Fractional Interior Differentiability of the Stress Velocities to Elastic Plastic Problems with Hardening

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In memoriam Enrico Magenes

Abstract

We consider the classical variational inequalities modeling elastic plastic problems with kinematic and isotropic hardening. For kinematic hardening, it is shown that the stress and strain velocities have interior fractional derivatives of order $1/2 - \delta$ in L^2 in space and time direction. For isotropic hardening, related weaker results hold.

Sunto

In questo articolo consideriamo un modello classico di deformazione elastoplastica con incrudimento cinematico e mostriamo che le derivate temporali del tensore degli sforzi, del tensore delle deformazioni e delle variabili interne possiedono derivate frazionarie di ordine $1/2 - \delta$ in L^2 in tutte le direzioni, nell'intero dominio spaziale, globalmente nel tempo.

La derivata frazionaria degli sforzi in direzione del tempo, cioe' del parametro di carico, esiste fino al bordo anche nel caso di incrudimento isotropo.

Key words Plasticity with hardening, isotropic and kinematic hardening, von Mises yield criterion, regularity of solutions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The domain Ω represents a solid body which undergoes an elastic plastic deformation, hence the case $n = 3$ is the natural application, however, the study of arbitrary dimensions $n \geq 2$ gives additional mathematical insight. Our aim is to prove a new regularity result for a classical variational inequality which models elastic plastic deformation with isotropic and kinematic hardening.

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Let us first fix some general notation. For $n \times n$-matrices τ, σ, the scalar product $\sigma : \tau$, Euclidean norm $|\sigma|$, the trace $\mathrm{tr}\sigma$ and the deviator σ_D are given by:

$$\sigma : \tau = \sum_{i,k=1}^{n} \sigma_{ik}\tau_{ik}, \quad |\sigma| = (\sigma : \sigma)^{1/2}, \quad \mathrm{tr}\sigma = \sum_i \sigma_{ii}, \quad \sigma_D = \sigma - \frac{\mathrm{tr}\sigma}{n}\mathbb{I},$$

where $\mathbb{I}$ is the unit matrix. By $\mathbb{R}^{n\times n}_{sym}$, we denote the set of all symmetric $n \times n$ matrices $\tau \in \mathbb{R}^{n\times n}$.

Under the influence of a volume force with density $f(t,x)$ and an external loading $p = p(t,x)$ there appear stresses $\sigma(t,x) \in \mathbb{R}^{n\times n}_{sym}$, where $x \in \Omega$ or $x \in \partial\Omega$. The parameter $t \in [0,T]$ is a so called loading parameter, but with some abuse of notation is often referred to as 'time'-variable. Assuming that the body is clamped in a region $\Gamma \subset \partial\Omega$, the balance of forces implies

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega, \qquad \nu \cdot \sigma = p \quad \text{on } \partial\Omega \setminus \Gamma, \tag{1.1}$$

where $\nu = \nu(x)$ is the outer unit normal vector at $x \in \partial\Omega$. For the formulation of the classical hardening problem we need a set of hardening variables $\xi = \xi(t,x)$, which is but a scalar function for isotropic hardening, or a symmetric tensor function -the so-called back stress- in the case of kinematic hardening. In addition we have a yield condition specified to

$$|\sigma_D| - \xi \le \kappa \quad \text{('isotropic hardening'), or} \tag{1.2}$$

$$|\sigma_D - \xi_D| \le \kappa \quad \text{('kinematic hardening'),} \tag{1.3}$$

respectively, where $\kappa > 0$ is a given constant.

In general the yield condition is formulated as

$$F(\sigma;\xi) \le 0$$

with a convex function F, but we confine us to the so called von-Mises-yield-condition (1.2) or (1.3). The problem is completed with constitutive laws, namely a stress-strain relation and a hardening law which describes the evolution of ξ, they are indicated in the formulae (1.17) and (1.18) below.

The proper mathematical formulation as a variational inequality involves some well known function spaces: The symbol $L^q(\Omega)$, with $1 \le q \le \infty$, denotes the usual Lebesgue-space, where we do not distinguish between scalar-, vector- , or tensor-valued functions as long as no confusion arises. In all cases we indicate the $L^2(\Omega)$-scalar product with brackets $(\cdot\,,\cdot)_\Omega$. For $T > 0$ and a given Banach space X (which is always a function space in the sequel), the symbol $L^q(0,T;X)$ stands for the Bochner space of measurable and q-summable functions defined on the interval $[0,T]$ with values in X. For $X = L^q(\Omega)$, we frequently shorten the notation to $L^p(L^q)$, if no confusion arises. The space of functions in $L^2(\Omega)$ with derivatives up to order $m \in \mathbb{N}$ is denoted by $H^m(\Omega)$, furthermore

$$H^1_\Gamma(\Omega) = \{\varphi \in H^1(\Omega,\mathbb{R}^n) \mid \varphi|_\Gamma = 0\}$$

the boundary condition has to be understood in the sense of traces, of course. The part $\Gamma \subset \partial\Omega$ is either void or a relatively open subset.

For convenience, we assume that the volume force density f, the external loading p and the initial value σ_0 fulfil the following regularity assumptions:

$$f, \dot{f} \in L^\infty(0,T;L^\infty(\Omega)), \ddot{f} \in L^1(0,T;L^2), \tag{1.4}$$
$$p, \dot{p} \in L^\infty(0,T;L^\infty(\partial\Omega)), \ddot{p} \in L^1(0,T;L^2(\partial\Omega)), \tag{1.5}$$

here the dot indicates the derivative with respect to the 'time' variable t.

Definition 1.1 (Admissible stresses and hardening variables) $\mathbb{K}(t)$ *is the set of all pairs* (τ, η) *with the following properties:*

$$\tau \in L^2(\Omega; \mathbb{R}^{n\times n}_{sym}), \eta \in L^2(\Omega, \mathbb{R}^m) \tag{1.6}$$

where τ *fulfills the balance of forces in the weak form:*

$$(\tau, \nabla\varphi)_\Omega = (f, \varphi)_\Omega + \int_{\partial\Omega} p\varphi \, do \quad \text{for all } \varphi \in H^1_\Gamma(\Omega). \tag{1.7}$$

In the case of isotropic *hardening we have* $m = 1$ *and:*

$$\eta \in L^2(\Omega; \mathbb{R}), \qquad |\tau_D| - \eta \leq \kappa, \tag{1.8}$$

in the case of kinematic *hardening we have* $m = n(n+1)/2$ [1] *and:*

$$\eta \in L^2(\Omega; \mathbb{R}^{n\times n}_{sym}), \qquad |\tau_D - \eta_D| \leq \kappa. \tag{1.9}$$

We assume that the hardening variables start at zero, that is $\xi(0) = 0$, while for the initial value σ_0 of the stresses and the pair $(\sigma_0, 0)$ we require

$$\sigma_0 \in H^2(\Omega), \quad (\sigma_0, 0) \in \mathbb{K}(0). \tag{1.10}$$

Finally, we need the so called compliance tensor or inverse elasticity tensor $A = (a^{\nu\mu}_{ik})$, a given symmetric tensor of rank four, and the hardening tensor $H \in \mathbb{R}^{m\times m}$. We assume that A and H satisfy the usual ellipticity condition

$$\tau : A\tau \geq c_0|\tau|^2, \quad H\eta \cdot \eta \geq c_1|\eta|^2 \tag{1.11}$$

for all $\tau \in \mathbb{R}^{n\times n}_{sym}$, $\eta \in \mathbb{R}^m$, respectively, with constants $c_0, c_1 > 0$. In order to limit the technical details we formally consider only the case of constant A and H here, however, all the results remain true if the entries of A and H are Lipschitz continuous functions on Ω, since this generalization will cause only pollution terms in the proofs. The tensors A and H have to fulfill the condition (1.11) uniformly in x in this case. Note that in the case of isotropic harding the term H is just a scalar function with $H(x) \geq c_1 > 0$ uniformly on Ω.

With this notation the classical variational inequality for isotropic or, respectively, kinematic hardening is the following

[1] In order to have a unified notation also in the calculations needed lateron we identify ξ with a vector of $\mathbb{R}^{n(n+1)/2}$

Hardening Problem: Let σ_0 be a given initial stress, such that $(\sigma, 0) \in \mathbb{K}(0)$. Find $\sigma \in L^\infty(L^2)$, $\xi \in L^\infty(L^2)$ such that

$$\dot\sigma \in L^2(L^2), \quad \dot\xi \in L^2(L^2) \tag{1.12}$$

$$(\sigma(t), \xi(t)) := (\sigma(t,\cdot), \xi(t,\cdot)) \in \mathbb{K}(t), \quad t \in [0,T] \tag{1.13}$$

$$\sigma(0) = \sigma_0, \quad \xi(0) = 0 \tag{1.14}$$

$$(A\dot\sigma, \sigma - \tau) + (H\dot\xi, \xi - \eta) \le 0 \quad \text{a.e. in } [0,T] \text{ for all } (\tau, \eta) \in \mathbb{K}(t). \tag{1.15}$$

This problem has a unique solution [9]. The inequality (1.15) contains the constitutive law, under additional regularity conditions it is equivalent to an a.e. point-wise equation provided that the so called safe load condition holds:

Definition 1.2 (Safe load condition) *There exist $\hat\sigma \in L^\infty(L^2)$, $\hat\xi \in L^\infty(L^2)$ with*

$$\begin{aligned} &\dot{\hat\sigma} \in L^\infty(L^2),\ \ddot{\hat\sigma} \in L^1(L^2),\ \dot{\hat\xi} \in L^\infty(L^2) && (1.16)\\ &(\hat\sigma(0), 0) \in \mathbb{K}(0),\ \hat\xi|_{t=0} = 0 \\ &(\hat\sigma(t,.), \hat\xi(t,.)) \in \mathbb{K}(t), \end{aligned}$$

and there exists a $\delta > 0$ such that

$$|\hat\sigma_D| - \xi \le \kappa - \delta \ \text{ or } |\hat\sigma_D - \hat\xi_D| \le \kappa - \delta, \text{ respectively.}$$

