LIBOR rate models, related derivatives and model calibration *

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Abstract

Based on Jamshidians framework, [8], a general strategy for the quasi-analytical valuation of large classes of LIBOR derivatives will be developed. As a special case we will address the quasi-analytical approximation formula for swaptions of Brace Gatarek and Musiela in [2] and show that a similar formula can be derived with Jamshidian's methods as well. As further applications we will study the *callable reverse floater* and the *trigger swap*. Then, we will study the thorny issues around simultaneous calibration of (low factor) LIBOR models to cap(let) and swaption prices in the markets. We will argue that a low factor market model cannot be calibrated to these prices in a stable way and propose an, in fact, many factor model with only *the same number of loading parameters as a two factor model,* but, with much better stability properties.

1 Introduction

Recently, several models for LIBOR rates and valuation methods for LIBOR rate related derivatives have appeared, e.g. Brace, Gatarek and Musiela (1997), [2], Jamshidian (1997), [8]. The advantage of these approaches is that they model the LIBOR rate process directly as the primary object in an arbitrage free way instead of deriving it from the term structure of instantaneous rates modelled in a HJM framework by Heath, Jarrow and Morton (1992), [7]. Whereas in Brace et al. [2] the LIBOR process was constructed in the numeraire measure induced by the continuously compounded spot rate serving as an instantaneous saving bond, Jamshidian, [8] showed that because of their payoff homogeneity LIBOR and swap derivatives can be priced and hedged in an arbitrage free framework of zero-coupon bonds without assuming the existence of an instantaneous saving bond.

In this sequel we study the valuation of fairly general LIBOR related derivatives in a LIBOR market model within the framework of Jamshidian, [8] and we discuss the calibration of the market model to market prices of cap(let)s and swaptions.

In section (2) we review some general arbitrage theory and general methods for derivative pricing developed in [8] and by using the results of section (2) we (re)derive in section (3) the dynamics of the general LIBOR process. The notion of LIBOR market models is introduced in section (3) as well.

Via a slight extension of an idea of Brace Gatarek and Musiela which has led to their swaption approximation formula in [2] we will derive in section (4) a multi-dimensional, log-normal approximation for the simultaneous distribution of different forward LIBORs, at different forward times and with respect to different forward (numeraire) measures. Next, we will show in (4) that it is possible to value large classes of LIBOR derivatives by quasi-analytical approximation formulas based on this log-normal approximation and two important classes are identified.

Applications will be given in section (5). In particular, we address the quasi-analytical swaption approximation formula of Brace et al., [2] and show that a similar formula can be derived with Jamshidian's methods as well. In (6) we will argue, however, that the rank 1 assumption with respect to the volatility correlation matrix in [2] turns out to be too restrictive when the resulting swaption formula is used for model calibration to a whole family of cap and swaption prices. Therefore, in (5) we also derive a multi-factor swaption approximation formula in a (Jamshidian) LIBOR market model. As further applications we tackle in (5) the *callable reverse floater* and the *trigger swap*.

In section (6) we study the calibration of market models to the prices of (liquidly traded) cap(let)s and swaptions and explain (at least partially) why low factor models are generally difficult to calibrate in a stable way. Therefore, as an alternative to low factor models, we propose via the identification of a special correlation structure a market model which is, in a sense, a many factor model, however, with the same number of model parameters as a two factor model. We will argue that this model has more ability to match the actual nature of LIBOR correlations in the markets and therefore the calibration of this model will be more stable.

2 Some arbitrage theory

We will review some definitions, methods and results on arbitrage theory and option pricing developed by Jamshidian, [8], in a self contained way. However, since we want to avoid too much bracket calculus and compensator analysis in this paper, we re-derive some important results in a somewhat different way.

2.1 Arbitrage free systems, self-financing trading strategies, complete markets

We fix some $\tau > 0$ large enough and consider a continuous trading economy on the interval $[0, \tau]$. Let $\mathcal{E}$ be the collection of continuous semi-martingales on $[0, \tau]$, with respect to a complete filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \le t \le \tau}, \mathbb{P})$ satisfying the usual conditions. Let further $\mathcal{E}_+ := \{X \in \mathcal{E} \mid X > 0\}$, $\mathcal{E}^n := \{X \mid X = (X_1, .., X_n),\ X_i \in \mathcal{E}\}$ etc. A price system $B \in \mathcal{E}^n$ on the probability space $(\Omega, (\mathcal{F}_t)_{0 \le t \le \tau}, \mathbb{P})$ will be called a *market*. We now recall some basic definitions from Jamshidian, [8].

Definition 2.1.1 (arbitrage) *The price system (market) $B \in \mathcal{E}^n$ is said to be* **arbitrage free** *(AF) if there exists a ξ, $\xi \in \mathcal{E}_+$ with $\xi_0 = 1$, such that ξB_i are martingales for all $1 \le i \le n$. The process ξ is called a state price deflator.*

Note that the *state price deflator* makes deflated prices martingales in the *actual measure*. See also Duffie, [5].

Definition 2.1.2 (self-financing trading strategies) *Let $B \in \mathcal{E}^n$ and $\theta = (\theta_1, .., \theta_n)$ be a vector of adapted B_i-integrable processes θ_i. Then, the pair (θ, B) is called a* **self-financing trading strategy** *(SFTS) if $\theta_t \cdot B_t = \theta_0 \cdot B_0 + \int_0^t \theta_s \cdot dB_s$ for all $0 \leq t \leq \tau$.*

Definition 2.1.3 (complete markets) *The price system $B \in \mathcal{E}^n$ on $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq \tau}, \mathbb{P})$ is called a* **complete market** *if for any T; $0 \leq T \leq \tau$ and any random variable $C_T \in \mathcal{F}_T$ (an $\mathcal{F}_T-$claim) there exists an SFTS (θ, B) such that $\theta_T \cdot B_T = C_T$.*

The following fundamental *completeness theorem* is essentially equivalent to related theorems in Delbaen and Schachermayer, [4] and in Harrison and Pliska, [6].

Theorem 2.1.4 (completeness) *An arbitrage free system $B \in \mathcal{E}^n$ on $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq \tau}, \mathbb{P})$ is* **complete** *if and only if there exists exactly one $\xi \in \mathcal{E}_+$ with $\xi_0 = 1$, such that ξB_i are martingales for all $1 \leq i \leq n$.*

2.2 Itô processes

The results obtained in this sequel are based on stochastic models for price systems governed by Itô processes; processes which can be represented by a stochastic Itô-integral. We will study this important class of price systems in more detail.

Let W be a $d-$dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \tau}, \mathbb{P})$, where $(\mathcal{F}_t)$ is the by W generated natural filtration $\mathcal{F}_t := \sigma\{W_s \mid 0 \leq s \leq t\}$. On the same probability space we consider a price system given by the Itô processes

$$\begin{aligned} \ln B_i \quad &: \ = \ln B_i(0) + \int_0^t (\mu_i - \frac{1}{2}|\sigma_i|^2) ds + \int_0^t \sigma_i \cdot dW, \\ \ln \xi \quad &: \ = \ln \xi(0) + \int_0^t (-r - \frac{1}{2}|\varphi|^2) ds - \int_0^t \varphi \cdot dW, \end{aligned} \tag{1}$$

where $r(t, \omega), \mu_i(t, \omega)$; $i = 1, .., n$ are scalar processes and $\varphi(t, \omega), \sigma_i(t, \omega)$; $i = 1, .., n$ are $d-$dimensional vector processes, all adapted and satisfying the usual requirements for the existence of the Itô integrals. So, by taking the exponential in (1) we have

$$\begin{aligned} B_i &= B_i(0) \exp\left[\int_0^t (\mu_i - \frac{1}{2}|\sigma_i|^2) ds + \int_0^t \sigma_i \cdot dW\right] \quad \text{and} \\ \xi &= \xi(0) \exp\left[\int_0^t (-r - \frac{1}{2}|\varphi|^2) ds - \int_0^t \varphi \cdot dW\right] \end{aligned} \tag{2}$$

and application of Itô's lemma to the processes (2) leads to the system of stochastic differential equations

$$\frac{dB_i}{B_i} = \mu_i dt + \sigma_i \cdot dW = \mu_i dt + \sum_{k=1}^d \sigma_{ik} dW_k,$$

$$\frac{d\xi}{\xi} = -rdt - \varphi \cdot dW = -rdt - \sum_{k=1}^{d} \varphi_k dW_k. \tag{3}$$

From the explicit representations in (2) it is easily seen that ξB_i are martingales for every i, whenever $-\frac{1}{2}|\sigma_i - \varphi|^2 = \mu_i - \frac{1}{2}|\sigma_i|^2 - \frac{1}{2}|\varphi|^2$ for every i, or equivalently,

$$\mu_i = r + \sigma_i \cdot \varphi, \qquad \text{for} \quad i = 1, .., n. \tag{4}$$

Hence the price system B is arbitrage free if there exist r and φ such that μ_i and σ_i satisfy (4) for every i. The vector process φ is called the *market price of risk.* Now, the following proposition follows from theorem (2.1.4), some linear algebra and a martingale representation argument for the second part.

Proposition 2.2.1 *Suppose for each (t, ω) the $n \times d$ matrix σ, defined by $\sigma[i, k] := \sigma_i[k]$ has constant rank q; $q \leq \min(n, d)$. Then we have,*

i) *For $q = n$, the market is arbitrage free but incomplete. In this case necessarily $d \geq n$.*
ii) *If $q = d = n - 1$ and $\mathbf{1} \notin \text{range}(\sigma)$, the market is arbitrage free and complete.*

2.3 Derivative pricing

Assume an arbitrage free price system $B \in \mathcal{E}^n$, $\xi \in \mathcal{E}_+$ and let $C_T \in \mathcal{F}_T$, $T \in [0, \tau]$ be a claim such that there exists an SFTS or hedging strategy (θ, B) with $\theta_T \cdot B_T = C_T$. Since $(\theta, \xi B)$ is also an SFTS, see Jamshidian, [8], it follows that

$$\xi_T C_T = \theta_T \cdot \xi_T B_T = \theta_0 \cdot \xi_0 B_0 + \int_0^T \theta_s \cdot d(\xi_s B_s)$$

and by the martingale property of the integral in the right-hand-side we find for $t < T$,

$$\mathbb{E}[\xi_T C_T \mid \mathcal{F}_t] = \theta_0 \cdot \xi_0 B_0 + \int_0^t \theta_s \cdot d(\xi_s B_s) = \theta_t \cdot \xi_t B_t.$$

Hence

$$\theta_t \cdot B_t, = \xi_t^{-1} \mathbb{E}[\xi_T C_T \,|\mathcal{F}_t].$$

In an incomplete market where the price deflator ξ is not unique it follows that the right-hand-side does not depend on the choice of ξ. On the other hand, if $(\tilde{\theta}, \tilde{B})$ is another hedging SFTS with $\tilde{\theta}_T \cdot \tilde{B}_T = C_T$, it follows that $\theta_t \cdot B_t = \tilde{\theta}_t \cdot \tilde{B}_t$ for any $t < T$. Hence, the two SFTS's have always the same price. Therefore, the time $t < T$ value of the claim C_T is properly defined by

$$C_t := \xi_t^{-1} \mathbb{E}[\xi_T C_T \mid \mathcal{F}_t]. \tag{5}$$

As a result, in a complete market where ξ is uniquely determined, any $\mathcal{F}_T-$measurable claim C_T can be hedged by an SFTS and the price C_t of this claim at a prior time $t < T$

is given by (5).

We introduce the notion of *numeraire measures* and will give different representations for the claim price C_t in (5) by using numeraire measure transformations.

