The Numéraire Portfolio, Asymmetric Information and Entropy

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Abstract

We study the relation between different forms of non-existence of arbitrage and the characteristics of the stochastic basis under the different filtrations. This is achieved through the analysis of the properties of the numéraire portfolio. Furthermore, we focus on the problem of calculating the additional logarithmic utility of the better informed investor in terms of the Shannon entropy of his additional information. We show that the expected logarithmic utility increment due to better information equals its Shannon entropy also in case of a pure jump basis with jumps that are quadratically hedgeable, and so extend a similar result known for bases consisting of continuous semimartingales.

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1 Introduction

One of the fundamental questions in mathematical finance is the existence of arbitrage in the market. In markets generated by semimartingales, the most common no arbitrage concept, *no free lunch with vanishing risk* (NFLVR), is shown, in [7] and [8], to be equivalent with the existence of an *equivalent sigma martingale measure*. Under this measure the dynamics of the market assets S discounted by a risk free bond B are seen to be sigma martingales. A number of less restrictive concepts of arbitrage have since been introduced, with the latest being the notion of *no unbounded profit with bounded risk* (NUPBR), see [11]. From a mathematical point of view, the requirement of the existence of an equivalent sigma martingale measure under (NFLVR) is weakened in a (NUPBR) market by the existence of a process W, such that any possible portfolio in the market that is discounted by W is a supermartingale under the original market measure. This process is called the numéraire and its existence, as is shown in [11], is linked to the characteristic triplet of the stochastic basis of the market.
The first aim of this paper is to find and explicit link between the characteristic triplet of the underlying semimartingales and the different notions of arbitrage in the market. Assuming

a weak form of the *structure condition*, first introduced in [14], and under additional (yet necessary) conditions, we obtain a representation of the underlying semimartingales that depends on the market price of risk (or information drift). Under this representation we show that the existence of arbitrage depends, firstly on the jump structure of the semimartingales and secondly on the integrability of the process describing the market price of risk.
Having studied this link, we extend our results to markets with asymmetric information. Furthermore, we study the utility advantage that a better informed investor may have in terms of the underlying market structure, and interpret it as in [2] by entropy notions such as the Shannon entropy in the case of logarithmic utility. In contrast to previous work this is to be achieved in a setting as general as possible, in the sense of [11].

The paper is organised as follows. The market set-up is explained in Section 2. The special semimartingales playing the role of underlying price dynamics for the market are discussed in Section 3. The link between characteristics of the underlying semimartingales and the different notions of arbitrage, i.e. the existence of the numéraire, is studied in Section 4. The advantages of our analysis is illustrated in the examples of Section 4.1, in which the explicit form of the numéraire portfolio can be given. With the ensuing descriptions especially of the related numéraire portfolios, we discuss in Section 5 the structure of the information drift of an (initially enlarged) filtration $\mathcal{G}$, and therefore the expected logarithmic utility advantage of the better informed investor. We are able to identify the extra expected logarithmic utility in a purely discontinuous setting, in which the squares of the jumps are hedgeable, with the Shannon entropy of the additional information, thereby extending this striking equality beyond the continuous case, see [3], [1].

Introductory remarks and notation

In the analysis hereafter the notation and results on semimartingales are based on [10]. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ satisfies the usual conditions. With $\mathcal{P}(\mathbb{R}^d)$ we denote the set of $\mathbb{R}^d$ valued predictable processes in the given probability space. For any adapted càdlàg process X we define the jump process $\Delta X = X - X_-$, where X_- denotes the left-hand limit of X.
Let $\pi \in \mathcal{P}(\mathbb{R}^d)$ and Y be a $d-$dimensional semimartingale, then $\pi \cdot Y = \int \pi dY$ denotes the stochastic integral whenever this is well defined. Furthermore, we define the quadratic covariation process of two semimartingales X, Y as $[X, Y] = XY - X_- \cdot Y - Y_- \cdot X$. If X, Y are locally square integrable martingales, then $[X, Y]$ is locally integrable and has a predictable compensator $\langle X, Y\rangle$.
Lastly, for a semimartingale X starting at zero, with $\mathcal{E}(X)$ we denote the Dolean-Dade exponential. The exponential has the form

$$\mathcal{E}(X) = \exp\left(X - \frac{1}{2}[X^c, X^c]\right)\Pi_{s\leq \cdot}\left(1 + \Delta X_s i\right)\exp\left(-\Delta X_s\right),$$

where X^c denotes the continuous part of the process, and satisfies the integral equation $Z = 1 + Z_- \cdot Y$.

2 Market set-up

We work in a market characterized by a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and the time horizon T is finite. The market consists of a risk free asset S^0 and d risky assets $S^1, \dots, S^d$. With no loss of generality we assume that $S^1, \ldots, S^d$ are strictly positive semimartingales and $S^0 = 1$. Therefore we may state that for every i there exists a semimartingale X^i, with $X_0^i = 0$ and $\Delta X^i > -1$ such that

$$S^i = S_0^i \mathcal{E}(X^i),$$

with $S_0^i > 0$.
With the $d+1$ assets of our market we create a portfolio W^π, where π denotes the investment strategy. As in [11], we impose a "credit limit" in order to avoid "doubling strategies". This limit is a uniform lower bound on the wealth process W^π, which we set equal to zero, i.e. we impose $W^\pi > 0$. Furthermore we normalize the initial value $W_0 = 1$.
Let $X = (X^1, \ldots, X^d)^*$, $\pi = (\pi^1, \ldots, \pi^d)$, with π^i denoting the proportion of the portfolio value invested in asset i, $i = 1, \ldots, d$, and $\pi^0 = 1 - \sum_{i=1}^d \pi^i$ denoting the proportion of the portfolio invested in the risk free asset. Then the dynamics of the portfolio satisfy the equation

$$\frac{dW_t^\pi}{W_t^\pi} = \sum_{i=1}^d \pi_t^i \frac{dS_t^i}{S_t^i} = \pi_t dX_t,$$

hence $W^\pi = \mathcal{E}(\pi \cdot X)$. For the latter to make sense the integral $\int_0^\cdot \pi_t dX_t$ has to be well defined. Furthermore from the credit limit W^π has to be positive. For these the set of *admissible portfolios*, denoted by $\mathcal{W}$, is defined as

$$\mathcal{W} = \left\{ W^\pi = \mathcal{E}(\pi \cdot X) \middle| \pi \in L(X) \text{ and } \pi \Delta X > -1 \right\},$$

with $L(X)$ denoting the set of $\mathbb{R}^d-$valued predictable processes that are integrable with respect to X. Of specific interest are the admissible portfolios that "outperform" any other portfolio in $\mathcal{W}$. More precisely we focus on the portfolios introduced by the following definition.

Definition 2.1 (i). An admissible portfolio W^π is called the *numéraire portfolio*, if the process $\frac{W^\rho}{W^\pi}$ is a supermartingale for every $W^\rho \in \mathcal{W}$.

(ii). An admissible portfolio W^π is called *(relative) growth optimal* (GOP), if

$$E\left[\log\left(\frac{W_T^\rho}{W_T^\pi}\right)\right] \leq 0$$

for all $W^\rho \in \mathcal{W}$.

(iii). An admissible portfolio W^π with $E[\ln W^\pi] < \infty$ is called *log-utility-optimal portfolio* if $E[\ln W^\rho] \leq E[\ln W^\pi]$ for every $W^\rho \in \mathcal{W}$.

The existence and properties of these optimal portfolios is closely related to the existence of different forms of arbitrage in the market, that are presented in the following definition.

Definition 2.2 [11] We consider the following types of arbitrage.

(i). A portfolio $W^\pi \in \mathcal{W}$ is said to generate an *arbitrage opportunity*, if it satisfies $P[W_T^\pi \geq 1] = 1$ and $P[W_T^\pi > 1] > 0$. If such a portfolio does not exist, we have *no arbitrage*(NA).

(ii). A sequence $(W^{\pi_n})_{n\in\mathbb{N}}$ of admissible portfolios is said to generate an *unbounded profit with bounded risk* (UPBR), if the collection of positive random variables $(W_T^{\pi_n})_{n\in\mathbb{N}}$ is unbounded in probability, i.e. if

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}} P[W_T^{\pi_n} > m] > 0.$$

If such a sequence does not exist, we say that there is *no unbounded profit with bounded risk* (NUPBR).

(iii). A sequence $(W^{\pi_n})_{n\in\mathbb{N}}$ of admissible portfolios is said to be a *free lunch with vanishing risk* (FLVR), if there exist an $\epsilon > 0$ and an increasing sequence $(\delta_n)_{n\in\mathbb{N}}$ with $0 \le \delta_n \uparrow 1$, such that $P[W_T^{\pi_n} > \delta_n] = 1$ as well as $P[W_T^{\pi_n} > 1+\epsilon] \ge \epsilon$. If such a sequence does not exist, we say that there is *no free lunch with vanishing risk* (NFLVR).

(iv). An admissible portfolio W^π is said to generate an *unbounded increasing profit* if the wealth process is increasing, i.e., if $P[W_s^\pi \le W_t^\pi, \forall \quad 0 \le s < t \le T] = 1$, and if $P[W_T^\pi > 1] > 0$. If such a portfolio does not exist, *no unbounded increasing profit* (NUIP) is said to hold.

The connection between the different forms of arbitrage is an interesting subject by itself, however for our purposes it suffices to consider only their hierarchical ordering. According to [11] and [7], we can state that (NUIP) is a weaker notion than (NUPBR) and (NA), which in turn are weaker notions than (NFLVR). Furthermore, (NFLVR) holds if and only if (NUPBR) and (NA) hold. However there is no apparent connection between (NA) and (NUPBR).

The link between the optimal portfolios and different forms of arbitrage, which has been studied in the aforementioned papers, has been summarised in [9], from where we have the following theorem, modulo some changes that fit our notation.

Theorem 2.1 For an $\mathbb{R}^d$-values semimartingale S, the following are equivalent:

(i). S satisfies (NUPBR)

(ii). The numéraire portfolio exists.

(iii). The growth-optimal portfolio exists.

Furthermore, the numéraire and the growth-optimal portfolio are unique and identical.
In the case that $\sup\{E[\log W_T^\pi] | W^\pi \in \mathcal{W} \text{ with } E[\log W_T^\pi] < \infty\} < \infty$, the above statements are equivalent to

(iv). The log-utility optimal portfolio exists. Furthermore it is unique and identical to the numéraire growth optimal portfolio.