By a theorem of Johnson [9] it is known that there exists a displacement

$$u \in L^\infty(0,T; H^{1,2}_\Gamma(\Omega, \mathbb{R}^n)) \text{ with } \dot u \in L^\infty(0,T; H^{1,2}_\Gamma(\Omega, \mathbb{R}^n))$$

and a multiplier

$$\dot\lambda \in L^\infty(0,T; L^2(\Omega, \mathbb{R}))$$

such that in the case of isotropic hardening, a.e. in $[0,T] \times \Omega$

$$\begin{cases} \frac{1}{2}(\nabla \dot u + \nabla \dot u^{\mathrm{T}}) & = A\dot\sigma + \dot\lambda \sigma_D |\sigma_D|^{-1} \\ 0 & = H\dot\xi - \dot\lambda \end{cases} \tag{1.17}$$

and in the case of kinematic hardening

$$\begin{cases} \frac{1}{2}(\nabla \dot u + \nabla \dot u^{\mathrm{T}}) & = A\dot\sigma + \dot\lambda(\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1} \\ 0 & = H\dot\xi - \dot\lambda(\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1}. \end{cases} \tag{1.18}$$

The multiplier function $\dot\lambda$ satisfies $\dot\lambda \ge 0$ a.e. and

$$\dot\lambda(|\sigma_D| - \kappa - \xi) = 0$$

in the case of isotropic hardening or

$$\dot\lambda(|\sigma_D - \xi_D| - \kappa) = 0,$$

respectively, in the case of kinematic hardening. This implies that $\dot{\lambda} = 0$ if $|\sigma_D| = 0$ in the case of isotropic hardening, and $\dot{\lambda} = 0$ if $|\sigma_D - \xi_D| = 0$ in the case of kinematic hardening, so (1.17) und (1.18) can be defined. Vice versa, from (1.17) and (1.18) one recovers the variational inequality (cf. [3] for a simple proof concerning the construction of λ). Equation (1.17) and (1.18) have the advantage, that they are defined point-wise and that the deformation velocity $\dot{u}$ appears explicitly.

Concerning interior regularity of the solution (σ, ξ) and the displacements the following results are known: If f is sufficiently regular, that is

$$f \in L^\infty(H^1_{\mathrm{loc}}), \tag{1.19}$$

then in the case of kinematic hardening [13]:

$$\sigma \in L^\infty(0,T;H^1_{\mathrm{loc}}),\ \xi \in L^\infty(0,T;H^1_{\mathrm{loc}}),\ u \in L^\infty(0,T;H^2_{\mathrm{loc}})$$

In the case of isotropic hardening the $L^\infty(H^1_{\mathrm{loc}})$-property is known only for σ and ξ cf. [13], while the $L^\infty(H^1_{\mathrm{loc}})$-property for ∇u is an interesting open problem. In [4] it was shown that $\nabla u \in L^\infty(L^6_{\mathrm{loc}})$, if $n = 3$. Up to now there are no regularity results for $\dot{\sigma}$, $\dot{\xi}$, besides a result of [5], where the inclusion $\dot{\sigma}, \dot{\xi} \in L^\infty(L^{2+2\delta})$ for small $\delta > 0$ is proved. For more general models describing elastic-plastic deformation with hardening see also [1, 2, 7]. These papers use the so called primal formulation, i.e the principal unknowns are the displacements rather than the stresses. Our main result states that in $[0,T] \times \Omega_0$, $\Omega_0 \subset\subset \Omega$, the functions $\dot{\sigma}$, $\dot{\xi}$ have fractional derivatives in time *and* space direction of order $\frac{1}{2} - \delta$, $\delta > 0$. In the kinematic case we obtain this regularity property also for $\nabla \dot{u}$, in the isotropic case we reach only $\nabla \dot{u} \in L^{\frac{8}{3}-\delta_1}(L^{\frac{8}{3}-\delta_1})$, $n = 3$, for all $\delta_1 > 0$ (cf Remark 4.4, [6]).

For the result in time direction, the method of the proof is related to a recent paper [6] of the authors about the Prandtl-Reuss problem, where fractional differentiability of order $\frac{1}{2} - \delta_1$ was achieved for the stress velocities. The ideas of this paper can be adapted to gain a similar result in the setting with hardening, as it is considered here, however, the method to achieve also the fractional differentiability of order $\frac{1}{2} - \delta_1$ with respect to space direction needs an additional consideration. This is the purpose of our paper.

A counterexample of D. Knees [10] indicates that our regularity result is optimal.

2 The main results

We formulate the main results, starting with the regularity in time: The stress velocities $\dot{\sigma}$ and the time derivatives $\dot{\xi}$ of the hardening variables ξ have fractional derivatives of order $\frac{1}{2}$ in time direction, in a weak sense. This result holds up to the boundary $\partial\Omega$, for arbitrary dimension n, and both for kinematic and isotropic hardening. We recall the notation of difference operators: Let e_i denote the i-th unit vector, for any $w = w(t,x)$, and $s > 0$ we put

$$\Delta^s_t w(t,x) = w(t+h,x) - w(t,x), \quad \Delta^s_i w(t,x) = w(t,x+se_i) - w(t,x). \tag{2.1}$$

Theorem 2.1 (Regularity in time) *Let the data f, p and σ_0 fulfill the regularity assumptions* (1.4), (1.5) *and* (1.10), *assume that the ellipticity condition* (1.11) *for the tensors A and H and further the safe load condition (cf. Def. 1.2) are satisfied. Then for the solution σ, ξ of the hardening problem introduced in Section 1, there holds the estimate*

$$h^{-2}\int_0^h\int_0^{T-h}\int_\Omega \left[|\Delta_t^s\dot\sigma|^2+|\Delta_t^s\dot\xi|^2\right]\,dx\,dt\,ds\le C \tag{2.2}$$

uniformly for $0<h<h_0$.

Remark 2.2 *The inequality* (2.2) *is a weak version of the Nikolskii - space property*

$$\sup_{0<h<h_0}\frac{1}{h}\int_0^{T-h}\int_\Omega \left[|\Delta_t^s\dot\sigma|^2+|\Delta_t^s\dot\xi|^2\right]\,dx\,dt\le C,$$

(which we do not prove), see the discussion in [6]. Theorem 2.1 implies that the Fourier-coefficients $c_m=c_m(x)$ of $\dot\xi$, $\dot\sigma$ in time-direction (cf. the proof of Lemma 5.3 for the definition) gain the following summability property [6, Lemma A.1]

$$\sum_{m=-\infty}^{\infty} m^{1-\delta}\int_\Omega |c_m(x)|^2dy\le C_\delta \quad \textit{for all } \delta>0.$$

In the case of kinematic hardening, Theorem 2.1 implies the fractional differentiability in time for the strain velocities:

Theorem 2.3 *Assume the requirements of Theorem 2.1 are met and the pair (σ,ξ) is the solution to the problem with kinematic hardening. Then the corresponding displacement field u satisfies the estimate*

$$h^{-2}\int_0^{T-h}\int_0^h\int_\Omega |\Delta_t^s\nabla\dot u|^2\,dx\,ds\,dt\le C \tag{2.3}$$

with a constant independent of $0<h\le h_0$.

Using the result for the time we can also prove that the velocities $\dot\sigma$, $\dot\xi$ of the stresses and the hardening parameters have fractional derivatives of order $\frac{1}{2}-\delta$ in space direction, where $\delta>0$ can be arbitrarily small. The derivatives have to be taken in the weak sense as outlined in Theorem 2.4 below. However, we have to *assume* that the strain velocities $\nabla\dot u$ have fractional derivatives of order $\frac{1}{2}$ in time direction in the sense of Theorem 2.3, that is we require that (2.3) is valid. In the *kinematic* case this is just the result of Theorem 2.3, while in the isotropic case the estimate (2.3) is not known yet.

Theorem 2.4 (Local regularity in space) *Assume that in addition to the requirements of Theorem 2.1 the regularity estimate* (2.3) *holds true for the solution pair (σ,ξ) of the* *hardening problem formulated in Section 1. Then for any $\delta > 0$, the velocities $\dot\sigma$, $\dot\xi$ have local fractional derivatives of order $1/2 - \delta$ in space direction, in the following sense*

$$\sup_{0\le h\le h_0} \frac{1}{h^{1-\delta}} \int\limits_0^{T-h} \int\limits_{\Omega_0} |\Delta_i^h \dot\sigma|^2 + |\Delta_i^h \dot\xi|^2 \, dx\, dt \le C, \;\; i = 1, \ldots, n \tag{2.4}$$

for any domain Ω_0 such that $\overline{\Omega}_0 \subset \Omega$ and $h_0 \le dist(\partial\Omega, \partial\Omega_0)$.

