3) and moving the terms with corresponding signs to the left respectively right hand side we deduce that

$$
\begin{align*}
& -p \int_{\Omega} \sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(\nabla u) \frac{\left(x \cdot \nabla u^{\alpha}\right)\left(x \cdot \nabla u^{\beta}\right)|x|^{d-p-2} \tau_{R}^{p-1} \tau_{R}^{\prime}}{|x|\left(|x|^{d-p}+\varepsilon\right)} d x \\
& \quad+\int_{\Omega} F(\nabla u) \frac{p \tau_{R}^{p}|x|^{d-p}+\varepsilon d \tau_{R}^{p}}{\left(|x|^{d-\gamma}+\varepsilon\right)^{2}} d x \\
& \quad+(d-p) \int_{\Omega} \sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(\nabla u) \frac{\left(x \cdot \nabla u^{\alpha}\right)\left(x \cdot \nabla u^{\beta}\right)|x|^{d-p-2} \tau_{R}^{p}}{\left(|x|^{d-p}+\varepsilon\right)^{2}} d x  \tag{3.4}\\
& =-p \int_{\Omega} F(\nabla u) \frac{|x| \tau_{R}^{p-1} \tau_{R}^{\prime}}{|x|^{d-p}+\varepsilon} d x+\int_{\Omega} \sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(\nabla u) \frac{\nabla u^{\alpha} \cdot \nabla u^{\beta} \tau^{p}}{|x|^{d-p}+\varepsilon} d x .
\end{align*}
$$

First, since $\tau_{R}$ is non-increasing we see, after using (1.10), that all integrals on the left hand side are nonnegative and the same holds also for the right hand side. Moreover, we can take easily the limit $\varepsilon \rightarrow 0_{+}$in all terms containing $\tau_{R}^{\prime}$ since we do not integrate over possible singularity. To handle also the second term on the right hand side, we use (1.8) and (1.9) to deduce that (note that here we use the fact that $\theta=0$ )

$$
\sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(\nabla u) \nabla u^{\alpha} \cdot \nabla u^{\beta}=F_{\eta}(\nabla u) \cdot \nabla u \leq p F(\nabla u)+C .
$$

Therefore,

$$
\sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(\nabla u) \frac{\nabla u^{\alpha} \cdot \nabla u^{\beta} \tau^{p}}{|x|^{d-p}+\varepsilon} \leq F(\nabla u) \frac{p \tau^{p}|x|^{d-p}+\tau_{R}^{p} p \varepsilon}{\left(|x|^{d-p}+\varepsilon\right)^{2}}+\frac{C \tau_{R}^{p}}{|x|^{d-p}+\varepsilon}
$$

and we see that since $d>p$ we can handle a crucial part of the second integral on the right hand side of (3.4) by the second integral on the left hand side of (3.4). Thus, using finally (1.5) and (1.10) we deduce from (3.4) that

$$
\begin{aligned}
& \int_{\Omega}\left(|\nabla u|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|x \cdot \nabla u|^{2} \frac{|x|^{d-p-2} \tau_{R}^{p}}{\left(|x|^{d-\gamma}+\varepsilon\right)^{2}} d x \\
& \leq C\left(d, p, \alpha_{0}, \alpha_{1}\right) \int_{\Omega}\left(|\nabla u|^{2}+\delta_{0}\right)^{\frac{p}{2}} \frac{|x| \tau_{R}^{p-1}\left|\tau_{R}^{\prime}\right|}{|x|^{d-p}+\varepsilon}+\frac{C \tau_{R}^{p}}{|x|^{d-p}+\varepsilon} d x .
\end{aligned}
$$

Now, it remains to let $\varepsilon \rightarrow 0_{+}$. For the limiting procedure in the integral on the left hand side one can use the standard monotone convergence theorem and the convergence in the integrals on the right hand side is easy, since in the first one we exclude from the integration domain the singularity at 0 and the second one is convergent and can be estimated by $C R^{p}$.

## 4. Caccioppoli inequality

In this section we use the structural assumption (1.7) to derive a local estimate for not-weighted gradient (Caccioppoli like inequality). To simplify and shorten the notation we formulate the result only for $x_{0} \equiv 0$ and all $R$ such that $B_{2 R}(0) \subset \Omega$. The proof for other $x_{0}$ is however the same. We also use the notation $B_{R}:=B_{R}(0)$ and $A_{R}:=A_{R}(0)$.

Lemma 4.1. Let $p \in(1, d)$ and $F$ satisfy (1.8), (1.9) and (1.10). Assume that $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ is a weak solution to (1.4). Then the following estimate holds

$$
\begin{equation*}
\int_{B_{R}} \frac{\left(|\nabla u|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|\nabla u|^{2}}{R^{d-p}} d x \leq C\left(I_{R}^{p}+R^{p}\right)+\frac{1}{4} \int_{A_{R}} \frac{\left(|\nabla u|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|\nabla u|^{2}}{(2 R)^{d-p}} d x \tag{4.1}
\end{equation*}
$$

where
$I_{R}^{p}:= \begin{cases}\left(\int_{A_{R}} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x|^{2}}{R^{d-p+2}} d x\right)^{\frac{1}{2}}\left(\int_{A_{R}} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p}{2}}}{R^{d-p}} d x\right)^{\frac{1}{2}} \quad \text { for } p \geq 2, \\ \left(\int_{A_{R}} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x|^{2}}{R^{d-p+2}} d x\right)^{\frac{1}{p}}\left(\int_{A_{R}} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p}{2}}}{R^{d-p}} d x\right)^{\frac{1}{p}} \quad \text { for } p<2 .\end{cases}$
Proof. Again for simplicity, we present the proof only for $b_{i j}=\delta_{i j}$ and $G \equiv 0$. The complete proof is given in Section 6. Thus, we test the $\alpha$-th equation in $(1.4)_{1}$ by $\left(u^{\alpha}-c^{\alpha}\right) \tau_{R}^{p}(|x|)$. After taking the sum over $\alpha=1, \ldots, N$ and integration by parts we find that

$$
\begin{equation*}
\int_{\Omega} F_{\eta} \cdot \nabla\left((u-c) \tau_{R}^{p}\right) d x=0 \tag{4.2}
\end{equation*}
$$

