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Closed-loop optimal experiment design: Solution via moment extension

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Abstract

We consider optimal experiment design for parametric prediction error system identification of linear time-invariant multiple-input multiple-output (MIMO) systems in closed-loop when the true system is in the model set. The optimization is performed jointly over the controller and the spectrum of the external excitation, which can be reparametrized as a joint spectral density matrix. We have shown in [18] that the optimal solution consists of first computing a finite set of generalized moments of this spectrum as the solution of a semi-definite program. A second step then consists of constructing a spectrum that matches this finite set of optimal moments and satisfies some constraints due to the particular closed-loop nature of the optimization problem. This problem can be seen as a moment extension problem under constraints. Here we first show that the so-called *central extension* always satisfies these constraints, leading to a constructive procedure for the optimal controller and excitation spectrum. We then show that, using this central extension, one can construct a broader set of parametrized optimal solutions that also satisfy the constraints; the additional degrees of freedom can then be used to achieve additional objectives. Finally, our new solution method for the MIMO case allows us to considerably simplify the proofs given in [18] for the single-input single-output case.

1 Introduction

Optimal experiment design for system identification has seen an intense development in the last decade. This advance was initiated by the appearance of modern convex optimisation methods in the nineties, most notably semi-definite programming. Accordingly, most of the recent work in optimal input design focusses on casting different input design problems as semi-definite programs. Once an optimization problem is available in the standard format of a semi-definite program, it can be solved by commercially or freely available solvers. One of the pioneering contributions introducing semi-definite programming into optimal input design for open loop identification was [25]. For further motivation and an extensive reference list we refer to [20].

However, converting optimisation problems into semi-definite programs is often far from trivial. Sometimes this is due to the NP-hardness of the problem. If a semi-definite description cannot be obtained, one usually tries to relax the problem in order to construct a semi-definite approximation. Often such a relaxation is easily at hand, but nothing about its quality is known.

In this paper we provide an optimal solution to a general class of optimal experiment design problems for the identification of parametric linear time-invariant (LTI) systems operating in closed loop. The degrees of freedom which are relevant for closed-loop experiment design problems are the power spectrum of the external excitation signal fed into the system and the feedback controller transfer function. Both can easily be converted into a joint power spectrum of some signals present in the loop. These spectra are frequency-dependent functions and as such infinite-dimensional objects. Their infinitely many degrees of freedom have to be condensed into a finite-dimensional vector of design variables. A semi-definite description of optimal experiment design problems in this class has for years been elusive.

Two basic approaches to the choice of the design variables can be distinguished in the literature. The first is based on a *finite dimensional approximation* of the joint spectrum, the second, often called *partial* *correlation approach*, is based on expressing the criterion and the constraints as a function of a finite number of linear functionals of the joint spectrum, called *generalized moments*. In both cases, the optimal experiment design problem is then transformed into a semi-definite program expressed in terms of the parameters of the finite dimensional approximation for the first approach, and the generalized moments for the second approach.

In [19] the finite dimensional approximation approach was used. A solution was obtained by first parametrizing the joint spectrum mentioned above using a Youla-Kucera parametrization to constrain the solution set to deliver a stabilizing closed loop controller, and then using a finite dimensional approximation of this joint spectrum. The finite set of design variables are obtained as the coefficients of a truncated series development of the input power spectrum and of the Youla parameter. The optimal design problem is then reduced to a convex optimization problem under linear matrix inequality (LMI) constraints over the coefficients of this finite dimensional approximation. Given that the solution space is restricted by the finite dimensional approximation, it leads to a suboptimal solution.

In [18] we provided an optimal solution based on the partial correlation approach. Our solution applies to a wide class of optimal design problems in which the criterion and the constraints are expressed as integral functions over the frequency range.

In this framework the criterion and the constraints can be expressed as linear functions of *a finite set of $n+1$ generalized moments*, which are linear functionals of the joint power spectrum. They become the design variables of the optimal design problem. The conditions on the vector of design variables to correspond to a realizable experiment design are then shown to be equivalent to the satisfaction of an LMI, possibly involving additional auxiliary variables. The optimal moment sequence is then obtained by solving a standard semi-definite program. Geometrically, the optimization is performed over a finite-dimensional *projection* of the infinite-dimensional cone of possible joint power spectra. The optimal finite moment sequence will then in general correspond to an infinite set of spectral density matrices rather than a single spectrum, and every possible spectrum is represented by some point in the cone generated by the finite set of optimal moments, thus resulting in a truly optimal solution.

The construction of a spectrum or a set of spectra whose first $n+1$ generalized moments coincide with the optimal moments that solve the semi-definite program is known as the *Carathéodory extension problem*. The case of scalar-valued moments has been well studied in the last century [8], [30], [2], [24], [21], [1]. The scalar theory can be generalized to the case of matrix-valued moments [27], [28], [3], [23], [11], [12]. The key result for solving the Carathéodory extension problem is the Carathéodory-Fejer theorem. This theorem implies that a given finite sequence of moments is indeed generated by a positive power spectrum if and only if it satisfies a certain LMI [22, Chapter VI, Theorem 4.1]. Such a spectrum can be represented in a number of equivalent ways. This includes the representation as a matrix-valued positive semi-definite measure on the unit circle, as an infinite sequence of moments, or as a Carathéodory function, i.e., a matrix-valued holomorphic function defined on the open unit disc whose Hermitian part is positive semi-definite. The representations can easily be transformed in one another [27, Section II].

The set of all possible infinite extensions of a finite moment sequence may be parametrized by an infinite sequence of complex numbers in the unit disc (in the scalar case) or complex contractive matrices (in the matrix case) [11, Theorem 1]. Here the first k matrices in the sequence define the first k undetermined moments of the extension, i.e., the first k moments which follow the $n+1$ moments given by the solution of the semi-definite program. In this way, fixing the contractive matrices one by one, the user can consecutively construct all moments of the extension. These matrices hence represent a *choice sequence*. The contractive matrices can be defined in different ways and carry different names, e.g., *Schur param-*

eters, *Szegö parameters*, *reflection parameters*, or *canonical moments* [1], [28], [27], [4]. In [10] it was shown that they are all essentially identical to the *Verblunsky coefficients*, see also [9] and [29, p.30] for a discussion.

The particular extension corresponding to the case when all Verblunsky coefficients vanish is called *central extension* [11], [12], [31, Section 3.6], and the measure on the unit circle which defines the corresponding positive semi-definite spectrum is called *central measure* [4, Remark 8.4, p.104]. In [11] it was shown that this measure can be characterized as the solution of an entropy minimization problem. In the scalar case this approach has been used in [7] to characterize all extensions with the same degree as the central extension. In [6] these results have been generalized to the matrix-valued case. If a non-degeneracy condition is satisfied, then the power spectrum defined by the central measure can be expressed in closed-form as a rational function with coefficients depending in an explicit manner on the problem data, i.e., on the optimal truncated moment sequence [27], [31].

A more compact way to parametrize the set of all possible extensions of a given finite moment sequence is via the representation of the extensions as Carathéodory functions. The set of all such functions which can be obtained from the finite moment sequence is given by a linear-fractional transformation (LFT) of a single parameter. This parameter takes values in the *Schur class*, i.e., the set of all holomorphic matrix-valued functions on the open unit disc which are contractive. The coefficients of the LFT depend explicitly on the problem data, i.e., the original finite moment sequence [5, Theorem 1.1]. The central extension corresponds to the case when the Schur function is identically zero. The Carathéodory function corresponding to the central measure is hence a rational function with coefficients depending explicitly on the problem data [13], [5, Theorem 1.3]. If this function is continuously extendible to the closed unit disc, then the power spectrum defined by the central measure is also rational.

The classical Carathéodory-Fejer theorem holds *only* if no restrictions are imposed on the spectrum other than to produce the truncated sequence of moments under consideration, and positivity. In other words, a finite sequence of moments can be extended to an infinite sequence of moments of a positive spectrum if and only if it satisfies the LMI condition, but no additional constraint on the moments of this extension can be guaranteed to be satisfied. However, in closed-loop optimal experiment design, where the controller is part of the design variables, constraints have to be imposed on the matrix-valued joint power spectrum under consideration. These constraints reflect the fact that the controller must produce a stable closed loop, and that the signals defining the joint power spectrum are not all part of the design variables, which implies that some elements of the joint spectrum are fixed. The constraints on the joint power spectrum translate into additional constraints on the infinite moment extensions in order for these extensions to define an admissible spectrum.