Definition 2.3.1 (numeraire measure) *Let $B \in \mathcal{E}^n$ be an arbitrage free price system $B \in \mathcal{E}^n$, $\xi \in \mathcal{E}_+$ and let ξA be a martingale, where $A > 0$. We will define the $A-$numeraire measure $\mathbb{P}_A$ as follows. Define the probability measure $\mathbb{P}_A$ by $\frac{d\mathbb{P}_A}{d\mathbb{P}} = M_A(\tau)$, where the martingale M_A is given by $M_A := \frac{\xi A}{A(0)}$.*

The following useful lemma is easy to prove.

Lemma 2.3.2 *If ξA and ξX are martingales then X/A is a $\mathbb{P}_A$ martingale.*

Indeed, for $0 \leq t, s$; $t + s \leq \tau$ we have

$$\begin{aligned} \mathbb{E}_A[\frac{X(t+s)}{A(t+s)} \mid \mathcal{F}_t] &= \frac{\mathbb{E}[M_A(\tau)\frac{X(t+s)}{A(t+s)} \mid \mathcal{F}_t]}{\mathbb{E}[M_A(\tau) \mid \mathcal{F}_t]} = \frac{\mathbb{E}[M_A(t+s)\frac{X(t+s)}{A(t+s)} \mid \mathcal{F}_t]}{M_A(t)} \\ &= \frac{\mathbb{E}[\xi(t+s)\frac{X(t+s)}{A(0)} \mid \mathcal{F}_t]}{\xi(t)\frac{A(t)}{A(0)}} = \frac{X(t)}{A(t)}. \end{aligned}$$

Now we can give an alternative representation for the option price (5) in terms of the B_i-numeraire.

Proposition 2.3.3 (option price in the B_i-numeraire) *Suppose that $B_i > 0$ for some fixed i and C_T is an option ($\mathcal{F}_T-$claim) which can be hedged with an SFTS. Then, we have*

$$\begin{aligned} C_t &= \xi_t^{-1}\mathbb{E}[\xi_T C_T \mid \mathcal{F}_t] \\ &= B_i(t)\mathbb{E}_{B_i}[\frac{C_T}{B_i(T)} \mid \mathcal{F}_t]. \end{aligned} \tag{6}$$

Proof Since ξC and ξB_i are martingales it follows from lemma (2.3.2) that C/B_i is a $\mathbb{P}_{B_i}$ martingale, hence $\frac{C_t}{B_i(t)} = \mathbb{E}_{B_i}[\frac{C_T}{B_i(T)} \mid \mathcal{F}_t]$.

Representation (2.3.3) turns out to be very useful in general and in particular in the case where B_i is a $T-$maturity zero coupon bond with $B_i(T) = 1$. Then, we get simply $C_t = B_i(t)\mathbb{E}_{B_i}[C_T \mid \mathcal{F}_t]$.

3 Jamshidian LIBOR rate models and LIBOR rate derivatives

3.1 The LIBOR rate process

Definition 3.1.1 (LIBOR processes) *Suppose* $B \in \mathcal{E}_+^n$ *and* $\delta_i > 0$ *for* $i = 1,..,n-1$. *The* $n-1$ *dimensional* **LIBOR process** $L \in \mathcal{E}_+^{n-1}$ *is defined by*

$$L_i := \delta_i^{-1}(\frac{B_i}{B_{i+1}} - 1).$$

Definition (3.1.1) is motivated by the practically important situation where for each i the B_i represents the price of a zero-coupon bond with face value 1 at maturity date T_i and where $\delta_i = T_{i+1} - T_i$ for $i = 1,..,n-1$. Then L_i thus defined is just the effective rate or LIBOR rate seen at time t over the period $[T_i, T_{i+1}]$.

We will now derive the dynamics of the LIBOR process L when the $B-$dynamics is given by the system (3) under the no-arbitrage condition (4). Inserting the explicit representations (2) for the B_i in (3.1.1) and applying Itô's lemma yields straightforwardly,

$$\begin{aligned}
dL_i = \delta_i^{-1}\tfrac{B_i(0)}{B_{i+1}(0)} d\exp\left[\textstyle\int_0^t (\mu_i - \mu_{i+1} - \tfrac{1}{2}|\sigma_i|^2 + \tfrac{1}{2}|\sigma_{i+1}|^2)ds + \int_0^t (\sigma_i - \sigma_{i+1})\cdot dW\right] = \\
\delta_i^{-1}\tfrac{B_i(0)}{B_{i+1}(0)} \exp\left[\textstyle\int_0^t (\mu_i - \mu_{i+1} - \tfrac{1}{2}|\sigma_i|^2 + \tfrac{1}{2}|\sigma_{i+1}|^2)ds + \int_0^t (\sigma_i - \sigma_{i+1})\cdot dW\right]\cdot \\
\cdot\{(\mu_i - \mu_{i+1} - \tfrac{1}{2}|\sigma_i|^2 + \tfrac{1}{2}|\sigma_{i+1}|^2)dt + (\sigma_i - \sigma_{i+1})\cdot dW\} + \\
+\tfrac{1}{2}\delta_i^{-1}\tfrac{B_i(0)}{B_{i+1}(0)} \exp\left[\textstyle\int_0^t (\mu_i - \mu_{i+1} - \tfrac{1}{2}|\sigma_i|^2 + \tfrac{1}{2}|\sigma_{i+1}|^2)ds + \int_0^t (\sigma_i - \sigma_{i+1})\cdot dW\right]\cdot \\
\cdot|\sigma_i - \sigma_{i+1}|^2 dt = \\
\delta_i^{-1}(1 + \delta_i L_i)(\mu_i - \mu_{i+1} + \sigma_{i+1}\cdot(\sigma_{i+1} - \sigma_i))dt + (\sigma_i - \sigma_{i+1})\cdot dW) = \\
\delta_i^{-1}(1 + \delta_i L_i)(\sigma_i - \sigma_{i+1})\cdot(dW + (\varphi - \sigma_{i+1})dt).
\end{aligned}$$

By the introduction of the absolute LIBOR volatilities

$$\beta_i := \delta_i^{-1}(1 + \delta_i L_i)(\sigma_i - \sigma_{i+1}) \tag{7}$$

and the drifted Brownian motions

$$dW^{(j)} := dW + (\varphi - \sigma_j)dt, \tag{8}$$

we may write

$$\begin{aligned}
dL_i &= \beta_i \cdot dW^{(i+1)} && (9)\\
&= \beta_i \cdot (dW + (\varphi - \sigma_n)dt - \sum_{j=i+1}^{n-1} (\sigma_j - \sigma_{j+1})dt) \\
&= -\sum_{j=i+1}^{n-1} \beta_i \cdot (\sigma_j - \sigma_{j+1})dt + \beta_i \cdot dW^{(n)} \\
&= -\sum_{j=i+1}^{n-1} \frac{\delta_j \beta_i \cdot \beta_j}{(1 + \delta_j L_j)} dt + \beta_i \cdot dW^{(n)}. && (10)
\end{aligned}$$

From definition (3.1.1) and lemma (2.3.2) it follows that L_i is a martingale with respect to the measure $\mathbb{P}_{B_{i+1}}$ and then from (8), (9) and general representation theorems for martingales it follows that $(W^{(i+1)} \mid 0 \leq t \leq T_i)$ is standard Brownian motion under $\mathbb{P}_{B_{i+1}}$, for $1 \leq i < n$. On the other hand, since $B_{j+1}/B_j = (1+\delta_j L_j)^{-1}$ is a martingale under $\mathbb{P}_{B_j}$, we can derive similarly that $(W^{(j)}(t) \mid 0 \leq t \leq T_j)$ is standard Brownian motion under $\mathbb{P}_{B_j}$, for $1 \leq j < n$. Combining we get,

Corollary 3.1.2 *for each $j = 1, \ldots, n$ the process*

$$W^{(j)}(t) = W(t) + \int_0^t (\varphi - \sigma_j) ds \quad 0 \leq t \leq T_j \wedge T_{n-1}$$

is a $d-$dimensional Brownian motion under the measure $\mathbb{P}_{B_j}$.

In practice it is more usual to deal with *relative* volatilities defined by $\gamma_i := \beta_i / L_i$. In terms of the γ_i, also called *the factor loadings*, (9) and (10) read in stead

$$\begin{aligned} dL_i &= L_i \gamma_i \cdot dW^{(i+1)} && (11) \\ &= -\sum_{j=i+1}^{n-1} \frac{\delta_j L_i L_j \gamma_i \cdot \gamma_j}{(1+\delta_j L_j)} dt + L_i \gamma_i \cdot dW^{(n)}. && (12) \end{aligned}$$

Remark 3.1.3 (numeraire notation) From now on we will use for the numeraire measure $\mathbb{P}_{B_j}$ the shorter notation $\mathbb{P}_j$ and for the related expectation we will write $\mathbb{E}_j$.

3.2 LIBOR market models

A very important LIBOR model is the so called *LIBOR market model* defined as follows.

Definition 3.2.1 (LIBOR market model) *The LIBOR model, (11) or (12), is called a* **market model** *if it is specified by a* **deterministic** *relative volatility structure, i.e. the factor loadings*

$$\gamma_i(t, \omega) =: \gamma_i(t), \qquad i = 1, .., n-1, \tag{13}$$

are assumed to be bounded deterministic functions of time t.

Remark 3.2.2 (existence of the LIBOR process) *In Jamshidian, [8], Th. 7.1, it is shown that for any volatility structure of the type $\gamma_i(t, L)$, where γ is bounded and locally Lipschitz in L, there exists an arbitrage free system of bond prices satisfying the* **zero coupon bond constraint** $B_i(T_i) = 1$ *for which the associated LIBOR process L is positive and satisfies (12). So, in particular, this holds for the market model (3.2.1).*

3.3 Valuation of LIBOR derivatives, forward transporting arguments

We now illustrate the importance of corollary (3.1.2) by the following example.

Example 3.3.1 Suppose we are given a tenor structure $0 < T_1 < T_2 < .. < T_n$ with intervals $\delta_i := T_{i+1} - T_i$ and an arbitrage free system $B \in \mathcal{E}_+^n$ of T_i maturity bonds B_i, for $i = 1, .., n$, with $B_i(T_i) = 1$. Let C be an option with payoff $C_{T_{i+1}}$ at T_{i+1}, which is a function of the LIBOR rate $L_i(T_i)$, say $C_{T_{i+1}} = f(L_i(T_i))$. Then, the measurability conditions of Th. (5.2) in [8] are satisfied and it follows that such an option can be hedged with an SFTS. Next, using the B_{i+1} numeraire, we get for the $t < T_i$ price of the option,

$$C_t = \xi_t^{-1}\mathbb{E}[\xi_{T_{i+1}} C_{T_{i+1}} \mid \mathcal{F}_t] = B_{i+1}(t)\mathbb{E}_{i+1}[\frac{C_{T_{i+1}}}{B_{i+1}(T_{i+1})} \mid \mathcal{F}_t] = B_{i+1}(t)\mathbb{E}_{i+1}[f(L_i(T_i)) \mid \mathcal{F}_t].$$

Now, in a market model where L has deterministic volatilities (13), we get by integrating (11), $L_i(T_i) = L_i(t) + \int_t^{T_i} L_i\gamma_i \cdot dW^{(i+1)}$, where $W^{(i+1)}$ is Brownian motion under $\mathbb{P}_{i+1}$. Hence, $\ln[L_i(T_i)/L_i(t)] = -\frac{1}{2}\int_t^T |\gamma_i|^2 ds + \int_t^{T_i} \gamma_i(t) \cdot dW^{(i+1)}$ and so $\ln[L_i(T_i)/L_i(t)]$ has a normal distribution under $\mathbb{P}_{i+1}$, with mean $-\frac{1}{2}\int_t^T |\gamma_i|^2 ds$ and variance $\int_t^T |\gamma_i|^2 ds$). A well known example is a $[T_i, T_{i+1}]-$*caplet* with strike K, defined by the payoff function $f(x) = \max(x - K, 0)$, for which we recover Black's market caplet formula in this way.