Next using (1.7), (1.9) (with $b_{i j}=\delta_{i j}$ and $\left.G \equiv 0\right)$ and the properties of $\tau_{R}$ we find that

$$
\begin{align*}
& \int_{B_{R}} \alpha_{0}\left(|\nabla u|^{2}+\delta_{0}\right)^{\frac{p-2}{2}}|\nabla u|^{2} d x \\
& \quad \leq-\int_{A_{R}} \sum_{\alpha, \beta=1}^{N} A^{\alpha \beta}(\nabla u)\left(\nabla u^{\beta} \cdot x\right)\left(u^{\alpha}-c^{\alpha}\right)|x|^{-1} \tau_{R}^{\prime} d x \tag{4.3}
\end{align*}
$$

Finally, using (1.10) and again the properties of $\tau_{R}$ we derive from (4.3) the following estimate

$$
\begin{align*}
& \int_{B_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u|^{2} d x \\
& \leq C\left(\alpha_{0}, \alpha_{1}\right) \int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x||u-c| \frac{\left|\tau_{R}^{\prime}\right|}{|x|} d x  \tag{4.4}\\
& \leq C R^{-2} \int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x||u-c| d x=: C R^{-2} I_{1} .
\end{align*}
$$

In what follows we set $c:=\left|A_{R}\right|^{-1} \int_{A_{R}} u d x$ and split the proof onto two parts.
Case $p \geq 2$ : In this case we can estimate $I_{1}$ by using the Hölder and Poincare inequalities as

$$
\begin{aligned}
I_{1}= & C \int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{4}}\left(\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{4}}|\nabla u \cdot x|\right)(|u-c|) d x \\
\leq & C\left(\int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p}{2}} d x\right)^{\frac{p-2}{2 p}}\left(\int_{A_{R}}|u-c|^{p} d x\right)^{\frac{1}{p}} \cdot \\
& \cdot\left(\int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x|^{2} d x\right)^{\frac{1}{2}} \\
\leq & C R\left(\int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p}{2}} d x\right)^{\frac{1}{2}}\left(\int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x|^{2} d x\right)^{\frac{1}{2}} \\
= & C R^{d-p+2}\left(\int_{A_{R}} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p}{2}}}{R^{d-p}} d x\right)^{\frac{1}{2}}\left(\int_{A_{R}} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x|^{2}}{R^{d-p+2}} d x\right)^{\frac{1}{2}}
\end{aligned}
$$

Thus, inserting these estimates into (4.4) and dividing the result by $R^{d-p}$ we arrive at (4.1).

Case $1<p<2$ : In this case we define $\alpha:=\frac{2}{p^{\prime}}<1$ and by using the Hölder inequality and the Poincare inequality we conclude that

$$
\begin{aligned}
I_{1} & \leq C \int_{A_{R}}\left(\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{4}}|\nabla u \cdot x|\right)^{\alpha}|\nabla u \cdot x|^{1-\alpha}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}+\frac{\alpha(2-p)}{4}}|u-c| d x \\
& \leq C R^{1-\alpha} \int_{A_{R}}\left(\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{4}}|\nabla u \cdot x|\right)^{\alpha}|u-c| d x \\
& \leq C R^{1-\alpha}\left(\int_{A_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x|^{2} d x\right)^{\frac{1}{p^{\prime}}}\left(\int_{A_{R}}|u-c|^{p} d x\right)^{\frac{1}{p}} \\
& \leq C R^{d-p+2}\left(\int_{A_{R}} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u \cdot x|^{2}}{R^{d-p+2}} d x\right)^{\frac{1}{p^{\prime}}}\left(\int_{A_{R}} \frac{|\nabla u|^{p}}{R^{d-p}} d x\right)^{\frac{1}{p}},
\end{aligned}
$$

which again, after substitution these relations into (4.4) and division the result by $R^{d-p}$, leads to (4.1).

## 5. Hole filling technique

In this section we combine the results from previous two sections and by using a generalized version of the hole-filling method we prove the following lemma that combined with estimate (3.1) lead to the statement of Theorem 1.2.

Lemma 5.1. There exists $\alpha>0$ such that any $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ satisfying (3.1) and (4.1) belongs to $\mathcal{C}^{\alpha}\left(\Omega^{\prime} ; \mathbb{R}^{N}\right)$ for any $\Omega^{\prime} \subset \subset \Omega$. Moreover, for all $x_{0} \in \Omega$ and any $R_{0}>0$ such that $B_{R_{0}}\left(x_{0}\right) \in \Omega$ we have

$$
\begin{equation*}
\int_{B_{R}\left(x_{0}\right)} \frac{|\nabla u|^{2}}{R^{d-p+p \alpha}} d x \leq C+C \int_{B_{R_{0}\left(x_{0}\right)}} \frac{|\nabla u|^{2}}{R_{0}^{d-p+p \alpha}} d x \quad \text { for all } 0<R<R_{0} / 2 . \tag{5.1}
\end{equation*}
$$

Proof. In order to shorten the proof we define

$$
\begin{aligned}
U_{R}\left(x_{0}\right) & :=\int_{B_{R}\left(x_{0}\right)} \frac{|\nabla u|^{p}}{R^{d-p}} d x \\
W_{R}\left(x_{0}\right) & :=\int_{B_{R}\left(x_{0}\right)} \frac{\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}\left|\nabla u \cdot\left(x-x_{0}\right)\right|^{2}}{\left|x-x_{0}\right|^{d-p+2}} d x
\end{aligned}
$$

Using this notation, we easily derive from (3.1) that (here we assume that $B_{2 R}\left(x_{0}\right) \in$ $\Omega$ )

$$
\begin{equation*}
W_{R}\left(x_{0}\right) \leq C R^{p}+C U_{2 R}\left(x_{0}\right) . \tag{5.2}
\end{equation*}
$$

Next, we define $q:=\min (2, p)$. With this definition, we can also easily estimate $I_{R}^{p}$ in Lemma 4.1 as

$$
\begin{equation*}
I_{R}^{p} \leq\left(W_{2 R}\left(x_{0}\right)-W_{R}\left(x_{0}\right)\right)^{\frac{1}{q^{\prime}}}\left(R^{\frac{p}{q}}+\left(U_{2 R}\left(x_{0}\right)\right)^{\frac{1}{q}}\right) \tag{5.3}
\end{equation*}
$$

Therefore, substituting this into (4.1) we find that

$$
\begin{equation*}
U_{R}\left(x_{0}\right) \leq C\left(W_{2 R}\left(x_{0}\right)-W_{R}\left(x_{0}\right)\right)^{\frac{1}{q^{\prime}}}\left(R^{\frac{p}{q}}+\left(U_{2 R}\left(x_{0}\right)\right)^{\frac{1}{q}}\right)+C R^{p}+\frac{1}{4} U_{2 R}\left(x_{0}\right) \tag{5.4}
\end{equation*}
$$