Remark 2.5 (Possible generalizations) *Apart from passing to Lipschitz continuous entries in the tensors A and H - as already mentioned in Section 1 it is also possible to consider more general flow rules for the hardening variables ξ. For example, one my add in (1.17), (1.18) globally Lipschitz continuous functions $g(t, x, \sigma, \xi)$. Terms of this type create pollution terms which can be treated via Gronwalls inequality. However, we are mostly interested in in the classical case as in Johnson [8], mainly because they present the essential mathematical difficulties.*

3 Penalty approximation

The hardening problem (1.12) can be approximated in several ways via penalty approximations. In this paper we follow the approach in [3, 4, 5]. We introduce the penalty potentials

$$G_\mu^{iso}(\sigma, \xi) = \frac{1}{2}\mu^{-1}[|\sigma_D| - (\kappa + \xi)]_+^2$$

in the case of isotropic hardening, and

$$G_\mu^{kin}(\sigma, \xi) = \frac{1}{2}\mu^{-1}[|\sigma_D - \xi_D| - \kappa]_+^2$$

in the case of kinematic hardening, where for any real valued function ϕ the expression $[\phi]_+ = \max(\phi, 0)$ is the positive part, and $\mu = 0$ a small parameter. Then we obtain

$$\begin{aligned} G_{1\mu}^{iso}(\sigma, \xi) &:= \frac{\partial}{\partial\sigma} G_\mu^{iso}(\sigma, \xi) = \mu^{-1}[|\sigma_D| - (\kappa + \xi)]_+ \sigma_D |\sigma_D|^{-1} \\ G_{2\mu}^{iso}(\sigma, \xi) &:= \frac{\partial}{\partial\xi} G_\mu^{iso}(\sigma, \xi) = -\mu^{-1}[|\sigma_D| - (\kappa + \xi)]_+, \end{aligned}$$

and in particular the (point-wise) relation

$$|G_{1\mu}^{iso}| = |G_{2\mu}^{iso}|. \tag{3.1}$$

For kinematic hardening we have

$$\begin{aligned} G_{1\mu}^{kin}(\sigma, \xi) &:= \frac{\partial}{\partial\sigma} G_\mu(\sigma, \xi) = \mu^{-1}[|\sigma_D - \xi_D| - \kappa]_+ (\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1} \\ G_{2\mu}^{kin}(\sigma, \xi) &:= \frac{\partial}{\partial\xi} G_\mu(\sigma, \xi) = -\mu^{-1}[|\sigma_D - \xi_D| - \kappa]_+ (\sigma_D - \xi_D)|\sigma_D - \xi_D|^{-1}, \end{aligned}$$

that is

$$G_{1\mu}^{kin} = -G_{2\mu}^{kin} \tag{3.2}$$

Note that for $|\sigma_D| = 0$ or $|\sigma_D - \xi_D| = 0$, the terms $G_{j\mu}^{\cdots}$ can be continuously extended by 0; for $G_{2\mu}^{iso}$ this follows from $\xi \geq 0$.

Due to our conventions in notation, the mathematical formulations of the penalty problem in the kinematic and the isotropic look the same, thus in the following we simply write $G_{j\mu}$ instead of $G_{1\mu}^{kin}$ or $G_{1\mu}^{iso}$, if the arguments run parallel. We formulate the penalty approximation of the hardening problem:

Definition 3.1 (Penalty problem) *Find $\sigma_\mu, \xi_\mu \in L^\infty(0,T;L^2(\Omega))$ such that $\dot\sigma_\mu, \dot\xi_\mu \in L^\infty(0,T;L^2(\Omega))$, the pairs (σ_μ, ξ_μ) fulfill the initial condition* (1.14) *and the balance of forces* (1.7) *in the weak form for almost every t, further*

$$(A\dot\sigma_\mu + G_{1\mu}(\sigma_\mu, \xi_\mu), \tau)_\Omega = 0 \tag{3.3}$$

for all $\tau \in L^2(\Omega, \mathbb{R}^{n\times n}_{sym})$ which satisfy $(\tau, \nabla\varphi)_\Omega = 0$ for all $\varphi \in H^1_\Gamma(\Omega; \mathbb{R}^n)$,

$$H\dot\xi_\mu + G_{2\mu}(\sigma_\mu, \xi_\mu) = 0 \tag{3.4}$$

By the L^2-Helmholtz decomposition theorem for symmetric tensors we may replace (3.3) by the point-wise equation

$$\frac{1}{2}(\nabla v + \nabla v^{\mathrm{T}}) = A\dot\sigma + G_{1\mu}(\sigma, \xi) \tag{3.5}$$

with the so called deformation velocity $v = \dot u$, $v \in H^1_\Gamma(\Omega, \mathbb{R}^n)$.

It is well known (see the discussion and references in [3]) that the penalty problem has a unique solution (σ_μ, ξ_μ). Moreover, we have the following estimates independent of $\mu \in (0, \mu_0]$, provided that the safe load condition holds:

$$\|\sigma_\mu\|_{L^\infty(L^2)} + \|\dot\sigma_\mu\|_{L^\infty(L^2)} + \|u_\mu\|_{L^\infty(L^2)} + \|\dot u_\mu\|_{L^\infty(L^2)} \leq C, \tag{3.6}$$

$$\|\xi_\mu\|_{L^\infty(L^2)} + \|\dot\xi_\mu\|_{L^\infty(L^2)} \leq C, \tag{3.7}$$

$$\|\nabla \dot u_\mu\|_{L^\infty(L^2)} \leq C \tag{3.8}$$

The estimates (3.6), (3.7) have been worked out with a related penalty term in [9]. It is a routie matter to adapt this proof to our case. An alternative reference is [12]. By an argument of Johnson [9] ('Johnson's trick'), involving the relations (3.1) and (3.2), one has (3.8). Compared to the Prandtl-Reuss law, which corresponds to $\xi = 0$, where only an $L^\infty(L^1)$-estimate is available, the estimate (3.8) makes the analysis much easier and gives better regularity results.

By monotonicity methods we have the convergence (see, e.g. [12]) $\sigma_\mu \to \sigma$, $\xi_\mu \to \xi$ strongly in $L^2(L^2)$ while $\nabla\dot u_\mu \rightharpoonup \nabla \dot u$, as $\mu \to 0$. Here (σ, ξ) is the solution of the original hardening problem. In [5, Sec. 3] it was shown that even

$$\dot\sigma_\mu \to \sigma,\ \dot\xi_\mu \to \xi \text{ strongly in } L^2(L^2), \tag{3.9}$$

which implies also $\nabla \dot{u}_\mu \to \nabla \dot{u}$ in the kinematic case. For the proof it is essential to know that (in the case of isotropic hardening)

$$G^{iso}_{1\mu} \rightharpoonup \dot{\lambda} \frac{\sigma_D}{|\sigma_D|}, \quad G^{iso}_{2\mu} \rightharpoonup -\dot{\lambda}$$

weakly in L^2, with a function $\dot{\lambda} \in L^\infty(L^2)$ enjoying the properties

$$\dot{\lambda} \geq 0, \quad \dot{\lambda} = 0 \ \text{ if } |\sigma_D| - \xi < \kappa.$$

This result was proved in [3], the case of kinematic hardening runs in a completely analogous way. Using the multiplier $\dot{\lambda}$, the constitutive equation can be written in the form (1.17).

Furthermore, there are local uniform estimates

$$\|\nabla \sigma_\mu\|_{L^\infty(L^2(\Omega_0))} + \|\nabla \xi_\mu\|_{L^\infty(L^2(\Omega_0))} \leq C_{\Omega_0}, \quad \Omega_0 \subset\subset \Omega, \tag{3.10}$$

cf. [13, 12]. In addition, in the case of kinematic hardening, one has $u_\mu \in L^\infty(H^2_{\text{loc}})$ and

$$\|u_\mu\|_{L^\infty(H^2(\Omega_0))} \leq C_{\Omega_0}, \quad \Omega_0 \subset\subset \Omega, \ \mu \to 0. \tag{3.11}$$

It is an interesting open problem to obtain (3.11) also in the case of isotropic hardening. In [4], the authors were only able to prove $\nabla u \in L^\infty(L^6_{\text{loc}})$, $n = 3$, in the latter case. Finally, it is known [5] that

$$\dot{\sigma} \in L^\infty(L^{2+2\delta}), \ \dot{\xi} \in L^\infty(L^{2+2\delta}), \ \nabla \dot{u} \in L^\infty(L^{2+2\delta}), \tag{3.12}$$

for some small $\delta > 0$.

4 The regularity in time

4.1 Auxiliary estimates

The proofs of the regularity results involve various auxiliary results for the penalty terms. We start with the proof that for almost every t, the penalty potential tends to 0 in $L^1(\Omega)$, as $\mu \to 0$.

Lemma 4.1 *Let σ_μ, ξ_μ be the solution of the penalty problem 3.1, where the data f, p and σ_0 fulfill the regularity assumptions* (1.4), (1.5) *and* (1.10). *Then*

$$\int_0^T \int_\Omega G_\mu(\sigma_\mu(t,x), \xi_\mu(t,x))\, dx\, dt \to 0 \quad \text{as } \mu \to 0,$$

in particular there exists a sequence $\mu_n \to 0$ such that [2]

$$\int_\Omega G_{\mu_n}(\sigma_{\mu_n}(t,x), \xi_{\mu_n}(t,x))\, dx \to 0 \quad \text{as } n \to \infty \quad \text{for a.e. } t \in [0,T].$$

[2] In the following we omit the explicit mentioning of a subsequence in order to keep the proofs as simple as possible.

Proof. We only give the details in the case of kinematic hardening, the isotropic case can be done in an analogous way. Using the pair $(\sigma_\mu - \sigma, \xi_\mu - \xi)$ as test functions in the relations (3.3) and (3.4) of the penalty problem and observing (3.2), we obtain

$$
\begin{aligned}
0 = & \int_0^T (A\dot\sigma_\mu, \sigma_\mu - \sigma)_\Omega + (H\dot\xi_\mu, \xi_\mu - \xi)_\Omega \, dt \\
& + \int_0^T \int_\Omega \mu^{-1} [|\sigma_{\mu D} - \xi_{\mu D}| - \kappa]_+ \frac{\sigma_{\mu D} - \xi_{\mu D}}{|\sigma_{\mu D} - \xi_{\mu D}|} (\sigma_{\mu D} - \sigma_D - \xi_{\mu D} + \xi_D) \, dx \, dt =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathcal{I}_1 = & \frac{1}{2} \int_0^T \frac{\partial}{\partial t} (A(\sigma_\mu - \sigma), \sigma_\mu - \sigma)_\Omega + \int_0^T (A\dot\sigma, \sigma_\mu - \sigma)_\Omega \, dt + \\
& \frac{1}{2} \int_0^T \frac{\partial}{\partial t} (H(\xi_\mu - \xi), \xi_\mu - \xi)_\Omega + \int_0^T (H\dot\xi, \xi_\mu - \xi)_\Omega \, dt \\
= & \frac{1}{2} \Big((A(\sigma_\mu - \sigma), \sigma_\mu - \sigma)_\Omega + (H(\xi_\mu - \xi), \xi_\mu - \xi)_\Omega \Big)\Big|_{t=T} + o(1),
\end{aligned}
$$