In [18] we have shown that the Carathéodory-Fejer theorem *also* holds for the type of structured generalized moment problem arising in closed-loop optimal optimal experiment design. Namely, if a finite sequence of moments satisfies the additional stability constraints, then the LMI condition given by the Carathéodory-Fejer theorem not only insures the existence of a general extension of this moment sequence, but the existence of an extension which also satisfies the constraints.

The proof of this main result in [18] had several drawbacks. First it was written for single-input single-output (SISO) systems, even though an extension to multiple-input multiple-output (MIMO) is easily obtained. More importantly, it proved the *existence* of an extension that satisfies the constraints on the joint spectrum, but it was not constructive. Finally, the proof was very long and complicated, as it relied on the partial positive definite matrix completion theorem from [16], which itself required to appeal to graph-theoretical properties of the Töplitz matrix made up of the generalized moments.

The present paper makes progress in several directions with respect to [18]. First we allow the system to have multiple inputs and outputs. Our main contribution is to show that the stability constraints are satisfied by the central extension, which under a non-degeneracy condition can be explicitly computed from the set of $n+1$ optimal moments. The central extension defines a unique power spectrum, which solves the optimal experiment design problem. Thus once the optimal truncated moment sequence has been obtained by solving the semi-definite program, an optimal joint power spectrum can be immediately written down in closed form, shortcutting the somewhat ad hoc and complicated recovery step in [18].

Our second main contribution is to show that the set of all extensions which satisfy the additional constraints on the joint power spectrum can also be parametrized by a choice sequence of contractive matrices. These matrices have a smaller size than the Verblunsky coefficients, because at each step, a part of the degrees of freedom given by the Verblunsky coefficient is fixed by the additional constraint on the corresponding moment. We may call these contractive matrices *restricted Verblunsky coefficients*. The central extension corresponds to the case when all restricted Verblunsky coefficients vanish. This result allows one to generate a finite-dimensional, explicitly parametrized family of optimal solutions by first fixing a finite number of restricted Verblunsky coefficients, constructing the corresponding finite moment extension, and then using the central extension of this already finitely extended moment sequence. In the simplest case one would extend the $n+1$ optimal moments with a family of an $(n+2)$-nd moment, parametrized by the corresponding restricted Verblunsky coefficient. The resulting $(n+2)$-tuples of moments then also satisfy the stability constraints. Computing the central extension for this extended family yields a parametrized family of admissible optimal spectra. This procedure can be repeated one step at a time, yielding a doubly infinite family of admissible optimal spectra, etc. These additional degrees of freedom can be used to satisfy additional performance criteria, constraints, or robustness properties that the user may want to inject into the problem.

Feasibility of the central extension actually implies the validity of the Carathéodory-Fejer theorem for the structured generalized moment problem. This allows us to significantly shorten the proof of this result given in [18]. For this reason, and in order to make the present contribution self-contained, we also provide the new proof of the structured Carathéodory-Fejer theorem here.

The remainder of the paper is organized as follows. In the next section we define the class of input design problems to be solved. In Section 3 we introduce the concepts of central extensions, central measures, Carathéodory functions and Verblunsky coefficients. Our main result is in Section 4, where we show the feasibility of the central extension for optimal closed-loop experiment design and parametrize the set of all feasible solutions by the choice sequence of restricted Verblunsky coefficients. In Section 5 we present a complete solution algorithm for the proposed class of problems, including a semi-definite description of the feasible set of truncated moment sequences. In Section 6 we illustrate via an example that even in the case where the Töplitz matrix made up of the $n+1$ optimal moments is singular, the central extension may produce an optimal spectrum that remains finite. In the Appendix we provide auxiliary results on a special case of the partial positive matrix completion problem.

2 Problem formulation

In this section we define the class of optimal experiment design problems treated in this paper. We intend to perform parametric prediction error identification of a MIMO LTI system in closed loop. The system dynamics is given by the relation

$$y = G_0(q)u + H_0(q)e, \tag{1}$$

where the signal u is of dimension m, and e, y are of dimension p. Here G_0 is the plant transfer function matrix, H_0 the noise transfer function matrix, q the forward-shift operator, e a vector-valued zero mean white noise with (co-)variance $\lambda_0 I_p$, I_k being the $k \times k$ identity matrix, u is the input vector, and y is the output vector of the system[1]. The transfer function matrices $G_0(z), H_0(z)$ are embedded in a model structure $G(z;\theta), H(z;\theta)$ and correspond to some true parameter value θ_0, $G_0(z) = G(z;\theta_0)$, $H_0(z) = H(z;\theta_0)$. We assume that the plant transfer function G_0 is stable, and the noise model H_0 is stable and inversely stable.

The parameter vector θ_0 is to be identified by an experiment, which consists in closing the loop according to the relation

$$u = -K(q)y + r, \tag{2}$$

where r is a quasistationary process of dimension m, and collecting a set of input-output data u, y. The design variables at our disposal are thus the power spectrum $\Phi_r(\omega)$ of the external vector-valued input signal r and the $m \times p$ matrix-valued feedback controller $K(q)$. The configuration of the identification experiment is schematically depicted in Fig. 1. The estimator $\hat{\theta}$ of the true parameter value θ_0 is then evaluated as the minimizer of some prediction error criterion. Our goal is to design an experiment by choosing the spectrum of an external input r and a controller K such that some cost function of Φ_r, K is minimized and some constraints on the pair (Φ_r, K) are satisfied.

Following [19], we first move from the quantities Φ_r, K to the spectra Φ_u, Φ_{ue}, which, as long as we work in the frequency domain and use formulas that are asymptotic in the number of data, yield an equivalent description of the experimental conditions. The power spectrum Φ_r of r and the controller K determine Φ_u, Φ_{ue} by the formulas

$$\begin{aligned}
\Phi_u(\omega) &= \lambda_0 (I_m + KG_0)^{-1} K H_0 H_0^* K^* (I_m + KG_0)^{-*} \\
&\quad + (I_m + KG_0)^{-1} \Phi_r(\omega) (I_m + KG_0)^{-*}, &(3)\\
\Phi_{ue}(\omega) &= -\lambda_0 (I_m + KG_0)^{-1} K H_0, &(4)
\end{aligned}$$

where the transfer functions on the right-hand side are evaluated at $z = e^{j\omega}$. By A^* we denote the complex conjugate transpose of the matrix A and by A^{-*} the inverse of A^*. On the other hand, Φ_r and K can be recovered from Φ_u, Φ_{ue} by the formulas

$$\begin{aligned}
\Phi_r &= (I_m + KG_0)(\Phi_u - \lambda_0^{-1} \Phi_{ue} \Phi_{ue}^*)(I_m + KG_0)^*, \\
K &= -\Phi_{ue} (\lambda_0 H_0 + G_0 \Phi_{ue})^{-1}. &(5)
\end{aligned}$$

Thus there is a one-to-one relationship between (Φ_r, K) and (Φ_u, Φ_{ue}). Parametrizing the experimental conditions by the joint power spectrum

$$\Phi_{\chi_0} = \begin{pmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{ue}^* & \lambda_0 I_p \end{pmatrix} \tag{6}$$

of the signals u, e instead of the quantities Φ_r, K has the advantage that the feasible set becomes convex, which is a prerequisite for a semi-definite representation [19]. The matrix Φ_{χ_0} is of size $(m + p) \times (m + p)$.

Within the framework of the partial correlation approach, the ultimate design variables are a finite set of moments of the joint power spectrum Φ_{χ_0}. Accordingly, the cost criterion and the constraints of the optimal input design problem have to be expressible in a tractable manner in terms of these moments. Apart from this compatibility requirement, we do not impose any condition on the cost criterion and the constraints.

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[1] For simplicity, we have assumed a white noise (co-)variance $\lambda_0 I_p$; however, our results apply equally well for any symmetric positive definite (co-)variance matrix Σ.