Remark 2.1 In this section we started by describing the assets in the market as semimartingales. This assumption can be omitted in markets generated by continuous price dynamics under the condition of finite logarithmic utility. In this setting it is proven by [3] that for simple buy and hold strategies, finiteness of the logarithmic utility implies that the continuous processes in the market are semimartingales, with no assumption on the existence of arbitrage. [12] elaborates on this by showing that finite utility not only implies that S is a semimartingale for any admissible trading strategy, but that it also has a canonical decomposition of the form $S = M + \alpha\langle M\rangle$, where M is a (local) martingale and α a square integrable predictable process. Furthermore it is proven that there exists a GOP that is given by W^α, i.e. by investing on S according to the strategy α. Hence from Theorem 2.1 we can conclude that finiteness of the logarithmic utility in this market implies (NUPBR), or even (NFLVR) if S satisfies some further technical conditions. However these nice properties do not translate to the non-continuous setting, as is illustrated in [12] by a counterexample. The authors show that finiteness of logarithmic utility not only does not imply a decomposition for the process as stated before, but not even that the semimartingale property of the underlying process S holds.

3 Semimartingale decomposition

Having introduced the setting of the market, in this section we turn our attention to the dynamics of the underlying semimartingale. More specifically we introduce new assumption that allow us to reach an explicit form of the numéraire .

3.1 Market price of risk

We assume that X is a d-dimensional special semimartingale, with the unique representation

$$X = M + L, \tag{1}$$

where $M = (M^1, \ldots, M^d)^*$ is a d-dimensional local square integrable martingale and $L = (L^1, \ldots, L^d)^*$ is a d-dimensional predictable process with finite variation. Hence $[X, X]$ is locally integrable, and the predictable process $\langle X \rangle = \langle M \rangle$ is well defined.
In the case of a market generated by a continuous semimartingale X, the existence of arbitrage is closely linked to the properties of the process L. A number of papers deals with this subject. In [7] the authors prove that if (NFLVR) holds then X is a semimartingale and $dL_t^i \ll d\langle X^i \rangle_t$ for $1 \leq i \leq d$. If theres exists a $d-$dimensional process α in $L^2(X)$ such that $dL_t = \alpha_t dX_t$, the market satisfies the *structure condition*(SC), see [14]. In the case of continuous semimartingale it has been proven in [9], that (NUPBR) is equivalent to the (SC). In the more general setting of discontinuous semimartingales, a weaker form of (SC) is necessary for (NUPBR) but not sufficient. This is denoted by (SC') and differs from (SC), in that α is assumed to only be a predictable $d-$dimensional process, see [9]. In order to illustrate this argument we need to introduce the notion of *immediate arbitrage*. The definition we provide is a slight modification of the one in [11], that fits our setting.

Definition 3.1 A strategy ξ is called an *immediate arbitrage opportunity*, if for all $t \in [0, T]$ it satisfies

$$\xi_t d\langle X^c \rangle_t = 0, \quad \xi_t \Delta X_t \geq 0 \text{ and } \xi_t dL_t \geq 0 \quad \mathbb{P} - a.s.$$

Immediate arbitrage is the weakest notion of arbitrage and its existence in the market leads to the violation of (NA) and (NUPBR), and consequently of (NFLVR).
The authors in [7] have proven that the market has no immediate arbitrage iff $dL_t^i \ll d\langle X^i \rangle_t$. In the case of discontinuous semimartingales, as is pointed out in [11], Remark 3.13, the condition $dL_t^i \ll d\langle X^i \rangle_t$ for $i = 1, \ldots, d$, is necessary for the absence of immediate arbitrage, and hence the absence of (UPBR) and (FLVR), but not sufficient. Therefore we introduce the following assumption.

Assumption 1 There exists a predictable process α with values in $\mathbb{R}^d$ such that $dL^i = \alpha^i d\langle X^i \rangle$ for $i = 1, \ldots, d$, i.e. (SC') is satisfied.

This assumption provides us with a process that captures the market price of risk. Moreover, it is not restrictive, since if it fails, there is already immediate arbitrage in the market and there is not much that we can say about it.

Moving on from the assumption of the existence and predictability of the market price of risk α, we come to the question of its integrability and its impact on arbitrage in the market. In the continuous case it is proven, see [3], that (NFLVR) is violated in case α is not integrable. As the next theorem illustrates, the integrability of α is only relevant, if the strategy α produces a positive portfolio $W^{\alpha} > 0$.

Theorem 3.1 Let α be the market price of risk such that $W_t^{\alpha} > 0$ P-a.s. for all $t \in [0, T]$. Then, if $P(\int_0^T \alpha_s d\langle X\rangle_s \alpha_s = \infty) > 0$, (NUPBR) is violated.

Proof
Since there is a positive probability that $\int_0^T \alpha_s d\langle X\rangle_s \alpha_s = \infty$, we have $\alpha \notin L(X)$. From Proposition 4.16 in [11] the non-integrability of α implies $P(W_T^{\alpha} = \infty) > 0$. This in turn implies that (NUPBR) is violated. •

Remark 3.1 In the case the . Hence, from the definition of $\mathcal{W}$ and the previous theorem, we conclude that: $\alpha \in \mathcal{W}$ iff $W_t^{\alpha} > 0$ $\mathbb{P}-$a.s. for all $t \in [0, T]$, and $\alpha \in L^2(X)$. Which implies that the (SC) holds true.

3.2 Characteristics of the market

Our aim is to study the existence of (UPBR) in the market and explicitly calculate the numéraire portfolio. For this reason we introduce further assumptions, that provide us with a more explicit form of the d-dimensional semimartingale X.

Assumption 2 The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is *quasi-left continuous* .

Assumption 3 The d-dimensional locally square integrable martingale M has the following representation:

$$M = M^c + H * (\mu - \eta),$$

where M^c is the continuous part of the martingale, μ is a d-dimensional random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, η is the d-dimensional compensator of μ, H is a d-dimensional predictable process that is in $G_{loc}(\mu)$[1] and $H * \mu = \int_0^{\cdot} \int_{\mathbb{R}^d} H(s, z)\mu(ds, dz)$.

Hence X is a quasi-left continuous semimartingale, with a characteristic triplet (B, C, η), where $dB = \alpha d\langle X\rangle$, $C = \langle M^c, M^c\rangle$.
From [10] Proposition II.2.9, the characteristics of X can take the form

$$\begin{aligned} C &= c \cdot A && (2) \\ \eta(dt, dz) &= \nu_t(dz) dA_t, && (3) \end{aligned}$$

where c is a predictable processes in $\mathbb{R}^{d\times d}$ and positive definite. A is a d-dimensional continuous predictable process in $\mathbb{R}^d$, with $A_0^i = 0$ for $i = 1, \dots, d$ and non decreasing paths.

[1]For a definition see Definition II.1.27 p.72 in [10].

Hence

$$d\langle X\rangle_t = \left(c_t + \int_{\mathbb{R}_0} H^2(t,z)\nu_t(dz)\right) dA_t \tag{4}$$

and

$$X_t = \int_0^t dM_s^c + \int_0^t \int_{\mathbb{R}_0} H(s,z)(\mu(dz,ds) - \nu_s(dz)dA_s) + \int_0^t \alpha_s \left(c_s + \int_{\mathbb{R}_0} H^2(s,z)\nu_s(dz)\right) dA_s$$

"Abusing" the notation in what follows, we denote by $(\alpha\left(c + H^2 \cdot \nu\right), c, H \cdot \nu)$ the "characteristics" of X.

Remark 3.2 The characteristic triplet (B, C, η) of any special semimartingale can be represented as in the system of equations (2) and (3). The condition of Y being quasi-left continuous in Assumption 2 is necessary for A to be a continuous process. This condition is introduced in order to ease the presentation and the analysis in the forthcoming sections, since the choice of a continuous A provides a tractable version of $\langle X\rangle$ as given in (4).

4 Numéraire portfolio.

So far we have introduced a market, the assets of which are driven by special semimartingales. We have also presented different notions of arbitrage and defined the numéraire portfolio. In this section we study the relationship between the characteristics of X and the existence of arbitrage in its various forms. More specifically we are interested in including portfolios that can potentially violate (NUPBR). For this reason we need to consider a larger class of portfolios than the admissible ones denoted by $\mathcal{W}$. This class is defined and studied in this section after some introductory results.

From this point onwards, to simplify notation and computations, we consider a market consisting of only one risky asset.

The only requirement that is imposed on the considered invest strategies π is the "credit limit", i.e. $W^\pi > 0$ $\mathbb{P}-$a.s. The following definition presents the interval in which these strategies live.

Definition 4.1 Let $t \in [0,T]$. Define the *max* fraction $\overline{\pi}_t$ and the *min* fraction $\underline{\pi}_t$ as

$$\underline{\pi}_t = \inf\{\pi_t | 1 + \pi_t H(t,z) > 0, \nu - \text{almost everywhere}\}$$

$$\overline{\pi}_t = \sup\{\pi | 1 + \pi_t H(t,z) > 0, \nu - \text{almost everywhere}\}.$$

The *limit* set of investment strategies $\tilde{\Pi}$ satisfying the "credit limit" is defined as $\tilde{\Pi} := \{\pi | \pi \in \mathcal{P}(\mathbb{R}) \text{ and } \pi_t \in [\underline{\pi}_t, \overline{\pi}_t], t \in [0,T]\}$ and the set of investment strategies π is defined as $\Pi := \{\pi | \pi \in \Pi \text{ and } 0 < 1 + \pi_t H(t,z) < \infty \quad \mathbb{P} - a.s., \text{ for all} t \in [0,T]\}$.

In the case that $\underline{\pi}$ and $\overline{\pi}$ are bounded processes the limit set $\tilde{\Pi}$ coincides with the set of investment strategies Π. However, since $\underline{\pi}, \overline{\pi}$ can take values in $\{\pm\infty\}$ or values that lead to a zero value portfolio, the set Π is included in $\tilde{\Pi}$.
Having introduced the space of investment strategies, in order to impose an optimality condition in this class we proceed with the study of the ratio of two portfolios in Π.