because the remaining integrals tend to 0 as $\mu \to \infty$ due to the weak convergence of σ_μ, ξ_μ. Note the first term on the right-hand side is nonnegative thanks to the positivity condition (1.11). The integrand of $\mathcal{I}_2$ can be treated as follows:

$$
\begin{aligned}
& \mu^{-1} [|\sigma_{\mu D} - \xi_{\mu D}| - \kappa]_+ \left(|\sigma_{\mu D} - \xi_{\mu D}| - \frac{\sigma_{\mu D} - \xi_{\mu D}}{|\sigma_{\mu D} - \xi_{\mu D}|} (\sigma_D - \xi_D) \right) \\
& \quad \geq \mu^{-1} [|\sigma_{\mu D} - \xi_{\mu D}| - \kappa]_+ \left(|\sigma_{\mu D} - \xi_{\mu D}| - |\sigma_D - \xi_D| \right) \\
& \quad \geq \mu^{-1} [|\sigma_{\mu D} - \xi_{\mu D}| - \kappa]_+ (|\sigma_{\mu D} - \xi_{\mu D}| - \kappa) = 2G_\mu^{kin} \geq 0,
\end{aligned}
$$

the last inequality holds, since $|\sigma_D - \xi_D| \leq \kappa$. Hence we have

$$
0 \leq \int_0^T \int_\Omega G_\mu^{kin} \, dx \, dt \leq -K + o(1),
$$

from which the assertion follows. $\square$

Now we establish estimates involving difference quotients. In addition to the notation (2.1) we use the expressions

$$
\begin{aligned}
& Iw(t,x) = w(t,x), \; E_t^h w(t,x) = w(t+h,x), \; D_t^h w(t,x) = \frac{1}{h} \Delta_t^h w(t,x), \text{ hence} \\
& \Delta_t^h w(t,x) = (E_t^h - I) w(t,x) = h D_t w(t,x).
\end{aligned}
$$

The uniform $L^\infty(H^1)$- and $H^1(L^2)$-estimates (3.6)– (3.8) imply certain estimates of the penalty term, which we want to fix. Recall that $G_{j\mu}$ is either $G_{j\mu}^{kin}$ or $G_{j\mu}^{iso}$.

Lemma 4.2 *Assume the hypotheses of Lemma 4.1, then*

$$\mathcal{P}_0 := \int\limits_0^{T-h} \big(D_t^h G_{1\mu}(\sigma_\mu, \xi_\mu), D_t^h \sigma_\mu\big)_\Omega + \big(D_t^h G_{2\mu}(\sigma_\mu, \xi_\mu), D_t^h \xi_\mu\big)_\Omega \, dt \le C$$

$$\textit{uniformly for} \quad 0 < h < h_0,\ 0 < \mu < \mu_0.$$

Proof. During this proof, we drop the index μ and simply write σ, ξ and u. We apply the operation D_t^h to (3.5) and (3.4) and use $D_t^h\sigma$ and $D_t^h\xi$, respectively, as a test function. Then we obtain

$$\begin{aligned} &\frac{1}{2}\int\limits_0^{T-h} \frac{\partial}{\partial t}\big(D_t^h\sigma, AD_t^h\sigma\big)_\Omega + \frac{\partial}{\partial t}\big(D_t^h\xi, HD_t^h\xi\big)_\Omega \, dt \ + \ \mathcal{P}_0 = \int\limits_0^{T-h} \big(D_t^h\nabla\dot u, D_t^h\sigma\big)_\Omega \, dt \\ &= \int\limits_0^{T-h} \big(D_t^h\nabla\dot u, D_t^h\big(\sigma - \hat\sigma\big)\big)_\Omega \, dt + \int\limits_0^{T-h} \big(D_t^h\nabla\dot u, D_t^h\hat\sigma\big)_\Omega \, dt =: \mathcal{I}_3 + \mathcal{I}_4, \end{aligned} \tag{4.1}$$

where $\hat\sigma$ is defined in the definition 1.2 of the safe load condition. Since $\dot u \in L^\infty(H^1_\Gamma(\Omega))$ we have $D_t^h\dot u(t) \in H^1_\Gamma(\Omega)$ for almost every t, hence the first integrand vanishes for almost every $t \in [0, T-h]$, since both σ and $\hat\sigma$ fulfill the balance of forces (1.7). We can transform the term $\mathcal{I}_4$ with the help of (1.7), too, then we integrate by parts with respect to t and obtain

$$\begin{aligned} \mathcal{I}_4 &= \int\limits_0^{T-h} \big(D_t^h\dot u, D_t^h f\big)_\Omega \, dt + \int\limits_0^{T-h}\int\limits_{\partial\Omega} D_t^h\dot u \cdot D_t^h p \, do \, dt \\ &= -\int\limits_0^{T-h} \big(D_t^h u, D_t^h \dot f\big)_\Omega \, dt \ + \ \big(D_t^h u, D_t^h f\big)_\Omega\Big|_0^{T-h} \\ &\quad - \int\limits_0^{T-h}\int\limits_{\partial\Omega} D_t^h u \cdot D_t^h \dot p_0 \, \mathrm{do} \, dt \ + \ \int\limits_{\partial\Omega} D_t^h u \cdot D_t^h p_0 \, \mathrm{do}\Big|_0^{T-h}. \end{aligned}$$

After a possible redefinition on a set of measure zero (in time) we have for all $t \in [0, T-h]$.

$$\|D_t^h u(t)\|_{L^2(\Omega)} \le \|\dot u\|_{L^\infty(0,T;L^2(\Omega))} \le C$$

due to (3.6). To estimate of the boundary integrals we need the trace theorem [11] in addition:

$$\|D_t^h u(t)\|_{L^2(\partial\Omega)} \le \|\dot u\|_{L^\infty(0,T;L^2(\partial\Omega))} \le \|\nabla\dot u\|_{L^\infty(0,T;L^2(\Omega))} \le C$$

due to (3.8). Using the assumptions on the data we see that all terms in $\mathcal{I}_4$ are bounded, thus (4.1) leads to

$$\begin{aligned}
&\mathcal{P}_0 + \frac{1}{2}\Big(\big(D_t^h\sigma, AD_t^h\sigma\big)_\Omega + \frac{1}{2}(D_t^h\xi, HD_t^h\xi\big)_\Omega\Big)\Big|_{t=T-h} \\
&\le\ C + \frac{1}{2}\Big(\big(D_t^h\sigma, AD_t^h\sigma\big)_\Omega + \frac{1}{2}\big(D_t^h\xi, HD_t^h\xi\big)_\Omega\Big)\Big|_{t=0} \\
&\le\ C\Big(1 + h^{-1}\int_0^h\int_\Omega |\dot\sigma|^2 + |\dot\xi|^2\,dx\,dt\Big) \le C
\end{aligned}$$

again since $\dot\sigma$ and $\dot\xi$ are bounded in $L^\infty(L^2)$. This finishes the proof of Lemma 4.2. $\square$

The following lemma serves as an auxiliary tool to control the quantity

$$\lim_{\mu\to 0} h^{-2}\int_0^h\int_0^{T-h} (\Delta_t^s\dot\sigma_\mu, A\dot\sigma_\mu)_\Omega + (\Delta_t^s\dot\xi_\mu, H\dot\xi_\mu)_\Omega\,dt\,ds,$$

which is needed in the proof of Theorem 2.1 to estimate the fractional derivative of $\dot\sigma$ and $\dot\xi$ in time direction. The proof as well as the arguments for lemma 4.3 below run analogously to the proof of a corresponding result concerning the Prandtl-Reuss law [6].

Lemma 4.3 *Let $G_{j\mu}(t,x) = G_{j\mu}(\sigma_\mu(t,x), \xi_\mu(t,x))$, then*

$$\mathcal{T}_0(t_1,t_2) =: \limsup_{\mu\to 0}\int_0^h\int_{t_1}^{t_2-h} (G_{1\mu}, \Delta_t^s\dot\sigma_\mu)_\Omega + (G_{2\mu}, \Delta_t^s\dot\xi_\mu)_\Omega\,dt\,ds\ \le\ 0$$

a.e. with respect to $t_1, t_2 \in [0,T]$, such that $0 \le t_1 \le t_2 - h \le T - h$.

Proof. We split

$$\begin{aligned}
\mathcal{T}_{0\mu} &:= \int_0^h\int_{t_1}^{t_2-h} \big(G_{1\mu}, \Delta_t^s\dot\sigma_\mu\big)_\Omega + \big(G_{2\mu}, \Delta_t^s\dot\xi_\mu\big)_\Omega\,dt\,ds \qquad (4.2)\\
&= \int_0^h\int_{t_1}^{t_2-h} \big(G_{1\mu}, E_t^s\dot\sigma_\mu\big)_\Omega + \big(G_{2\mu}, E_t^s\dot\xi_\mu\big)_\Omega\,dt\,ds - \int_0^h\int_{t_1}^{t_2-h}\int_\Omega \frac{\partial}{\partial t}G_\mu\,dx\,dt\,ds \\
&=:\ \mathcal{I}_5 + \int_0^h\int_\Omega G_\mu\,dx\Big|_{t_1}^{t_2-h} ds = \mathcal{I}_5 + h\int_\Omega G_\mu\,dx\Big|_{t_1}^{t_2-h}
\end{aligned}$$

The last term tends to zero as $\mu \to 0$ due to Lemma 4.1, a.e. for $t_1, t_2 \in [0,T]$, $t_2 - h \ge t_1$. For the first term we use the identities $\int_0^h E_t^s\dot\sigma_\mu\,ds = \Delta_t^h\sigma_\mu$, $\int_0^h E_t^s\dot\xi_\mu\,ds = \Delta_t^h\xi_\mu$, hence

$$\mathcal{I}_5 = \int_{t_1}^{t_2-h} (G_{1\mu}, \Delta_t^h\sigma_\mu)_\Omega + (G_{2\mu}, \Delta_t^h\xi_\mu)_\Omega\,dt.$$