A useful technique for the valuation of a LIBOR derivative, specified by several payoff's at different tenors, is the method of *forward transported cash flows,* [8]. Consider, as in example (3.3.1), a tenor structure (T_i) together with an arbitrage free system B of T_i-maturity bonds B_i. Let further C be an option contract which specifies for each i, $1 \le i \le n-1$ at date T_{i+1} a payoff C_i, where C_i is supposed to be measurable with respect to the LIBOR process L up to time T_i. For the valuation of the option contract it is equivalent to deliver instead of the cash at T_{i+1}, an amount $C_i/B_n(T_{i+1})$ of B_n bonds which in turn guarantees a "forward transported" cash payment of $C_i/B_n(T_{i+1})$ at time T_n. Since the T_n payoff $C_i/B_n(T_{i+1}) = C_i\frac{B_{i+1}(T_{i+1})}{B_n(T_{i+1})}$ is measurable with respect to the LIBOR process up to T_{n-1} for every i, the option contract may be defined equivalently by a single aggregated payoff $C(T_n)$ at time T_n, which is L measurable up to T_{n-1}. Then, by Th. (5.2) in [8] and the constraint $B_n(T_n) = 1$, it follows that the option value at $t < T_1$ is given by

$$C(t) = B_n(t)\mathbb{E}_n(C(T_n)).$$

This value can be computed, at least in principal, by Monte Carlo simulation, for instance, by simulation of the SDE (12) in the $\mathbb{P}_n-$measure.

4 Approximate valuation of LIBOR derivatives in a LIBOR market model

For a LIBOR market model (3.2.1), where the volatilities γ_j are deterministic functions, we will design a procedure for the valuation of a large class of LIBOR derivatives based on a log-normal approximation of the LIBOR process L in this model. In particular, this procedure can be applied to the European swaption and thus covers the swaption approximation formula developed in Brace et al., [2].

4.1 Log-normal approximations of forward LIBOR rates

We consider a market model for the forward LIBOR rates L_j, $j = 1, .., n-1$, for a fixed tenor structure $0 < T_1 < \cdots < T_n$. For l, m with $1 \le m \le l < n$, the integrated version of

(11) reads

$$L_l(T_m) = L_l(t) \exp \int_t^{T_m} \left[\frac{-|\gamma_l|^2}{2} ds + \gamma_l \cdot dW^{(l+1)} \right] \tag{14}$$

and for any $i \in \{m, .., n\}$, $0 \leq t \leq T_m$, we derive from (7) and (8),

$$\begin{aligned} dW^{(l+1)} &= dW^{(i)} - \sum_{j=l+1}^{n-1} \frac{\delta_j L_j \gamma_j}{1+\delta_j L_j} dt + \sum_{j=i}^{n-1} \frac{\delta_j L_j \gamma_j}{1+\delta_j L_j} dt \\ &= dW^{(i)} + \sum_{j=m}^{l} \frac{\delta_j L_j \gamma_j}{1+\delta_j L_j} dt - \sum_{j=m}^{i-1} \frac{\delta_j L_j \gamma_j}{1+\delta_j L_j} dt, \end{aligned} \tag{15}$$

where an empty sum is defined to be 0. We thus have,

$$\begin{gathered} L_l(T_m) = \\ L_l(t) \exp \int_t^{T_m} \left(\frac{-|\gamma_l|^2}{2} ds + \sum_{j=m}^{l} \frac{\delta_j L_j \gamma_j \cdot \gamma_l}{1+\delta_j L_j} ds - \sum_{j=m}^{i-1} \frac{\delta_j L_j \gamma_j \cdot \gamma_l}{1+\delta_j L_j} ds + \gamma_l \cdot dW^{(i)} \right). \end{gathered} \tag{16}$$

In [2], Brace Gatarek and Musiela study a continuous family of forward LIBOR rates $K(t,T)$, $T \geq t$, over the period $[T, T+\delta]$, for fixed $\delta > 0$ and they derive the $t-$dynamics of K in the risk neutral measure from a Heath Jarrow and Morton framework. Next, a log-normal volatility structure for K is assumed and a first order approximation for L combined with a certain rank 1 assumption is used to derive a tractable approximation formula for the European swaption. In this sequel, where we employ Jamshidian's framework, we obtain an analogous approximation by approximating the processes L_j under the integral in (16) by their initial values $L_j(t)$, thus yielding a log-normal approximation for the distribution of $L_l(T_m)$, $m \leq l < n$ under the measure $\mathbb{P}_i$, $m \leq i \leq n$. So we obtain,

$$\begin{aligned} L_l(T_m) &\approx L_l(t) \exp \int_t^{T_m} \left(\frac{-|\gamma_l|^2}{2} ds + \gamma_l \cdot dW^{(i)} \right) \cdot \\ &\cdot \exp \left(\sum_{j=m}^{l} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)} \int_t^{T_m} \gamma_j \cdot \gamma_l ds - \sum_{j=m}^{i-1} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)} \int_t^{T_m} \gamma_j \cdot \gamma_l ds \right) \\ &= L_l(t) \exp \left(\sum_{j=m}^{l} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)} \Delta_{jl} - \sum_{j=m}^{i-1} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)} \Delta_{jl} \right) \cdot \\ &\cdot \exp \left(-\tfrac{1}{2} \Delta_{ll} + \int_t^{T_m} \gamma_l \cdot dW^{(i)} \right), \end{aligned} \tag{17}$$

where we have introduced the deterministic quantities

$$\Delta_{jl}^{(m)}(t) := \Delta_{jl}^{(m)} := \int_t^{T_m} \gamma_j \cdot \gamma_l ds; \quad j, l \in \{m, \ldots, n-1\}. \tag{18}$$

For $m \leq l < n$ and $m - 1 \leq k < n$, we introduce further

$$\mu_{lk}^{(m)}(t) := \mu_{l,k}^{(m)} := \sum_{j=m}^{l} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)} \Delta_{jl}^{(m)} - \sum_{j=m}^{k} \frac{\delta_j L_j(t)}{1+\delta_j L_j(t)} \Delta_{jl}^{(m)} - \frac{1}{2} \Delta_{ll}^{(m)}, \tag{19}$$

hence $\mu_{ll}^{(m)} = -\Delta_{ll}^{(m)}/2$ and a path-wise approximation for the LIBOR process is given by

$$\ln \frac{L_l(T_m)}{L_l(t)} \approx \mu_{l,i-1}^{(m)} + \int_t^{T_m} \gamma_l \cdot dW^{(i)}, \tag{20}$$

under the measure $\mathbb{P}_i$, $m \leq i \leq n$. Since $(W^{(i)}(s) \mid t \leq s \leq T_m)$ is Brownian motion under $\mathbb{P}_i$, $m \leq i \leq n$, at time t the joint distribution under $\mathbb{P}_i$ of the forward log-LIBOR rates may be well approximated as a multivariate normal distribution with,

$$\mathbb{E}_i[\ln \frac{L_l(T_m)}{L_l(t)}] = \mu^{(m)}_{l,i-1}(t), \tag{21}$$

$$Cov[\ln \frac{L_l(T_m)}{L_l(t)}, \ln \frac{L_{l'}(T_{m'})}{L_{l'}(t)}] = \int_t^{T_m \wedge T_{m'}} \gamma_l \cdot \gamma_{l'} ds = \Delta^{(m \wedge m')}_{ll'}(t), \tag{22}$$

where $m \leq l < n$, $m' \leq l' < n$, $(m \vee m') \leq i \leq n$. For the forward LIBOR correlations we thus get

$$\rho[\ln \frac{L_l(T_m)}{L_l(t)}, \ln \frac{L_{l'}(T_{m'})}{L_{l'}(t)}] = \frac{\Delta^{(m \wedge m')}_{ll'}(t)}{\sqrt{\Delta^{(m)}_{ll}(t)}\sqrt{\Delta^{(m')}_{l'l'}(t)}}. \tag{23}$$

For *constant* loadings γ_j this yields

$$\rho[\ln \frac{L_l(T_m)}{L_l(t)}, \ln \frac{L_{l'}(T_{m'})}{L_{l'}(t)}] = \frac{\gamma_l \cdot \gamma_{l'}(T_m \wedge T_{m'} - t)}{|\gamma_l||\gamma_{l'}|\sqrt{T_m - t}\sqrt{T_{m'} - t}},$$

which reads for $m \leq m'$,

$$\rho[\ln \frac{L_l(T_m)}{L_l(t)}, \ln \frac{L_{l'}(T_{m'})}{L_{l'}(t)}] = \frac{\gamma_l \cdot \gamma_{l'}}{|\gamma_l||\gamma_{l'}|}\sqrt{\frac{T_m - t}{T_{m'} - t}}; \quad m \leq m'; \quad m \leq l; \quad m' \leq l'. \tag{24}$$

Note that the correlations between the forward log-LIBORS do not depend on the choice of the measure $\mathbb{P}_i$, but the drifts do.

4.2 A general class of LIBOR derivatives

We specify a general class $\mathcal{C}$ of LIBOR derivatives and we will map out a strategy for the valuation of these derivatives. As usual we consider a tenor structure $(T_i)_{1 \leq i \leq n}$ together with an arbitrage free system B of T_i maturity bonds B_i. First, we start with the introduction of a subclass $\mathcal{C}_0$; $\mathcal{C}_0 \subset \mathcal{C}$.

Definition 4.2.1 ($\mathcal{C}_0$) *A derivative contract C_0 belongs to the class $\mathcal{C}_0$ when it specifies for each $1 \leq j < n$ a payoff C_j at time T_{j+1} via an explicitly given function f_j of the forward LIBORs $L_l(T_m)$, $1 \leq m \leq l < n$, $m \leq j$. So, $C_j =: f_j(L_l(T_m);\ 1 \leq m \leq l < n,\ m \leq j)$.*

Examples of $\mathcal{C}_0-$derivatives are the *cap, swap, trigger swap* and the *reverse floater*, which are studied in this sequel. Next, we define the larger class $\mathcal{C} \supset \mathcal{C}_0$ as the family of derivatives C which are (generalized) "callable" $\mathcal{C}_0$ options with maturity T_1 in the sense of the following definition.

Definition 4.2.2 ($\mathcal{C}$) *A derivative contract C belongs to the class $\mathcal{C}$ when it is specified by a payoff $\psi(C_0(T_1))$ at T_1 for a certain derivative $C_0 \in \mathcal{C}_0$ and some real valued reward function $\psi:\ y \to \psi(y)$.*

The most important case is where $\psi(y) = \max(y, 0)$ and thus C_0 will be "called" at T_1 whenever its value is positive, or in other terms, the holder of a contract C has the right to enter into a contract C_0 at T_1. Examples of $\mathcal{C}-$derivatives are the *swaption* and the *callable reverse floater*, also studied in this sequel.

Now, the log-normal approximation method for the forward LIBOR rates in a market model, combined with the forward transporting technique, provides us with an, in principle, feasible strategy for the valuation of $\mathcal{C}_0$ and $\mathcal{C}$ derivatives.