In the following we combine estimates (5.2) and (5.4) to deduce (5.1). We proceed by using a modified hole-filling technique. We see that the good "hole-filling" term, i.e., the term $\left(W_{2 R}-W_{R}\right)$ is only in the inequality (5.4). Hence, this inequality has to play more important role than (5.2). Therefore, we multiply (5.2) by $\varepsilon>0$ small, which will be chosen later, multiply (5.4) by 4 and sum the result to get

$$
\begin{align*}
4 U_{R}\left(x_{0}\right)+\varepsilon W_{R}\left(x_{0}\right) \leq C & \left(W_{2 R}\left(x_{0}\right)-W_{R}\left(x_{0}\right)\right)^{\frac{1}{q^{\prime}}}\left(R^{\frac{p}{q}}+\left(U_{2 R}\left(x_{0}\right)\right)^{\frac{1}{q}}\right)  \tag{5.5}\\
+ & C R^{p}+(C \varepsilon+1) U_{2 R}\left(x_{0}\right)
\end{align*}
$$

Next applying the Young inequality on the first term on the right hand side and moving the resulting term $W_{R}\left(x_{0}\right)$ onto the left hand side, we get (here $C$ still denotes some generic constant)

$$
\begin{equation*}
4 U_{R}\left(x_{0}\right)+(\varepsilon+C) W_{R}\left(x_{0}\right) \leq C W_{2 R}\left(x_{0}\right)+C R^{p}+(C \varepsilon+2) U_{2 R}\left(x_{0}\right) \tag{5.6}
\end{equation*}
$$

Note, that up to now, $\varepsilon$ was arbitrary, while $C$ is some generic constant independent of $\varepsilon$. Thus, if we set $\varepsilon:=C^{-1}$ and divide (5.6) by 4 we get

$$
\begin{equation*}
U_{R}\left(x_{0}\right)+\frac{C^{-1}+C}{4} W_{R}\left(x_{0}\right) \leq \frac{C}{4} W_{2 R}\left(x_{0}\right)+C R^{p}+\frac{3}{4} U_{2 R}\left(x_{0}\right) . \tag{5.7}
\end{equation*}
$$

Thus, setting finally

$$
Z_{R}\left(x_{0}\right):=U_{R}\left(x_{0}\right)+\frac{C^{-1}+C}{4} W_{R}\left(x_{0}\right)
$$

we see from (5.7) that

$$
\begin{equation*}
Z_{R}\left(x_{0}\right) \leq \gamma Z_{2 R}\left(x_{0}\right)+C R^{p} \tag{5.8}
\end{equation*}
$$

with $\gamma<1$ given as

$$
\gamma:=\max \left(\frac{3}{4}, \frac{C}{C^{-1}+C}\right)
$$

Thus, defining $\alpha:=\min \left(2^{-1},-p^{-1} \ln _{2} \gamma\right)$ and dividing (5.8) by $R^{\alpha p}$ we see that

$$
\frac{Z_{R}\left(x_{0}\right)}{R^{\alpha p}} \leq \frac{Z_{2 R}\left(x_{0}\right)}{(2 R)^{\alpha p}}+C R^{p(1-\alpha)}
$$

and therefore for all $R<R_{0} / 2$ we obtain

$$
\frac{Z_{R}\left(x_{0}\right)}{R^{\alpha p}} \leq C\left(\frac{Z_{R_{0}}\left(x_{0}\right)}{R_{0}^{\alpha p}}+R^{p(1-\alpha)}\right)
$$

and we see that (5.1) follows. Consequently, using the Morrey lemma we find that $u \in \mathcal{C}^{\alpha}\left(\Omega^{\prime}\right)$ that completes the proof.

## 6. CONCLUDING REMARKS

In this final section, we first show how to generalize the procedure introduced in Sections 3-4 to cover also the case with general $\theta \in(0,1), b$ and $G$ (these parameters appear in the assumptions (1.7)-(1.10). Next, we also show how our result can be extended to the regularity up to boundary and also onto the case when $F$ and consequently the matrix $b$ is $x$-dependent. To summarize we give a sketch of the proofs of the following theorems.
Theorem 6.1. Let $p \in(1, \infty), \Omega$ be Lipschitz domain and $F$ satisfy (1.5)-(1.10). Assume in addition that $F$ is $\alpha_{0}$-Hölder continuous w.r.t. $x$ in the following sense

$$
\begin{align*}
|F(x, \eta)-F(y, \eta)| & \leq C|x-y|^{\alpha_{0}}(1+|\eta|)^{p} \\
\left|F_{\eta}(x, \eta)-F_{\eta}(y, \eta)\right| & \leq C|x-y|^{\alpha_{0}}(1+|\eta|)^{p-1} \tag{6.1}
\end{align*}
$$

and assume that there is $\alpha_{1}>0$ such that

$$
\begin{equation*}
\sup _{x_{0} \in \Omega} \sup _{R>0} \int_{B_{R}\left(x_{0}\right) \cap \Omega}|\nabla v|^{p} R^{p-d-\alpha_{1} p} d x<\infty . \tag{6.2}
\end{equation*}
$$

Then there exists $\alpha>0$ such that the unique weak solution $u$ to (1.4) belongs to $\mathcal{C}^{\alpha}\left(\Omega ; \mathbb{R}^{N}\right)$.

The proof of this theorem is based on the estimates for $F$ being $x$-independent that are valid up to the boundary, see Subsection 6.3, on generalization of Lemma 3.1 and 4.1, and on using Campanato-like comparison technique for problems with $x$ dependent coefficients, see Subsection 6.4.1. Note that in order to be able to apply Campanato-like method, we require convexity of $F$, assumption (1.5), to guarantee solvability of (1.4).