Lemma 4.1 Let X be a semimartingale with characteristics $(\alpha\left(c+H^2\cdot\nu\right), c, H\cdot\nu)$, such that $\Delta X > -1$. Then for $\pi\in\Pi$ we have

$$\begin{aligned} W^\pi &= \exp(\pi X^c + [\ln(1+\pi X)] * \mu) \\ &\times \exp\left\{\left(-\frac{1}{2}\pi(\pi-2\alpha)c + [\alpha\pi H^2 - \pi H] * \nu\right)\cdot A\right\}, \end{aligned}$$

and for any $\rho\in\Pi$

$$\begin{aligned} d\frac{W_t^\rho}{W_t^\pi} &= \frac{W_{t-}^\rho}{W_{t-}^\pi}\left\{(\rho_t-\pi_t)dX_t^c + (\rho_t-\pi_t)\int_{\mathbb{R}_0}\frac{\pi_t H(t,z)}{1+\pi_t H(t,z)}\tilde{\mu}(dz,dt)\right. \\ &+ \left.(\pi_t-\rho_t)\left((\pi_t-\alpha_t)c_t + \int_{\mathbb{R}_0}\left(\frac{\pi_t H^2(t,z)}{1+\pi_t H(t,z)} - \alpha_t H^2(t,z)\right)\nu_t(dz)\right)dA_t\right\}, t\in[0,T]. \end{aligned}$$

Proof

For $\pi\in\Pi$ we have

$$\begin{aligned} W^\pi &= \mathcal{E}(\pi X) \\ &= \exp\left(\pi X - \frac{1}{2}\pi^2\langle X^c, X^c\rangle\right)\Pi_{s\leq\cdot}(1+\pi\Delta X_s)\exp(-\pi\Delta X_s) \\ &= \exp\left(\pi X^c + [\pi H]*(\mu-\nu) + \alpha\pi(c+[H^2]*\nu)\cdot A - \frac{1}{2}\pi^2 c\cdot A\right) \\ &\times \exp([\ln(1+\pi H)]*\mu - [\pi H]*\mu) \\ &= \exp(\pi X^c + [\ln(1+\pi H)]*\mu) \\ &\times \exp\left\{\left(-\frac{1}{2}\pi(\pi-2\alpha)c + [\alpha\pi H^2 - \pi H]*\nu\right)\cdot A\right\} \end{aligned}$$

For $\rho\in\Pi$ we therefore have

$$\begin{aligned} \frac{W^\rho}{W^\pi} &= \exp\left((\rho-\pi)X^c + \left[\ln\frac{1+\rho H}{1+\pi H}\right]*\mu\right) \\ &\times \exp\left\{\left(-\frac{1}{2}(\rho-\pi)(\rho+\pi-2\alpha)c + [(\rho-\pi)(\alpha H^2 - H)]*\nu\right)\cdot A\right\}. \end{aligned}$$

Applying Itô's formula the dynamics of the portfolio are given by

$$
\begin{aligned}
d\frac{W_t^\rho}{W_t^\pi} &= \frac{W_{t-}^\rho}{W_{t-}^\pi}\Big\{(\rho_t-\pi_t)dX_t^c + \int_{\mathbb{R}_0}\left(\frac{1+\rho_t H(t,z)}{1+\pi_t H(t,z)}-1\right)\mu(dz,dt) + \frac{1}{2}(\rho_t-\pi_t)^2 c_t dA_t \\
&- (\rho_t-\pi_t)\left(\frac{1}{2}(\rho_t+\pi_t-2\alpha_t)c_t c_t - \int_{\mathbb{R}_0}\left(\alpha_t H^2(t,z)-H(t,z)\right)\nu_t(dz)\right)dA_t\Big\} \\
&= \frac{W_{t-}^\rho}{W_{t-}^\pi}\Big\{(\rho_t-\pi_t)dX_t^c + \int_{\mathbb{R}_0}\frac{(\rho_t-\pi_t)H(t,z)}{1+\pi_t H(t,z)}\mu(dz,dt) \\
&+ (\rho_t-\pi_t)\left((\alpha_t-\pi_t)c_t + \int_{\mathbb{R}_0}\left(\alpha_t H^2(t,z)-H(t,z)\right)\nu_t(dz)\right)dA_t\Big\} \\
&= \frac{W_{t-}^\rho}{W_{t-}^\pi}\Big\{(\rho_t-\pi_t)dX_t^c + \int_{\mathbb{R}_0}\frac{(\rho_t-\pi_t)H(t,z)}{1+\pi_t H(t,z)}\tilde{\mu}(dz,dt) + (\rho_t-\pi_t)(\alpha_t-\pi_t)c_t dA_t \\
&+ (\rho_t-\pi_t)\int_{\mathbb{R}_0}\left(\alpha_t H^2(t,z)-H(t,z)+\frac{H(t,z)}{1+\pi_t H(t,z)}\right)\nu_t(dz)dA_t\Big\} \\
&= \frac{W_{t-}^\rho}{W_{t-}^\pi}\Big\{(\rho_t-\pi_t)dX_t^c + (\rho_t-\pi_t)\int_{\mathbb{R}_0}\frac{H(t,z)}{1+\pi_t H(t,z)}\tilde{\mu}(dz,dt) \\
&+ (\pi_t-\rho_t)\left((\pi_t-\alpha_t)c_t + \int_{\mathbb{R}_0}\left(\frac{\pi_t H^2(t,z)}{1+\pi_t H(t,z)}-\alpha_t H^2(t,z)\right)\nu_t(dz)\right)dA_t\Big\},
\end{aligned}
$$

$t \in [0,T]$. •

Having the explicit form of the ratio $\frac{W^\rho}{W^\pi}$, we want to find, if it exists, a portfolio that has the greatest returns in the market relative to any possible investment strategy in Π. For this reason the main object of interest from the last lemma is the drift of the ratio process $\frac{W^\rho}{W^\pi}$, namely

$$D_t(\rho_t,\pi_t) = (\pi_t-\rho_t)\left((\pi_t-\alpha_t)c_t + \int_{\mathbb{R}_0}\left(\frac{\pi_t H^2(t,z)}{1+\pi_t H(t,z)}-\alpha_t H^2(t,z)\right)\nu_t(dz)\right), \quad t\in[0,T].$$

More specifically, we are interested in the existence of a portfolio W^π such that the drift term is negative for any $\rho \in \Pi$ at any fixed point $(t,\omega) \in [0,T]\times\Omega$.

Definition 4.2 A portfolio W^π is called Π-optimal if $\pi \in \Pi$ and

$$D_t(\rho_t,\pi_t) \le 0 \quad \mathbb{P}-a.s. \text{ for all } t\in[0,T] \text{ and } \rho\in\Pi.$$

As a example, we study the case when the jump measure is trivial, i.e. $\nu_t = 0 \quad \mathbb{P}-a.s.$ for all $t \in [0,T]$. Then the drift has the form

$$D_t(\rho_t,\pi_t) = (\pi_t-\rho_t)(\pi_t-\alpha_t)c_t, \quad t\in[0,T].$$

Assuming that $\alpha \in \Pi$, the Π-optimal portfolio clearly is the one that follows the strategy α. If $\alpha \in L(X)$, then W^α is not only the Π-optimal portfolio but also the numéraire. Furthermore, from Remark 3.1 it follows that $\frac{1}{W^\alpha}$ is a martingale and the density of an equivalent martingale measure, implying (NFLVR) in the market. Otherwise, from Theorem 3.1 portfolio W^α takes advantage of arbitrage opportunities in the market, leading to the violation of (NUPBR).

In general the jump measure is not trivial, hence we need to study the functions

$$E_t(\pi_t) = \int_E \left(\frac{\pi_t H^2(t,z)}{1+\pi_t H(t,z)} - \alpha_t H^2(t,z) \right) \nu_t(dz),$$

and

$$F_t(\pi_t) = (\pi_t - \alpha_t)c_t + E_t(\pi_t), \quad t \in [0,T].$$

Both $x \mapsto E_t(x)$ and $x \mapsto F_t(x)$ are increasing functions, a property that is critical for the analysis in the sequel.

Let $\tilde{\Pi} = \Pi$, i.e. $\underline{\pi}, \overline{\pi} \in \Pi$, and fix $(\omega, t) \in \Omega \times [0,T]$. Then:

1. If $0 < E_t(\underline{\pi}_t)$, $E_t(\pi_t) > 0$ holds for any $\pi_t \in [\underline{\pi}_t, \overline{\pi}_t]$. Hence the sign of $F_t(\cdot)$ depends on the market price of risk α_t:
   (a) If $\alpha_t < \underline{\pi}_t$, then $F_t(\pi_t) > 0$ for any $\pi_t \in [\underline{\pi}_t, \overline{\pi}_t]$. For the $\Pi-$optimal portfolio to exist we need to have $D_t(\rho_t, \pi_t) < 0$ for every $\rho_t \in [\underline{\pi}_t, \overline{\pi}_t]$, which makes $\pi_t = \underline{\pi}_t$ the $\Pi-$optimal strategy. The analysis hereafter follows the same logic.
   (b) If $\underline{\pi}_t \leq \alpha_t$, since the function $F_t(\cdot)$ is increasing, the following cases are possible:
      i. If $F_t(\underline{\pi}_t) > 0$, the $\Pi-$optimal strategy is $\pi_t = \underline{\pi}_t$.
      ii. If $F_t(\overline{\pi}_t) < 0$, the $\Pi-$optimal strategy is $\pi_t = \overline{\pi}_t$.
      iii. Otherwise, F takes both positive and negative values in $\pi_t \in [\underline{\pi}_t, \overline{\pi}_t]$, hence the $\Pi-$optimal strategy is the unique solution of the equation $F_t(\pi_t) = 0$.
2. If $E_t(\underline{\pi}_t) \leq 0 \leq E_t(\overline{\pi}_t)$, the drift behaves as follows:
   (a) If $\alpha_t < \underline{\pi}_t$, then $F_t(\overline{\pi}_t) \geq 0$. The sign of $F_t(\underline{\pi}_t)$ is crucial for the possible scenarios. Since $F_t(\cdot)$ is an increasing function, there exist two cases
      i. If $F_t(\underline{\pi}_t) \leq 0 \leq F_t(\overline{\pi}_t)$ the equation $F_t(\pi_t) = 0$ has a solution in $[\underline{\pi}_t, \overline{\pi}_t]$, which is also the $\Pi-$optimal strategy.
      ii. If $F_t(\underline{\pi}_t) > 0$ the $\Pi-$optimal strategy is $\underline{\pi}_t$.
   (b) If $\underline{\pi}_t \leq \alpha_t \leq \overline{\pi}_t$ the conclusion is the same as in (a).i).
   (c) If $\overline{\pi}_t < \alpha_t$, then $F_t(\underline{\pi}_t) \leq 0$. Again the sign of $F_t(\overline{\pi}_t)$ is crucial. Since $F_t(\cdot)$ is an increasing function, there exist two cases
      i. If $F_t(\underline{\pi}_t) \leq 0 \leq F_t(\overline{\pi}_t)$ the equation $F_t(\pi_t) = 0$ has a solution in $[\underline{\pi}_t, \overline{\pi}_t]$, which is also the $\Pi-$optimal strategy.
      ii. If $F_t(\overline{\pi}_t) < 0$ the $\Pi-$optimal strategy is $\overline{\pi}_t$.
3. $E_t(\overline{\pi}_t) < 0$.
   In this case $E_t(\pi_t) < 0$ for all $\pi \in [\underline{\pi}_t, \overline{\pi}_t]$. Then we have the following cases:
   (a) Let $\alpha_t > \overline{\pi}_t$, then the $\Pi-$optimal strategy is given by $\pi_t = \overline{\pi}_t$.
   (b) $\overline{\pi}_t \geq \alpha_t$.
   Since the function $F_t(\cdot)$ is increasing, we face the following cases.
      i. Let $F_t(\underline{\pi}_t) > 0$. Then the $\Pi-$optimal strategy is given by $\pi_t = \underline{\pi}_t$.
      ii. Let $F_t(\overline{\pi}_t) < 0$. Then the $\Pi-$optimal strategy is $\pi_t = \overline{\pi}_t$.
      iii. Otherwise, there exist a solution of the equation $F_t(\pi_t) = 0$.