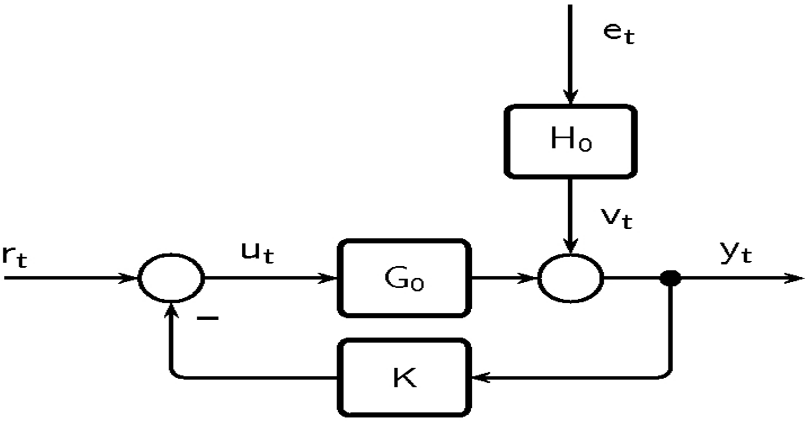

Figure 1: Experimental setup

Assumption 1. *There exist integers $N \geq 0$, $n \geq s \geq 0$ and a polynomial $d(z) = \sum_{l=0}^{s} d_l z^l$ of degree s with the following properties. The coefficients d_l are real, obey $d_0 \neq 0$, $d_s \neq 0$, and the polynomial $d(z)$ has all roots outside the closed unit disk. Define $(m+p) \times (m+p)$ matrices*

$$m_k = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{1}{|d(e^{j\omega})|^2} \Phi_{\chi 0}(\omega) e^{jk\omega} \, d\omega \tag{7}$$

for integral k. Then the constraints of the input design problem can be written as a linear matrix inequality

$$\exists \, x_1, x_2, \dots, x_N : \quad \mathcal{A}(m_0, m_1, \dots, m_n, x_1, x_2, \dots, x_N) \succeq 0 \tag{8}$$

in the elements of the $n+1$ matrices m_k, $k = 0, \dots, n$, and N additional auxiliary variables x_l, $l = 1, \dots, N$, and the cost function of the input design problem is given by a linear function

$$f_0(m_0, m_1, \dots, m_n, x_1, x_2, \dots, x_N) = \sum_{k=0}^{n} \langle C_k, m_k \rangle + \sum_{l=1}^{N} c_l x_l, \tag{9}$$

where C_k are fixed matrices, and c_l are fixed reals.

Here $\langle A, B \rangle = trace(AB^T)$ is the usual scalar product in the space of matrices. The matrices m_k defined by (7) are called the generalized moments of the spectrum $\Phi_{\chi 0}$. Note that the moments m_k are real and obey the relation $m_k = m_{-k}^T$.

In [17],[18] we presented a semi-definite description of the set of finite moment sequences $(m_0, \dots, m_n)$ corresponding to valid experiment designs. This allows to obtain the optimal truncated moment sequence $(m_0, \dots, m_n)$ by solving a semi-definite program.

Under some mild assumptions the asymptotic in the number of data average per data sample information matrix of the experiment is given by [26]

$$\overline{M} = \frac{1}{2\pi\lambda_0} \sum_{k=1}^{p} \int_{-\pi}^{+\pi} F_k(e^{j\omega}) \Phi_{\chi 0}(\omega) F_k^*(e^{j\omega}) \, d\omega, \tag{10}$$

where the l-th row of the matrix F_k is given by the k-th row of the matrix $[H_0^{-1} G'_{\theta^l}(\theta_0), \; H_0^{-1} H'_{\theta^l}(\theta_0)]$. Here G'_{θ^l}, H'_{θ^l} denote the gradients of $G(z;\theta), H(z;\theta)$ with respect to the l-th entry of the parameter vector θ. If the model structure is rational, then (10) is affine in the moment matrices $m_0, m_1, \dots, m_n$ for a suitably chosen polynomial $d(z)$. In addition, most experiment design criteria are formulated as scalar functions of $\overline{M}$. Therefore, Assumption 1 covers a wide variety of problem formulations in closed-loop optimal experiment design, see also [25],[20],[19]. In particular, all classical designs (D-optimal, A-optimal, L-optimal etc.) subject to variance constraints on the signals fall within the framework of Assumption 1.

3 Central extensions

In this section we introduce the concept of moment extensions, and in particular, central extensions. Before we focus on the generalized moments (7) of the structured power spectrum (6), we will first consider the case of moment sequences of general power spectra. First we shall consider different ways to represent a positive semi-definite power spectrum in Subsection 3.1. Then the set of all possible moment extensions and its parametrizations is considered in Subsection 3.2. In Subsection 3.3 we introduce the central extension, which is a particular moment extension, under the assumption of a certain non-degeneracy condition. Finally, we consider the central extension in the general case in Subsection 3.4.

3.1 Representations of power spectra

Let $\Phi(\omega)$ be an integrable 2π-periodic matrix-valued complex-Hermitian positive semi-definite function of size $l \times l$, possibly containing a singular part consisting of Dirac δ-functions. Define the *moments* of Φ by

$$m_k = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Phi(\omega) e^{jk\omega}\, d\omega. \tag{11}$$

Note that $m_{-k} = m_k^*$. Then the block-Töplitz matrices

$$T_k = \begin{pmatrix} m_0 & m_1^* & \ddots & m_{k-1}^* & m_k^* \\ m_1 & m_0 & \ddots & m_{k-2}^* & m_{k-1}^* \\ \ddots & \ddots & \ddots & & \\ m_k & m_{k-1} & \ddots & m_1 & m_0 \end{pmatrix} \tag{12}$$

are positive semi-definite for all $k \geq 0$. On the other hand, given an infinite sequence of matrices m_k, $k \in \mathbb{Z}$, satisfying $m_{-k} = m_k^*$ and such that all block-Töplitz matrices T_k, $k \geq 0$, are positive semi-definite, there exists a unique positive semi-definite function $\Phi(\omega)$ producing the matrices m_k as in (11) [27, Theorem 1]. Note that if $\Phi(-\omega) = \Phi(\omega)^T$, then all moments m_k are real, and the complex conjugate transpose in (12) becomes the ordinary transpose.

There exist other representations of the function $\Phi(\omega)$ than by its infinite moment sequence. One of these is the *Carathéodory function*

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\omega} + z}{e^{j\omega} - z} \Phi(\omega) d\omega, \tag{13}$$

which is an analytic function defined on the open unit disc such that its Hermitian part $\frac{1}{2}(F(z) + F^*(z))$

is positive semi-definite and $F(0)$ is Hermitian. The spectrum can be recovered from F as the limit

$$\Phi(\omega) = \lim_{r \to 1-} \frac{1}{2}(F(re^{j\omega}) + F^*(re^{j\omega})). \tag{14}$$

If Φ has a singular part, then the limit has to be understood in the sense of a distribution [27, Section II]. The Carathéodory function $F(z)$ can be also determined from the moment sequence by the Taylor expansion $F(z) = m_0 + 2\sum_{k=1}^{\infty} m_{-k} z^k$.

3.2 Moment extensions

An obvious necessary condition for a *finite* sequence $m_0, \dots, m_n$ of $l \times l$ matrices to be extendable to an infinite sequence $m_0, \dots, m_n, m_{n+1}, \dots$ which can be obtained from some positive semi-definite function Φ by formula (11) is that the block-Töplitz matrix T_n is positive semi-definite, $T_n \succeq 0$. The Carathéodory-Fejer theorem (see, e.g., [22, Chapter VI, Theorem 4.1]) states that this is also a sufficient condition. We shall call such infinite sequences $m_0, \dots, m_n, m_{n+1}, \dots$ an (infinite) *extension* of the finite sequence $m_0, \dots, m_n$. Since the condition $T_k \succeq 0$ implies $T_{k'} \succeq 0$ for all $k' \leq k$, it makes also sense to speak of extensions by a finite number $m_{n+1}, \dots, m_{n'}$ of matrices. The sequence $m_0, \dots, m_n, m_{n+1}, \dots, m_{n'}$ is a finite extension of the sequence $m_0, \dots, m_n$ if and only if $T_{n'} \succeq 0$.