For a $\mathcal{C}_0$ derivative, the option value at $t < T_1$ is given by,

$$\begin{aligned} C_0(t) &= \sum_{j=1}^{n-1} B_{j+1}(t)\mathbb{E}_{j+1}[f_j(L_l(T_m); \quad 1 \leq m \leq l < n, \; m \leq j) \mid \mathcal{F}_t] \\ &= B_n(t)\sum_{j=1}^{n-1} \mathbb{E}_n[\frac{1}{B_n(T_{j+1})} f_j(L_l(T_m); \quad 1 \leq m \leq l < n, \; m \leq j) \mid \mathcal{F}_t] \\ &=: B_1(t)\Psi(L_l(t); \; 1 \leq l < n)., \end{aligned} \tag{25}$$

where the claim value relative to the B_1-bond is denoted by Ψ and is LIBOR measurable indeed. Hence, after making the log-normal approximations in a market model the valuation of C_0 or, equivalently, the identification of Ψ in general comes down to the computation of multivariate Gaussian integrals. In several cases, however, the problem reduces considerably. E.g. for a cap each term in (25) leads to a well known Black-Scholes expression, see (3.3.1) and for the reverse floater we get something similar, see (38). In the case of a trigger swap the involved multi-dimensional integrals can be done by faster routines when a special correlation structure is imposed on the LIBOR model, see section (6) and Curnow, Dunnett, [3].

Next, for the valuation of a $\mathcal{C}$ derivative we thus get,

$$\begin{aligned} C(t) &= B_1(t)\mathbb{E}_1[\psi(C_0(T_1)) \mid \mathcal{F}_t] \\ &= B_1(t)\mathbb{E}_1[\psi \circ \Psi(L_l(T_1); \; l \geq 1) \mid \mathcal{F}_t] \end{aligned} \tag{26}$$

So, if the valuation problem for the C_0 option is solved, i.e. the function Ψ is identified, the value of C is obtained, in principal, by multivariate normal integrals again. However, if the γ_i are calibrated and if the time span $T_n - T_1$ of the tenors is not too long, in practice only the first few eigenvalues of the LIBOR-correlation matrix

$$\rho[\ln \frac{L_l(T_1)}{L_l(t)}, \ln \frac{L_{l'}(T_1)}{L_{l'}(t)}] \tag{27}$$

are significantly positive and so, by a low rank (eg. rank 1) approximation of this matrix, $L(T_1)$ can be approximated by a $\mathbb{R}^{n-1}$ valued random variable $L(T_1)(\zeta)$, where ζ is a low dimensional (e.g. a scalar) standard normal random variable under $\mathbb{P}_1$. Hence, we thus get a low-factor (e.g. one-factor) approximation for the value of $C(t)$ in (26), where only the computation of a low dimensional (e.g. scalar) Gaussian integral is needed.

5 Applications

As a first application of the method presented in section (4.2) we consider in section (5.1) the European swaption and in section (5.2) we will sketch the route which leads to a multi-factor approximation formula which covers the results of Brace et al. in [2]. Then, subsequently, in section (5.3) we will tackle the callable reverse floater and in section (5.4) the trigger swap.
Through the whole section (5) we assume a tenor structure $(T_i)_{1\leq i\leq n}$ as before.

5.1 European swaption

A $[T_1, T_n]-$swap on a certain principal is a contract to pay a fixed rate κ and to receive spot LIBOR at the settlement dates $T_2, .., T_n$. The present value of this contract for a \$1 principal is equal to

$$\begin{aligned} Swap(t) &:= \sum_{j=1}^{n-1} B_{j+1}(t)\mathbb{E}_{j+1}[(L_j(T_j)-\kappa)\delta_j|\mathcal{F}_t] \\ &= B_1(t) - B_n(t) - \kappa\sum_{k=2}^{n}\delta_{k-1}B_k, \qquad t < T_1, \end{aligned} \tag{28}$$

since L_j is a $\mathbb{P}_{j+1}-$martingale, however, (28) also follows by a simple portfolio argument. Now the *swap rate* $S(t)$ is defined as *that* fixed rate κ for which $Swap(t) = 0$. Hence,

$$S(t) = \frac{B_1(t) - B_n(t)}{\sum_{k=2}^{n}\delta_{k-1}B_k(t)}. \tag{29}$$

A *swaption contract* with maturity T_1, strike κ and principal \$1 gives the right to contract at T_1 to pay a fixed coupon κ and receive the $T1-$swap rate at the settlement dates $T_2, .., T_n$. As, equivalently, one can contract for receiving spot LIBOR instead of the $T1-$swaprate, according to (25) and (26), the price of the swaption at $t < T_1$ can be given by

$$Swpn(t) = B_1(t)\mathbb{E}_1\left[\left(\sum_{j=1}^{n-1} B_{j+1}(T_1)\mathbb{E}_{j+1}[(L_j(T_j)-\kappa)\delta_j \mid \mathcal{F}_{T_1}]\right)^+ |\mathcal{F}_t\right],$$

where $(\cdot)^+ := \max(\cdot, 0)$. By using the martingale property again, this simplifies to

$$Swpn(t) = B_1(t)\mathbb{E}_1\left[\left(\sum_{j=1}^{n-1} B_{j+1}(T_1)[(L_j(T_1)-\kappa)\delta_j]\right)^+ |\mathcal{F}_t\right]. \tag{30}$$

In terms of the swap rate the expression between brackets in (30) is equal to

$$(S(T_1)-\kappa)\sum_{k=2}^{n}\delta_{k-1}B_k(T_1),$$

which is positive whenever $S(T_1) > \kappa$. Hence, by denoting the $\mathcal{F}_{T_1}$ measurable event $\{S(T_1) > \kappa\}$ with A, for (30) we may write

$$\begin{aligned} Swpn(t) &= \sum_{j=1}^{n-1} B_1(t)\mathbb{E}_1\left[1_A B_{j+1}(T_1)(L_j(T_1)-\kappa)\delta_j \;|\mathcal{F}_t\right] \\ &= \sum_{j=1}^{n-1} B_{j+1}(t)\mathbb{E}_{j+1}\left[1_A(L_j(T_1)-\kappa)\delta_j \;|\mathcal{F}_t\right], \end{aligned} \tag{31}$$

where we changed numeraires again for the second expression.

The representation (31) for the swaption price is completely general in the sense that it represents the option price in any arbitrage free model of T_i maturity bonds B_i. Note that (31) is similar to a representation derived in Brace et al. [2], however, (31) is derived without assuming an instantaneous saving bond numeraire and thus even holds when the market is incomplete.

5.2 Multi-factor swaption approximation

Starting out with (31) we can now mimic the procedure of Brace, Gatarek and Musiela in [2] and derive an analogous swaption approximation formula for a Jamshidian market model. However, in [2] there is made a rank 1 approximation with respect to a covariance matrix of forward LIBORS and in section (6) we will argue that this assumption is too restrictive when the resulting formula is used for certain calibration purposes, see conclusion (6.1.2). Therefore, we will redo the procedure in [2] in short in Jamshidians terms while we drop the rank 1 assumption and thus obtain a more general result.

The set A in (31) can be characterized further as

$$\begin{gathered} A = \{S(T_1) > \kappa\} = \left\{\frac{1-B_n(T_1)}{\sum_{k=2}^{n}\delta_{k-1}B_k(T_1)} > \kappa\right\} = \\ \{B_n(T_1) + \sum_{k=2}^{n}\kappa\delta_{k-1}B_k(T_1) < 1\} = \{\sum_{k=2}^{n} c_k B_k(T_1) < 1\} = \\ \left\{\sum_{k=2}^{n} c_k \left(\prod_{l=1}^{k-1}(1+\delta_l L_l(T_1))\right)^{-1} < 1\right\}, \end{gathered}$$

where we have introduced the constants $c_k := \kappa\delta_{k-1}$ for $2 \le k < n$ and $c_n := 1 + \kappa\delta_{n-1}$.

We now assume a LIBOR market model (3.2.1). Let Y be the $\mathcal{F}_{T_1}-$measurable random $(n-1)-$vector defined by

$$Y_l := \ln\frac{L_l(T_1)}{L_l(t)},$$

for $l = 1,..,n-1$. By taking $m = 1$ in (21) and (22) we have with respect to $\mathbb{P}_{j+1}$, conditional $\mathcal{F}_t$,

$$\mathbb{E}_{j+1}[Y_l] \quad = \quad \mu_{lj}^{(1)} \quad \text{and} \tag{32}$$

$$Cov[Y_l, Y_{l'}] \quad = \quad \int_t^{T_1} \gamma_l \cdot \gamma_{l'}\, ds = \Delta_{ll'}^{(1)}. \tag{33}$$

Since in practice only the first few eigenvectors of the matrix $\Delta^{(1)}$ are significantly positive, we assume that for a fixed r; $1 \le r < n$, the matrix $\Delta^{(1)}$ admits a decomposition

$$\Delta^{(1)} =: \Gamma\Gamma^T$$

for an $(n-1) \times r$ matrix Γ. In fact, if $\lambda_1 > .. > \lambda_r > 0$ are the non-zero eigenvalues of $\Delta^{(1)}$ and $g_1, .., g_r$ are corresponding orthonormal eigenvectors satisfying $g_i \cdot g_j = \delta_{ij}$ we can take $\Gamma_{ip} = \sqrt{\lambda_p} g_p[i]$, where $1 \le i < n$ and $1 \le p \le r$. Hence, for Y we write

$$Y_l = \mu_{lj}^{(1)} + \sum_{p=1}^{r} \Gamma_{lp}\varsigma_p$$

where $\varsigma := [\varsigma_p;\ 1 \le p \le r]$ is a random vector with standard normal $\mathcal{N}(0, I_r)$ distribution under the measure $\mathbb{P}_{j+1}$ and the indicator function of the set A now reads

$$1_A \ = 1_{\left\{1-\sum_{k=2}^{n} c_k \left(\prod_{l=1}^{k-1}(1+\delta_l L_l(t)\exp[\mu_{lj}^{(1)}+\sum_{p=1}^{r}\Gamma_{lp}\varsigma_p]\ )\right)^{-1} \ge 0\right\}}.$$

Next, we introduce the function

$$f_j(z) =: f_j(z_1, .., z_r) := 1 - \sum_{k=2}^{n} c_k \left(\prod_{l=1}^{k-1}(1 + \delta_l L_l(t)\exp(\mu_{lj}^{(1)} + \sum_{p=1}^{r}\Gamma_{lp}z_p))\right)^{-1}$$

and for $i = 0, .., n-1$ the $\mathbb{R}^r-$column vectors d_i by $d_0 \equiv 0$ and

$$d_i[p] := \sum_{j=1}^{i} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)}\Gamma_{jp}, \qquad i = 1, \ldots n-1,\ p = 1, \ldots, r. \tag{34}$$

It follows that

$$\begin{aligned}\mu_{lj}^{(1)} - \mu_{lk}^{(1)} &= -\textstyle\sum_{i=1}^{j} \frac{\delta_i L_i(t)}{1+\delta_i L_i(t)} \sum_{p=1}^{r} \Gamma_{ip}\Gamma_{lp} + \sum_{i=1}^{k} \frac{\delta_i L_i(t)}{1+\delta_i L_i(t)} \sum_{p=1}^{r} \Gamma_{ip}\Gamma_{lp} \\ &= \Gamma(d_k - d_j)\,[l].\end{aligned}$$