Due to the convexity of G_μ we have

$$G_{1\mu} : \Delta_t^h \sigma_\mu + G_{2\mu} \cdot \Delta_t^h \xi_\mu \ \le \ \Delta_t^h G_\mu$$

and hence

$$\mathcal{I}_5 \ \le \ \int_{t_1}^{t_2-h} \int_\Omega \Delta_t^h G_\mu \, dx \, dt.$$

For fixed $h > 0$, the latter term tends to 0 a.e. with respect to $t_1, t_2 \in [0, T]$, as $\mu \to 0$, here we used Lemma 4.1 again. This proves Lemma 4.3. □

4.2 Proof of Theorem 2.1

Our approach here is very similar to the arguments used in [6]. Due to the strong convergence (3.9) of $\dot\sigma_\mu$ and $\dot\xi_\mu$, we have for any fixed h:

$$\int_0^h \int_0^{T-h} \int_\Omega |\Delta_t^s \dot\sigma|^2 + |\Delta_t^s \dot\xi|^2 \, dx \, dt \, ds = \lim_{\mu\to 0} \int_0^h \int_0^{T-h} \int_\Omega |\Delta_t^s \dot\sigma_\mu|^2 + |\Delta_t^s \dot\xi_\mu|^2 \, dx \, dt \, ds. \tag{4.3}$$

In order to control the term on the right-hand side we choose $\int_0^h \Delta_t^s \dot\sigma_\mu ds$ and $\int_0^h \Delta_t^s \dot\xi_\mu ds$ as test-functions in (3.5) and (3.4), then using the notation $\mathcal{T}_{0\mu} = \mathcal{T}_{0\mu}(t_1, t_2)$ from (4.2), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2-h} \int_0^h (A\dot\sigma_\mu, \Delta_t^s \dot\sigma_\mu)_\Omega \, ds \, dt \ + \ \int_{t_1}^{t_2-h} \int_0^h (H\dot\xi_\mu, \Delta_t^s \dot\xi_\mu)_\Omega \, ds \, dt \ + \ \mathcal{T}_{0\mu} \\ &= \int_{t_1}^{t_2-h} \int_0^h (\nabla \dot u_\mu, \Delta_t^s \dot\sigma_\mu)_\Omega \, ds \, dt =: \mathcal{I}. \end{aligned} \tag{4.4}$$

Next we recall an elementary identity, which holds for $\tau(t) \in \mathbb{R}^{n\times n}_{sym}$, and any symmetric tensor A,

$$A\tau : \Delta_t^s \tau = -\frac{1}{2} A \Delta_t^s \tau : \Delta_t^s \tau + \frac{1}{2} \Delta_t^s (A\tau : \tau), \tag{4.5}$$

if τ and A are scalar functions, this is even simpler. Relation (4.5) turns (4.4) into

$$\begin{aligned} \mathcal{L}(\dot\sigma_\mu, \dot\xi_\mu) &=: \int_{t_1}^{t_2-h} \int_0^h (A\Delta_t^s \dot\sigma_\mu, \Delta_t^s \dot\sigma_\mu)_\Omega + (H\Delta_t^s \dot\xi_\mu, \Delta_t^s \dot\xi_\mu)_\Omega \, ds \, dt \\ &= \int_{t_1}^{t_2-h} \int_0^h \int_\Omega \Delta_t^s (A\dot\sigma_\mu : \dot\sigma_\mu) + \Delta_t^s (H\dot\xi_\mu : \dot\xi_\mu) \, dx \, ds \, dt \ + 2\mathcal{T}_{0\mu} - 2\mathcal{I} := \mathcal{R}_\mu \end{aligned}$$

Since $\mathcal{L}$ defines a lower semi-continuous functional with respect to the $L^2(L^2)$-norm, the weak convergence of $(\dot\sigma_\mu, \dot\xi_\mu)$ implies

$$\mathcal{L}(\dot\sigma, \dot\xi) \le \liminf_{\mu\to 0} \mathcal{L}(\dot\sigma_\mu, \dot\xi_\mu),$$

note, that here even the limit exists since we have the strong convergence (3.9). Due to the positivity conditions for A and H (and Fubini's theorem) we get

$$\lim_{\mu\to 0} \int_0^h \int_{t_1}^{t_2-h} \int_\Omega |\Delta_t^s \dot\sigma_\mu|^2 + |\Delta_t^s \dot\xi_\mu|^2 \, dx\, dt\, ds \le C_0 \lim_{\mu\to 0} \mathcal{L}(\dot\sigma_\mu, \dot\xi_\mu), \tag{4.6}$$

hence the assertion (2.2) is true, if we show

$$\limsup_{\mu\to 0} \mathcal{R}_\mu \le Ch^2. \tag{4.7}$$

To estimate the first summand of $\mathcal{R}_\mu$, we put

$$\varphi(t) = \int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \, dx$$

and use the following argument

$$\Big| \int_{t_1}^{t_2-h} \int_0^h \Delta_t^s \varphi(t)\, ds\, dt \Big| = \Big| \int_0^h \int_{t_2-h}^{t_2-h+s} \varphi(t)\, dt\, ds - \int_0^h \int_{t_1}^{t_1+s} \varphi(t)\, dt\, ds \Big| \le 2\|\varphi\|_{L^\infty}\ h^2,$$

due to the bounds (3.6) for $\|\dot\sigma_\mu\|_{L^\infty(L^2)}$ and (3.7) for $\|\dot\xi_\mu\|_{L^\infty(L^2)}$. Note that the constant here is also independent from t_1 and t_2. Using the safe load from Definition 1.2 we rewrite the term $\mathcal{I}$ (cf. (4.4)):

$$\begin{aligned}
\mathcal{I} &= \int_{t_1}^{t_2-h} \int_0^h \big(\nabla \dot u_\mu, \Delta_t^s(\dot\sigma_\mu - \dot{\hat\sigma})\big)_\Omega \, ds\, dt + \int_{t_1}^{t_2-h} \int_0^h \big(\nabla \dot u_\mu, \Delta_t^s \dot{\hat\sigma}\big)_\Omega \, ds\, dt \\
&= \int_0^h \int_{t_1}^{t_2-h} \big(\nabla \dot u_\mu, \Delta_t^s \dot{\hat\sigma}\big)_\Omega \, dt\, ds,
\end{aligned}$$

since $\operatorname{div}(\dot\sigma_\mu - \dot{\hat\sigma}) = 0$ (and by using Fubini's theorem). Unlike in the proof for the Prandtl Reuss case we may already use the bound $\|\nabla \dot u_\mu\|_{L^\infty(L^2)} \le C_T$, together with the assumption (1.16) for $\hat\sigma$ this implies

$$\begin{aligned}
|\mathcal{I}| &\le \int_0^h \int_0^{T-h} \|\nabla \dot u_\mu\|_{L^2(\Omega)} \|D_t^s \dot{\hat\sigma} ds\|_{L^2(\Omega)} \, dt\, ds \\
&\le \|\nabla \dot u_\mu\|_{L^\infty(L^2)}\ h \int_0^h \int_0^T \|\ddot{\hat\sigma}\|_{L^2(\Omega)} \, dt\, ds \le C(T)h^2
\end{aligned}$$

where $C(T)$ is independent of $\mu \le \mu_0$ and $h < T$, and of t_1 and t_2, of course. From Lemma 4.3 we get then (4.7) for almost all t_1 and t_2, where C is independent of h and t_1, t_2. Since the integrals in (4.6) depend continuously on t_1 and t_2, we can pass to the limit $t_1 \to 0$, and $t_2 \to T$, which finishes the proof. $\square$

4.3 Proof of Theorem 2.3

Using the penalty equations (3.5) and (3.4) together with (3.2), we obtain in the case of kinematic hardening

$$\frac{1}{2}(\nabla u_\mu + (\nabla u_\mu)^\top) = A\dot{\sigma}_\mu + H\dot{\xi}_\mu.$$

Hence Theorem 2.3 follows from Theorem 2.1 and Korn's inequality. $\square$

5 The regularity in space direction

5.1 Auxiliary inequalities for the penalty terms

In the next lemma we derive a local bound for the spatial difference quotients of the penalty term similar to Lemma 4.2. Since we prove only local regularity in space direction we fix a localization function ζ with compact support in Ω such that $\nabla\zeta$ is Lipschitz continuous. We use difference and shift operators similar as in Section 4. Let e_j be the j-th unit-vector in $\mathbb{R}^n$, and $h > 0$. Apart from the notation Δ_i^h introduced in (2.1) we use also

$$E_j^h w(t,x) = w(t, x + he_j), \quad D_j^h = \frac{1}{h}\Delta_j^h.$$

Lemma 5.1 *Let (σ_μ, ξ_μ) be the solution to the penalty problem introduced in Section 3, then the following estimate holds*

$$\mathcal{P}_j := \int_0^T \big(D_j^h G_{1\mu}(\sigma_\mu, \xi_\mu), \zeta^2 D_j^h \sigma_\mu\big)_\Omega + \big(D_j^h G_{2\mu}(\sigma_\mu, \xi_\mu), \zeta^2 D_j^h \xi_\mu\big)_\Omega \, dt \le C$$

uniformly for $0 < \mu < \mu_0$ and $0 < h < h_0$, where $h_0 <$ dist (supp $\zeta, \partial\Omega$).