We first parameterize all extensions of the finite sequence $m_0, \dots, m_n$ by *one* additional matrix m_{n+1}. We have the following result, where we comment that $m_{-k} = m_k^T$ for all k.

Theorem 1. *Let $m_0, \dots, m_n$ be a sequence of real $l \times l$ matrices such that the block-Töplitz matrix T_n defined by* (12) *is positive semi-definite. Then the $l \times l$ matrix m_{n+1} extends the sequence $m_0, \dots, m_n$ in such a way that $T_{n+1} \succeq 0$ if and only if it can be written as*

$$m_{n+1} = \begin{pmatrix} m_{-n} \\ \vdots \\ m_{-1} \end{pmatrix}^T T_{n-1}^{\dagger} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} + \left(m_0 - \begin{pmatrix} m_{-n} \\ \vdots \\ m_{-1} \end{pmatrix}^T T_{n-1}^{\dagger} \begin{pmatrix} m_{-n} \\ \vdots \\ m_{-1} \end{pmatrix} \right)^{1/2} \Delta_{n+1} \left(m_0 - \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}^T T_{n-1}^{\dagger} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \right)^{1/2}$$

with Δ_{n+1} a real $l \times l$ matrix satisfying $\sigma_{\max}(\Delta_{n+1}) \leq 1$, where $T_{n-1}^{\dagger}$ denotes the pseudo-inverse of T_{n-1}.

Proof. The matrices $m_0, \dots, m_n$ partially specify the entries of the block-Töplitz matrix T_{n+1}. By the condition $T_n \succeq 0$ this partially specified matrix is partial positive semi-definite. The claim of the theorem now follows by application of Lemma 2 in the Appendix. $\square$

In the complex case Theorem 1 is equivalent to [31, Theorem 3.4.1] or [4, Theorem 2.11b]. The contractive matrix Δ_{n+1} will be called *Verblunsky coefficient* [9]. It has been shown in [10] that up to a possible sign change it is equal to the Schur or Szegö parameters, which are contractive matrices defined in a different way [1], [27], [28].

A longer extension $m_0, \dots, m_{n'}$ of the sequence $m_0, \dots, m_n$ can be obtained step by step. We proceed by first choosing a contractive matrix Δ_{n+1} and calculating the next moment m_{n+1} from it. Then we choose a matrix Δ_{n+2} and compute m_{n+2}. Note that m_{n+2} then depends also on Δ_{n+1} via its dependence on m_{n+1}. Then we choose Δ_{n+3} and so on, until the final choice of $\Delta_{n'}$ which determines the last moment matrix $m_{n'}$ of the extension. In this way, all extensions $m_0, \dots, m_{n'}$ can be parametrized by $n' - n$ contractive $l \times l$ matrices Δ_k, $k = n+1, \dots, n'$. In the same way, an infinite extension is determined by an infinite sequence of matrices $\Delta_{n+1}, \Delta_{n+2}, \dots$, and the set of all such extensions is parametrized by all such sequences. Note, however, that in the case when the block-Töplitz matrices T_k are degenerate different choices of the matrices Δ_k can lead to the same extension. In the extreme case, all sequences of Δ_k lead to the same, unique, extension. This happens if and only if the resulting spectrum $\Phi(\omega)$ is discrete [5, Theorem 6.7].

A more compact way to parameterize the set of all extensions of a finite sequence $m_0, \dots, m_n$ is via the Carathéodory function (13). In order to formulate this result, we need a couple of definitions. Let the positive semi-definite $l \times l$ matrices L, R be given by

$$L = \left(m_0 - \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}^T T_{n-1}^\dagger \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \right)^{1/2}, \quad R = \left(m_0 - \begin{pmatrix} m_{-n} \\ \vdots \\ m_{-1} \end{pmatrix}^T T_{n-1}^\dagger \begin{pmatrix} m_{-n} \\ \vdots \\ m_{-1} \end{pmatrix} \right)^{1/2}.$$

For $k \geq 1$, define the $l \times (k+1)l$ matrix-valued polynomial

$$U_k(z) = \begin{pmatrix} z^k I_l & z^{k-1} I_l & \cdots & I_l \end{pmatrix} \tag{15}$$

and the lower-triangular block-Töplitz matrix

$$S_k = \begin{pmatrix} m_0 & 0 & \dots & 0 \\ 2m_1 & m_0 & & 0 \\ \vdots & & \ddots & 0 \\ 2m_k & \dots & 2m_1 & m_0 \end{pmatrix}.$$

Note that $T_k = \frac{1}{2}(S_k + S_k^*)$. Let the polynomials a_n, b_n, c_n, d_n be given by

$$\begin{aligned} a_n(z) &= m_0 + z U_{n-1}(z) S_{n-1} T_{n-1}^\dagger \begin{pmatrix} m_n & \dots & m_1 \end{pmatrix}^*, \\ b_n(z) &= I_l - z U_{n-1}(z) T_{n-1}^\dagger \begin{pmatrix} m_n & \dots & m_1 \end{pmatrix}^*, \\ c_n(z) &= m_0 + z^n \begin{pmatrix} m_{-1} & \dots & m_{-n} \end{pmatrix} T_{n-1}^\dagger S_{n-1} U_{n-1}^T(z^{-1}), \\ d_n(z) &= I_l - z^n \begin{pmatrix} m_{-1} & \dots & m_{-n} \end{pmatrix} T_{n-1}^\dagger U_{n-1}^T(z^{-1}). \end{aligned} \tag{16}$$

These are formally polynomials of degree n. For a polynomial $f(z)$ which is formally of degree n, define the reciprocal polynomial $\tilde{f}^{[n]}(z) = z^n f^*(1/\bar{z})$.

Proposition 1. *[5, Theorem 1.1] Let $m_0, \dots, m_n$ be a finite sequence of $l \times l$ matrices such that the block-Töplitz matrix* (12) *satisfies $T_n \succeq 0$. Then the Carathéodory function* (13) *obtained from an infinite extension of the sequence $m_0, \dots, m_n$ has the general form*

$$\begin{aligned} F(z) &= \left(a_n(z) - z \tilde{c}_n^{[n]}(z) L^\dagger \phi(z) R \right) \left(b_n(z) + z \tilde{d}_n^{[n]}(z) L^\dagger \phi(z) R \right)^{-1} \\ &= \left(d_n(z) + z L \phi(z) R^\dagger \tilde{b}_n^{[n]}(z) \right)^{-1} \left(c_n(z) - z L \phi(z) R^\dagger \tilde{a}_n^{[n]}(z) \right), \end{aligned}$$

where $\phi(z)$ is an arbitrary Schur function of size $l \times l$, i.e., an analytic function on the open unit disc which is contractive. Moreover, the denominator matrices are invertible. □

The function $F(z)$ is hence a matrix-valued LFT of the Schur function $\phi(z)$, with coefficients given by polynomials which are explicit functions of the moments $m_0, \dots, m_n$. For a given Schur function ϕ, the spectrum $\Phi(\omega)$ can be recovered from F by the limit (14).

3.3 Central extension in the regular case

In this subsection we introduce a special moment extension, the *central extension*. Let $m_0,\dots,m_n$ be a finite sequence of $l \times l$ matrices. Following [27], in this subsection we consider only the case when the matrix T_n constructed from this sequence is positive definite, $T_n \succ 0$. We return to the general case $T_n \succeq 0$ in the next subsection.

Following [27], define the $l \times l$ matrix-valued polynomial

$$A_n(z) = U_n(z)T_n^{-1}U_n^T(0) = \sum_{k=0}^{n} A_n^k z^k.$$

The matrix coefficient A_n^k of z^k is given by the $(n+1-k, n+1)$-th $l \times l$ block of the inverse T_n^{-1}. Note also that $A_n(0) = A_n^0$ is positive definite.

