A condition for weak disorder for
directed polymers in random environment

Matthias Birkner

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A condition for weak disorder for directed polymers in random environment

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Abstract

We give a sufficient criterion for the weak disorder regime of directed polymers in random environment, which extends a well-known second moment criterion. We use a stochastic representation of the size-biased law of the partition function.

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We consider the so-called directed polymer in random environment, being defined as follows: Let $p(x, y) = p(y - x)$, $x, y \in \mathbb{Z}^d$ be a shift-invariant, irreducible transition kernel, $(S_n)_{n \in \mathbb{N}_0}$ the corresponding random walk. Let $\xi(x, n), x \in \mathbb{Z}^d, n \in \mathbb{N}$ be i.i.d. random variables satisfying

\[ \mathbb{E}[\exp(\beta \xi(x, n))] < \infty \quad \text{for all } \beta \in \mathbb{R}, \tag{1} \]

we denote their cumulant generating function by

\[ \lambda(\beta) := \log \mathbb{E}[\exp(\beta \xi(x, n))]. \tag{2} \]

We think of the graph of $S_n$ as the (directed) polymer, which is influenced by the random environment generated by the $\xi(x, n)$ through a reweighting of paths with

\[ e_n := e_n(\xi, S) := \exp \left( \sum_{j=1}^n \beta \xi(S_j, j) - \lambda(\beta) \right), \]

that is, we are interested in the random probability measures on path space given by

\[ \mu_n(ds) = \frac{1}{Z_n} \mathbb{E}[e_n 1(S \in ds) | \xi(\cdot, \cdot)], \]

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1Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin
Germany
Email: birkner@wias-berlin.de
where the normalising constant (or partition function) is given by

\[ Z_n = \mathbb{E}[e_n | \xi] = \sum_{s_1, \ldots, s_n \in \mathbb{Z}^d} \prod_{j=1}^n p(s_{j-1}, s_j) \exp \left( \sum_{k=1}^n \beta \xi(s_k, k) - \lambda(\beta) \right). \]

Note that \((Z_n)\) is a martingale, and hence converges almost surely. This model has been studied by many authors, see e.g. [2] and the references given there. It is known that the behaviour of \(\mu_n\) as \(n \to \infty\) depends on whether \(\lim_n Z_n > 0\) or \(\lim_n Z_n = 0\). One speaks of \textit{weak disorder} in the first, and of \textit{strong disorder} in the second case.

Our aim here is to give a condition for the weak disorder regime.

Let \((S_n)\) and \((S'_n)\) be two independent \(p\)-random walks starting from \(S_0 = S'_0 = 0\), and let \(V := \sum_{n=1}^\infty 1(S_n = S'_n)\) be the number of times the two paths meet. Define

\[ \alpha_* := \sup \{ \alpha \geq 1 : \mathbb{E}[\alpha^V | S'_n] < \infty \text{ almost surely} \}. \tag{3} \]

**Proposition 1** If \(\lambda(2\beta) - 2\lambda(\beta) < \log \alpha_*\), then

\[ \lim_{n \to \infty} Z_n > 0 \text{ almost surely,} \]

that is, the directed polymer is in the \textit{weak disorder regime}.

Note that Proposition 1 implicitly requires that the difference random walk \(S - S'\) be transient, for otherwise we would have \(\log \alpha_* = 0\), but we also have \(\lambda(2\beta) - 2\lambda(\beta) \geq 0\) by convexity. For symmetric simple random walk in dimension \(d = 1, 2\) we have \(Z_n \to 0\) almost surely for any \(\beta \neq 0\), see [2], Thm. 2.3 (b).

Observe that

\[ \alpha_* \geq \alpha_2 := \sup \{ \alpha \geq 1 : \mathbb{E}[\alpha^V] < \infty \} = \frac{1}{1 - \mathbb{P}_{(0,0)}(S_n \neq S'_n \text{ for } n \geq 1)}. \]

An easy calculation shows that \((Z_n)\) is an \(L^2\)-bounded martingale iff \(\lambda(2\beta) - 2\lambda(\beta) < \log \alpha_2\), cf. e.g. [2], equation (1.8) and the paragraph below it on p. 707 and the references given there (note that for symmetric simple random walk, \(\mathbb{P}_{(0,0)}(S_n \neq S'_n \text{ for } n \geq 1) = \mathbb{P}_0(S_n \neq 0 \text{ for } n \geq 1) =: q\)).