Proof. This proof contains also the arguments for the H^1_{loc}-regularity of σ. Just like in the proof of Lemma 4.2 we drop the index μ here. We apply the operation D_j^h to (3.5), (3.4), and then, similar as in the proof of Lemma 4.2, test the first relation with $\zeta^2 D_j^h \sigma$ and the second with $\zeta^2 D_j^h \xi$, then for any $T_1 \le T$ we have

$$\frac{1}{2}\int_0^{T_1} \frac{\partial}{\partial t}(D_j^h\sigma, \zeta^2 A D_j^h \sigma)_\Omega + \frac{\partial}{\partial t}(D_j^h\xi, \zeta^2 H D_j^h \xi)_\Omega \, dt + \mathcal{P}_j = \int_0^{T_1} (D_j^h \nabla\dot{u}, \zeta^2 D_j^h \sigma)_\Omega \, dt. \quad (5.1)$$

We integrate by parts in the right-hand side and use $-\operatorname{div}\sigma = f$, which leads to

$$\int_0^{T_1} (D_j^h \nabla \dot u, \zeta^2 D_j^h \sigma)_\Omega \, dt = -\int_0^{T} (D_j^h \dot u, \zeta^2 f)_\Omega \, dt - 2\int_0^{T} (D_j^h \dot u, \zeta \nabla\zeta D_j^h \sigma)_\Omega \, dt =: \mathcal{I}_1 + \mathcal{I}_2. \quad (5.2)$$

Since we have the uniform bound (3.8) for $\|\nabla \dot u\|_{L^\infty(L^2(\operatorname{supp}\zeta))}$ as $\mu \to 0$, we obtain also $\|D_j^h \dot u\|_{L^\infty(L^2(\operatorname{supp}\zeta))} \le C$, uniformly in $0 < \mu < \mu_0$, $0 < h < h_0$. Therefore the term $\mathcal{I}_1$ turns out to be bounded due to the regularity assumptions for f. Using this argument once more, Hölder's and Young's inequalities imply

$$|\mathcal{I}_2| \le C_\zeta \big(1 + \int_0^T \int_\Omega \zeta^2 |D_j^h \sigma|^2 \, dx \, dt\big). \quad (5.3)$$

Now we evaluate the left-hand side of (5.1). Since $\xi(0) = 0$, we have

$$\begin{aligned} &\frac{1}{2}\int_0^T \frac{\partial}{\partial t}(D_j^h \sigma, \zeta^2 A D_j^h \sigma)_\Omega + \frac{\partial}{\partial t}(D_j^h \xi, \zeta^2 H D_j^h \xi)_\Omega \, dt \\ &\quad = \frac{1}{2}\Big((D_j^h \sigma A, \zeta^2 D_j^h \sigma)_\Omega + (D_j^h \xi, \zeta^2 H D_j^h \xi)_\Omega\Big)\Big|_{t=T_1} - (D_j^h \sigma^0 A, \zeta^2 D_j^h \sigma^0)_\Omega \end{aligned}$$

The convexity of the penalty potential implies $\mathcal{P}_j \ge 0$, hence the bounds for $\mathcal{I}_1$ and (5.3) for $\mathcal{I}_2$ together with the positivity of A and H lead to the inequality

$$\begin{aligned} \int_\Omega \zeta^2 |D_j^h \sigma(T_1)|^2 \, dx &\le (D_j^h \sigma, \zeta^2 A D_j^h \sigma)_\Omega\Big|_{t=T_1} + (D_j^h \xi, \zeta^2 H D_j^h \xi)_\Omega\Big|_{t=T_1} + 2\mathcal{P}_j \\ &\le C\big(1 + \int_\Omega |D_j^h \sigma_0|^2 \zeta^2 dx + \int_0^{T_1}\int_\Omega \zeta^2 |D_j^h \sigma|^2 \, dx \, dt\big). \end{aligned}$$

Since $\sigma_0 \in H^1(\Omega)$, now Gronwall's inequality implies that $|\mathcal{I}_2|$ is bounded independent of μ, but this implies also the desired bound for $\mathcal{P}_j$. $\Box$

We also need a result corresponding to Lemma 4.3 for difference quotients of $\dot\sigma_\mu$ in space direction. We recall that

$$\big(E_t^s E_i^h - I\big)\, w(t,x) = w(t+s, x+he_i) - w(t,x).$$

Lemma 5.2 *Fix $h_0 > 0$ such that supp $E_i^h \zeta \subset \Omega$ for any h with $0 \le h \le h_0$. Then there exists a constant $\tilde C > 0$, independent of h, t_1, and t_2 such that*

$$\limsup_{\mu\to 0} \int_0^h \int_{t_1}^{t_2-h} \int_\Omega \Big[G_{1\mu} : (E_t^s E_i^h - I)\dot\sigma_\mu \;+\; G_{2\mu} : (E_t^s E_i^h - I)\dot\xi_\mu\Big]\, \zeta^2 \, dx \, dt \, ds \le \tilde C h^2,$$

for almost every t_1, t_2 such that $0 \le t_1 \le t_2 - h \le T - h$.

Proof. We set

$$\mathcal{T}_{i\mu} = \mathcal{T}_{i\mu}(t_1, t_2) = \int\limits_0^h \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \Big[G_{1\mu} : (E_t^s E_i^h - I)\dot{\sigma}_\mu \ + \ G_{2\mu} : (E_t^s E_i^h - I)\dot{\xi}_\mu \Big] \zeta^2 \, dx \, dt \, ds$$

and decompose $\mathcal{T}_{i\mu} = \mathcal{T}_{i\mu}^1 - \mathcal{T}_{i\mu}^2$ where

$$\begin{aligned} \mathcal{T}_{i\mu}^1 \ &= \int\limits_0^h \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \Big[G_{1\mu} : E_t^s E_i^h \dot{\sigma}_\mu \ + \ G_{2\mu} : E_t^s E_i^h \dot{\xi}_\mu \Big] \zeta^2 \, dx \, dt \, ds, \\ \mathcal{T}_{i\mu}^2 \ &= \int\limits_0^h \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \Big(G_{1\mu} : \dot{\sigma}_\mu + G_{2\mu} : \dot{\xi}_\mu \Big) \zeta^2 \, dx \, dt \, ds \\ &= \int\limits_0^h \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \frac{d}{dt} G_\mu \zeta^2 \, dx \, dt \, ds = h \int\limits_\Omega G_\mu \zeta^2 \, dx \Big|_{t_1}^{t_2-h} \end{aligned}$$

Lemma 4.1 implies

$$\lim_{\mu \to 0} \mathcal{T}_{i\mu}^2 \ = \ 0 \quad \text{a.e. with respect to } t_1, t_2, \tag{5.4}$$

such that $0 \le t_1 \le t_2 - h \le T - h$. To analyze $\mathcal{T}_{i\mu}^1$ we perform the integration with respect to s, then we add and subtract the terms $G_{1\mu} : \sigma_\mu$ and $G_{2\mu} : \xi_\mu$ in order to achieve an additional splitting:

$$\begin{aligned} \mathcal{T}_{i\mu}^1 = & \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \big[G_{1\mu} : (E_t^h E_i^h \sigma_\mu - E_i^h \sigma_\mu) \ + \ G_{2\mu} : (E_t^h E_i^h \xi_\mu - E_i^h \xi_\mu) \big] \zeta^2 \, dx \, dt \\ = & \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \big[G_{1\mu} : (E_t^h E_i^h \sigma_\mu - \sigma_\mu) \ + \ G_{2\mu} : (E_t^h E_i^h \xi_\mu - \xi_\mu) \big] \zeta^2 \, dx \, dt \\ & - h \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \big[G_{1\mu} : D_i^h \sigma_\mu \ + \ G_{2\mu} : D_i^h \xi_\mu \big] \zeta^2 \, dx \, dt =: \mathcal{T}_{i\mu}^{1a} + \mathcal{T}_{i\mu}^{1b}. \end{aligned}$$

We first will get rid of $\mathcal{T}_{i\mu}^{1b}$ and rewrite

$$\begin{aligned} \mathcal{T}_{i\mu}^{1b} = & - h \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \big[E_i^h G_{1\mu} : D_i^h \sigma_\mu + E_i^h G_{2\mu} D_i^h \xi_\mu \big] \zeta^2 \, dx \, dt \ + \\ & + h^2 \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \big[D_i^h G_{1\mu} : D_i^h \sigma_\mu + D_i^h G_{2\mu} D_i^h \xi_\mu \big] \zeta^2 \, dx \, dt \end{aligned}$$

Since the last integral is bounded according to Lemma 5.1 we conclude

$$\mathcal{T}^{1b}_{i\mu} \;\le \tilde{C}\, h^2 - h \int\limits_{t_1}^{t_2-h} \int\limits_{\Omega} \left[E^h_i G_{1\mu} : D^h_i \sigma_\mu \;+\; E^h_i G_{2\mu} D^h_i \xi_\mu\right] \zeta^2 \, dx\, dt$$

Now we exploit the convexity of G_μ again, which leads to

$$-\left(E^h_i G_{1\mu} : D^h_i \sigma_\mu \;+\; E^h_i G_{2\mu} D^h_i \xi_\mu\right) \;\le\; h^{-1}\left(G_\mu - E^h_i G_\mu\right) \;=-\; D^h_i G_\mu$$

This implies

$$\mathcal{T}^{1b}_{i\mu} \;\le \tilde{C} h^2 - h \int\limits_{t_1}^{t_2-h} \int\limits_{\Omega} D^h_i G_\mu \zeta^2 \, dx,$$

here the last summand tends to zero as $\mu \to 0$ due to Lemma 4.1. Thus we arrive at

$$\limsup_{\mu\to\infty} \mathcal{T}^{1b}_{i\mu} \;\le \tilde{C}\, h^2 \tag{5.5}$$

A similar argument works for $\mathcal{T}^{1a}_{i\mu}$: From the convexity of G_μ we get

$$\mathcal{T}^{1a}_{i\mu} \;\le\; \int\limits_{t_1}^{t_2-h} \int\limits_{\Omega} [E^h_t E^h_i G_\mu - G_\mu] \zeta^2 \, dx\, dt. \tag{5.6}$$

while the right-hand side term tends to zero as $\mu \to 0$, h fixed, a.e. with respect to t_1, t_2, again due to Lemma 4.1. Collecting (5.4), (5.5), (5.6), we arrive at $\lim\limits_{\mu\to\infty} \mathcal{T}_{i\mu} \;\le\; \tilde{C}\, h^2$ a.e. which is the statement of Lemma 5.2. $\square$

5.2 Testing the strain velocity

We have to prepare one additional auxiliary estimate for the regularity result in space direction.