We thus get the following relationship for the f_j,

$$f_j(z) = f_i(z + d_i - d_j).$$

If G_j is the region in $\mathbb{R}^r$ defined by

$$G_j := \{z \in \mathbb{R}^r |\ f_j(z) \ge 0\},$$

then clearly

$$G_j = d_j - d_1 + G_1,$$

where as usual the set $x + A$ is defined by $\{x + a|\ a \in A\}$. We may also write,

$$G_j = d_j + G_0,$$

where

$$G_0 := -d_1 + G_1 = -d_1 +$$
$$\left\{ z \in \mathbb{R}^r \mid 1 - \sum_{k=2}^{n} c_k \left(\prod_{l=1}^{k-1} (1 + \delta_l L_l(t) \exp(\Gamma(z + d_l - d_1)\,[l] - \tfrac{1}{2}\Delta_{ll}^{(1)})) \right)^{-1} > 0 \right\}$$
$$= \left\{ z \in \mathbb{R}^r \mid 1 - \sum_{k=2}^{n} c_k \left(\prod_{l=1}^{k-1} (1 + \delta_l L_l(t) \exp(\Gamma(z + d_l)\,[l] - \tfrac{1}{2}\Delta_{ll}^{(1)})) \right)^{-1} > 0 \right\}.$$

Finally, by substituting the above expressions in (31) we derive straightforwardly the swaption approximation formula,

$$\begin{aligned} Swpn(t) \approx \textstyle\sum_{j=1}^{n-1} \delta_j B_{j+1}(t) L_j(t) \int_{-\Gamma_j + d_j + G_0} \phi^{(r)}(z) dz + \\ - \textstyle\sum_{j=1}^{n-1} \kappa \delta_j B_{j+1}(t) \int_{d_j + G_0} \phi^{(r)}(z) dz, \end{aligned} \tag{35}$$

where $\phi^{(r)}$ is the $r-$dimensional normal density given by

$$\phi^{(r)}(z) := \frac{1}{(2\pi)^{r/2}} \exp[-\frac{|z|^2}{2}].$$

For $r > 1$, the multi-factor case, (35) can be easily implemented by Monte Carlo simulation of the $r-$dimensional standard normal distribution. If we assume $r = 1$, as in [2], Γ becomes a column vector and the d_j are now scalars. For the integration we get simply $G_0 = [z_0, \infty[$ where z_0 is the unique root of the equation

$$1 - \sum_{k=2}^{n} c_k \left(\prod_{l=1}^{k-1} (1 + \delta_l L_l(t) \exp(\Gamma_l (z + d_l) - \frac{1}{2}\Delta_{ll}^{(1)})) \right)^{-1} = 0$$

and now (35) simplifies to

$$\begin{aligned} Swpn(t) \approx \\ \textstyle\sum_{j=1}^{n-1} \delta_j B_{j+1}(t) L_j(t) \mathcal{N}(-z_0 - d_j + \Gamma_j) - \textstyle\sum_{j=1}^{n-1} \kappa \delta_j B_{j+1}(t) \mathcal{N}(-z_0 - d_j). \end{aligned} \tag{36}$$

In fact, this formula is equivalent with theorem 3.2 in Brace et al., [2].

5.3 Callable reverse floater

Let $K, K' > 0$. A *reverse floater* (RF) contracts for receiving $L_i(T_i)$ while paying $\max(K - L_i(T_i), K')$ at time T_{i+1} for $i = 1, .., n-1$, with respect to a unit principal.
A *callable reverse floater* (CRF) is an option to enter into a reverse floater at T_1. The option will be exercised at T_1 when the value of the reverse floater at T_1 is positive.
For the reverse floater, the $T_{i+1}-$cashflows are given by

$$\begin{aligned} C_{T_{i+1}} &:= \delta_i L_i(T_i) - \delta_i \max(K - L_i(T_i), K') \\ &= \delta_i (L_i(T_i) - K') - \delta_i \max(K - K' - L_i(T_i), 0). \end{aligned}$$

So, the aggregated forward transported payoffs at T_n are equal to $\sum_{i=1}^{n-1} \frac{C_{T_{i+1}}}{B_n(T_{i+1})}$ and according to (25), for the T_1 value of the RF we have

$$RF(T_1) = B_n(T_1)\mathbb{E}_n \left(\sum_{i=1}^{n-1} \frac{C_{T_{i+1}}}{B_n(T_{i+1})} | \mathcal{F}_{T_1} \right). \tag{37}$$

Next, by (26), for the $t < T_1$ value of the CRF we get

$$\begin{aligned} CRF(t) &:= B_1(t)\mathbb{E}_1 \left[B_n(T_1)\mathbb{E}_n \left(\sum_{i=1}^{n-1} \frac{C_{T_{i+1}}}{B_n(T_{i+1})} | \mathcal{F}_{T_1} \right)^+ | \mathcal{F}_t \right] \\ &= B_n(t)\mathbb{E}_n \left[\mathbb{E}_n \left(\sum_{i=1}^{n-1} \frac{C_{T_{i+1}}}{B_n(T_{i+1})} | \mathcal{F}_{T_1} \right)^+ | \mathcal{F}_t \right]. \end{aligned}$$

Let us assume $K' = 0$, so $C_{T_{i+1}} := \delta_i L_i(T_i) - \delta_i \max(K - L_i(T_i), 0)$ and consider some special cases. When, for example, $K \approx 2L_i(0)$, the probability that LIBORS exceed K within the period $[0, T_n]$ can be neglected in practice if the time period T_n is not too long. So, when the option is called the cashflows are practically given by $C_{T_{i+1}} := 2\delta_i(L_i(T_i) - K/2)$ and we see that the option is basically a swaption on a doubled principal with strike rate $K/2$. If, however, $K \approx L_i(0)$ we may neglect the possibility that LIBORS fall below $K/2$ and practically speaking the $C_{T_{i+1}} := \delta_i L_i(T_i) - \delta_i \max(K - L_i(T_i), 0)$ will be surely positive for every i and so the option will be exercised in any case, yielding a cashflow equal to the difference of the LIBOR rate on a forward loan with unit principal and the cashflow of a floor over the period $[T_1, T_n]$ with strike K. Therefore, in this situation the valuation of the CRF involves the valuation of a floor. We thus observe that the CRF has *both* cap/floor *and* swaption characteristics.

We will continue with the valuation of the RF and CRF in the special case where $K' = 0$. The general case goes in a similar way. From (37) and the payoff specifications we have for the reverse floater,

$$\begin{gathered} RF(t) = B_n(t)\mathbb{E}_n \left[\sum_{i=1}^{n-1} \frac{\delta_i L_i(T_i) - \delta_i \max(K - L_i(T_i), 0)}{B_n(T_{i+1})} \mid \mathcal{F}_t \right] = \\ B_n(t)\mathbb{E}_n \left[\sum_{i=1}^{n-1} \frac{\delta_i L_i(T_i)}{B_n(T_{i+1})} \mid \mathcal{F}_t \right] - B_n(t)\mathbb{E}_n \left[\sum_{i=1}^{n-1} \frac{\delta_i \max(K - L_i(T_i), 0)}{B_n(T_{i+1})} | \mathcal{F}_t \right] =: \\ (1) - (2). \end{gathered}$$

(1) simplifies to $\sum_{i=1}^{n-1} B_{i+1}(t)\mathbb{E}_{i+1} \left[\frac{\delta_i L_i(T_i)}{B_{i+1}(T_{i+1})} | \mathcal{F}_t \right] = \sum_{i=1}^{n-1} B_i(t) - B_{i+1}(t) = B_1(t) - B_n(t)$, whereas (2) is equal to the price of a floor with strike K over $[T_1, T_n]$. Hence, for the reverse floater price we get

$$RF(t) = B_1(t) - B_n(t) - \sum_{i=1}^{n-1} B_{i+1}(t)\mathbb{E}_{i+1}[\delta_i \max(K - L_i(T_i), 0) | \mathcal{F}_t], \tag{38}$$

which can be evaluated analytically in a LIBOR market model since in a market model the terms in the sum can be expressed by Black-type formulas, e.g. see example (3.3.1).

It is clear that $RF(t)$ is non-increasing as function of K. Let $K^*(t)$ be such that the value of the RF contract is zero. Then, by (26), we get for the price of the CRF the equivalent representations

$$\begin{aligned} CRF(t) &:= B_1(t)\mathbb{E}_1\left[RF(T_1)^+|\mathcal{F}_t\right] = B_n(t)\mathbb{E}_n\left[\tfrac{RF(T_1)^+}{B_n(T_1)}|\mathcal{F}_t\right] = \\ B_1(t)\mathbb{E}_1 &\left[RF(T_1)1_{[K^*(T_1)>K]}|\mathcal{F}_t\right] = B_n(t)\mathbb{E}_n\left[\tfrac{RF(T_1)1_{[K^*(T_1)>K]}}{B_n(T_1)}|\mathcal{F}_t\right], \end{aligned} \tag{39}$$

which give, at least in principal, a Monte Carlo procedures for the option price of the CRF. However, we will analyze (39) further in order to get more tractable approximations. Substitution of (38) in (39) gives

$$\begin{aligned} CRF(t) &= B_1(t)\mathbb{E}_1\left[1_{[K^*(T_1)>K]}|\mathcal{F}_t\right] - B_1(t)\mathbb{E}_1\left[B_n(T_1)1_{[K^*(T_1)>K]}|\mathcal{F}_t\right] + \\ -B_1(t)\mathbb{E}_1 &\left[1_{[K^*(T_1)>K]}\textstyle\sum_{i=1}^{n-1} B_{i+1}(T_1)\mathbb{E}_{i+1}[\delta_i \max(K - L_i(T_i), 0)|\mathcal{F}_{T_1}]|\mathcal{F}_t\right] \\ &=: (I) - (II) - (III). \end{aligned}$$

By numeraire changes we get

$$(II) = B_1(t)\mathbb{E}_1\left[B_n(T_1)1_{[K^*(T_1)>K]}|\mathcal{F}_t\right] = B_n(t)\mathbb{E}_n\left[1_{[K^*(T_1)>K]}|\mathcal{F}_t\right] \quad \text{and}$$

$$\begin{aligned} (III) &= \textstyle\sum_{i=1}^{n-1} B_{i+1}(t)\mathbb{E}_{i+1}\left[1_{[K^*(T_1)>K]}\mathbb{E}_{i+1}[\delta_i \max(K - L_i(T_i), 0)|\mathcal{F}_{T_1}]|\mathcal{F}_t\right] \\ &=: \textstyle\sum_{i=1}^{n-1} B_{i+1}(t)\mathbb{E}_{i+1}\left[1_{[K^*(T_1)>K]}F_i(T_1, K)|\mathcal{F}_t\right], \end{aligned}$$

where F_i is defined such that $B_{i+1}(t)F_i(t, K)$ is the price of a floorlet with strike K over the period $[T_i, T_{i+1}]$. Resuming, we have

$$\begin{aligned} CRF(t) &:= B_1(t)\mathbb{E}_1\left[1_{[K^*(T_1)>K]}|\mathcal{F}_t\right] - B_n(t)\mathbb{E}_n\left[1_{[K^*(T_1)>K]}|\mathcal{F}_t\right] + \\ &- \textstyle\sum_{i=1}^{n-1} B_{i+1}(t)\mathbb{E}_{i+1}\left[1_{[K^*(T_1)>K]}F_i(T_1, K)|\mathcal{F}_t\right]. \end{aligned} \tag{40}$$

So far, the expression (40) for the price of the callable reverse floater is still completely general.