The second result we present here does not require the convexity assumption (1.5), but since we are not able to use the comparison technique (because we do not know whether the solution to problem with freezed coefficients does exist) we need to "generalize" the notion for Noether equation due to the possible dependence
of $F$ on the spatial variable $x$. Hence similarly as in the introduction, we say that a weak solution $u$ to (1.4) satisfies the Noether equation provided that it satisfies in a sense of distribution the following identities

$$
\begin{align*}
\sum_{i=1}^{d} \sum_{\alpha=1}^{N} D_{i}\left(F_{\eta_{i}^{\alpha}}(x, \nabla u) \cdot D_{k} u^{\alpha}\right)+\frac{\partial F(x, \nabla u)}{\partial x_{k}}  \tag{6.3}\\
=D_{k} F(x, \nabla u) \quad \text { for all } k=1, \ldots, d .
\end{align*}
$$

Note that these equations can be formally derive from (1.4) by multiplying the $\alpha$-th equation by $D_{k} u^{\alpha}$, summing over $\alpha=1, \ldots, N$ and by using the potential structure of the equation. Here it is also evident that due to the presence of the derivative of $F$ w.r.t. $x$ in (6.3) we have to impose a stronger assumption on the regularity of $F$ than the one supposed for convex $F$ in Theorem 6.1, see assumption (6.1). Thus the second main result of the paper is the following.
Theorem 6.2. Let $\Omega$ be a Lipschitz domain, $p \in(1, \infty)$ and $F$ satisfy (1.3), (1.7)(1.10) and

$$
\begin{align*}
\left|\frac{\partial F(x, \eta)}{\partial x}\right| & \leq C(1+|\eta|)^{p}  \tag{6.4}\\
\left|\frac{\partial F_{\eta}(x, \eta)}{\partial x}\right| & \leq C(1+\eta)^{p-1}
\end{align*}
$$

Assume that the boundary data $v$ satisfies (6.2). Then there exists $\alpha>0$ such that any weak solution $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ to (1.4) satisfying the generalized Noether equation (6.3) belongs to $\mathcal{C}^{\alpha}\left(\Omega ; \mathbb{R}^{N}\right)$ and we have

$$
\|u\|_{\mathcal{C}^{\alpha}(\Omega)} \leq C(\Omega, v)
$$

In the rest of this section we give a sketch how one can generalize the procedure developed in the previous sections to obtain Theorem 6.1 and 6.2. First, we give the full proof of Lemma 4.1 and 3.1 in Subsection 6.1 and 6.2. Then in Subsection 6.3 we show how the estimates can be extended up to the boundary for the case $F$ being $x$-independent. Finally, in Subsection 6.4 we present the method that is used to obtain the Hölder continuity of the solution $u$ to (1.4) for $F$ depending on the spatial variable $x$.
6.1. Sketch of the proof of Lemma 4.1 for general constant $b$. Here the essential change is due to the presence of general $b$ and $G$ in (1.9). To be able to derive the estimate (4.1) we first modify our test function. For this purpose, we define a modified scalar product in $\mathbb{R}^{d}$ as

$$
(u, v)_{b}:=\sum_{i, j=1}^{d} b_{i j} v_{i} v_{j} .
$$

Note that due to the assumption $(1.10)_{2}$ it is the scalar product that gives the norm $|v|_{b}$ the is equivalent to standard one. To capture such differences also in localization we introduce

$$
\begin{aligned}
& B_{R}\left(x_{0}\right):=\left\{x \in \mathbb{R}^{d} ;\left|x-x_{0}\right|_{b}<R\right\}, \\
& A_{R}\left(x_{0}\right):=B_{2 R}\left(x_{0}\right) \backslash B_{R}\left(x_{0}\right) .
\end{aligned}
$$

Here we want to point out that the estimates (3.1) and (4.1) are valid with the only taking the correct "deformed" balls $B_{R}$ and annuli $A_{R}$, replacing $|\nabla u \cdot x|^{2}$ by
$(\nabla u, x)_{b}^{2}$. Having this notation, we use the same test function as in Section 4 to derive the inequality similar to (4.3)

$$
\begin{equation*}
\int_{B_{R}}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u|^{2} d x \leq-C \int_{\Omega} \sum_{\alpha=1}^{N} \sum_{i=1}^{d} F_{\eta_{i}^{\alpha}}(\nabla u)\left(u^{\alpha}-c^{\alpha}\right) D_{i} \tau_{R}^{p}(|x|) d x \tag{6.5}
\end{equation*}
$$

Using (1.9) we see that there are now two terms on the right hand side of (6.5) that have to be handled. The second term that appears is $G$ but since $\theta<1$ (see $(1.10)_{3}$ ) it can be easily handled by using the Hölder and the Young inequality and assuming that $c^{\alpha}$ is the mean value of $u^{\alpha}$ over $A_{R}$ as

$$
\int_{\Omega}|G|\left|u^{\alpha}-c^{\alpha}\right|\left|\nabla \tau_{R}^{p}(|x|)\right| d x \leq \frac{R^{d-p}}{4} \int_{A_{R}} \frac{\left(1+|\nabla u|^{2}\right)^{\frac{p-2}{2}}|\nabla u|^{2}}{(2 R)^{d-p}} d x+C R^{d}
$$

The first term involving the matrix $A$ then takes the form

$$
\int_{A_{R}} \sum_{\alpha, \beta=1}^{N} \sum_{i, j=1}^{d} A^{\alpha \beta}(\nabla u) b_{i j} D_{j} u^{\beta}\left(u^{\alpha}-c^{\alpha}\right) p \tau_{R}^{p-1} x_{i}|x|^{-1} \tau_{R}^{\prime}(|x|) d x
$$

Therefore, using the definition of $(\cdot, \cdot)_{b}$ we can continue as in Section 4, we have to only replace the standard scalar product in the term involving $\nabla u \cdot x$ by the deformed one $(\nabla u, x)_{b}$.
6.2. Sketch of the proof of Lemma 3.1 for general $x$-independent $b$ and $G$.

Here we again works with deformed norm $|x|_{b}$ to capture the presence of $b$ in (1.9). Moreover, because of the presence of the pollution term $G$ in (1.9), we do not get directly the estimate (3.1) but we have to iterate. We would also like to remind that the estimates are taken over different balls and we also replace $|x \cdot \nabla u|^{2}$ by $(x, \nabla u)_{b}^{2}$ in (3.1). Thus, in (2.3) we use as a test function the following $\psi$

$$
\psi(x):=\frac{x \tau_{R}^{p}\left(|x|_{b}\right)}{|x|_{b}^{d-\gamma}}, \gamma \in[p, d)
$$