Remark 4.1 As is obvious from the previous analysis in the case $\tilde{\Pi} = \Pi$, there exists a $\Pi-$optimal portfolio for any $(\omega, t) \in \Omega \times [0, T]$. However, this does not imply the existence of a numéraire in the market. The latter depends on the integrability of the $\Pi-$optimal strategy.

Remark 4.2 In the special case in which $\underline{\pi}, \overline{\pi} \in L(X)$, the optimal strategy belongs to the set of admissible ones, making the optimal portfolio also the numéraire.

The results of this analysis are summarized in the following theorems, after additional notation is introduced.
We define the following predictable subsets of $\Omega \times [0, T]$:

$$\mathcal{I} = \left\{(t, \omega) | F_t(\underline{\pi}_t) \leq 0 \leq F_t(\overline{\pi}_t)\right\}$$

$$\underline{\mathcal{I}} = \left\{(t, \omega) | F_t(\underline{\pi}_t) = 0\right\}$$

$$\overline{\mathcal{I}} = \left\{(t, \omega) | F_t(\overline{\pi}_t) = 0\right\}$$

The following theorem is the first main result of this paper.

Theorem 4.1 Let X be a special semimartingale with characteristic triplet $(\alpha \left(c + H^2 \cdot \nu\right), c, H \cdot \nu)$, $\underline{\pi}, \overline{\pi} \in \Pi \cap L(X)$. Then there exist a numéraire portfolio $W_T^{\pi} < \infty$, hence (NUPBR) is satisfied. Moreover,

(i). If $\mathcal{I}^c$ has measure T, then the fraction π_t invested in the numéraire at time t takes values in $\{\underline{\pi}_t, \overline{\pi}_t\}$ for all $t \in [0, T]$. Furthermore, $\frac{1}{W^{\pi}}$ is a strict supermartingale.

(ii). If $\mathcal{I}$ has measure T, then the fraction π_t invested in the numéraire at time t is the solution of $F_t(\pi) = 0$ for all $t \in [0, T]$. Furthermore, $\frac{1}{W^{\pi}}$ is the density of an equivalent local martingale measure implying that (NFLVR) is also satisfied.

(iii). Let $\alpha_t \in [\underline{\pi}_t, \overline{\pi}_t]$ for all $t \in [0, T]$. Then W^{α} is the numéraire portfolio and

(a) if X is a continuous semimartingale, (NFLVR) is satisfied and $\frac{1}{W^{\alpha}}$ is the density of the equivalent martingale measure;

(b) if $E(\alpha_t) = 0$, $P \times dt$-a.s. , (NFLVR) is satisfied and there exists an equivalent minimal martingale measure $\mathbb{Q}$, such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{1}{W^{\alpha}}$.

Proof
The fact that the numéraire exists and (NUPBR) is satisfied follows from Remarks 4.1 and 4.2.
Part (i) follows from cases 1(a), 1(b),i), 1(b),ii), 2(a),ii), 2(c),ii), 3(a), 3(b),i), 3(b),ii).
Part (ii) follows from the combination of cases 1(b),iii), 2(a),i), 2(b), 2(c),i), 3(b),iii).
Part $(iii), (a)$ follows from the pre-existing analysis. Part$(iii), (b)$ is a combination of part 2,(b), the Remark 3.1 and the definition of the Fölmer-Schweizer minimal martingale measure.
•

The following theorem covers the case in which $\underline{\pi}$ and/or $\overline{\pi}$ are not integrable.

Theorem 4.2 Let X be a special semimartingale with characteristic triplet $(\alpha \left(c + H^2 \cdot \nu\right), c, H \cdot \nu)$ and $\underline{\pi}, \overline{\pi} \in \Pi$.

(i). If $\overline{\pi}, \underline{\pi}$ are not in $L(X)$ and $\mathcal{I}^c$ or $\underline{\mathcal{I}} \cup \overline{\mathcal{I}}$ has measure T, then (NUPBR) is violated.

(ii). If $\overline{\pi}$ (resp. $\underline{\pi}$) is not in $L(X)$ and $\underline{\mathcal{I}}$ (resp.$\overline{\mathcal{I}}$) has measure T, then (NUPBR) is violated.

Proof
This follows from Theorem 3.1 and the cases of the analysis of the drift, where $\overline{\pi}$ or $\underline{\pi}$ is selected as an optimal strategy. •

The last case that we would like to explore is what happens to the drift of $\frac{W^\rho}{W^\pi}$, when Π is a strict subset of the limit strategy set $\tilde{\Pi}$. In this case there exist $t \in [0,T]$ such that $\lim_{\pi_t \to \underline{\pi}_t}(1+\pi_t H(t,z)) = 0$ and/or $\lim_{\pi_t \to \overline{\pi}_t}(1+\pi_t H(t,z)) = \infty$ $\mathbb{P}-$a.s.. The analysis of when the drift $D_t(\rho_t, \pi_t)$ is negative for $\rho \in \Pi$ and $\pi \in \tilde{\Pi}$, follows the same steps as in the case $\pi \in \Pi$, with the only exception being that we now need to study the behaviour of the limits $\lim_{\pi \to \underline{\pi}_t} F(\pi)$ and $\lim_{\pi \to \overline{\pi}_t} F(\pi)$. The conclusions are also the same modulo that there is no $\Pi-$optimal strategy in the case when only in the limit strategies $\underline{\pi}_t, \overline{\pi}_t$ the drift is non-positive. Let us define

$$\mathcal{J} = \Big\{(t,\omega)|\lim_{\pi \to \underline{\pi}_t} F_t(\pi_t) \leq 0 \leq \lim_{\pi \to \overline{\pi}_t} F_t(\pi_t)\Big\}$$

$$\underline{\mathcal{J}} = \Big\{(t,\omega)|\lim_{\pi \to \underline{\pi}_t} F_t(\pi_t) = 0\Big\}$$

$$\overline{\mathcal{J}} = \Big\{(t,\omega)|\lim_{\pi \to \overline{\pi}_t} F_t(\pi_t) = 0\Big\}.$$

Then we have the following corollary.

Corollary 4.1 Let X be a special semimartingale with characteristic triplet $(\alpha\left(c + H^2 \cdot \nu\right), c, H \cdot \nu)$ and $\underline{\pi}, \overline{\pi} \in \tilde{\Pi}$.

i). If $\underline{\pi}, \overline{\pi}$ are not in Π and $\mathcal{I}^c$ or $\underline{\mathcal{I}} \cup \overline{\mathcal{I}}$ has a positive measure, then there exists no $\Pi-$optimal strategy and (NUPBR) is violated.

(ii). If $\overline{\pi}$ (resp. $\underline{\pi}$) is not in $\tilde{\Pi}$ and $\underline{\mathcal{J}}$ (resp.$\overline{\mathcal{J}}$) has a positive measure, then there exists no $\Pi-$optimal strategy and (NUPBR) is violated.

Remark 4.3 In the previous analysis we started with the characteristics of the underlying semimartingales and found an explicit link between them and various forms of arbitrage. The question could also be reversed, as is in the case of [13], where the authors assume (NFLVR) and find necessary conditions on the characteristics of the semimartingales.

4.1 Examples

In the following examples we examine the properties of the characteristics of X and their relationship to arbitrage properties.

Example 4.1 From Karatzas and Kardaras[11] Let us assume that $S_t = \mathcal{E}(N_t)$, where N is a Poisson process with intensity $\lambda = 1$. The market is characterized by the triplet $(1,0,1)$ and the range of the $\tilde{P}i$ investement strategies is $[-1,+\infty]$ for all $t \in [0,T]$. Furthermore, the market price of risk is $\alpha_t = 1$ for all $t \in [0,T]$, and we have $E_t(\pi_t) = \frac{-1}{1+\pi_t}$, where $\pi_t \in (-1,+\infty)$. Clearly $E_t(\cdot)$ is strictly negative with $\lim_{\pi_t \to +\infty} E_t(\pi_t) = 0$. Thus we are in case (ii) of Corollary 4.1, and we conclude that (NUPBR) is violated.