We now assume a LIBOR market model and proceed with the derivation of an approximation formula for the CRF in such a model. Since in a market model F can be expressed as a Black-type formula, the relative price RF/B_1 can be considered as an explicitly known function Ψ of $L(t)$ and K,

$$\begin{aligned} RF(t) &=: B_1(t)\Psi(L_1(t), .., L_{n-1}(t); K) \\ &= B_1(t) - B_n(t) - \sum_{i=1}^{n-1} B_{i+1}(t)F_i(T_1, K). \end{aligned}$$

Further, since Ψ is decreasing in K we have

$$1_{[K^*(T_1)>K]} = 1_{[\Psi(L_1(T_1), .., L_{n-1}(T_1), K)>0]}.$$

Just as for the swaption approximation in (5.1) we assume a rank$-r$ decomposition, $1 \le r < n$ of the matrix $\Delta^{(1)}$ in (22) again. However, for simplicity, we now only show the derivation of a CRF-approximation formula for the case $r = 1$. The multi-factor case $r > 1$ can be derived similarly along the lines which has led to (35). We thus assume that for some column vector $\Gamma \geq 0$ we have

$$\Delta^{(1)} = \Gamma\Gamma^T.$$

For a fixed $i \in \{1, \dots, n\}$ we approximate the forward LIBORs at T_1 by

$$L_l(T_1) = L_l(t) \exp Y_l = L_l(t) \exp[\mu^{(1)}_{l,i-1} + \Gamma_l \varsigma], \qquad 1 \le l < n,$$

under the measure $\mathbb{P}_i$, where the real variable ς is, conditional $\mathcal{F}_t$, under $\mathbb{P}_i$ normal $\mathcal{N}(0,1)$ distributed. Next, we introduce the functions h_i by

$$h_i(\varsigma) := \Psi(L_1(t) \exp[\mu^{(1)}_{1,i-1} + \Gamma_1 \varsigma], .., L_{n-1}(t) \exp[\mu^{(1)}_{n-1,i-1} + \Gamma_{n-1} \varsigma], K).$$

It is easy to see that $h_i(-\infty) = -\sum_{j=1}^{n-1} \delta_j K$ and $h_i(\infty) = 1$. Moreover, since $\partial\Psi/\partial L_j > 0$ for every j, there is a unique ς_i for which $h_i(\varsigma_i) = 0$. Hence, we may write

$$1_{[K^*(T_1)>K]} = 1_{[\varsigma > \varsigma_i]}.$$

We define the scalars d_i like in (34), where we take $r = 1$ and in the same way it follows that $\mu^{(1)}_{li} - \mu^{(1)}_{lk} = \Gamma_l(d_k - d_i)$ and so $h_i(\varsigma) = h_1(\varsigma - d_{i-1})$, hence

$$\varsigma_i = \varsigma_1 + d_{i-1}.$$

Now we return to the price of the CRF given by (40) and we abbreviate this expression by $CRF(t) = (*) - (**) - (***)$ and work out the terms separately. In the sum (***), each $F_i(T_1, K)$ can be expressed as a Black formula involving $L_i(T_1) = L_i(t) \exp[\Gamma_i \varsigma - \frac{\Gamma_i^2}{2}]$ in the measure $\mathbb{P}_{i+1}$. In particular,

$$F_i(T_1, K) = \mathbb{E}_{i+1}[\delta_i \max(K - L_i(T_i), 0) | \mathcal{F}_{T_1}] =$$
$$\delta_i K \mathcal{N}\left(\frac{-\ln \frac{L_i(T_1)}{K} + \frac{1}{2}\int_{T_1}^{T_i} |\gamma_i|^2 ds}{\sqrt{\int_{T_1}^{T_i} |\gamma_i|^2 ds}}\right) - \delta_i L_i(T_1) \mathcal{N}\left(\frac{-\ln \frac{L_i(T_1)}{K} - \frac{1}{2}\int_{T_1}^{T_i} |\gamma_i|^2 ds}{\sqrt{\int_{T_1}^{T_i} |\gamma_i|^2 ds}}\right)$$

and can be re-expressed as a function of ς, say $F_i(\varsigma)$,

$$\begin{aligned} F_i(\varsigma) \quad : \quad &= \delta_i K \mathcal{N}\left(\frac{-\ln \frac{L_i(t)}{K} - \Gamma_i \varsigma + \frac{\Gamma_i^2}{2} + \frac{1}{2}\int_{T_1}^{T_i} |\gamma_i|^2 ds}{\sqrt{\int_{T_1}^{T_i} |\gamma_i|^2 ds}}\right) + \\ & -\delta_i L_i(T_1) \mathcal{N}\left(\frac{-\ln \frac{L_i(t)}{K} - \Gamma_i \varsigma + \frac{\Gamma_i^2}{2} - \frac{1}{2}\int_{T_1}^{T_i} |\gamma_i|^2 ds}{\sqrt{\int_{T_1}^{T_i} |\gamma_i|^2 ds}}\right). \end{aligned}$$

It follows that the $i-$th term in (***) is approximately equal to $B_{i+1}(t)\int_{\varsigma>\varsigma_{i+1}} F_i(\varsigma)\phi(\varsigma)d\varsigma$. Together with the approximations for (*) and (**) and using $\varsigma_{i+1}=\varsigma_1+d_i$ we now have the following (one-factor) approximation formula for (40).

$$CRF(t)=B_1(t)\mathcal{N}(-\varsigma_1)-B_n(t)\mathcal{N}(-\varsigma_1-d_{n-1})-\sum_{i=1}^{n-1}B_{i+1}(t)\int_{\varsigma_1+d_i}^{\infty}F_i(\varsigma)\phi(\varsigma)d\varsigma,$$

where the integrals can be computed by quadrature.

5.4 Trigger swap

The trigger swap is a contract of type $\mathcal{C}_0$ which is specified as follows. At the first tenor T_i for which $L_i(T_i)>K_i$, the counter party has to enter into a swap with fixed coupon κ over the remaining period $[T_i,T_n]$. If we define the index τ by $\tau:=\min_{1\le p<n}\{p\,|\,L_p(T_p)>K_p\}$, by forward transporting arguments the $t<T_1$ price of the trigger swap can be represented by

$$\begin{aligned}Trswp(t)&=B_n(t)\mathbb{E}_n\left[\sum_{j=\tau}^{n-1}\tfrac{1}{B_n(T_{j+1})}(L_j(T_j)-\kappa)\delta_j\mid\mathcal{F}_t\right]=\\ &B_n(t)\mathbb{E}_n\left[\sum_{p=1}^{n-1}1_{[\tau=p]}\sum_{j=p}^{n-1}\tfrac{1}{B_n(T_{j+1})}(L_j(T_j)-\kappa)\delta_j\mid\mathcal{F}_t\right]=\\ &B_n(t)\mathbb{E}_n\left[\sum_{p=1}^{n-1}1_{[\tau=p]}\sum_{j=p}^{n-1}\mathbb{E}_n\left[\tfrac{1}{B_n(T_{j+1})}(L_j(T_j)-\kappa)\delta_j\mid\mathcal{F}_{T_p}\right]\mid\mathcal{F}_t\right],\end{aligned}\tag{41}$$

since T_τ is a stopping time. We proceed by changing numeraires in (41),

$$\begin{aligned}Trswp(t)&=B_n(t)\mathbb{E}_n\left[\sum_{p=1}^{n-1}1_{[\tau=p]}\tfrac{1}{B_n(T_p)}\sum_{j=p}^{n-1}B_{j+1}(T_p)\mathbb{E}_{j+1}\left[(L_j(T_j)-\kappa)\delta_j|\mathcal{F}_{T_p}\right]\mid\mathcal{F}_t\right]\\ &=B_n(t)\mathbb{E}_n\left[\sum_{p=1}^{n-1}1_{[\tau=p]}\tfrac{1}{B_n(T_p)}\sum_{j=p}^{n-1}B_{j+1}(T_p)(L_j(T_p)-\kappa)\delta_j\mid\mathcal{F}_t\right],\end{aligned}$$

where is used that L_j is a $\mathbb{P}_{j+1}-$martingale. Next, by plugging in the definition of LIBOR,

$$Trswp(t)=\sum_{p=1}^{n-1}B_n(t)\mathbb{E}_n\left[1_{[\tau=p]}\frac{1}{B_n(T_p)}\left(1-B_n(T_p)-\kappa\sum_{j=p}^{n-1}B_{j+1}(T_p)\delta_j\right)\mid\mathcal{F}_t\right].\tag{42}$$

and by changing numeraires again we get the following expression for the value of the trigger swap

$$\begin{aligned}Trswp(t)&=\sum_{p=1}^{n-1}B_p(t)\mathbb{E}_p\left[1_{[\tau=p]}|\mathcal{F}_t\right]-\sum_{p=1}^{n-1}B_n(t)\mathbb{E}_n\left[1_{[\tau=p]}|\mathcal{F}_t\right]+\\ &-\sum_{p=1}^{n-1}\sum_{j=p}^{n-1}\kappa\delta_jB_{j+1}(t)\mathbb{E}_{j+1}\left[1_{[\tau=p]}|\mathcal{F}_t\right].\end{aligned}\tag{43}$$

Remark 5.4.1 If all the K_p are zero, we have $\tau=1$ with probability 1 and we get a swap contract which swaps LIBOR against a fixed coupon κ. Indeed, by next setting (43) equal to zero we yield the usual swap rate again.

Remark 5.4.2 *Using the swap rate formula (29) for the $[T_p,T_n]$ swap rate $S_{p,n}$ and changing to the annuity numeraires $\mathbb{P}_{p,n}$ defined by the annuity $B_{p,n}:=\sum_{j=p}^{n-1}B_{j+1}\delta_j$ we get from*

(42) and numeraire change another interesting representation for the trigger swap,

$$\begin{aligned} Trswp(t) &= \textstyle\sum_{p=1}^{n-1} B_n(t)\mathbb{E}_n\left[1_{[\tau=p]}\frac{1}{B_n(T_p)}B_{p,n}(T_p)(S_{p,n}(T_p)-\kappa)\mid\mathcal{F}_t\right] \\ &= \textstyle\sum_{p=1}^{n-1} B_{p,n}(t)\mathbb{E}_{p,n}\left[1_{[\tau=p]}(S_{p,n}(T_p)-\kappa)\mid\mathcal{F}_t\right]. \end{aligned}$$

Moreover, from lemma (2.3.2) we see that $S_{p,n}$ is a martingale under $\mathbb{P}_{p,n}$ and follows even a driftless geometrical Brownian motion under $\mathbb{P}_{p,n}$ in a swap market model. See Jamshidian, [8]. However, the simultaneous distribution of $S_{p,n}$ and τ under this annuity measure is a mess and therefore we rather stick to the LIBOR measure representation (43).

For the computation of the trigger swap we need to get hold of the conditional probabilities

$$\mathbb{E}_i\left[1_{[\tau=p]}|\mathcal{F}_t\right],$$

for $i=p,..,n$; $p=1,..,n-1$. The $\mathcal{F}_{T_p}$ measurable trigger event $[\tau=p]$ depends on the LIBOR history up to T_p and we have

$$[\tau=p]=\{L_p(T_p)>K_p\}\cap\bigcap_{1\le j<p}\{L_j(T_j)\le K_j\} \tag{44}$$

for $p=1,..,n-1$, with the usual convention that an intersection of subsets of Ω over an empty index set is equal to Ω itself. For the log-LIBORs (44) reads

$$[\tau=p]=\left\{\{\ln\frac{L_p(T_p)}{L_p(t)}>\ln\frac{K_p}{L_p(t)}\}\cap\bigcap_{1\le j<p}\{\ln\frac{L_j(T_j)}{L_j(t)}\le\ln\frac{K_j}{L_j(t)}\}\right\}. \tag{45}$$

Now we recall the normal approximations for the log-LIBOR distributions under the different measure $\mathbb{P}_i$ in section (4.1), yielding the moments (21), (22). Hence, the distribution of $[\ln\frac{L_l(T_l)}{L_l(t)}]_{l=1,..,p}$, under a fixed $\mathbb{P}_i$, $i\in\{p,..,n\}$, conditional $\mathcal{F}_t$, is in this approximation $p-$variate normal, $\mathcal{N}(\eta^{(p,i-1)},\Theta^{(p)})$, where

$$\eta_l^{(p,i-1)}:=\mu_{l,i-1}^{(l)}\qquad l=1,..,p\quad\text{and} \tag{46}$$

$$\Theta_{l_1l_2}^{(p)}:=\Delta_{l_1l_2}^{(l_1\wedge l_2)}=\qquad l_1,l_2=1,..,p \tag{47}$$

It is important to note that even in the case of a one factor model the matrices $\Theta^{(p)}$ are now generally of full rank p due to the fact that we are now dealing with LIBORS at *different* tenors instead of LIBORS at a fixed maturity T_1 as in the previous applications.