Lemma 5.3 *Let ζ be a localization function as in Lemma 5.1, and let $h_0 > 0$ be fixed such that $h_0 < dist\,(supp\,\zeta, \partial\Omega)$. Then*

$$h^{\delta-2} \int\limits_0^h \int\limits_0^{T-h} \left(\nabla \dot{u}\,,\, \zeta^2 (E^s_t E^h_i - I)\dot{\sigma}\right)_\Omega dt\, ds \le C \tag{5.7}$$

uniformly in $h \in (0, h_0]$.

Proof. Step 1. We denote

$$\mathcal{S} = \int\limits_0^h \int\limits_0^{T-h} \left(\nabla \dot{u}\,,\, \zeta^2 (E^s_t E^h_i - I)\dot{\sigma}\right)_\Omega dt\, ds = \mathcal{S}^1 + \mathcal{S}^2,$$

where

$$\begin{aligned}
\mathcal{S}^1 :&= \int_0^h \int_0^{T-h} \big(\nabla \dot{u}\,,\, \zeta^2 E_t^s (E_i^h - I)\dot{\sigma}\big)_\Omega \,dt\,ds = \int_0^h \int_0^{T-h} \big(\nabla \dot{u}\,,\, \zeta^2 E_t^s \Delta_i^h \dot{\sigma}\big)_\Omega \,dt\,ds, \\
\mathcal{S}^2 :&= \int_0^h \int_0^{T-h} \big(\nabla \dot{u}\,,\, \zeta^2 (E_t^s - I)\dot{\sigma}\big)_\Omega \,dt\,ds = \int_0^h \int_0^{T-h} \big(\nabla \dot{u}\,,\, \zeta^2 \Delta_t^s \dot{\sigma}\big)_\Omega \,dt\,ds.
\end{aligned} \tag{5.8}$$

In the next two steps we will show, that the first term can be estimated uniformly even with a factor h^{-2}, while for the second term we can only reach $h^{\delta-2}|\mathcal{S}^2| \leq C$.

Step 2. Estimates for $|\mathcal{S}^1|$.

To this end, we integrate by parts in the term $\mathcal{S}^1$, then use the relation $-\operatorname{div}\sigma = f$, end up with

$$\mathcal{S}^1 = -\int_0^h \int_0^{T-h} \big(\dot{u}\zeta^2\,,\, E_t^s \Delta_i^h \dot{f}\big)_\Omega \,dt\,ds - \int_0^h \int_0^{T-h} \big(\dot{u}\nabla\zeta^2\,,\, \Delta_t^s \dot{\sigma}\big)_\Omega \,dt\,ds =: \mathcal{S}^{11} + \mathcal{S}^{12}.$$

Moving the operator Δ_i^h from $\dot{f}$ to $\dot{u}\zeta^2$ yields

$$\mathcal{S}^{11} = -\int_0^h \int_0^{T-h} \big(\Delta_i^{-h}(\dot{u}\zeta^2)\,,\, E_t^s \dot{f}\big)_\Omega \,dt\,ds.$$

Since

$$\|\Delta_i^{-h}(\dot{u}\zeta^2)\|_{L^\infty(L^2)} = h\|D_j^{-h}\big(\dot{u}\zeta^2\big)\|_{L^\infty(L^2)} \leq C\big(\|\dot{u}\|_{L^\infty(L^2)} + \|\nabla\dot{u}\|_{L^\infty(L^2)}\big)h,$$

the uniform estimates (3.6), (3.8) together with the assumption $\dot{f} \in L^\infty(L^2)$ (cf (1.4)) lead to

$$|\mathcal{S}^{11}| \leq h \int_0^h \|\dot{f}\|_{L^1(L^2)} \|D_i^{-h}(\dot{u}\zeta^2)\|_{L^\infty(L^2)}\,ds \leq C_T h^2,$$

where K_T is independent of $0 < \mu \leq \mu_0$, and $0 < h \leq h_0$. A similar argument works for the summand $\mathcal{S}^{12}$, hence, again with (3.6) and (3.8),

$$\begin{aligned}
|\mathcal{S}^{12}| &= \left| \int_0^h \int_0^{T-h} \big(\Delta_i^{-h}(\dot{u}\nabla\zeta^2)\,,\, E_t^s \dot{\sigma}\big)_\Omega \,dt\,ds \right| \\
&\leq C\ h \int_0^h \|\dot{\sigma}\|_{L^1(L^2)} \|D_i^{-h}(\dot{u}\nabla\zeta^2)\|_{L^\infty(L^2)}\,ds \leq C_T\ h^2.
\end{aligned}$$

Step 3. Estimates for $|\mathcal{S}^2|$.

To show that this quantity is bounded by $Ch^{2-\delta}$, it is not enough to use $\nabla\dot{u} \in L^\infty(L^2)$

together with (2.2), because then we are left with the term $h^{1/2-\delta}$. Instead we need to exploit (2.2) and (2.3), which will be done via Fourier analysis. To this end we extend the functions $\zeta^2\nabla\dot{u}$ and $\dot{\sigma}$ by zero to the symmetric time interval $[-2T, 2T]$, that is for $t \in [-2T, 2T]$, we set

$$y(t,\cdot) = \begin{cases} \zeta^2(\cdot)\nabla\dot{u}(t,\cdot), & t \in [0,T] \\ 0 & \text{else} \end{cases}, \qquad z(t,\cdot) = \begin{cases} \dot{\sigma}(t,\cdot), & t \in [0,T] \\ 0 & \text{else} \end{cases}.$$

Then both y and z have jumps at $t = 0$ and $t = T$. However, since we know that $\nabla\dot{u} \in L^\infty(L^2)$, $\dot{\sigma} \in L^\infty(L^2)$, we obtain even

$$\begin{aligned}
&\int_0^h \int_{-2T}^{2T} \int_\Omega |\Delta_t^s y|^2 + |\Delta_t^s z|^2 \, dx\, dt\, ds = \int_0^h \int_{-s}^{0} \int_\Omega |E_t^s y|^2 + |E_t^s z|^2 \, dx\, dt\, ds \\
&\quad + \int_0^h \int_0^{T-h} \int_\Omega |\Delta_t^s y|^2 + |\Delta_t^s z|^2 \, dx\, dt\, ds + \int_0^h \int_{T-h}^{T} \int_\Omega |\Delta_t^s y|^2 + |\Delta_t^s z|^2 \, dx\, dt\, ds \\
&\le C \left(\int_0^h \int_0^{T-h} \int_\Omega |\Delta_t^s y|^2 + |\Delta_t^s z|^2 \, dx\, dt\, ds + 3h^2(\|\nabla\dot{u}\|^2_{L^\infty(L^2)} + \|\dot{\sigma}\|^2_{L^\infty(L^2)}) \right).
\end{aligned}$$

The same splitting leads to

$$\begin{aligned}
\mathcal{S}^2 &= \int_0^h \int_0^{T-h} \left(y, \Delta_t^s z\right)_\Omega dt\, ds \\
&= \int_0^h \int_{-2T}^{2T} \left(y, \Delta_t^s z\right)_\Omega dt\, ds - \left(\int_0^h \int_{-s}^{0} \left(y, E_t^s z\right)_\Omega dt\, ds + \int_0^h \int_{T-h}^{T} \left(y, \Delta_t^s z\right)_\Omega dt\, ds \right) \\
&=: \mathcal{S}^3 + \mathcal{I}_{rem},
\end{aligned} \tag{5.9}$$

where again $|\mathcal{I}_{rem}| \le C\, h^2$ independent of h. Now let $b_m, c_m \in L^2(\Omega)$ be the Fourier coefficients of y and z, respectively, that is

$$y(t,x) = \sum_{m=-\infty}^{\infty} b_m(x) \exp(\frac{im\pi}{2T} t), \quad z(t,x) = \sum_{m=-\infty}^{\infty} c_m(x) \exp(\frac{im\pi}{2T} t).$$

In [6, Lemma A.1] it was shown that

$$\sum_{m=-\infty}^{\infty} \|b_m\|^2_{L^2(\Omega)} m^{1-\delta} \le C(T) \sup_{0<h\le h_0} h^{-2} \int_0^h \int_{-2T}^{2T} \|\Delta_t^s y\|_{L^2(\Omega)} \, dt\, ds, \tag{5.10}$$

the same inequality holds with b_m replaced by c_m, and y by z, of course. The term $\mathcal{S}^3$ can be estimated as follows:

$$
\begin{aligned}
|\mathcal{S}^3| &\le \Big| \sum_{m=-\infty}^{\infty} (b_m, c_m)_\Omega \Big| \int_0^h |\exp(\frac{im\pi}{2T}s) - 1|\, ds \\
&\le \sum_{m=-\infty}^{\infty} \|b_m\|_{L^2(\Omega)} \|c_m\|_{L^2(\Omega)} \int_0^h \big(2 - 2\cos(\frac{m\pi}{2T}s)\big)^{1/2}\, ds. \qquad (5.11)
\end{aligned}
$$

Furthermore, elementary calculations lead to

$$
h^{\delta-1}\big(2 - 2\cos(\frac{m\pi}{2T}s)\big)^{1/2} = h^{\delta-1}\big|\sin(\frac{m\pi}{2T}s)\big| = \frac{m^{1-\delta}}{(mh)^{1-\delta}}\big|\sin(\frac{m\pi}{2T}s)\big| \le Cm^{1-\delta},
$$

where $C = C(T)$ is independent of h. Indeed, for $hm \ge T$ we can use $|\sin \cdots| \le 1$, while for $hm \le T$ we have

$$
(mh)^{\delta-1}\big|\sin(\frac{m\pi}{2T}s)\big| \le (mh)^{\delta-1}\big|\sin(\frac{m\pi}{2T}h)\big| \le (mh)^{\delta} \le T^{\delta},
$$

since $0 \le s \le h$. Now from (5.11) and (5.10) it follows

$$
\begin{aligned}
h^{\delta-2}|\mathcal{S}^3| &\le C \sum_{m=-\infty}^{\infty} \|b_m\|_{L^2(\Omega)}\ \|c_m\|_{L^2(\Omega)}\ m^{1-\delta} \\
&\le C\Big(\sup_h h^{-2} \int_0^h \int_0^{T-h} \int_\Omega |\Delta_t^s \nabla \dot{u}|^2\, dx\, dt\, ds\Big)^{1/2} \Big(\sup_h h^{-2} \int_0^h \int_0^{T-h} \int_\Omega |\Delta_t^s \dot{\sigma}|^2\, dx\, dt\, ds\Big)^{1/2}
\end{aligned}
$$

which is bounded independent of h due to Theorem 2.1 and 2.3 (or rather by assumption in the case of isotropic hardening).