Example 4.2 From Becherer [4] This example is a continuous time version of ex. 6 in [4]. Let $S_t = \Pi_{s\leq t} Y_s$, where $t \in [0,T]$ and Y is lognormally distributed, $\log Y \sim \mathcal{N}(\mu, \sigma^2)$. The semimartingale that generates the market is given by

$$X_t = \int_0^t \int_{\mathbb{R}_0} (e^z - 1)\tilde{\mu}(dz, ds) + \left(e^{\mu+\frac{\sigma^2}{2}} - 1\right) t,$$

with the characteristic triplet $\left(e^{\mu+\frac{\sigma^2}{2}} - 1, 0, \int_{\mathbb{R}_0}(e^z-1)\nu(dz)\right)$, where ν is the density of the standard normal distribution. It follows that the market price of risk is given by $\alpha_t = \frac{e^{\mu+\frac{\sigma^2}{2}}-1}{(e^{\sigma^2}-1)e^{\mu+\sigma^2}+\left(e^{\mu+\frac{\sigma^2}{2}}-1\right)^2}$. There is no short sale in the market, hence the range of the Π-optimal strategies is $[0,1]$. Under these assumptions the conditions of Theorem 4.1 are satisfied. This implies that (NUPBR) is satisfied and a numéraire portfolio exists. Since this is a pure jump market, we study the properties of $E_t(\cdot)$. We have $E_t(0) = 1 - e^{\mu+\frac{\sigma^2}{2}}$ and $E_t(1) = e^{-\mu+\frac{\sigma^2}{2}} - 1$, $t \in [0,T]$.
If $\mu \leq -\frac{\sigma^2}{2}$, $E_t(0) > 0$ and $\alpha_t < 0 = \underline{\pi}_t$ for all $t \in [0,T]$. Hence we are in case 1,(i), or case (i) of Theorem 4.1, which implies that the optimal strategy, which also describes the numéraire, is given by $\underline{\pi}_. = 0$, and the numéraire is a strict supermartingale.
For $-\frac{\sigma^2}{2} \leq \mu \leq \frac{\sigma^2}{2}$, since $E_t(0) < 0 < E_t(1)$ for all $t \in [0,T]$ we are in case (ii) of Theorem 4.1, the numéraire portfolio exists and $\frac{1}{W^\pi}$ is a martingale.

For $\mu \leq \frac{\sigma^2}{2}$, we are in case 3,(b),ii), since $E_t(1) < 0$ and $\overline{\pi}_t > \alpha_t$ for all $t \in [0,T]$. This implies that the Π−optimal strategy is $\overline{\pi}_. = 1$ and the numéraire is a strict supermartingale.

Example 4.3 Christensen-Platen[6]
Here we consider a one dimensional version of the setting in [6]. The market asset satisfies the sde

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= \left(\theta_t^2 + \int_E \frac{\psi^2(t,z)}{1-\psi(t,z)}\nu(dz)\right) dt + \theta_t dW_t \\ &+ \int_E \frac{\psi(t,z)}{1-\psi(t,z)}\tilde{\mu}(dz,dt),\end{aligned}$$

where θ is a predictable and square integrable process, $\psi(\cdot,\cdot)$ is predictable and $\psi(t,z) < 1$ a.e.. Furthermore, the Lévy measure ν is finite.
In this case $\alpha_t = \frac{\theta_t^2+\int_E \frac{\psi^2(t,z)}{1-\psi(t,z)}\nu(dz)}{\theta_t^2+\int_E \frac{\psi^2(t,z)}{(1-\psi(t,z))^2}\nu(dz)}$, the characteristics are given by $\left(\left(\theta_t^2 + \int_E \frac{\psi^2(t,z)}{1-\psi(t,z)}\nu(dz)\right), \theta_t^2, \int_E \frac{\psi(t,z)}{1-\psi(t,z)}\nu(dz)\right)$, and the range of Π−optimal strategies is $[0,1]$. Then

$$\begin{aligned}F_t(\pi_t) &= (\pi_t - 1)\theta_t^2 + \int_E \left(\frac{\pi_t\left(\frac{\psi(t,z)}{1-\psi(t,z)}\right)^2}{1+\pi_t\frac{\psi(t,z)}{1-\psi(t,z)}} - \frac{\psi^2(t,z)}{1-\psi(t,z)}\right)\nu(dz) \\ &= (\pi_t - 1)\left\{\theta_t^2 + \int_E \frac{\psi^2(t,z)}{1+(1-\pi_t)\psi(t,z)}\nu(dz)\right\}.\end{aligned}$$

Hence it is easy to see that we are in case (ii) of Theorem 4.1 , and the numéraire portfolio is given by $\pi_t = 1$. In this case $\frac{1}{S^\pi}$ and $\frac{S^\rho}{S^\pi}$ are local martingales for all $\rho \in \mathcal{W}$.

5 Enlarged filtration

In this section we are interested in identifying the difference in return due to asymmetric information. The classical approach to this problem compares the logarithmic utilities under different information structures. To this end, under the assumption of finite logarithmic utilities, we calculate the additional logarithmic utility of a trader with larger information flow $\mathcal{G}$ than the rest of the market, possessing information described by a smaller filtration $\mathcal{F} \subset \mathcal{G}$. Optimal logarithmic utility is linked to the existence of a GOP and in essence to the existence of a numéraire, see Theorem 2.1. For this reason subsection 5.1 summarizes results on the link between the the optimal logarithmic utility of the portfolio and the numéraire. In subsection 5.2 the characteristics of the underlying semimartingale X under $\mathcal{G}$ are derived, the available results on the relationship between the characteristics of X and the existence of the numéraire portfolio are extended to the setting in the large filtration $\mathcal{G}$. In a final step we aim at comparing the additional logarithmic utility with the relative entropy of the filtrations. From [2] we know that in a continuous semimartingale framework the extra logarithmic utility of an insider is equal to the Shannon entropy of his additional information. This property also holds true in markets with purely discontinuous semimartingale basis under further assumptions.

5.1 Log-utility

The description of the logarithmic utility under (NFLVR) involves the set of (local) equivalent martingale measure, and in the extended framework of (NUPBR) the set of supermartinale densities. The definition of these sets is taken from [4].

Definition 5.1 1. With $\mathcal{M}$ we denote the set of all probability measures Q, such that $Q \sim P$ and W^ρ is a Q-local martingale for any $W^\rho \in \mathcal{W}$.

2. The set of all P-supermartingale densities is denoted by

$$\mathcal{SM} := \{Z \big| Z \geq 0, Z_0 = 1, ZW^\rho \text{ is a } P\text{-supermartingale for all } W^\rho \in \mathcal{W}\}.$$

Then the following basic results hold.

Proposition 5.1 Let (NUPBR) be satisfied and $u < \infty$. Then there exists a numéraire portfolio $W^\pi \in \mathcal{W}$(i.e. a (GOP)), that satisfies

$$\begin{aligned} E\left[\log W^\pi\right] &= \sup_{W^\rho \in \mathcal{W}} E\left[\log W^\rho\right] \\ &= \inf_{Z \in \mathcal{SM}} E\left[\log \frac{1}{Z_T}\right] \end{aligned}$$

Furthermore, if (NFLVR) holds, we have

$$E\left[\log W^\pi\right] = \inf_{\mathbb{Q} \in \mathcal{M}} H(\mathbb{P}|\mathbb{Q}).$$

Lemma 5.1 Let (NUPBR) hold and $u < \infty$. Then the return of the (GOP) for a market with characteristics $\left(\alpha\left(c + H^2 \cdot \nu\right), c, H \cdot \nu\right)$ is given by

$$\begin{aligned} E[\log W_T^\pi] &= E\left[\int_0^T -\frac{1}{2}\left(\pi_t^2 - 2\alpha_t\right) c_t dA_t\right] \\ &+ E\left[\int_E \left(\ln(1 + \pi_t H(t,z)) + \pi_t H(t,z)(\alpha_t H(t,z) - 1)\right) \nu_t(dz) dA_t\right]. \end{aligned}$$

5.2 Asymmetric filtration

To describe the additional logarithmic utility, in this subsection start in the following enlargement of filtrations setting. Let $\mathcal{G}$ be a filtration such that $\mathcal{F} \subset \mathcal{G}$. We work under the following assumption concerning the decomposition of the underlying X in the larger filtration.

Assumption 4 X is a quasi-left-continuous semimartingale under $\mathcal{G}$ and has the representation,

$$X = N + \beta \cdot \langle X, X \rangle,$$

where N is a local square integrable martingale with respect to the filtration $\mathcal{G}$ and β is a predictable process with respect to $\mathcal{G}$.

In the previous sections, under Assumptions 1 and 2 we have deduced the characteristics of X with respect to $\mathcal{F}$, studied their relationship with arbitrage properties, and evaluated the optimal logarithmic utility in Lemma 5.1. To extend this to the enlarged filtration framework we determine the characteristics of X under $\mathcal{G}$ in the following theorem.

Theorem 5.1 Let X be a semimartingale with characteristics $(\alpha(c + H^2 \cdot \nu), \cdot c, H \cdot \nu)$ with respect to the filtration $\mathcal{F}$. Let $\mathcal{G}$ be a filtration such that $\mathcal{F} \subseteq \mathcal{G}$. Then the characteristic triplet of X under $\mathcal{G}$ is given by $(\beta(c + H^2 \cdot \nu), c, H[1 - (\alpha - \beta)H] \cdot \nu)$.

Proof
From the representations of X under the different filtrations we have

$$\begin{aligned} N &= M + (\alpha - \beta) \cdot \langle X, X \rangle \\ &= M^c + [H] * (\mu - \nu \cdot A) + (\alpha - \beta)\left(c \cdot A + [H^2] * \nu \cdot A\right) \\ &= M^c + (\alpha - \beta) c \cdot A + [H] * \mu - [H\,(1 - (\alpha - \beta)H)] * \nu \cdot A. \end{aligned}$$

Using orthogonality arguments the result follows. •

From the previous theorem, we conclude that the structure of the jump size with respect to the original filtration is preserved in the enlarged filtration. Hence the following lemma is immediate.

Lemma 5.2 The limit set of investment strategies with respect to the filtration $\mathcal{G}$ coincides with the set of limit strategies $\tilde{\Pi}$ under $\mathcal{F}$.

The lemma implies that also the set of investment strategies under $\mathcal{G}$ coincides with the set of investment strategies Π under $\mathcal{F}$.