Note that this is only formal and to do it rigorously one has to mollify such $\psi$. But since we already showed the rigorous procedure in Section 3 we omit it here. We can again get by using the definition of $\psi$ and $|x|_{b}$ that
$D_{j} \psi_{i}=\frac{\delta_{i j} \tau_{R}^{p}\left(|x|_{b}\right)}{|x|_{b}^{d-\gamma}}-(d-\gamma) \frac{x_{i} \sum_{k} b_{j k} x_{k} \tau_{R}^{p}\left(|x|_{b}\right)}{|x|_{b}^{d-p+2}}+p \frac{x_{i} \sum_{k} b_{j k} x_{k} \tau_{R}^{p-1}\left(|x|_{b}\right) \tau_{R}^{\prime}\left(|x|_{b}\right)}{|x|_{b}^{d-p+1}}$
$\operatorname{div} \psi=\frac{\gamma \tau_{R}^{p}\left(|x|_{b}\right)}{|x|_{b}^{d-\gamma}}+p \frac{\tau_{R}^{p-1}\left(|x|_{b}\right) \tau_{R}^{\prime}\left(|x|_{b}\right)}{|x|_{b}^{d-p+1}}$.
In what follows we omit the presence of terms involving derivative of $\tau_{R}$ since such terms can be always moved onto the right hand side and are of the same form as the right hand side of (3.1). Thus, we deduce the inequality similar to (3.4) that
has the form

$$
\begin{align*}
& \gamma \int_{\Omega} F(\nabla u)|x|_{b}^{\gamma-d} \tau_{R}^{p} d x \\
& \quad+(d-\gamma) \int_{\Omega} \sum_{\alpha, \beta=1}^{N} \sum_{i, j, k, l=1}^{d} A^{\alpha \beta}(\nabla u) b_{i k} b_{j l} D_{k} u^{\beta} D_{j} u^{\alpha} x_{i} x_{l}|x|_{b}^{p-d-2} \tau_{R}^{p} d x  \tag{6.7}\\
& =\int_{\Omega} F_{\eta}(\nabla u) \cdot \nabla u|x|_{b}^{\gamma-d} \tau_{R}^{p} d x+\int_{\Omega}|G||\nabla u||x|_{b}^{\gamma-d} \tau_{R}^{p} d x+\text { pollution. }
\end{align*}
$$

Thus, using (1.8) and (1.10) we get for some $\eta<1$

$$
\begin{align*}
& (\gamma-p) \int_{\Omega}|\nabla u|^{p}|x|_{b}^{\gamma-d} \tau_{R}^{p} d x \\
& \quad+(d-\gamma) \int_{\Omega}\left(\delta_{0}+|\nabla u|^{2}\right)^{\frac{p-2}{2}}(\nabla u, x)_{b}^{2}|x|_{b}^{p-d-2} \tau_{R}^{p} d x  \tag{6.8}\\
& \quad \leq C \int_{\Omega}|\nabla u|^{\eta p}|x|_{b}^{\gamma-d} \tau_{R}^{p} d x+\text { pollution. }
\end{align*}
$$

Therefore we see that for any $\gamma>p$ we get an estimate

$$
\begin{equation*}
\int_{\Omega}|\nabla u|^{p}|x|_{b}^{\gamma-d} \tau_{R}^{p} d x \leq C(\gamma-p)^{-K} R^{p}+\text { small terms } \tag{6.9}
\end{equation*}
$$

which explodes as $\gamma \rightarrow p_{+}$. Consequently, we can now set $\gamma=p$ in (6.8) and with help of (6.9) (now with setting $\gamma:=p+\varepsilon$ with $0<\varepsilon \ll 1$ ) and the Young inequality is it easy to bound the right side of (6.8) in terms of right hand side of (3.1).
6.3. Estimates up to the boundary. Here, we give a brief explanation why the method works up to the boundary. First, to derive Caccioppoli inequality also for balls closed to the boundary one can replace the test function for such balls by $\left(u^{\alpha}-v^{\alpha}\right) \tau_{R}$. It is a possible test function (having zero trace) and also the Poincare inequality is valid. All we need is to assume that also $v$ belongs to sufficiently good Morrey space.

More delicate is obtaining the estimates based on the Noether equation. To illustrate that it is possible we consider that we are interested in estimating the integrals over balls centered at the point of flat boundary with zero boundary data. Then one can generalize such procedure to general domains by standard procedure. Hence assume that that the boundary is given as $\left\{x \in \mathbb{R}^{d} ; x_{d}=0\right\}$. We would like to show the estimate (3.1) on $B_{R}(0) \cap \Omega$ Thus, we multiply (1.4) by $|x|^{p-d} \sum_{i} x_{i} D_{i} u^{\alpha}$. That means we would like to test (1.4) by $\sum_{i} x_{i} D_{i} u^{\alpha} \tau_{R}$ (we omit the weight $|x|^{p-d}$ here). Then we would like to integrate by part, but in this case it is easy, since $\tau \sum_{i=1}^{d} x_{i} D_{i} u=0$ on the boundary (we use the fact that $x_{d}=0$ and the fact that tangential derivatives of $u$ are zero due to zaro Dirichlet data) and therefore the boundary integral vanishes. Hence we get by using the potential that

$$
\int_{\mathbb{R}^{d-1} \times \mathbb{R}_{+}} \sum_{\alpha=1}^{N} \sum_{i, j=1}^{d} F_{\eta^{\alpha_{i}}}(\nabla u) D_{i}\left(x_{j} \tau_{R}\right) D_{j} u^{\alpha}+\nabla F(\nabla u) \cdot x \tau_{R} d x=0
$$

Now, we would like to integrate by parts also in the second terms, but since $\tau_{R} x \cdot n$ is zero on the boundary, it is easy and the boundary integral again vanishes. So one can easily continue in the same way as in Section 3.
6.4. $F$ dependent on $x$. Here we shortly discuss the possible dependence of $F$ on the spatial variable $x$. We need, similarly as in the introduction, split the discussion onto two parts, first when $F$ is convex and one can use Campanato-like procedure and second when $F$ is not convex and some additional hypothesis are required.
6.4.1. Convex $F$ and Campanato technique. In this subsection we consider $F$ satisfying (1.5) and (1.7)-(1.10), where we allow a possible dependence on $x$. Then we would like to show that the unique (uniqueness follows from convexity of $F$ ) solution to (1.4) satisfies (1.19) with $\varepsilon=d-p$. From this the Hölder continuity follows by using the Morrey lemma. Thus, to show it we freeze the coefficient in $F$ and define $\tilde{F}$ as

$$
\tilde{F}(\eta):=F\left(x_{0}, \eta\right)
$$

Then we consider a problem

$$
\begin{align*}
-\operatorname{div} \tilde{F}_{\eta}(\nabla w) & =0 \text { in } B_{R}\left(x_{0}\right),  \tag{6.10}\\
w & =u \text { on } \partial B_{R}\left(x_{0}\right) .
\end{align*}
$$