In a once calibrated market model the covariance matrices $\Theta^{(p)}$ and drifts $\eta^{(p,i-1)}$ are directly available. So, if we denote the $r-$dimensional normal density with drift vector $\eta\in\mathbb{R}^r$ and correlation matrix $G\in\mathbb{R}^{r\times r}$ by $n_r(z_1,..,z_r;\eta,G)$, we thus find by (45) for $i=p,..,n$,

$$\begin{aligned}\mathbb{E}_i\left[1_{[\tau=p]}|\mathcal{F}_t\right]=\int_{-\infty}^{\ln\frac{K_1}{L_1(t)}-\eta_1^{(p,i-1)}}dz_1..\int_{-\infty}^{\ln\frac{K_{p-1}}{L_{p-1}(t)}-\eta_{p-1}^{(p,i-1)}}dz_{p-1}\int_{\ln\frac{K_p}{L_p(t)}-\eta_p^{(p,i-1)}}^{\infty}dz_p\cdot \\ \cdot n_p(z_1,..,z_p;0,\Theta^{(p)})\end{aligned} \tag{48}$$

Note. For $p = 1$, integrals over $z_1, .., z_{p-1}$ have to be interpreted as 1.

By substituting the expressions (48) in (43) we have established an approximation algorithm for the trigger swap in a Jamshidian market model. Moreover, when a special correlation structure is imposed on the model, the multi-dimensional integrals can be done by faster routines because of the constant integration bounds. See section (6) and Curnow, Dunnett, [3].

6 Simultaneous calibration of LIBOR market models to caps and swaptions, special correlation structures

When dealing with LIBOR rate models the calibration of the factor loadings γ_i is always a main issue. In a *general* LIBOR model, given by (11) or (12), the $\gamma_i{}'s$ even represent fairly arbitrary processes. In a market model, however, the $\gamma_i{}'s$ are deterministic and in (6.1) we will see that a market model with *constant* $\gamma_i{}'s$ is already quite rich, in the sense that it contains enough degrees of freedom for simultaneous valuation of a large family of caps and swaptions. Since these *plain vanilla* options are liquidly traded in the markets, their prices can be considered as "correct" to some extent and can be used as benchmarks for calibration of the LIBOR model. A once calibrated model can be used subsequently for the valuation of exotic options such as the trigger swap or the callable reverse floater, along the lines explained in the previous sections.

6.1 Constant factor loadings

We assume a tenor structure $0 < T_1 < \cdots < T_n$ as usual and now consider a LIBOR market model with constant factor loadings γ_i. From example (3.3.1) it follows that in this model the price of a $[T_i, T_{i+1}]-$caplet can be given by a Black-Scholes formula, involving an input volatility $|\gamma_i|$ and an input "risk-free rate" equal to zero. See e.g. [1]. As a consequence, the norms $|\gamma_i|$ of the $\gamma_i{}'s$ are already determined by the market caplet prices as being the implied $[T_i, T_{i+1}]-$caplet volatilities. However, the individual components of the γ_i, the γ_{ik}; $k = 1, ..d$ which reflect the correlation structure of the increments of forward LIBORs *cannot* be recovered from the caplet prices at all. But, clearly, the swaption prices *do* depend on this specific correlation structure and are thus plausible candidates for further calibration of the model or the recovering of the γ_{ik}. We note that the total of different caplet and swaption prices on the given tenor structure has the number $n(n-1)/2$. Since any orthogonal transformation applied to an $\mathbb{R}^d-$Brownian motion leads to an equivalent Brownian motion with the same distribution, multiplication of the matrix $[\gamma_{ik}]$ on the right with a $d \times d$ orthogonal matrix gives an equivalent market model. So, in fact, the essential model parameters to be calibrated are the $n(n-1)/2$ inner products $\gamma_i \cdot \gamma_j$ rather than the γ_i itself. Because this number is just equal to the total of caplet and swaption prices, we conclude the following.

Conclusion 6.1.1 *A market model with constant loadings is determined by the inner products $\gamma_i \cdot \gamma_j$ and thus contains just enough dergrees of freedom to be calibrated to a complete system of cap(let) and swaption prices on the given tenor structure.*

Unfortunately, however, calibrating a market model with full rank volatility matrix $[\gamma_i \cdot \gamma_j]$ is very difficult in practice and, besides, generally the system of cap and swaption prices is only partially given. Therefore, as an alternative, we consider the calibration of a lower factor market model, where the volatility matrix may have lower rank and where possibly a lower number of market prices are given. Assume a market model is given with an $(n-1) \times (n-1)$ covariance matrix Δ,

$$\Delta := [\gamma_i \cdot \gamma_j], \quad \text{where } 1 \leq rank(\Delta) < n.$$

From ordinary matrix theory it follows that the nonnegative definite symmetric matrix Δ of rank r admits a decomposition

$$\Delta = \Gamma\Gamma^T, \quad \text{for an (n} - 1) \times \text{r matrix } \Gamma. \tag{49}$$

This decomposition is not unique, for two such decompositions $\Gamma\Gamma^T = \tilde{\Gamma}\tilde{\Gamma}^T$, there exists an orthogonal $r \times r$ matrix Q such that $\tilde{\Gamma} = \Gamma Q$. However, if the submatrix $\Delta_r = [\Delta_{ij}]_{1 \leq i,j \leq r}$ has already rank r, there exists a *unique lower matrix* Γ, i.e. $\Gamma_{kl} = 0$ for $l > k$ with $\Gamma_{kk} > 0$ for $k = 1, .., r$ such that (49) holds. If $rank(\Delta_r) < r$ then, for a suitable permutation matrix Q, the matrix $Q\Delta Q^T$ has a unique 'lower matrix' decomposition (49). In this way we observe that there are in fact

$$(n-1) + (n-2) + .. + (n-r) = \frac{1}{2} r(2n - r - 1) \tag{50}$$

essential parameters in the model to calibrate and we thus need to sort out properly $\frac{1}{2}r(2n-r-1)$ caps and swaptions for the calibration. For example, in a one factor model where $d = r = 1$, there are only $n-1$ parameters to calibrate with. Indeed, the one factor model is completely determined by the $n-1$ implied caplet volatilities.

Let us now try to calibrate a LIBOR market model with *constant* factor loadings to both cap and swaption prices by using the *rank* 1 swaption approximation formula (36) in section (5.1). For constant $\gamma_i{}'s$ we get from (18),

$$\Delta_{jl}^{(m)}(t) = \gamma_j \cdot \gamma_l (T_m - t) \tag{51}$$

and the assumption

$$rank(\Delta_{jl}^{(1)}(t)) = rank(\gamma_j \cdot \gamma_l (T_1 - t)) = 1,$$

used in the derivation of (36), implies that there is a *constant* column vector $\lambda := [\lambda_1, .. , \lambda_{n-1}]^T$ such that

$$\gamma_j \cdot \gamma_l = \lambda_j \lambda_l; \qquad 1 \leq j, l < n.$$

As a consequence, for the dynamics of the LIBOR process L, for instance in the $\mathbb{P}_n$ measure given by (12), we now get

$$dL_i = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_i L_j \lambda_i \lambda_j}{(1+\delta_j L_j)} dt + L_i \gamma_i \cdot dW^{(n)}. \tag{52}$$

However, (52) can be described by a single *scalar* $\mathbb{P}_n-$Brownian motion $w^{(n)}$ in a complete equivalent way,

$$dL_i = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_i L_j \lambda_i \lambda_j}{(1+\delta_j L_j)} dt + L_i \lambda_i dw^{(n)}.$$

Hence by a one-factor model where, since $\lambda_i = |\gamma_i|$, the factor loadings are already determined by the cap-prices and there is no freedom left for further calibration to swaption prices. From the above we see that the rank one assumption on Δ in [2] is in essence the assumption of a one-factor model and we conclude the following.

Conclusion 6.1.2 *Simultaneous calibration of a LIBOR market model with constant factor loadings γ_{ik} to the prices of caplets and swaptions by using the rank* 1 *swaption approximation formula (36) is not possible.*

Of course, one might oppose that one should use time dependent $\gamma' s$ instead of constants in order to generate more degrees of freedom. However, then conclusion (6.1.2) still indicates that this would result very likely in a model for which the calibration to swaption prices behaves instable. Therefore, one should rather use multi-rank swaption formulas, where the choice of the rank depends on the number of swaption prices one wants to calibrate to, although the implementation will not be easy and stability problems still may occur for reasons explained in (6.3).

6.2 Implied LIBOR correlations from the cap and swaption markets

We next present another way of calibrating a market model to caplet and swaption prices. In fact, it is a method for recovering the correlation structure of instantaneous forward LIBOR increments from the cap/swaption markets and is widely used by interest rate traders and described in the more practical oriented financial literature, e.g. Rebonato, [10]. However, also in this method simplifying approximations are involved and the substantiating arguments used in the literature are generally rather vague. Therefore, we will study below the implied correlation method in more detail by using *bracket calculus* from stochastic analysis. See, e.g. [9].

With respect to a usual tenor structure $\{T_j\}$; $j = 1, .., n$, we consider $[T_p, T_q]-$swaps, for $1 \le p < q \le n$. The swap rate at time t is denoted by $S(t, T_p, T_q) =: S_{p,q}(t)$ and given by

$$S_{p,q}(t) \quad = \quad \frac{B_p(t) - B_q(t)}{\sum_{k=p+1}^{q} \delta_{k-1} B_k(t)} = \frac{\sum_{i=p+1}^{q} \delta_{i-1} B_i(t) L_{i-1}(t)}{\sum_{k=p+1}^{q} \delta_{k-1} B_k(t)}$$

$$=: \sum_{i=p}^{q-1} w_i(t) L_i(t), \tag{53}$$

where $t < T_p$ and the $w_i(t) := \frac{\delta_i B_{i+1}(t)}{\sum_{k=p+1}^{q} \delta_{k-1} B_k(t)}$ are weight factors which satisfy $\sum_{i=p}^{q-1} w_i = 1$. In differential form we get

$$dS_{p,q} := \sum_{i=p}^{q-1} w_i dL_i + \sum_{i=p}^{q-1} L_i dw_i + \sum_{i=p}^{q-1} d\langle L_i, w_i \rangle,$$

from which we derive by using some bracket calculus,

$$\begin{gathered} d\langle S_{p,q} \rangle := d\langle S_{p,q}, S_{p,q} \rangle = \\ \textstyle\sum_{i,j=p}^{q-1} w_i w_j L_i L_j (d\langle \ln L_i, \ln L_j \rangle + 2d\langle \ln w_i, \ln L_j \rangle + \\ d\langle \ln w_i, \ln w_j \rangle) \end{gathered} \tag{54}$$

Now, in practice, it turns out that compared to the behaviour of the L_i the behaviour of the weight factors w_i is rather smooth and therefore, in a good approximation, we assume that their quadratic variation processes are identically zero and the differentials in (54) involving the w_i can thus be neglected. This yields

$$\begin{aligned} d\langle S_{p,q} \rangle = S_{p,q}^2 d\langle \ln S_{p,q}, \ln S_{p,q} \rangle &\approx \textstyle\sum_{i,j=p}^{q-1} w_i w_j L_i L_j d\langle \ln L_i, \ln L_j \rangle \\ &= \textstyle\sum_{i,j=p}^{q-1} w_i w_j L_i L_j \gamma_i \cdot \gamma_j dt \end{aligned} \tag{55}$$

and after introducing the relative volatility process $\sigma_{p,q}$ for the swap rate by $d\langle \ln S_{p,q} \rangle =: \sigma_{p,q}^2 dt$, we get

$$\begin{aligned} S_{p,q}^2 \sigma_{p,q}^2 &\approx \sum_{i,j=p}^{q-1} w_i w_j L_i L_j \gamma_i \cdot \gamma_j \\ &= \sum_{i,j=p}^{q-1} w_i w_j L_i L_j |\gamma_i||\gamma_j| \rho_{ij}, \end{aligned} \tag{56}$$

where the correlation matrix ρ is defined by $\rho_{ij} := \gamma_i \cdot \gamma_j / |\gamma_i||\gamma_j|$.