Define the $l \times l$ matrix-valued function

$$\Phi(\omega) = A_n(e^{j\omega})^{-*}A_n(0)A_n(e^{j\omega})^{-1}. \tag{17}$$

Note that Φ is rational when considered as a function of $z = e^{j\omega}$ on the unit circle. The order of the components in the matrix U_n in (15) differs from that in [27, eq. (9)] because the definition (11) is different from [27, eq. (7)]. By [27, Theorem 6] the polynomial $A_n(z)$ has no zeros in the closed unit disk, by [27, Theorem 3] the function Φ is positive definite at all ω, and by [27, Theorem 9] the matrices $m_0,\dots,m_n$ are the first $n+1$ moments of Φ.

Let $m_{n+1}, m_{n+2}, \dots$ denote the subsequent moments of Φ, defined as in (11). Then the infinite sequence $m_0, m_1, \dots, m_n, m_{n+1}, \dots$ is an extension of the original finite sequence $m_0,\dots,m_n$. This extension is called the *central extension*. If the matrices $m_0,\dots,m_n$ are real, then the coefficients A_n^k are also real, and $\Phi(-\omega) = \Phi(\omega)^T$. In this case all moments of the central extension will be real. By [27, Theorem 9] the central extension of the sequence $m_0,\dots,m_n,m_{n+1},\dots,m_{n'}$ coincides with the central extension of $m_0, m_1, \dots, m_n$ for every $n' \geq n$.

The advantage of the central extension is that the corresponding spectrum has the comparatively simple explicit expression (17) as a function of the moments $m_0,\dots,m_n$, and it is given by a rational function. However, this holds only if the non-degeneracy condition $T_n \succ 0$ is satisfied. In the next subsection we will consider a generalization to the case of positive semi-definite matrices T_n.

3.4 Central extension in the general case

In Subsection 3.2 we have seen that in the regular case every extension of a finite sequence $m_0,\dots,m_n$ is determined by the choice of a sequence of contractive $l \times l$ matrices $\Delta_{n+1}, \Delta_{n+2}, \dots$. In [31, Section 3.6] it has been shown that the central extension, as defined in the previous subsection, corresponds to a specific choice of these matrices, namely $\Delta_k = 0$ for all $k \geq n+1$.

One might then *define* the central extension in the case of a singular matrix T_n by the relation $\Delta_k = 0$, $k \geq n+1$ [4, Def. 2.12]. However, in this case the central measure $\Phi(\omega)$ does not have the nice representation (17) anymore. Nevertheless, one can still give a closed-form expression for the Carathéodory function (13) defined by the central measure.

Proposition 2. *[13, Prop. 2.2, Theorem 2.3], [5, Theorem 1.3] Let $m_0,\dots,m_n$ be a finite sequence of $l \times l$ matrices such that the block-Töplitz matrix* (12) *satisfies $T_n \succeq 0$. Then the Carathéodory function*

(13) *obtained from the central extension of the sequence* $m_0, \dots, m_n$ *is given by the rational functions*

$$F(z) = a_n(z)b_n^{-1}(z) = d_n^{-1}(z)c_n(z),$$

where a_n, b_n, c_n, d_n *are the polynomials defined in* (16). ☐

The central measure can then be recovered from the Carathéodory function $F(z)$ by the limit (14). If the rational function F has poles on the unit circle, then the corresponding spectrum Φ might have a singular part, and the limit is to be considered in the sense of a distribution. Otherwise Φ is just the restriction of the Hermitian part of F on the unit circle and is also rational.

4 Moment extensions for closed-loop experiment design

In this section we return to our optimal closed-loop experiment design problem described in Assumption 1. In Subsection 4.1 we describe the constraints on the infinite generalized moment sequence $m_0, \dots, m_n, \dots$ which result from the particular structure (6) of the joint spectrum and the constraint (4) on Φ_{ue}. We show that these constraints impose linear relations between s successive moments, where s is the degree of $d(z)$. In Subsection 4.2 we determine necessary and sufficient conditions such that a finite moment sequence $m_0, \dots, m_n$ is extendable to an infinite moment sequence satisfying these specific constraints. We do this by showing that the central extension is a suitable infinite extension. In particular, we can use the central extension of the truncated moment sequence $(m_0, \dots, m_n)$ to recover the joint power spectrum (6) which realizes the sequence according to formula (7). In Subsection 4.3 we parameterize all infinite extensions corresponding to valid experiment designs by a choice sequence of *restricted Verblunsky coefficients*. The central extension corresponds to the case when all restricted Verblunsky coefficients are zero.

Throughout this section, the moments $m_0, \dots, m_n, \dots$ are defined by formula (7). This means that the m_k are the *generalized* moments of the joint power spectrum $\Phi_{\chi 0}$. Since in Section 3 the moments have been defined by formula (11), the power spectrum $\Phi(\omega)$ from this section has to be identified with the quotient $\frac{1}{|d(e^{j\omega})|^2}\Phi_{\chi 0}(\omega)$.

4.1 Structure of the infinite moment sequence

In this subsection we deduce linear relations between the moments $m_0 = m_0^T, m_1, \dots, m_n, \dots$ from the particular structure of the power spectrum Φ_{χ_0} in (7). Set $m_{-k} = m_k^T$ and partition the $l \times l$ matrix moments m_k into 4 blocks $m_{k,11}, m_{k,12}, m_{k,21}, m_{k,22}$, according to the partition of $\mathbb{R}^l$ into a sum $\mathbb{R}^m \oplus \mathbb{R}^p$. The moment matrices m_k depend on the spectra Φ_u, Φ_{ue}, which in turn determine the experimental conditions. However, as a result of the constraints (3), (4) and (6), not all pairs (Φ_u, Φ_{ue}), and hence not all sequences $(m_0, \dots, m_n, \dots)$, correspond to valid experiment designs.

From (7) it follows that

$$m_{k,22} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega \tag{18}$$

for all $k \in \mathbb{Z}$. The positivity of the joint power spectrum Φ_{χ_0} implies by the Carathéodory-Fejer theorem that the block-Töplitz matrix

$$T_k = \begin{pmatrix} m_0 & m_1^T & \ddots & m_{k-1}^T & m_k^T \\ m_1 & m_0 & \ddots & m_{k-2}^T & m_{k-1}^T \\ \ddots & \ddots & \ddots & & \\ m_k & m_{k-1} & \ddots & m_1 & m_0 \end{pmatrix} \tag{19}$$

is positive semi-definite for all $k \geq 0$. Further, the transfer functions from the signals r, e to the signals u, y are stable. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle. Then the function $f_{ue} : \mathbb{T} \to \mathbb{C}^{m \times p}$, defined by the cross spectrum Φ_{ue} by means of $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$, can be extended to a holomorphic function outside of the unit disc, including the point at infinity (compare also [19]). From

$$m_{k,12} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{1}{d(e^{j\omega})} \frac{\Phi_{ue}(\omega)}{d(e^{-j\omega})} e^{jk\omega} \, d\omega$$

it follows that

$$\sum_{i=0}^{s} d_i m_{k+i,12} = \frac{1}{2\pi j} \int_{\mathbb{T}} \frac{f_{ue}(z)}{d(z^{-1})} z^{k-1} \, dz.$$

Since all zeros of $d(z^{-1})$ are in the open unit disc, the ratio $f_{ue}(z)/d(z^{-1})$ is also holomorphic outside of the unit disc. It follows that $\sum_{i=0}^{s} d_i m_{k+i,12} = 0$ for all $k < 0$, and hence

$$\sum_{i=0}^{s} d_i m_{k-i,21} = 0 \tag{20}$$

for all $k > 0$. Similarly it follows that the matrices (18) satisfy

$$\sum_{i=0}^{s} d_i m_{k-i,22} = 0 \tag{21}$$

for all $k > 0$. The next result shows that these relations are also sufficient.

Theorem 2. *Let $m_0 = m_0^T, \dots, m_n, \dots$ be an infinite sequence of real $l \times l$ matrices, and set $m_{-k} = m_k^T$, $k > 0$. Then the sequence $m_0, \dots, m_n, \dots$ is generated by formula* (7) *from a joint power spectrum $\Phi_{\chi 0}$ as in* (3),(4),(6) *if and only if $T_k \succeq 0$ for all $k \geq 0$, and relations* (18),(20) *hold for all $k \in \mathbb{Z}$ and $k > 0$, respectively.*

Proof. The only if part has been demonstrated above. Let us show the if part.