Since $\tilde{F}$ is $x$ independent we can use Theorem 1.1 to deduce that there is $\alpha>0$ independent of $x_{0}$ and $R$ such that

$$
\begin{equation*}
\int_{B_{r}\left(x_{0}\right)} \frac{|\nabla w|^{p}}{r^{d-p}} d x \leq C\left(\frac{r}{R}\right)^{\alpha}\left(\int_{B_{R}\left(x_{0}\right)} \frac{|\nabla u|^{p}}{R^{d-p}} d x+R^{p}\right) . \tag{6.11}
\end{equation*}
$$

Next, since $w=u$ on $\partial B_{R}\left(x_{0}\right)$ we use (1.4) and (6.10) to get

$$
\int_{B_{R}\left(x_{0}\right)}\left(F_{\eta}(x, \nabla u)-\tilde{F}_{\eta}(\nabla w)\right) \cdot \nabla(u-w) d x=0 .
$$

Consequently, by using a simple algebraic manipulation we observe that
(6.12)
$\int_{B_{R}\left(x_{0}\right)}\left(\tilde{F}_{\eta}(\nabla u)-\tilde{F}_{\eta}(\nabla w)\right) \cdot \nabla(u-w) d x \leq \int_{B_{R}\left(x_{0}\right)}\left|F_{\eta}(x, \nabla u)-\tilde{F}_{\eta}(\nabla u)\right||\nabla(u-w)| d x$.
Next, to handle the integral on the right hand side we make an assumption on the Hölder continuity of $F$ w.r.t. $x$, namely we assume that there is $\alpha_{0}>$ such that

$$
\begin{equation*}
\left|F_{\eta}\left(x_{1}, \eta\right)-F_{\eta}\left(x_{2}, \eta\right)\right| \leq C\left|x_{1}-x_{2}\right|^{\alpha_{0}}\left(1+|\eta|^{p-1}\right) . \tag{6.13}
\end{equation*}
$$

Consequently, using (6.11) with $r=R$ we can bound the integral on the right hand side of (6.12) as

$$
\begin{equation*}
\int_{B_{R}\left(x_{0}\right)}\left|F_{\eta}(x, \nabla u)-\tilde{F}_{\eta}(\nabla u)\right||\nabla(u-w)| d x \leq C R^{\alpha_{0}}\left(R^{d}+\int_{B_{R}\left(x_{0}\right)}|\nabla u|^{p} d x\right) \tag{6.14}
\end{equation*}
$$

Using convexity of $F$, see (1.5), and consequently monotonicity of $F_{\eta}$ we can decrease the integration domain on the left hand side of (6.12) and by using (6.14) we get

$$
\begin{equation*}
\int_{B_{r}\left(x_{0}\right)}\left(\tilde{F}_{\eta}(\nabla u)-\tilde{F}_{\eta}(\nabla w)\right) \cdot \nabla(u-w) d x \leq C R^{\alpha_{0}}\left(R^{d}+\int_{B_{R}\left(x_{0}\right)}|\nabla u|^{p} d x\right) \tag{6.15}
\end{equation*}
$$

Next, using (1.7) onto the term on the right hand side, moving all terms depending on $u$ onto the right hand side and using the Young inequality we finally deduce that

$$
\begin{equation*}
\int_{B_{r}\left(x_{0}\right)}|\nabla u|^{p} d x \leq C R^{\alpha_{0}}\left(R^{d}+\int_{B_{R}\left(x_{0}\right)}|\nabla u|^{p} d x\right)+C \int_{B_{r}\left(x_{0}\right)}|\nabla w|^{p} d x . \tag{6.16}
\end{equation*}
$$

Finally, multiplying by $r^{p-d}$ and using (6.11) onto the last term on the right hand side of (6.16), we find that

$$
\begin{equation*}
\int_{B_{r}\left(x_{0}\right)} \frac{|\nabla u|^{p}}{r^{d-p}} d x \leq C\left(R^{\alpha_{0}}\left(\frac{R}{r}\right)^{d-p}+\left(\frac{r}{R}\right)^{\alpha}\right)\left(R^{p}+\int_{B_{R}\left(x_{0}\right)} \frac{|\nabla u|^{p}}{R^{d-p}} d x\right) \tag{6.17}
\end{equation*}
$$

Therefore, by using standard technique we see that there exists $\alpha_{1}>0$ such that $u \in \mathcal{C}^{\alpha_{1}}\left(\Omega ; \mathbb{R}^{N}\right)$.
6.4.2. Non-convex $F$. Since we do not assume in general such assumptions that would guarantee the existence of solution even for $F$ independent of $x$ we are not able to use Campanato technique for comparing two solutions (they may not exist). However, it is possible to use directly our approach. The only change is that in the Noether equation the new term involving the derivative of $F$ w.r.t. $x$ appears. On the other hand since we apply the derivative on $F$ we gain a better power in the weight, namely $|x|^{p-d+1}$ instead of $|x|^{p-d}$. Therefore assuming that $\left|F_{x}(x \nabla u)\right| \leq C(1+F(x, \nabla u))$ we wee that this terms is again the lower order one and whole procedure can be used.

## Appendix A. Estimates for potentials given by (1.11)-(1.13)

Here, we show under which assumptions on $a$ and $H$ the potentials (1.11)-(1.13) satisfy the assumptions of Theorem 1.1 and Theorem 1.2. We start with the easiest case (1.12). Hence, for the potential given by (1.12), a simple computation gives

$$
\begin{aligned}
\frac{\partial F(\eta)}{\partial \eta_{i}^{\alpha}} & =p(Q(\eta, \eta))^{\frac{p-2}{2}} \sum_{\beta=1}^{N} \sum_{j=1}^{d} a^{\alpha \beta} H_{i j} \eta_{j}^{\beta} \\
\frac{\partial^{2} F(\eta)}{\partial \eta_{i}^{\alpha} \partial \eta_{j}^{\beta}} & =p(Q(\eta, \eta))^{\frac{p-4}{2}}\left(Q(\eta, \eta) a^{\alpha \beta} H_{i j}+(p-2) \sum_{\gamma, \delta=1}^{N} \sum_{k, l=1}^{d} a^{\alpha \gamma} a^{\beta \delta} \beta H_{i k} H_{j l} \eta_{k}^{\gamma} \eta_{l}^{\delta}\right)
\end{aligned}
$$

Having such identities it is easy to check the validity of (1.5) and (1.7)-(1.10). Similarly, we obtain for the first derivative of $F$ given by (1.11) that

$$
\frac{\partial F(\eta)}{\partial \eta_{i}^{\alpha}}=|\eta|^{p-4}\left(\sum_{\beta=1}^{N} \sum_{j=1}^{d} 2 a^{\alpha \beta} H_{i j} \eta_{j}^{\beta}|\eta|^{2}+(p-2) Q(\eta, \eta) \eta_{i}^{\alpha}\right) .
$$