Along with the LIBOR market model we now also assume a *SWAP market model* with constant loadings $\sigma_{p,q}$, see [8], although, in fact, we *cannot* have *both* deterministic LIBOR volatilities *and* deterministic swap volatilities! So, again an approximation. Because the $\sigma_{p,q}$ can now be identified as the implied Black volatilities quoted in the markets via the swaption prices and the $|\gamma_i|$ are quoted via the cap(let) prices as well, there are in principal just enough equations in (56) to solve for the unkowns ρ_{ij}. However, when the market provides not enough quotes and we thus have to many unkowns, we need to come up with sensible improvisations. For instance, we could apply certain regularization techniques. See e.g. [12].

6.3 Special correlation structures

Consider a market model with constant loadings γ_j, which has only a few factors, say $d = 2$ or $d = 3$. Although the norms $|\gamma_j|$ can be easily identified as the implied caplet volatilities, the stable calibration of the components γ_{jk} turns out to be a perennial problem in practice. This stability problem can be explained, at least partially, by an intrinsic problem concerning the correlation structure of any low factor model. To see this, we consider again the correlation structure (23) of the forward LIBORS at a fixed tenor, say at T_1,

$$\rho[\ln \frac{L_l(T_1)}{L_l(t)}, \ln \frac{L_{l'}(T_1)}{L_{l'}(t)}] = \frac{\gamma_l \cdot \gamma_{l'}}{|\gamma_l||\gamma_{l'}|}.$$

It is observed in practice that for a fixed l the correlation decays more or less like a negative power of l' or maybe even like an negative exponential when l'; $l' \geq l$ increases. Besides, it is observed that for fixed p the correlation between L_l and L_{l+p} *increases* when l increases. Now, in particular, the kind of *decay behaviour* is actually not consistent with the decay behaviour resulting from a two or three factor market model, where the number of Brownian motions is two or three. In the later models the correlations are inclined to decay more or less like a *cosine* function of $l - l'$, due to the low number of factors, respectively the low rank of the covariance matrix Δ. This intrinsic problem of any low factor model also discussed in [10] is best illustrated by a very simple example (6.3.1) below and will be a main cause of occurring instability when one tries to calibrate such models to market prices of caps and swaptions simultaneously, as arbitrage free market prices of swaptions will be consistent with market LIBOR correlations.

Example 6.3.1 In a two factor model, $d = 2$, the γ_i can be represented as

$$\gamma_i =: |\gamma_i|(\cos \phi_i, \sin \phi_i)$$

yielding correlations

$$\rho_{ij} = \cos(\phi_i - \phi_j).$$

Now suppose, for instance, that $n = 20$ and that the market tells us the correlations $\rho_{1,j}$ behave like $\rho_{1j} = 18/(17 + j)$, thus falling down from 1 to 0.5. Then, if we calibrate this two-factor model, i.e. the ϕ_i, to these correlations it is easily seen that, as an immediate consequence, the correlations $\rho_{j,19}$ have to be $\rho_{j,19} = \frac{9}{17+j} + \frac{\sqrt{3}}{2}\sqrt{1 - \left(\frac{17}{18} + \frac{j}{18}\right)^{-2}}$, see figure (1). However, the behaviour of the correlations $\rho_{j,19}$ in figure (1) is clearly *not* consistent with their real behaviour in the market which should look more or less the same as $\rho_{1,j}$, mirrored at $j = 10$.

As a solution for this intrinsic low factor calibration problem we propose an alternative market model by the identification of a natural form for the correlation structure which matches the correlation behaviour in practice directly, but, only involves a relatively small number of essential parameters, in fact, the same number as in a two factor market model.

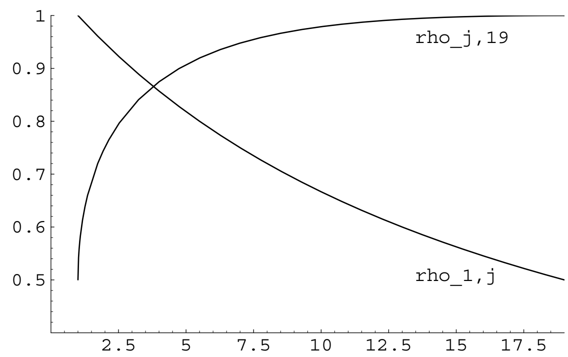

Figure 1:

Assumption 6.3.2 (special correlation structure) *For a sequence* $b = (b_1, .., b_{n-1})$ *with* $|b_l|$ *and* $|b_l/b_{l+p}|$ *nondecreasing in* l, *we postulate a correlation structure of the form,*

$$\frac{\gamma_l \cdot \gamma_{l'}}{|\gamma_l||\gamma_{l'}|} = \frac{b_{l\wedge l'}}{b_{l\vee l'}}.$$

A trivial example for b is $b_l = l$. Under assumption (6.3.2) we have for the log-LIBORS,

$$\rho[\ln \frac{L_l(T_m)}{L_l(t)}, \ln \frac{L_{l'}(T_{m'})}{L_{l'}(t)}] = \frac{b_l}{b_{l'}} \sqrt{\frac{T_m - t}{T_{m'} - t}}; \qquad m \leq m'; \quad l \leq l'; \quad m \leq l; \quad m' \leq l'. \quad (57)$$

The matrix (57) is indeed nonnegative in l, l' and in general of full rank and thus specifies, in fact, a many factor market model. However, the number of degrees of freedom is the same as in a two factor model and example (6.3.3) below shows that a market model based on (57) has much more potential to describe LIBOR correlations realistically. Indeed, because of the extra condition it is also covered that the correlation between L_i and L_{i+p} *increases* with i.

Example 6.3.3 *Consider the increasing sequence* b *with*

$$b_l = \exp{(\beta l^\alpha)},$$

for $\beta > 0$ *and* $0 < \alpha < 1$. *Then, indeed* b_l/b_{l+p} *increases to* 1 *as* $l \to \infty$ *and, e.g. if we take* $n = 20$, $\beta = 0.1$ *and* $\alpha = 0.8$, *we observe realistic behaviour of the functions* $j \to \rho_{i,j}$ *for various* i, *see figure (2).*

Besides, due to this special correlation structure, in several situations such as in the trigger swap formula the involved multi-variate normal probabilities and expectations can be evaluated by faster routines, see [3]. For the calibration of this model we can take the norms $|\gamma_i|$ from the implied cap(let) volatilities and then calibrate the b_i's against a suitable chosen

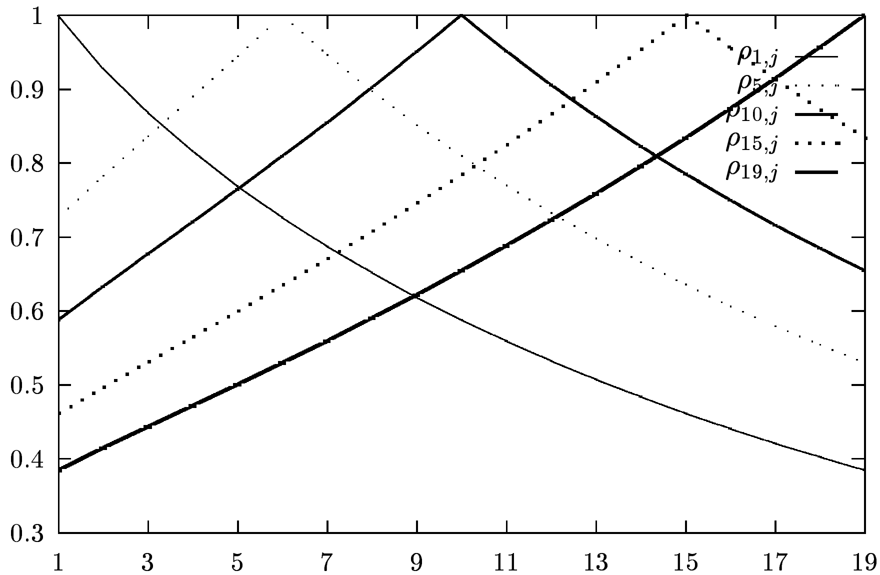

Figure 2: $j \to \rho_{i,j}$; $i = 1, 5, 10, 15, 19$ *for* $b_k = e^{0.1 * k^{0.8}}$

set of swaptions, e.g. by using swaption approximation formulas, Monte Carlo methods or, maybe more practically, by using (56) from the cap/swaption market, yielding

$$S_{p,q}^2 \sigma_{p,q}^2 \approx \sum_{i=p}^{q-1} w_i^2 L_i^2 |\gamma_i|^2 + 2 \sum_{i,j=p,\, i<j}^{q-1} w_i w_j L_i L_j |\gamma_i| |\gamma_j| \frac{b_i}{b_j}. \tag{58}$$

In principle the system (58) is over-determined but we may choose a suitable set of implied swaption volatilities and then solve for the parameters b_i or, alternatively, we may calibrate b as a least square solution of (58). Note finally that the choice of $b = (1, .., 1)$ gives the one factor model again and in this sense we can see the model (57) as an alternative depart from the one factor model, in fact, to a many factor model but with the dimensionality of a two factor model!

7 Simulation experiments and statistical tests

Statistical tests by O. Kurbanmuradov have shown that the distribution of the log-normal LIBOR approximations in the $\mathbb{P}_n$ measure are hardly distinguishable from the LIBOR distribution simulated by true Monte Carlo of the SDE (12). However, LIBOR simulation by the approximate distribution is considerably faster than Monte Carlo simulation of the SDE.

The general swaption formula (31) is tested by Monte Carlo simulation of the log-normal LIBOR approximations with a correlation structure of the type (6.3.3). It turned out that, in contrast to correlation parameters of two or three factor models, the parameters β and

α behave stable with respect to the price of the swaption. Besides, it is shown that the one-factor approximation (36), which corresponds to $\beta = 0$, may differ substantially from the general formula (31) when $\beta > 0$.

In a subsequent paper we will study the calibration of these many factor models with low dimensional correlation structures to the cap/swaption markets in more detail. Also we will improve the Monte Carlo methods by variance reduction techniques such as control variates and importance sampling for SDE's, see e.g. [11].

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