Proposition 5.2 Let X be as in Theorem 5.1, such that $\Delta X > -1$. Then for $\pi \in \Pi$ we have

$$\begin{aligned} W^\pi &= \exp\left(\pi N^c + [\ln(1 + \pi H)] * \mu\right) \\ &\times \exp\left\{\left(-\frac{1}{2}\pi(\pi - 2\beta)c + [\pi H(\alpha H - 1)] * \nu\right) \cdot A\right\}, \end{aligned}$$

and for any $\rho \in \Pi$

$$\frac{d\frac{W^\rho}{W^\pi}}{\frac{W^\rho}{W^\pi}} = (\pi-\rho)\left\{(\pi-\beta)c + \left[\frac{(\pi+\alpha-\beta)H^2}{1+\pi H} - \alpha H^2\right] * \nu\right\} dA$$
$$+ \quad (\rho-\pi)dN^c + \left[\frac{(\rho-\pi)H}{1+\pi H}\right] * \tilde{\mu}^{\mathcal{G}}, \quad t \in [0,T].$$

Proof
For $\pi \in \Pi$ we have

$$\begin{aligned}
W^\pi &= \mathcal{E}(\pi X) \\
&= \exp\left(\pi N^c + [\pi H] * \mu - [\pi H(1-(\alpha-\beta)H)] * \nu + \beta\pi(c + [H^2] * \nu)\cdot A - \frac{1}{2}\pi^2 c \cdot A\right) \\
&\times \exp([\ln(1+\pi H)] * \mu - [\pi H] * \mu) \\
&= \exp(\pi N^c + [\ln(1+\pi X)] * \mu) \\
&\times \exp\left\{\left(-\frac{1}{2}\pi(\pi-2\beta)c + [\pi H(\alpha H - 1)] * \nu\right)\cdot A\right\}.
\end{aligned}$$

Let $\rho \in \Pi$. Then

$$\begin{aligned}
\frac{W^\rho}{W^\pi} &= \exp\left((\rho-\pi)N^c + \left[\ln\frac{1+\rho H}{1+\pi H}\right] * \mu\right) \\
&\times \exp\left\{\left(-\frac{1}{2}(\rho-\pi)(\rho+\pi-2\beta)c + [(\rho-\pi)H(\alpha H-1)] * \nu\right) A\right\},
\end{aligned}$$

and by applying Itô's rule we have

$$\begin{aligned}
d\frac{W_t^\rho}{W_t^\pi} &= \frac{W_{t-}^\rho}{W_{t-}^\pi}\left\{(\rho_t-\pi_t)dX_t^c + \int_{\mathbb{R}_0}\left(\frac{1+\rho_t H(t,z)}{1+\pi_t H(t,z)} - 1\right)\mu(dz,dt) + \frac{1}{2}(\rho_t-\pi_t)^2 c_t dA_t\right. \\
&- \left.(\rho_t-\pi_t)\left(\frac{1}{2}(\rho_t+\pi_t-2\beta_t)c_t - \int_{\mathbb{R}_0} H(t,z)\left(\alpha_t H(t,z)-1\right)\nu_t(dz)\right) dA_t\right\} \\
&= \frac{W_{t-}^\rho}{W_{t-}^\pi}\left\{(\rho_t-\pi_t)dX_t^c + \int_{\mathbb{R}_0}\frac{(\rho_t-\pi_t)H(t,z)}{1+\pi_t H(t,z)}\mu(dz,dt)\right. \\
&+ \left.(\rho_t-\pi_t)\left((\beta_t-\pi_t)c_t + \int_{\mathbb{R}_0} H(t,z)\left(\alpha_t H(t,z)-1\right)\nu_t(dz)\right) dA_t\right\} \\
&= \frac{W_{t-}^\rho}{W_{t-}^\pi}\left\{(\rho_t-\pi_t)dX_t^c + \int_{\mathbb{R}_0}\frac{(\rho_t-\pi_t)H(t,z)}{1+\pi_t H(t,z)}\tilde{\mu}^{\mathcal{G}}(dz,dt) + (\rho_t-\pi_t)(-\pi_t+\beta_t)c_t dA_t\right. \\
&+ \left.(\rho_t-\pi_t)\int_{\mathbb{R}_0}\left(\alpha_t H^2(t,z) - H(t,z) + \frac{H(t,z)(1-(\alpha_t-\beta_t)H(t,z))}{1+\pi_t H(t,z)}\right)\nu_t(dz)dA_t\right\} \\
&= \frac{W_{t-}^\rho}{W_{t-}^\pi}\left\{(\rho_t-\pi_t)dX_t^c + (\rho_t-\pi_t)\int_{\mathbb{R}_0}\frac{\pi_t H(t,z)}{1+\pi_t H(t,z)}\tilde{\mu}^{\mathcal{G}}(dz,dt)\right. \\
&+ \left.(\pi_t-\rho_t)\left((\pi_t-\beta_t)c_t + \int_{\mathbb{R}_0}\left(\frac{(\pi_t+\alpha_t-\beta_t)H^2(t,z)}{1+\pi_t H(t,z)} - \alpha_t H^2(t,z)\right)\nu_t(dz)\right) dA_t\right\}.
\end{aligned}$$

From Theorem 5.1 we know the characteristics of X under $\mathcal{G}$. Following the same steps as in Proposition 4 we obtain the desired results. •

Hence the drift under $\mathcal{G}$ is given by

$$D_t^*(\rho_t) = (\pi_t - \rho_t)\left\{(\pi_t - \beta_t)c + \left[\frac{(\pi_t + \alpha_t - \beta_t)H^2(t,z)}{1+\pi_t H(t,z)} - \alpha_t H^2(t,z)\right] * \nu\right\}, \quad t \in [0,T].$$

As in section 4 we introduce the functions

$$E_t^*(\pi_t) = \int_E \left(\frac{(\pi_t + \alpha_t - \beta_t)H^2(t,z)}{1+\pi_t H(t,z)} - \alpha_t H^2(t,z)\right)\nu_t(dz),$$

and

$$F_t^*(\pi_t) = (\pi_t - \beta_t)c_t + E_t^*(\pi_t), \quad t \in [0,T].$$

To proceed with the analysis, we introduce the following assumption.

Assumption 5 The information drifts α, β satisfy $1 + (\beta_t - \alpha_t)H(t,z) > 0$ P-a.s. for all $t \in [0,T]$.[2]

Under this assumption, the functions $x \mapsto E_t^*(x)$ and $x \mapsto F_t^*(x)$ are increasing.
Using the characteristic triplet under $\mathcal{G}$ and the properties of the functions $E_t^*(\cdot), F_t^*(\cdot)$, the analysis of the drift is identical with the one under under $\mathcal{F}$, and the results transfer accordingly. In case the jump measure is trivial, i.e. $\nu_t = 0 \quad \mathbb{P}-a.s.$ for all $t \in [0,T]$, the optimal portfolio is the one that follows strategy β. If $\beta \in L(X)$, then W^β is the numéraire, $\frac{1}{W^\beta}$ is a martingale and the density of an equivalent martingale measure, implying (NFLVR) in the market. Otherwise, the portfolio W^β takes advantage of arbitrage opportunities in the market, leading to the violation of (NUPBR).
Let $\underline{\pi}, \overline{\pi} \in \Pi$ and fix $(\omega, t) \in \Omega \times [0,T]$. Then we have

1. if $E_t^*(\underline{\pi}_t)$ and
   (a) $\beta_t < \underline{\pi}_t$, the optimal strategy is given by $\pi_t = \underline{\pi}_t$.
   (b) $\underline{\pi}_t \leq \beta_t$, then
      i. for $F_t^*(\underline{\pi}_t) > 0$ the optimal strategy is described by $\pi_t = \underline{\pi}_t$,
      ii. for $F_t^*(\overline{\pi}_t) < 0$, the optimal strategy is $\pi_t = \overline{\pi}_t$,
      iii. otherwise, the optimal strategy is the unique solution of the equation $F_t^*(\pi_t) = 0$.
2. If $E_t^*(\underline{\pi}_t) \leq 0 \leq E_t^*(\overline{\pi}_t)$ and
   (a) $\beta_t < \underline{\pi}_t$, then
      i. for $F_t^*(\overline{\pi}_t) \leq 0 \leq F_t^*(\overline{\pi}_t)$ the optimal strategy is the unique solution of the equation $F_t^*(\pi_t) = 0$,
      ii. if $F_t^*(\underline{\pi}_t) > 0$ the optimal strategy is $\underline{\pi}_t$.
   (b) $\underline{\pi}_t \leq \beta_t \leq \overline{\pi}_t$, the conclusion is the same as in (a),i).
   (c) $\overline{\pi}_t < \beta_t$, then
      i. if $F_t^*(\underline{\pi}_t) \leq 0 \leq F_t^*(\overline{\pi}_t)$ the optimal strategy is the unique solution of the equation $F_t^*(\pi_t) = 0$,

[2] As will become evident in the next section, Assumption 5 is also necessary for the definition of the entropy and hence not restrictive.

ii. if $F_t^*(\overline{\pi}) < 0$ the optimal strategy is $\overline{\pi}_t$.

3. If $E_t^*(\overline{\pi}_t) < 0$ and

   (a) $\beta_t > \overline{\pi}_t$, the optimal strategy is $\pi_t = \overline{\pi}_t$,

   (b) $\overline{\pi}_t \geq \beta_t$,

      i. for $F_t^*(\underline{\pi}_t) > 0$, the optimal strategy is $\pi_t = \underline{\pi}_t$,

      ii. for $F_t^*(\overline{\pi}_t) < 0$,

      iii. otherwise, the optimal strategy is the unique solution of the equation $F_t^*(\pi_t) = 0$.

In analogy to section 4 we define the following predictable subsets of $\Omega \times [0, T]$:

$$\mathcal{I}^* = \Big\{(t,\omega)|F_t^*(\underline{\pi}_t) \leq 0 \leq F_t^*(\overline{\pi}_t)\Big\},$$

$$\underline{\mathcal{I}}^* = \Big\{(t,\omega)|F_t^*(\underline{\pi}_t) = 0\Big\},$$

$$\overline{\mathcal{I}}^* = \Big\{(t,\omega)|F_t^*(\overline{\pi}_t) = 0\Big\}.$$

$$\mathcal{J}^* = \Big\{(t,\omega)| \lim_{\pi\to\underline{\pi}_t} F_t^*(\pi_t) \leq 0 \leq \lim_{\pi\to\overline{\pi}_t} F_t^*(\pi_t)\Big\},$$

$$\underline{\mathcal{J}}^* = \Big\{(t,\omega)| \lim_{\pi\to\underline{\pi}_t} F_t^*(\pi_t) = 0\Big\},$$

$$\overline{\mathcal{J}}^* = \Big\{(t,\omega)| \lim_{\pi\to\overline{\pi}_t} F_t^*(\pi_t) = 0\Big\}.$$

We have the following result about the existence of numéraire portfolios.

Theorem 5.2 Let X be a special semimartingale as in Theorem 5.1.

1. If the market price of risk β satisfies $W_t^\beta > 0, \quad \mathbb{P} - a.s.$ for all $t \in [0, T]$, and $P(\int_0^T \beta_s^2 d\langle X\rangle_s) > 0$ (NUPBR) is violated.