It is again evident that (1.3) and (1.7)-(1.8) are satisfied. However, we also see that (1.9) and consequently also (1.10) holds only if $H_{i j}=h \delta_{i j}$, where $h$ is a positive constant. Indeed, defining

$$
\begin{aligned}
A^{\alpha \beta} & :=|\eta|^{p-4}\left(2 h a^{\alpha \beta}|\eta|^{2}+(p-2) Q(\eta, \eta) \delta^{\alpha \beta}\right), \\
b_{i j} & :=\delta_{i j}
\end{aligned}
$$

we see that (1.9) holds. Note that even under this restriction, we treat really the non-diagonal structure, where the coupling is achieved in vectorial components of $u$. Similarly as before we can ask wether $F$ satisfies also (1.5). Thus, computing
the second derivatives we get

$$
\begin{aligned}
\frac{\partial^{2} F(\eta)}{\partial \eta_{i}^{\alpha} \partial \eta_{j}^{\beta}} & =(p-4)|\eta|^{p-6} \eta_{j}^{\beta}\left(\sum_{\gamma=1}^{N} \sum_{k=1}^{d} 2 a^{\alpha \gamma} H_{i k} \eta_{k}^{\gamma}|\eta|^{2}+(p-2) Q(\eta, \eta) \eta_{i}^{\alpha}\right) \\
& +|\eta|^{p-4}\left(2 a^{\alpha \beta} H_{i j}|\eta|^{2}+\sum_{\gamma=1}^{N} \sum_{k=1}^{d} 4 a^{\alpha \gamma} H_{i k} \eta_{k}^{\gamma} \eta_{j}^{\beta}+(p-2) Q(\eta, \eta) \delta^{\alpha \beta} \delta_{i j}\right. \\
& \left.+2(p-2) \sum_{\gamma=1}^{N} \sum_{k=1}^{d} a^{\beta \gamma} H_{j k} \eta_{k}^{\gamma} \eta_{i}^{\alpha}\right)
\end{aligned}
$$

Consequently, we observe that (we use the notation $(\eta, \mu):=\eta \cdot \mu$ )

$$
\begin{aligned}
\sum_{\alpha, \beta=1}^{N} & \sum_{i, j=1}^{d} \frac{\partial^{2} F(\eta)}{\partial \eta_{i}^{\alpha} \partial \eta_{j}^{\beta}} \mu_{i}^{\alpha} \mu_{j}^{\beta}=(p-4)|\eta|^{p-6}(\eta, \mu)\left(2 Q(\eta, \mu)|\eta|^{2}+(p-2) Q(\eta, \eta)(\eta, \mu)\right) \\
& +|\eta|^{p-4}\left(2 Q(\mu, \mu)|\eta|^{2}+2 p Q(\eta, \mu)(\eta, \mu)+(p-2) Q(\eta, \eta)|\mu|^{2}\right) \\
& =(p-4)(p-2)|\eta|^{p-6}(\eta, \mu)^{2} Q(\eta, \eta) \\
& +|\eta|^{p-4}\left(2 Q(\mu, \mu)|\eta|^{2}+4(p-2) Q(\eta, \mu)(\eta, \mu)+(p-2) Q(\eta, \eta)|\mu|^{2}\right)
\end{aligned}
$$

and we see that the function $F$ is convex only for proper a choice of $p$ 's and $Q$ 's, or to be more precise, $F$ is convex if and only if the scalar product $Q(\eta, \mu)$ is closed to the scalar product $(\eta, \mu)$ (how "closed" depends on $p$ ).

The last example (1.13) can be treated similarly as (1.11)-(1.12) and we skip the computation here. We only recall that $F$ given by (1.13) satisfy (1.5) and (1.7)-(1.8). In addition this potential satisfies also (1.9)-(1.10) provided that either $a_{1}^{\alpha \beta}=c a_{2}^{\alpha \beta}$ or $H_{1}^{i j}=h H_{2}^{i j}$ for some positive constants $c, h$.

## References

[1] Alain Bensoussan and Jens Frehse. Regularity results for nonlinear elliptic systems and applications, volume 151 of Applied Mathematical Sciences. Springer-Verlag, Berlin, 2002.
[2] Ennio De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3), 3:25-43, 1957.
[3] Ennio De Giorgi. Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. Boll. Un. Mat. Ital. (4), 1:135-137, 1968.
[4] J. D. Eshelby. The force on an elastic singularity. Philos. Trans. Roy. Soc. London. Ser. A., 244:84-112, 1951.
[5] Lawrence C. Evans. Partial regularity for stationary harmonic maps into spheres. Arch. Rational Mech. Anal., 116(2):101-113, 1991.
[6] Martin Fuchs. Topics in the calculus of variations. Advanced Lectures in Mathematics. Friedr. Vieweg \& Sohn, Braunschweig, 1994.
[7] Mariano Giaquinta. Multiple integrals in the calculus of variations and nonlinear elliptic systems, volume 105 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1983.
[8] Mariano Giaquinta and Stefan Hildebrandt. Calculus of variations. I, volume 310 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1996. The Lagrangian formalism.
[9] S. Hildebrandt and K.-O. Widman. Variational inequalities for vector-valued functions. $J$. Reine Angew. Math., 309:191-220, 1979.
[10] R. J. Knops and C. A. Stuart. Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity. Arch. Rational Mech. Anal., 86(3):233-249, 1984.
[11] Olga A. Ladyzhenskaya and Nina N. Uraltseva. Linear and quasilinear elliptic equations. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York, 1968.
[12] Giuseppe Mingione. Regularity of minima: an invitation to the dark side of the calculus of variations. Appl. Math., 51(4):355-426, 2006.
[13] J. Nash. Continuity of solutions of parabolic and elliptic equations. Amer. J. Math., 80:931954, 1958.
[14] Jindřich Nečas. Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity. In Theory of nonlinear operators (Proc. Fourth Internat. Summer School, Acad. Sci., Berlin, 1975). Akademie-Verlag, Berlin, 1977.
[15] Emmy Noether. Invariant variation problems. Transport Theory Statist. Phys., 1(3):186-207, 1971. Translated from the German (Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1918, 235-257).
[16] K. R. Rajagopal. On implicit constitutive theories. Appl. Math., 48(4):279-319, 2003.
[17] K. R. Rajagopal. The elasticity of elasticity. Z. Angew. Math. Phys., 58(2):309-317, 2007.
[18] K. R. Rajagopal and A. R. Srinivasa. On the response of non-dissipative solids. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 463(2078), 2007.
[19] Klaus Steffen. Parametric surfaces of prescribed mean curvature. In Calculus of variations and geometric evolution problems (Cetraro, 1996), volume 1713 of Lecture Notes in Math., pages 211-265. Springer, Berlin, 1999.
[20] Andrzej Trautman. Noether equations and conservation laws. Comm. Math. Phys., 6:248261, 1967.
[21] K. Uhlenbeck. Regularity for a class of non-linear elliptic systems. Acta Math., 138(3-4):219240, 1977.
[22] Vladimír Šverák and Xiaodong Yan. Non-Lipschitz minimizers of smooth uniformly convex functionals. Proc. Natl. Acad. Sci. USA, 99(24):15269-15276, 2002.