If \(S - S'\) is transient and \(p\) satisfies

\[ \sup_{n,x} \frac{p_n(x)}{\sum_y p_n(y)p_n(-y)} < \infty \tag{4} \]

then we have

\[ \alpha_* = 1 + \left( \sum_{n=1}^\infty \exp \left( -H(p_n) \right) \right)^{-1} > \alpha_2, \tag{5} \]

where \(p_n(x) := \mathbb{P}_0(S_n = x)\) is the \(n\)-step transition probability of a \(p\)-random walk, and \(H(p_n) = -\sum_x p_n(x) \log(p_n(x))\) is its entropy, see [1], Thm. 5. Note that (4) is automatically satisfied if a local central limit theorem holds for \(p\), in particular, it
holds for symmetric simple random walk. Thus, Proposition 1 is an extension of the second moment condition (1.8) in [2].

Let $\hat{Z}_n$ have the size-biased law of $Z_n$, i.e.

$$\mathbb{E}[f(\hat{Z}_n)] = \mathbb{E}[Z_n f(Z_n)]$$

for any bounded, measurable $f$. The proof of Proposition 1 hinges on the representation of the sie-biased law given in the following lemma.

**Lemma 1** Let $(S'_n)$ be a $p$-random walk starting from $S'_0 = 0$, let $\xi(x, n)$ be as above, and let $\hat{\xi}(x, n)$, $x \in \mathbb{Z}^d$, $n = 1, 2, \ldots$ be an i.i.d. sequence with a tilted law given by

$$\mathbb{E}[f(\hat{\xi})] = e^{-\lambda(\beta)} \mathbb{E}[\exp(\beta \xi)f(\xi)] \text{ for any bounded } f : \mathbb{R}_+ \to \mathbb{R}.$$

Let

$$\tilde{Z}_n := \mathbb{E}\left[ \exp \left( \sum_{j=1}^{n} (1(S_j = S'_j)\hat{\xi}(S_j, j) + 1(S_j \neq S'_j)\xi(S_j, j) - \lambda(\beta)) \right) \right]_{S', \xi(\cdot, \cdot), \hat{\xi}}.$$

Then $\tilde{Z}_n$ and $\hat{Z}_n$ have the same distribution.

**Proof.** Note that $\tilde{Z}_n$ is a function of $S'$, $\xi$ and $\hat{\xi}$, namely

$$\hat{Z}_n = \sum_{s_1, \ldots, s_n \in \mathbb{Z}^d} \prod_{j=1}^{n} p(s_{j-1}, s_j) \times \exp \left( \sum_{j=1}^{n} (1(S_j = S'_j)\hat{\xi}(s_j, j) + 1(S_j \neq S'_j)\xi(s_j, j) - \lambda(\beta)) \right).$$

We have by definition for a bounded $f : \mathbb{R}_+ \to \mathbb{R}$

$$\mathbb{E}[f(\hat{Z}_n)] = \mathbb{E}[Z_n f(Z_n)]$$

for any bounded, measurable $f$. The proof of Proposition 1 hinges on the representation of the sie-biased law given in the following lemma.
Proof of Proposition 1. As \( \mathbb{P}(Z_{\infty} > 0) \in \{0, 1\} \) by Kolmogorov's 0–1 law (see e.g. (1.7) in [2]), the proposition will be proved if we can show that under the given condition, the sequence \( Z_n, n \in \mathbb{N} \) is uniformly integrable. This, in turn, is equivalent to tightness of the sequence \( \tilde{Z}_n \), see e.g. Lemma 9 in [1]. We see from Lemma 1 that this is equivalent to whether the family \( \mathcal{L}(\tilde{Z}_n), n \in \mathbb{N} \), is tight. Let us denote by \( \alpha := \mathbb{E}\exp(\beta \tilde{\xi} - \lambda(\beta)) = \exp(\lambda(2\beta) - 2\lambda(\beta)) \), then

\[
\mathbb{E}[\tilde{Z}_n | S'] = \mathbb{E}\left[ \alpha^{\#\{1 \leq i \leq n : S_i = S'_i\}} | S' \right],
\]

hence \( \alpha < \alpha_* \) implies \( \sup_n \mathbb{E}[\tilde{Z}_n | S'] < \infty \) almost surely, which in particular shows that the family of laws \( \mathcal{L}(\tilde{Z}_n) \) is tight. \( \square \)

Remark. Note that we obtain a sufficient condition for weak disorder by averaging out \( \xi(\cdot, \cdot) \) and \( \tilde{\xi}(\cdot, \cdot) \) in the construction of \( \tilde{Z}_n \) given in Lemma 1. In order to obtain a sharp criterion one would have to analyse the distribution of \( \tilde{Z}_n \) itself. Unfortunately, this seems a rather hard problem.

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