Assume that $T_k \succeq 0$ for all $k \geq 0$, and relations (18),(20) hold. We have to show that the moment sequence $m_0, \dots, m_n, \dots$ is generated by some joint power spectrum $\Phi_{\chi 0}$ such that its lower right $p \times p$ subblock is given by $\lambda_0 I_p$, as required in (6), and its upper right $m \times p$ subblock is a stable transfer function. This allows to construct the controller and external input spectrum K, Φ_r in (3),(4) by virtue of (5), obtaining a stable control loop.

By [27, Theorem 1] there exists a unique positive semi-definite power spectrum $\Phi(\omega)$ which produces the moment sequence $m_0, \dots, m_n, \dots$ as in (11). Set $\Phi_{\chi 0}(\omega) = |d(e^{j\omega})|^2 \Phi(\omega)$. Then (7) holds.

Let $\Phi_{\chi 0,22}$ be the $p \times p$ lower right subblock of $\Phi_{\chi 0}$. Relations (7) and (18) imply that $\int_{-\pi}^{+\pi} \frac{e^{jk\omega}}{|d(e^{j\omega})|^2} (\Phi_{\chi 0,22}(\omega) - \lambda_0 I_p) d\omega = 0$ for all k. Again from [27, Theorem 1] it then follows that $\Phi_{\chi 0,22}(\omega) = \lambda_0 I_p$.

Denote the upper right $m \times p$ subblock of $\Phi_{\chi 0}$ by Φ_{ue}. Relation (20) implies $\sum_{i=0}^{s} d_i m_{k+i,12} = 0$ for all $k < 0$. Writing this out, we obtain $\int_{-\pi}^{+\pi} \frac{\Phi_{ue}(\omega)}{d(e^{-j\omega})} e^{jk\omega}\, d\omega = 0$ for all $k < 0$. It follows that the function $\tilde{f}_{ue} : \mathbb{T} \to \mathbb{C}^{m \times p}$ defined by $\tilde{f}_{ue}(e^{j\omega}) = \frac{\Phi_{ue}(\omega)}{d(e^{-j\omega})}$ can be extended to a holomorphic function outside of the unit disc, including the point at infinity. The product $f_{ue}(z) = \tilde{f}_{ue}(z) d(z^{-1})$ is then a holomorphic extension of the function $f_{ue} : \mathbb{T} \to \mathbb{C}^{m \times p}$ defined by $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$. Thus Φ_{ue} represents a stable transfer function, which concludes the proof. □

4.2 Feasibility of the central extension

In this subsection we consider *finite* sequences $m_0 = m_0^T, m_1, \dots, m_n$ of real $l \times l$ matrices and their central extensions in relation to Theorem 2. Set $m_{-k} = m_k^T$ for $k = 1, \dots, n$.

In order for the finite sequence $(m_0, \dots, m_n)$ to be extendable to an infinite sequence $m_0, \dots, m_n, \dots$ satisfying the conditions of Theorem 2, it must clearly satisfy the following necessary conditions:

$$\begin{aligned} T_n &\succeq 0, && (22)\\ m_{k,22} &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega, \qquad k = 0, \dots, n, && (23)\\ \sum_{i=0}^{s} d_i m_{k-i,21} &= 0, \qquad k = 1, \dots, n. && (24) \end{aligned}$$

In [18, Theorem 1] we have shown for the SISO case that conditions (22)—(24) are also sufficient to guarantee the existence of a positive semi-definite joint power spectrum (6), satisfying $\Phi_{\chi 0}(\omega) = \Phi_{\chi 0}(-\omega)^T$, such that Φ_{ue} represents a stable transfer function, which reproduces the truncated moment sequence $(m_0, \dots, m_n)$ by formula (7). This proof extends without modifications also to the MIMO case considered here. The result [18, Theorem 1] is, however, non-constructive, because it does not yield an explicit power spectrum $\Phi_{\chi 0}$, but merely proves its existence.

We will now give a constructive proof by showing that the explicit power spectrum obtained by virtue of the central extension yields a feasible optimal experiment.

Theorem 3. *Let $m_0 = m_0^T, m_1, \dots, m_n$ be a finite sequence of real $l \times l$ matrices, and set $m_{-k} = m_k^T$ for $k = 1, \dots, n$. Assume that conditions* (22)—(24) *hold. Then the central extension of the sequence $(m_0, \dots, m_n)$ satisfies the conditions of Theorem* 2.

Proof. The condition $T_k \succeq 0$ is fulfilled for all $k \geq 0$ because the central extension is by definition a positive semi-definite moment extension. It remains to show the equality conditions (18),(20) for $k > n$.

This can be done by induction over k. Indeed, the central extension $m_0, \dots, m_n, m_{n+1}, \dots$ of the finite sequence $(m_0, \dots, m_n)$ coincides with the central extension of the finite sequence $(m_0, \dots, m_n, m_{n+1})$. Suppose we are able to show that the moment matrix m_{n+1} satisfies the conditions (18),(20) for $k = n + 1$. Incrementing n by one and repeating the reasoning will then prove the conditions for $k = n + 2$. Repeating the process, we prove the conditions for all $k > n$.

We shall hence consider the case $k = n + 1$. Note that

$$\sum_{i=0}^{s} d_i \left(\frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{j(n+1-i)\omega}\, d\omega \right) = \frac{1}{2\pi j} \int_{\mathbb{T}} \frac{\lambda_0 I_p}{d(e^{j\omega})} e^{jn\omega}\, de^{j\omega} = 0,$$

because the integrand in the second integral can be extended to a function which is holomorphic inside the unit disc. It follows that (18) is valid for $k = n + 1$ if and only if (21) is valid for $k = n + 1$.

But the validity of (20),(21) for $k = n + 1$ follows from Lemmas 2 and 3 in the Appendix. Indeed, set $A = m_0$, $B = \begin{pmatrix} m_1^T & \dots & m_n^T \end{pmatrix}$, $C = T_{n-1}$, $D^T = \begin{pmatrix} m_{n,21} & m_{n,22} & \dots & m_{1,21} & m_{1,22} \end{pmatrix}$, $E = m_{0,22}$, $X^T = \begin{pmatrix} m_{n+1,21} & m_{n+1,22} \end{pmatrix}$. Then the assumptions of Lemma 2 are satisfied by virtue of the condition $T_n \succeq 0$. The relation $X = BC^\dagger D$ follows from the definition of the central extension in Subsection 3.4. Let further F^T consist of the last p rows of the $l \times (n+1)l$ matrix $\begin{pmatrix} 0 & 0 & \cdots & 0 & d_s I_l & d_{s-1} I_l & \cdots & d_0 I_l \end{pmatrix}$. Then the relation $\begin{pmatrix} C & D \end{pmatrix} F = 0$ follows from (20),(21) for $k = 1, \dots, n$. It then follows from Lemma 3 that $\begin{pmatrix} B & X \end{pmatrix} F = 0$ which is equivalent to (20),(21) for $k = n + 1$. This completes the proof. □

Theorem 4. *Let $m_0 = m_0^T, m_1, \dots, m_n$ be a finite sequence of real $l \times l$ matrices, and set $m_{-k} = m_k^T$ for $k = 1, \dots, n$. Then $(m_0, \dots, m_n)$ is extendable to an infinite sequence $m_0, \dots, m_n, \dots$ satisfying the conditions of Theorem 2 if and only if conditions* (22)—(24) *hold.*

Proof. The only if part follows from the fact that the conditions in Theorem 2 imply (22)—(24). The if part follows from Theorem 3. □

Theorem 4 identifies (22)—(24) as the conditions on a finite sequence $m_0 = m_0^T, m_1, \dots, m_n$ of real $l \times l$ matrices to be realizable as a truncated sequence of generalized moments as in formula (7), with the joint power spectrum Φ_{χ_0} defining valid experimental conditions by virtue of (5),(6). This allows us to rewrite experiment design problems satisfying Assumption 1 as a semi-definite program satisfying the constraints (22)—(24), which will be accomplished in Section 5.

In the case when the block-Toeplitz matrix T_n is positive definite we have the following main result.