2. If $\underline{\pi}, \overline{\pi} \in \Pi \cap L(X)$, there exist a numéraire portfolio $W_T^\pi < \infty$, hence (NUPBR) is satisfied. Moreover,

   (i) If $(\mathcal{I}^*)^c$ has measure T, then the fraction π_t invested in the numéraire at time t takes values in $\{\underline{\pi}_t, \overline{\pi}_t\}$ for all $t \in [0, T]$. Furthermore, $\frac{1}{W^\pi}$ is a strict supermartingale.

   (ii) If $\mathcal{I}^*$ has measure T, then the fraction π_t invested in the numéraire at time t is the solution of $F_t^*(\pi) = 0$ for all $t \in [0, T]$. Furthermore, $\frac{1}{W^\pi}$ is a martingale implying that (NFLVR) is also satisfied.

   (iii) Let $\beta_t \in [\underline{\pi}_t, \overline{\pi}_t]$ for all $t \in [0, T]$. Then W^β is the numéraire portfolio and

      (a). if X is a continuous semimartingale, (NFLVR) is satisfied and $\frac{1}{W^\beta}$ is the density of the equivalent martingale measure.

      (b). If $E(\beta_t) = 0$, $P \times dt$-a.s. , (NFLVR) is satisfied and there exists an equivalent minimal martingale measure $\mathbb{Q}$, such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{1}{W^\beta}$.

3. Let $\underline{\pi}, \overline{\pi} \in \Pi$

   (a). If $\overline{\pi}, \underline{\pi}$ are not in $L(X)$ and $(\mathcal{I}^*)^c$ or $\underline{\mathcal{I}}^* \cup \overline{\mathcal{I}}^*$ has positive measure, then (NUPBR) is violated.

   (b). If $\overline{\pi}$ (resp. $\underline{\pi}$) is not in $L(X)$ and $\underline{\mathcal{I}}^*$ (resp.$\overline{\mathcal{I}}^*$) has a positive measure, then (NUPBR) is violated.

4. Let $\underline{\pi}, \overline{\pi} \in \tilde{\Pi}$.

   i). If $\underline{\pi}, \overline{\pi}$ are not in Π and $(\mathcal{J}^*)^c$ or $\underline{\mathcal{J}}^* \cup \overline{\mathcal{J}}^*$ has a positive measure, then there exists no $\Pi-$optimal strategy and (NUPBR) is violated.

   (ii). If $\overline{\pi}$ (resp. $\underline{\pi}$) is not in $\tilde{\Pi}$ and $\underline{\mathcal{J}}^*$ (resp.$\overline{\mathcal{J}}^*$) has a positive measure, then there exists no $\Pi-$optimal strategy and (NUPBR) is violated.

Proof
The arguments are equivalent the proofs of Theorem 4.1, Theorem 4.2, Corollary 4.1 and Theorem 3.1 . •

Proposition 5.3 Let X be a semimartingale with characteristic triplet $(\alpha\langle X\rangle, C, H\eta)$ with respect to a filtration $\mathcal{F}$ where (NUPBR) holds, and $\mathcal{G}$ a filtration such that (NUPBR) holds and $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in [0,T]$. Furthermore, if W^π and W^ρ are the numéraire portfolios under $\mathcal{F}$ and $\mathcal{G}$ respectively, the difference in return is given by

$$\begin{aligned} u^{\mathcal{G}} - u^{\mathcal{F}} &= E\left[\int_0^T \left(-\frac{1}{2}(\pi_t(\pi_t - 2\beta_t) - \rho_t(\rho_t - 2\alpha_t))\right) c_t dA_t\right] \\ &+ E\left[\int_0^T \left((\pi_t - \rho_t)H(t,z)(\alpha_t H(t,z) - 1) + \ln\frac{1+\pi_t H(t,z)}{1+\rho_t H(t,z)}\right.\right. \\ &+ \left.\left. (\beta_t - \alpha_t)H(t,z)\ln(1+\pi_t H(t,z))\right)\nu_t(dz)dA_t\right]. \end{aligned}$$

Proof
We have

$$\begin{aligned} E[\log W_T^\pi] &= E\left[\int_0^T \pi_t N_t^c dt + \int_0^T\int_{\mathbb{R}_0} \log(1+\pi_t H(t,z))\mu(dz,dt)\right] \\ &+ E\left[-\int_0^T \frac{1}{2}\pi_t(\pi_t - 2\beta_t)c_t dA_t + \int_0^T\int_{\mathbb{R}_0} \pi_t H(t,z)(\alpha_t H(t,z)-1)\nu_t(dz)dA_t\right] \\ &= E\left[\int_0^T \pi_t N_t^c dt + \int_0^T\int_{\mathbb{R}_0} \log(1+\pi_t H(t,z))\tilde{\mu}(dz,dt)\right] \\ &+ E\left[-\int_0^T \frac{1}{2}\pi_t(\pi_t - 2\beta_t)c_t dA_t + \int_0^T\int_{\mathbb{R}_0} [\pi_t H(t,z)(\alpha_t H(t,z)-1)\nu_t(dz)dA_t\right] \\ &+ E\left[\int_0^T\int_{\mathbb{R}_0} (1-(\alpha_t-\beta_t)H(t,z))\log(1+\pi_t H(t,z))\nu_t(dz)dA_t\right] \\ &= E\left[-\int_0^T \frac{1}{2}\pi(\pi_t - 2\beta_t)c_t dA_t\right] + E\left[\int_0^T\int_{\mathbb{R}_0} \beta_t H(t,z))\log(1+\pi_t H(t,z))\nu_t(dz)dA_t\right] \\ &+ E\left[\int_0^T\int_{\mathbb{R}_0} [(\alpha_t H(t,z)-1)\{\pi_t H(t,z) - \log(1+\pi_t H(t,z))\}\nu_t(dz)dA_t\right]. \end{aligned}$$

Combining the previous formula with Lemma 5.1, the result follows.

•

5.3 Entropy

In this section we describe the entropy of the additional information that a larger filtration provides with respect to a smaller one. To simplify our presentation we assume that $\mathcal{G}$ is obtained by an initial enlargement of $\mathcal{F}$.

Under the Assumption 2 the local semimartingale M generates the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let $(\mathcal{F}^0_t)_{t\in[0,T]}$ be a filtration the σ-algebras of which are countably generated, and under which M is a local martingale. Assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is the smallest filtration containing $(\mathcal{F}^0_t)_{t\in[0,T]}$ and satisfying the usual conditions. Also, let $(\mathcal{G}^0_t)_{t\in[0,T]}$ be a filtration with countably generated σ-algebras, and $(\mathcal{G}_t)_{t\in[0,T]}$ the smallest filtration satisfying the usual conditions and containing $\mathcal{F}_t$, i.e. $\mathcal{G}_t \supset \mathcal{F}_t$ for all $t \geq 0$.

The introduction of the smaller filtrations $\mathcal{F}^0, \mathcal{G}^0$ is a necessary condition for the regularity of the conditional probability $P_t(\cdot, A)$ given $\mathcal{F}^0_t$, where $A \in \mathcal{G}^0_T, t \in [0,T]$. From [2] and [1] it follows that $P_\cdot(\omega, A)$ is a $(\mathcal{F}^0_t)$-martingale. And by the martingale representation property, for $t \in [0,T]$, $P_t(\cdot, A)$ has the form

$$P_t(\cdot, A) = P_0(A) + \int_0^t \gamma_s(\cdot, A) dM^c_s + \int_0^t \int_{\mathbb{R}_0} \delta_u(z, \cdot, A)\tilde{\mu}(dz, dt), \tag{5}$$

where γ, δ are predictable processes belonging to $L^2(P)$ and $L^2(P \otimes \eta)$ respectively. To continue our analysis, we introduce the following assumption.

Assumption 6 For $0 \leq t \leq T$ let $\gamma_t(\omega, \cdot)|_{\mathcal{G}^0_t}$ and $\delta_t(z, \omega, \cdot)|_{\mathcal{G}^0_t}$ be signed measures on $\mathcal{G}^0_t$, such that

$$\gamma_t(\omega, \cdot)|_{\mathcal{G}^0_t} << P_t(\omega, \cdot)|_{\mathcal{G}^0_t}, \quad P - a.s.,$$

and

$$\delta_t(z, \omega, \cdot)|_{\mathcal{G}^0_t} << P_t(\omega, \cdot)|_{\mathcal{G}^0_t} \quad P \otimes \eta - a.s.$$

Theorem 5.3 Under Assumption 6 there exist $\mathcal{F}_t \otimes \mathcal{G}_t$ predictable processes

$$c_t(\omega, \omega') = \frac{\gamma_t(\omega, d\omega')}{P_t(\omega, d\omega')}\Big|_{\mathcal{G}^0_t} \quad P - a.s.,$$

and

$$d_t(z, \omega, \omega') = \frac{\delta_t(z, \omega, d\omega')}{P_t(\omega, d\omega')}\Big|_{\mathcal{G}^0_t} \quad P \otimes \pi - a.e.$$

Furthermore $c_t(\omega, \omega) = \beta_t(\omega) - \alpha_t(\omega)$ and $d_t(z, \omega, \omega) = (\beta_t(\omega) - \alpha_t(\omega))H(t, z, \omega)$ P-a.s.

Proof

From the beginning of this subsection we know that the information drift for the continuous part of the semimartingale is given by $\beta - \alpha$ and that for $t \in [0,T]$ $\frac{\eta^{\mathcal{G}}_t(dz,\omega)}{\eta_t(dz,\omega)} = 1 + (\beta_t(\omega) - \alpha_t(\omega))H(t, z, \omega)$. Then using the orthogonality of M^c and μ the result follows easily from Lemma 2.3 and Theorem 2.6 in [2], as well as Lemma 2.5 and Theorem 2.6 from [1]. •

The preceding theorem is instrumental in the computation of *additional information*, that is introduced in the following.

Definition 5.2 Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$ and P, Q two probability measures on $\mathcal{F}$. Then we define the *relative entropy* of P with respect to Q on the sigma field $\mathcal{A}$ by

$$\mathcal{H}_{\mathcal{A}}(P||Q) = \begin{cases} \int \log \frac{dP}{dQ} dP, & if \quad P \ll Q, \\ \infty \quad else. \end{cases}$$

Moreover, the *additional information* of $\mathcal{A}$ relative to the filtration $(\mathcal{F}_u)$ on $[s,t]$, where $0 \le s < t \le T$, is defined by

$$H_{\mathcal{A}}(s,t) = \int \mathcal{H}_{\mathcal{A}}(P_t(\omega,\cdot)||P_s(\omega,\cdot))dP(\omega).$$

The explicit form of $H_{\mathcal{G}}$ is provided by the following Lemma.