5.3 Proof of Theorem 2.4

We use the test function $\zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu$ in (3.3) and test the relation (3.4) with $\zeta^2(E_t^s E_i^h - I)\dot{\xi}_\mu$. We sum the relations and integrate from $t = t_1$ to $t = t_2 - h$, this yields

$$
\begin{aligned}
&\int_0^h \int_{t_1}^{t_2-h} \big(A\dot{\sigma}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu\big)_\Omega + \big(H\dot{\xi}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\xi}_\mu\big)_\Omega\, dt\, ds + \mathcal{T}_{i\mu}(t_1, t_2) = \\
&\qquad \int_0^h \int_{t_1}^{t_2-h} \big(\nabla \dot{u}_\mu, \zeta^2(E_t^s E_i^h - I)\dot{\sigma}_\mu\big)_\Omega\, dt\, ds =: \mathcal{S}_\mu(t_1, t_2), \qquad (5.12)
\end{aligned}
$$

where $\mathcal{T}_{i\mu}$ is the quantity coming from the penalty term and has the same meaning as in Lemma 5.2. Similar as in (4.5), for symmetric tensors A (with constant entries) and $\tau = \tau(t, x)$ we have the relation

$$
A\tau : (E_t^s E_i^h - I)\tau = \frac{1}{2}(E_t^s E_i^h - I)(A\tau : \tau) - \frac{1}{2}A(E_t^s E_i^h - I)\tau : (E_t^s E_i^h - I)\tau. \qquad (5.13)
$$

Put

$$\mathcal{R}_\mu(t_1,t_2) := \frac{1}{2}\int\limits_0^h \int\limits_{t_1}^{t_2-h} \int\limits_\Omega \Big[(E_t^s E_i^h - I)(A\dot\sigma_\mu : \sigma_\mu) + (E_t^s E_i^h - I)(H\dot\xi_\mu : \dot\xi_\mu)\Big]\zeta^2\, dx\, dt\, ds,$$

then (5.13) turns (5.12) into

$$\begin{aligned}
&\mathcal{L}_\mu(t_1,t_2) := \\
&\int\limits_0^h \int\limits_{t_1}^{t_2-h} (A(E_t^s E_i^h - I)\dot\sigma_\mu, \zeta^2(E_t^s E_i^h - I)\dot\sigma_\mu)_\Omega + (H(E_t^s E_i^h - I)\dot\xi_\mu, \zeta^2(E_t^s E_i^h - I)\dot\xi_\mu)_\Omega\, dt\, ds \\
&= \mathcal{R}_\mu(t_1,t_2) + \mathcal{T}_{i\mu}(t_1,t_2) - \mathcal{S}_\mu(t_1,t_2). \qquad (5.14)
\end{aligned}$$

Due to the strong convergence [3] $\dot\sigma_\mu \to \dot\sigma$, $\dot\xi_\mu \to \dot\xi$ and $\nabla\dot u_\mu \to \nabla\dot u$ in $L^2(L^2)$ we obtain for any fixed h with $0 < h \le h_0$:

$$\begin{aligned}
&\lim_{\mu\to 0} \mathcal{L}_\mu(t_1,t_2) =: \mathcal{L}(t_1,t_2) = \qquad (5.15)\\
&\frac{1}{2}\int\limits_0^h \int\limits_{t_1}^{t_2-h} (A(E_t^s E_i^h - I)\dot\sigma, \zeta^2(E_t^s E_i^h - I)\dot\sigma)_\Omega + (H(E_t^s E_i^h - I)\dot\xi, \zeta^2(E_t^s E_i^h - I)\dot\xi)_\Omega\, dt\, ds,
\end{aligned}$$

$$\lim_{\mu\to 0} \mathcal{S}_\mu(t_1,t_2) =: \mathcal{S}(t_1,t_2) = \int\limits_0^h \int\limits_{t_1}^{t_2-h} (\nabla\dot u, \zeta^2(E_t^s E_i^h - I)\dot\sigma)_\Omega\, dt\, ds \qquad (5.16)$$

for all t_1, t_2 with $0 \le t_1 < t_2 - h \le T - h$. To estimate the term $\mathcal{R}_\mu(t_1,t_2)$, we use the following argument: If $\omega \in L^\infty(L^1)$, then for all t_1, t_2 such that $0 \le t_1 \le t_2 - h \le T - h$ it follows

$$\Big|\int\limits_{t_1}^{t_2-h}\int\limits_\Omega \zeta^2(E_t^s E_i^h - I)\omega\, dx\, dt\Big| \le Ch\|\omega\|_{L^\infty(L^1)}. \qquad (5.17)$$

Indeed, rewriting the integral we find

$$\begin{aligned}
&\Big|\int\limits_{t_1}^{t_2-h}\int\limits_\Omega \zeta^2(E_t^s E_i^h - I)\omega\, dx\, dt\Big| = \Big|\int\limits_{t_1}^{t_2-h}\int\limits_\Omega \zeta^2(E_t^s - I)E_i^h\omega + \zeta^2(E_i^h - I)\omega\, dx\, dt\Big| \\
&\le \Big|\int\limits_{t_1+s}^{t_2-h+s}\int\limits_\Omega \zeta^2 E_i^h\omega\, dx\, dt - \int\limits_{t_1}^{t_2-h}\int\limits_\Omega \zeta^2 E_i^h\omega\, dx\, dt\Big| + \Big|\int\limits_{t_1}^{t_2-h}\int\limits_\Omega \zeta^2(E_i^h - I)\omega\, dx\, dt\Big| \\
&\le \Big|\int\limits_{t_1}^{t_1+s}\int\limits_\Omega \zeta^2 E_i^h\omega\, dx\, dt\Big| + \Big|\int\limits_{t_2-h}^{t_2-h+s}\int\limits_\Omega \zeta^2 E_i^h\omega\, dx\, dt\Big| + \Big|\int\limits_{t_1}^{t_2-h}\int\limits_\Omega \zeta^2(E_i^h - I)\omega\, dx\, dt\Big| \\
&\le C\, h\, \|\omega\|_{L^\infty(L^1)},
\end{aligned}$$

[3] Like in the proof of Theorem 2.1, in the first term it suffices to use the weak convergence and lower semi-continuity.

here we also used $|(E_i^h - I)\zeta| \le c\,h$.
Hence, taking also the integration over s into account, it follows

$$|\mathcal{R}_\mu(t_1,t_2)| \le C\,h^2\,(\|\dot\sigma_\mu\|^2_{L^\infty(L^2)} + \|\dot\xi_\mu\|^2_{L^\infty(L^2)}),$$

where C is independent of μ and h. With Lemma 5.2 it follows now for almost all t_1, t_2, such that $0 \le t_1 \le t_2 - h \le T - h$:

$$\mathcal{L}(t_1,t_2) \le \limsup_{\mu\to 0} \mathcal{T}_{i\mu}(t_1,t_2) + \limsup_{\mu\to 0} |\mathcal{R}_\mu(t_1,t_2)| - \mathcal{S}(t_1,t_2) \le Ch^2 - \mathcal{S}(t_1,t_2),$$

where the constant depends neither on h nor on t_1, t_2. Since $\mathcal{L}$ and $\mathcal{S}$ depend continuously on t_1 and t_2, we obtain

$$\mathcal{L}(0,T) \le Ch^2 + |\mathcal{S}(0,T)|. \tag{5.18}$$

Due to the positivity assumptions on A and H we have

$$\mathcal{L}(0,T) \ge C \int_0^h \int_0^{T-h} \int_\Omega \big(\, |(E_t^s E_i^h - I)\dot\sigma|^2 + |(E_t^s E_i^h - I)\dot\xi|^2 \,\big)\ \zeta^2\,dx\,dt\,ds.$$

Now we apply the argument

$$\begin{aligned}
|(E_t^s E_i^h - I)\dot\sigma|^2 &= |(E_t^s E_i^h - E_i^h + E_i^h - I)\dot\sigma|^2 \ \ge\ \frac{7}{8}|\Delta_i^h\dot\sigma|^2 - \frac{1}{8}|\Delta_t^s E_i^h\dot\sigma|^2,\\
|(E_t^s E_i^h - I)\dot\xi|^2 &\ \ge\ \frac{7}{8}|\Delta_i^h\dot\xi^2|^2 - \frac{1}{8}|\Delta_t^s E_i^h\dot\xi|^2.
\end{aligned}$$

We combine this with (5.18), use the translation invariance of integrals and arrive at

$$\begin{aligned}
\int_0^h \int_0^{T-h} \int_\Omega (|\Delta_i^h\dot\sigma|^2 + |\Delta_i^h\dot\xi|^2)\zeta^2\,dx\,dt\,ds &= h \int_0^{T-h} \int_\Omega (|\Delta_i^h\dot\sigma|^2 + |\Delta_i^h\dot\xi|^2)\zeta^2\,dx\,dt\\
&\le ch^2 + |\mathcal{S}(0,T)| + \tilde C \int_0^h \int_0^{T-h} \int_\Omega (|\Delta_t^s\dot\sigma|^2 + |\Delta_t^s\dot\xi|^2)\zeta^2\,dx\,dt\,ds.
\end{aligned}$$

Finally we multiply this inequality by $h^{\delta-2}$, then the assertion follows from Lemma 5.3 and Theorem 2.1. $\square$

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