Theorem 5. *Let $(m_0, \dots, m_n)$ be a $(n+1)$-tuple of real $l \times l$ matrices satisfying $m_0 = m_0^T$, and define $m_{-k} = m_k^T$ for all $k = 1, \dots, n$. Suppose that these matrices satisfy conditions* (23),(24)*, and $T_n \succ 0$. Then the rational power spectrum $\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2 \cdot \Phi(\omega)$, where $\Phi(\omega)$ is given by* (17) *as an explicit function of $m_0, \dots, m_n$, satisfies the following properties: it is of the form* (6)*, positive definite, satisfies $\Phi_{\chi_0}(\omega) = \Phi_{\chi_0}(-\omega)^T$, its upper right block Φ_{ue} represents a stable transfer function, and it reproduces the truncated moment sequence $(m_0, \dots, m_n)$ by formula* (7)*.*

Proof. The theorem follows from Theorem 2, Theorem 4, and the explicit formula (17) for the power spectrum corresponding to the central extension in case that T_n is invertible. □

We shall conclude by giving an explicit formula for the transfer function Φ_{ue} in the non-degenerate case. By (23),(24) the last p rows of the $l \times (n+1)l$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & d_s I_l & d_{s-1} I_l & \cdots & d_0 I_l \end{pmatrix} T_n$$

are given by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \sum_{i=0}^s d_i m_{-i,21} & \sum_{i=0}^s d_i m_{-i,22} \end{pmatrix}.$$

Recall that the last l rows of the inverse T_n^{-1} are given by $\begin{pmatrix} (A_n^n)^T & (A_n^{n-1})^T & \dots & A_n^0 \end{pmatrix}$. It follows that

$$\begin{pmatrix} 0 & d_k I_p \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^s d_i m_{-i,21} & \sum_{i=0}^s d_i m_{-i,22} \end{pmatrix} (A_n^k)^T,$$

where we put $d_k = 0$ for $k > s$ by convention. Multiplying by z^k and summing over k, we obtain after transposition

$$\begin{pmatrix} 0 \\ d(z)I_p \end{pmatrix} = A_n(z) \begin{pmatrix} \sum_{i=0}^s d_i m_{i,12} \\ \sum_{i=0}^s d_i m_{i,22} \end{pmatrix}. \tag{25}$$

The upper right $m \times p$ block Φ_{ue} of $\Phi_{\chi_0}(\omega)$ then equals

$$\begin{aligned} \begin{pmatrix} d(e^{j\omega})I_m \\ 0 \end{pmatrix}^* \Phi(\omega) \begin{pmatrix} 0 \\ d(e^{j\omega})I_p \end{pmatrix} &= \begin{pmatrix} d(e^{j\omega})I_m \\ 0 \end{pmatrix}^* A_n(e^{j\omega})^{-*} A_n^0 \begin{pmatrix} \sum_{i=0}^s d_i m_{i,12} \\ \sum_{i=0}^s d_i m_{i,22} \end{pmatrix} \\ &= d(e^{-j\omega}) \begin{pmatrix} I_m \\ 0 \end{pmatrix}^T A_n(e^{-j\omega})^{-T} \begin{pmatrix} 0 \\ d_0 I_p \end{pmatrix}. \end{aligned} \tag{26}$$

Here we used (17), (25) for the first relation and the constant term in (25) for the second one.

4.3 Parametrization of all feasible extensions

In Theorem 1 of Subsection 3.2 we have given the general form of the extended moment m_{n+1} in terms of the Verblunsky parameter Δ_{n+1}. However, this extension does not take account of the constraints (18), (20) imposed by the closed-loop setup of the experiment design problem. Here we present a parametrization of all feasible extensions, i.e. extensions that are compatible with these constraints.

Let $(m_0, \dots, m_n)$ be a finite sequence of real $l \times l$ matrices satisfying conditions (22)—(24). The previous subsection dealt with a specific infinite moment extension of $(m_0, \dots, m_n)$ satisfying the conditions of Theorem 2, namely the central extension. In this subsection we shall parameterize all extensions satisfying (18), (20)Ê in terms of a choice sequence.

First we determine all real $l \times l$ matrices m_{n+1} such that the block-Toeplitz matrix T_{n+1} is positive semi-definite and relations (18),(20) hold for $k = n + 1$. By virtue of $d_0 \neq 0$ the p lower rows of m_{n+1} are uniquely determined by the equivalent relations (20),(21) for $k = n + 1$. Namely, we have

$$m_{n+1,2\alpha} = -d_0^{-1} \sum_{i=1}^s d_i m_{n+1-i,2\alpha}, \qquad \alpha = 1, 2.$$

The upper m rows of m_{n+1} can be parameterized by virtue of Lemma 4 of the Appendix. Namely, set $A = E = m_0$, $B = \begin{pmatrix} m_1^T & \dots & m_n^T \end{pmatrix}$, $C = T_{n-1}$, $D^T = \begin{pmatrix} m_n & \dots & m_1 \end{pmatrix}$, $X_1^T = \begin{pmatrix} m_{n+1,11} & m_{n+1,12} \end{pmatrix}$, $X_2^T = \begin{pmatrix} m_{n+1,21} & m_{n+1,22} \end{pmatrix}$. Let the matrices D, E be partitioned as in Lemma 4. The relation $X_2 = BC^\dagger D_2$ then follows from the definition of the central extension in Subsection 3.4. By Lemma 4, the matrix X_1 containing the remaining blocks of m_{n+1} is parameterized as in (28) of that lemma by a contractive $l \times m$ matrix $\hat{\Delta}$. We will denote this matrix by $\hat{\Delta}_{n+1}$ and call it *restricted Verblunsky parameter*.

Having determined the moment m_{n+1} by the choice of the restricted Verblunsky parameter $\hat{\Delta}_{n+1}$, we may proceed in an analogous manner to the definition of the next moment m_{n+2} by the choice of the restricted Verblunsky parameter $\hat{\Delta}_{n+2}$. In this way, all the infinite moment extensions of the sequence $(m_0, \dots, m_n)$ which satisfy the conditions of Theorem 2 can be parameterized by the infinite choice sequence $\hat{\Delta}_{n+1}, \hat{\Delta}_{n+2}, \dots$ of contractive $l \times m$ matrices.

By Lemma 5 in the Appendix, the choice $\hat{\Delta}_k = 0$ for all $k > n$ leads to the central extension of the sequence $(m_0, \dots, m_n)$. In the same way, the choice $\hat{\Delta}_{k'} = 0$ for all $k' > n + k$ leads to the central extension of the sequence $(m_0, \dots, m_n, m_{n+1}, \dots, m_{n+k})$. Here the moments $m_{n+1}, \dots, m_{n+k}$ are parameterized by the remaining k free restricted Verblunsky parameters $\hat{\Delta}_{n+1}, \dots, \hat{\Delta}_{n+k}$. In this way, we obtain a set of infinite moment extensions which is parameterized algebraically by the klm elements of these matrices.

Note that if only the first parameter $\hat{\Delta}_{n+1}$ is free, while the other parameters are fixed to zero, then T_{n+1} is affine in $\hat{\Delta}_{n+1}$. By Proposition 2 the Carathéodory function associated to the joint power spectrum $\Phi_{\chi 0}$ is then rational in $\hat{\Delta}_{n+1}$.

5 Solution algorithm

In this section we outline a general scheme for the solution of optimal experiment design problems satisfying Assumption 1. The scheme consists of two steps. First we find the optimal truncated moment sequence by solving a semi-definite program, and then we recover the experimental conditions, i.e., the power spectrum Φ_r of the external input and the controller K from this moment sequence.