Lemma 5.3 The additional information of $\mathcal{G}_t^0$ relative to the filtration $(\mathcal{F}_u)$ on $[s,t]$ is given by

$$\begin{aligned} H_{\mathcal{G}_t^0}(s,t) &= E\Big[\int_s^t \frac{(\beta_t-\alpha_t)^2}{2} d\langle M^c\rangle_t \\ &+ \int_s^t \int_{\mathbb{R}_0} (\beta_t-\alpha_t)H(t,z)\nu(dz)dA_t \\ &+ (1+(\beta_t-\alpha_t)H(t,z))\ln\left(1+(\beta_t-\alpha_t)H(t,z)\right)\nu(dz)dA_t\Big]. \end{aligned}$$

Proof
Using Itô's rule for semimartingales we get

$$\begin{aligned} d\ln P_t(\cdot,A) &= \frac{\gamma_t}{P_t(\cdot,A)}dM_t^c - \frac{\gamma_t^2}{2P_t(\cdot,A)^2}d\langle M^c\rangle_t \\ &+ \int_{\mathbb{R}_0} \left[\ln(P_t(\cdot,x)+\delta_t(z)) - \ln P_t(\cdot,A)\right]\mu(dz,dt) + \int_{\mathbb{R}_0} \frac{\delta_t(z)}{P_t(\cdot,A)}\nu(dz)dA_t \\ &= \frac{\gamma_t}{P_t(\cdot,A)}dN_t^c + \int_{\mathbb{R}_0} \ln\left[1+\frac{\delta_t(z)}{P_t(\cdot,A)}\right]\hat{\mu}(dz,dt) \\ &+ \frac{\gamma_t}{P_t(\cdot,A)}\left(\beta_t-\alpha_t-\frac{\gamma_t}{2P_t(\cdot,A)}\right)d\langle M^c\rangle_t \\ &+ \int_{\mathbb{R}_0}\left[\frac{\delta_t(z)}{P_t(\cdot,A)} + (1+(\beta_t-\alpha_t)H(t,z))\ln\left(1+\frac{\delta_t(z)}{P_t(\cdot,A)}\right)\right]\nu(dz)dA_t. \end{aligned}$$

Since N^c and $\hat{\mu}$ are local martingales under $(\mathcal{G}_t)$ we have

$$\begin{aligned} E\left[P_t(\cdot,A)\log\frac{P_t(\cdot,A)}{P_s(\cdot,A)}\right] &= E\Big[\int_s^t \mathbf{1}_A \frac{\gamma_t}{P_t(\cdot,A)}\left(\beta_t-\alpha_t-\frac{\gamma_t}{2P_t(\cdot,A)}\right)d\langle M^c\rangle_t \\ &+ \int_s^t \mathbf{1}_A \int_{\mathbb{R}_0}\left[\frac{\delta_t(z)}{P_t(\cdot,A)} + (1+(\beta_t-\alpha_t)H(t,z))\ln\left(1+\frac{\delta_t(z)}{P_t(\cdot,A)}\right)\right]\nu(dz)dA_t\Big]. \end{aligned}$$

From Theorem 5.3 we infer

$$\begin{aligned} E\left[P_t(\cdot,A)\log\frac{P_t(\cdot,A)}{P_s(\cdot,A)}\right] &= E\Big[\int_s^t \mathbf{1}_A\frac{(\beta_t-\alpha_t)^2}{2}d\langle M^c\rangle_t \\ &+ \int_s^t \mathbf{1}_A \int_{\mathbb{R}_0}(\beta_t-\alpha_t)H(t,z)\nu(dz)dA_t \\ &+ (1+(\beta_t-\alpha_t)H(t,z))\ln\left(1+(\beta_t-\alpha_t)H(t,z)\right)\nu(dz)dA_t\Big]. \end{aligned}$$

Using the same steps as in the proof of Lemma 5.3 of [2] we reach our result. •

In the case of a continuous market, as explained in [2], the additional information $H_{\mathcal{G}}(0,T)$ equals the expected logarithmic utility increment between the two filtrations, since $u^{\mathcal{G}} - u^{\mathcal{F}} = E[\int_0^T \frac{(\beta_t - \alpha_t)^2}{2} dt]$. However, in the case of a stochastic basis with both a continuous and a jump component this is not obvious.

5.4 Purely discontinuous semimartingales

The expected logarithmic utility increment, as was noted before, is not always equal to the entropy of the additional information. However, as the next theorem shows, in purely discontinuous markets in which the jumps are hedgeable, the equality holds.

Theorem 5.4 Let X be a quasi-left continuous semimartingale under the filtration $\mathcal{F}$, with characteristic triplet $(\alpha H^2 \cdot \nu, 0, H \cdot \nu)$, such that $1 - \alpha_t H(t,z) > 0$ $\mathbb{P} - a.s.$ for all $t \in [0,T]$ and $\alpha \in \Pi$. Then the Π-optimal portfolio strategy is $\pi_t = \frac{\alpha_t}{1-\alpha_t H(t,z)}$. Furthermore, for a filtration $\mathcal{G} \supseteq \mathcal{F}$, where X has the characteristic triplet $(\beta H^2 \cdot \nu, 0, H[1 + (\beta - \alpha)H] \cdot \nu)$, the Π-optimal portfolio strategy is given by $\rho_t = \frac{\beta_t}{1-\alpha_t H(t,z)}$. If $\alpha, \beta \in L(X)$, then

$$u^{\mathcal{G}} - u^{\mathcal{F}} = H_{\mathcal{G}}(0,T).$$

Proof
Given the characteristic triplet under $\mathcal{F}$, from section 4 we have

$$F_t(\pi_t) = E_t(\pi_t) = \int_E \left(\frac{\pi_t H^2(t,z)}{1 + \pi_t H(t,z)} - \alpha_t H^2(t,z) \right) \nu_t(dz).$$

Clearly $E_t\left(\frac{\alpha_t}{1-\alpha_t H(t,z)}\right) = 0$ $P - a.s.$ for all $t \in [0,T]$, so we are in case 2. of the analysis. The optimal portfolio strategy is then given by $\pi = \frac{\alpha}{1-\alpha H}$, and W^π satisfies

$$d\frac{1}{W^\pi} = \frac{-\int_{\mathbb{R}_0} \alpha_t H(t,z) d\tilde{\mu}}{W^\pi}.$$

If $\alpha \in L(X)$, then W^π is a local martingale and is both the numéraire portfolio and the density of the minimal martingale measure. The expected logarithmic utility of W^π is given by

$$\begin{aligned} u^{\mathcal{F}} = E[\ln W_T^\pi] &= E\left[-[\ln(1+\pi H)] * \tilde{\mu}\right] + E\left[[\pi\alpha H^2 - \pi H + \ln(1+\pi H)] * \nu\right] \\ &= -E\left[\int_0^T \int_{\mathbb{R}_0} \left[\alpha_t H(t,z) + \ln\left(1 - \alpha_t H(t,z)\right)\right] \nu(dz) dA_t\right]. \end{aligned}$$

For a larger filtration $\mathcal{G}$ we have

$$F_t^*(\pi_t) = E_t^*(\pi_t) = \int_E \left(\frac{(\pi_t + \alpha_t - \beta_t) H^2(t,z)}{1 + \pi_t H(t,z)} - \alpha_t H^2(t,z) \right) \nu_t(dz).$$

From Assumption 5 $\rho = \frac{\beta}{1-\alpha H}$ is in Π. Furthermore $E_t^*(\rho_t) = 0$ and

$$d\frac{1}{W_t^\rho} = \frac{1}{W_t^\rho} \int_{\mathbb{R}_0} \frac{\beta_t H(t,z)}{1 + (\beta_t - \alpha_t) H(t,z)} d\bar{\mu}.$$

If $\beta \in L(X)$ the solution of the previous equation is a local martingale, hence $\frac{1}{W^\rho}$ is the density of martingale measure, and logarithmic utility given by

$$u^{\mathcal{G}} = E[\ln W_T^\rho] \quad = \quad E\left[[(1+(\beta-\alpha)H)(\ln(1+(\beta-\alpha)H) - \ln(1-\alpha H)) - \beta H] * \nu\right].$$

Note that

$$\int_0^T \int_{\mathbb{R}_0} \ln(1-\alpha_t H(t,z))\bar{\mu} - \int_0^T \int_{\mathbb{R}_0} \ln(1-\alpha_t H(t,z))\tilde{\mu} = \int_0^T \int_{\mathbb{R}_0} (\beta_t - \alpha_t)\ln(1-\alpha_t)H(t,z)\nu(dz)dA_t.$$

Hence

$$E\left[\int_0^T \int_{\mathbb{R}_0} (\beta_t - \alpha_t)\ln(1-\alpha_t)H(t,z)\nu(dz)dA_t\right] = 0.$$

We have

$$\begin{aligned} u^{\mathcal{G}} - u^{\mathcal{F}} &= E\left[\int_0^T \int_{\mathbb{R}_0} (1+(\beta-\alpha)H(t,z))\left[\ln(1+(\beta-\alpha)H(t,z)) - \ln(1-\alpha H(t,z))\right]\nu(dz)dA_t\right] \\ &+ E\left[\int_0^T \int_{\mathbb{R}_0} \left[-(\beta_t-\alpha_t)H(t,z) + \ln(1-\alpha_t H(t,z)\right]\nu(dz)dA_t\right] \\ &= E\left[\int_0^T \int_{\mathbb{R}_0} \Big\{(1+(\beta-\alpha)H(t,z))\ln\Big(1+(\beta-\alpha)H(t,z)\Big) - (\beta_t-\alpha_t)H(t,z)\Big\}\nu(dz)dA_t\right]. \end{aligned}$$

Hence under these assumptions we recover the result of the continuous market, namely that the expected logarithmic utility increment is equal to the Shannon entropy of the additional information. •

Remark 5.1 From 5.3 onwards we have assumed that the filtration $\mathcal{G}$ is an initial enlargement of $\mathcal{F}$. This assumption can be relaxed to include progressive enlargements, as is shown in [2]. However, this exceeds the scope of this paper.

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