Mathematical Institute, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 18675 Praha 8, Czech Republic

E-mail address: mbul8060@karlin.mff.cuni.cz
Institute for applied mathematics, Department of applied analysis, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: erdbeere@iam.uni-bonn.de

Bestellungen nimmt entgegen:

Sonderforschungsbereich 611
der Universität Bonn
Endenicher Allee 60
D-53115 Bonn

Telefon: 0228/73 4882
Telefax: 0228/737864
E-Mail: astrid.link@ins.uni-bonn.de http://www.sfb611.iam.uni-bonn.de/

## Verzeichnis der erschienenen Preprints ab No. 460

460. Brenier, Yann; Otto, Felix; Seis, Christian: Upper Bounds on Coarsening Rates in Demixing Binary Viscous Liquids
461. Bianchi, Alessandra; Bovier, Anton; loffe, Dmitry: Pointwise Estimates and Exponential Laws in Metastable Systems Via Coupling Methods
462. Basile, Giada; Bovier, Anton: Convergence of a Kinetic Equation to a Fractional Diffusion Equation; erscheint in: Review Markov Processes and Related Fields
463. Bartels, Sören; Roubíček, Tomáš: Thermo-Visco-Elasticity with Rate-Independent Plasticity in Isotropic Materials Undergoing Thermal Expansion
464. Albeverio, Sergio; Torbin, Grygoriy: The Ostrogradsky-Pierce Expansion: Probability Theory, Dynamical Systems and Fractal Geometry Points of View
465. Capella Kort, Antonio; Otto, Felix: A Quantitative Rigidity Result for the Cubic to Tetragonal Phase Transition in the Geometrically Linear Theory with Interfacial Energy
466. Philipowski, Robert: Stochastic Particle Approximations for the Ricci Flow on Surfaces and the Yamabe Flow
467. Kuwada, Kazumasa; Philipowski, Robert: Non-explosion of Diffusion Processes on Manifolds with Time-dependent Metric; erscheint in: Mathematische Zeitschrift
468. Bacher, Kathrin; Sturm, Karl-Theodor: Ricci Bounds for Euclidean and Spherical Cones
469. Bacher, Kathrin; Sturm, Karl-Theodor: Localization and Tensorization Properties of the Curvature-Dimension Condition for Metric Measure Spaces
470. Le Peutrec, Dorian: Small Eigenvalues of the Witten Laplacian Acting on p-Forms on a Surface
471. Wirth, Benedikt; Bar, Leah; Rumpf, Martin; Sapiro, Guillermo: A Continuum Mechanical Approach to Geodesics in Shape Space
472. Berkels, Benjamin; Linkmann, Gina; Rumpf, Martin: An SL (2) Invariant Shape Median
473. Bartels, Sören; Schreier, Patrick: Local Coarsening of Triangulations Created by Bisections
474. Bartels, Sören: A Lower Bound for the Spectrum of the Linearized Allen-Cahn Operator Near a Singularity
475. Frehse, Jens; Löbach, Dominique: Improved Lp-Estimates for the Strain Velocities in Hardening Problems
476. Kurzke, Matthias; Melcher, Christof; Moser, Roger: Vortex Motion for the Landau-Lifshitz-Gilbert Equation with Spin Transfer Torque
477. Arguin, Louis-Pierre; Bovier, Anton; Kistler, Nicola: The Genealogy of Extremal Particles of Branching Brownian Motion
478. Bovier, Anton; Gayrard, Véronique: Convergence of Clock Processes in Random Environments and Ageing in the p-Spin SK Model
479. Bartels, Sören; Müller, Rüdiger: Error Control for the Approximation of Allen-Cahn and Cahn-Hilliard Equations with a Logarithmic Potential
480. Albeverio, Sergio; Kusuoka, Seiichiro: Diffusion Processes in Thin Tubes and their Limits on Graphs
481. Arguin, Louis-Pierre; Bovier, Anton; Kistler, Nicola: Poissonian Statistics in the Extremal Process of Branching Brownian Motion
482. Albeverio, Sergio; Pratsiovyta, Iryna; Torbin, Grygoriy: On the Probabilistic, Metric and Dimensional Theories of the Second Ostrogradsky Expansion
483. Bulíček, Miroslav; Frehse, Jens: $C^{\alpha}$-Regularity for a Class of Non-Diagonal Elliptic Systems with p-Growth

[^0]:    2000 Mathematics Subject Classification. 35J60,49N60.
    Key words and phrases. Nonlinear elliptic systems, regularity, Noether equation, Hölder continuity.

    Miroslav Bulíček thanks to Collaborative Research Center (SFB) 611 and to Jindřich Nečas Center for Mathematical Modeling, the project LC06052 financed by MŠMT for their support. Jens Frehse acknowledges Jindřich Nečas Center for Mathematical Modeling, the project LC06052 financed by MŠMT.

[^1]:    ${ }^{1}$ We restrict ourselves only onto the case when $p<d$ and $N>1$. Indeed, for $p>d$ the continuity follows from the embedding theorem and the critical case $p=d$ can be usually solved by various techniques. The scalar case $N=1$ is also an exceptional one for which in most cases the regularity is known. We refer to famous paper of De Giorgi [2] for the case $p=2$ (see also [13]) and to [11] for the case $p \neq 2$.

[^2]:    ${ }^{2}$ In fact onto the right hand side of (1.7) could be added the term $-C$ to cover also a nondissipative processes. But since it is only an easy generalization, we omit it in this paper.
    ${ }^{3}$ Splitting conditions refers to the fact that $A^{\alpha \beta}$ does not depend on $i, j$. Recall, the indexes $i, j$ correspond to derivatives w.r.t. $x_{i}, x_{j}$ respectively.