Apart from the constraints following from the formulation of the particular problem instance under consideration, the moment sequence $(m_0,\dots,m_n)$ has to satisfy conditions (22)—(24). Condition (22) amounts to a linear matrix inequality. Condition (23) determines the blocks $m_{k,22}$ explicitly, while condition (24) yields linear relations on the blocks $m_{k,21}$. The optimal experiment design problem defined in Assumption 1 is thus turned into the following semi-definite program.

$$\min_{m_k,x_k}\left(\sum_{k=0}^{n}\langle C_k,m_k\rangle+\sum_{k=1}^{N}c_kx_k\right) \tag{27}$$

with respect to the constraints

$$\mathcal{A}(m_0,m_1,\dots,m_n,x_1,x_2,\dots,x_N)\succeq 0,$$

$$m_{k,22}=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{\lambda_0 I_p}{|d(e^{j\omega})|^2}e^{jk\omega}\,d\omega,\quad k=0,\dots,n,$$

$$\sum_{i=0}^{s}d_i m_{k-i,21}=0,\quad k=1,\dots,n,$$

$$T_n=\begin{pmatrix} m_0 & m_1^T & \ddots & m_n^T\\ m_1 & m_0 & \ddots & m_{n-1}^T\\ \ddots & \ddots & \ddots & \ddots\\ m_n & m_{n-1} & \ddots & m_0\end{pmatrix}\succeq 0,$$

where $m_{-k}=m_k^T$. By solving this semi-definite program, the user obtains the optimal truncated moment sequence $(m_0,\dots,m_n)$ and the optimal value of the cost function.

If the matrix T_n corresponding to the solution happens to be positive definite, then Theorem 5 allows to explicitly recover the joint power spectrum (6) by the explicit formula

$$\Phi_{\chi 0}(\omega)=|d(e^{j\omega})|^2\cdot A(e^{j\omega})^{-*}A(0)A(e^{j\omega})^{-1},$$

where $A(z)=U(z)T_n^{-1}U^T(0)$ and $U(z)=\begin{pmatrix} z^nI_l & z^{n-1}I_l & \cdots & I_l\end{pmatrix}$. Alternatively, the upper right $m\times p$ block Φ_{ue} of $\Phi_{\chi 0}$ can be obtained by the explicit formula (26). The power spectrum Φ_r and the controller K may then be recovered from Φ_{ue} and the upper left $m\times m$ block Φ_u by formulas (5).

If the matrix T_n happens to be singular, then Φ_{χ_0} can still be recovered as a rational function with possibly a singular part as outlined in Subsection 3.3. We shall give an example in the next section when the singular part is absent despite the singularity of T_n.

As is often the case in optimal experiment design, the calculation of the optimal experimental conditions requires knowledge of the transfer functions G_0, H_0 to be identified. This obstacle can be circumvented by performing a preliminary identification experiment and/or applying an iterative procedure, using the estimates from the previous iteration for the design of the experimental conditions in the current one.

6 Examples

Example 1

In this first example, we illustrate the construction of the central extension on the basis of a moment matrix made up of the moments m_0 and m_1. We also show that even when the moment matrix is singular, the spectrum defined by this central extension remains finite. Consider the moment matrix

$$T = \begin{pmatrix} 1 & 0 & a & c \\ 0 & 1 & -c & b \\ a & -c & 1 & 0 \\ c & b & 0 & 1 \end{pmatrix}.$$

We have $\det T = (c^2+ab-1-a+b)(c^2+ab-1+a-b)$, and $T \succeq 0$ if and only if $\max(|a|,|b|) \leq 1$ and $c^2 + ab + |a-b| \leq 1$. The polynomial $A(z)$ is given by

$$\begin{aligned} A(z) &= \frac{1}{(c^2+ab-1-a+b)(c^2+ab-1+a-b)} \cdot \\ &\left\{ z \begin{pmatrix} bc^2 + a(b^2-1) & c^3 + c(ab-1) \\ -c^3 - c(ab-1) & ac^2 + b(a^2-1) \end{pmatrix} \right. \\ &\left. + \begin{pmatrix} 1-b^2-c^2 & c(a-b) \\ c(a-b) & 1-a^2-c^2 \end{pmatrix} \right\}, \end{aligned}$$

its inverse by

$$\begin{aligned} A^{-1}(z) &= \frac{1}{z^2(c^2+ab) - z(a+b) + 1} \cdot \\ &\left\{ z \begin{pmatrix} 1-a^2-c^2 & -c^3 - c(ab-1) \\ c^3 + c(ab-1) & bc^2 + a(b^2-1) \end{pmatrix} \right. \\ &\left. + \begin{pmatrix} 1-a^2-c^2 & -c(a-b) \\ -c(a-b) & 1-b^2-c^2 \end{pmatrix} \right\}. \end{aligned}$$

The roots of the polynomial $z^2(c^2+ab) - z(a+b) + 1$ are given by $z = \frac{a+b\pm\sqrt{(a-b)^2-4c^2}}{2(c^2+ab)}$. The spectrum (17) of the central extension is given by

$$\begin{aligned}
\Phi_{11}(\omega) &= \frac{(1-a^2)(1+b^2)-c^4-2abc^2+2(ba^2+ac^2-b)\cos\omega}{(e^{2j\omega}(c^2+ab)-e^{j\omega}(a+b)+1)(e^{-2j\omega}(c^2+ab)-e^{-j\omega}(a+b)+1)}, \\
\Phi_{12}(\omega) &= \frac{-2j(c^3+c(ab-1))\sin\omega}{(e^{2j\omega}(c^2+ab)-e^{j\omega}(a+b)+1)(e^{-2j\omega}(c^2+ab)-e^{-j\omega}(a+b)+1)}, \\
\Phi_{22}(\omega) &= \frac{(1+a^2)(1-b^2)-c^4-2abc^2+2(ab^2+bc^2-a)\cos\omega}{(e^{2j\omega}(c^2+ab)-e^{j\omega}(a+b)+1)(e^{-2j\omega}(c^2+ab)-e^{-j\omega}(a+b)+1)}.
\end{aligned}$$

However, even if $(c^2+ab-1-a+b)(c^2+ab-1+a-b)=0$, implying that T is singular, the expression $e^{2j\omega}(c^2+ab)-e^{j\omega}(a+b)+1$ does not become zero in general. Hence the spectrum Φ remains finite.

For the values $a = 0.831471050378134$, $b = 0.584414659119109$, $c = 0.516739526518758$ for which the matrix T becomes singular, we have computed $\Phi(\omega)$ according to the formula above. Figures 2, 3 and 4 show, respectively, the plots of $|\Phi_{11}|, Im\Phi_{12}$ and $|\Phi_{22}|$. These plots show that, even in this so-called *degenerate case* where T is singular, the feasible optimal spectrum constructed using the central extension remains finite.

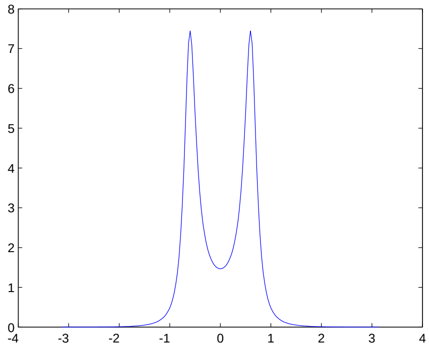

Figure 2: $|\Phi_{11}|$

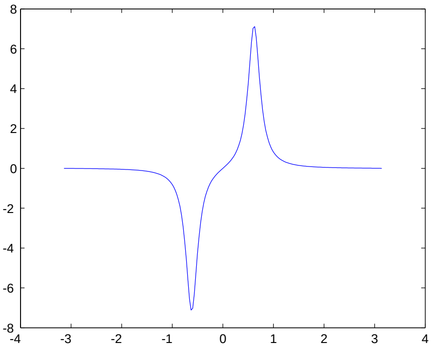

Figure 3: $Im\Phi_{12}$

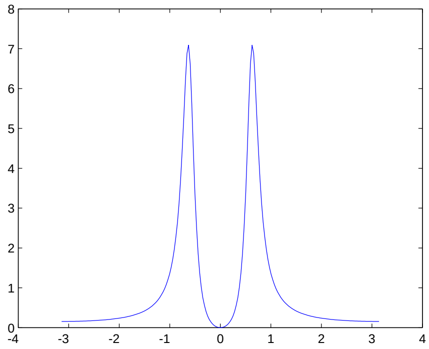

Figure 4: $|\Phi_{22}|$

Example 2

In the second example we consider an optimal experiment design problem applied to the identification of a stable plant $G = \frac{\theta_1 z^{-1}}{1+\theta_2 z^{-1}}$ with $|\theta_2| < 1$, $H = 1$. We wish to minimize the output power while achieving a fixed information matrix. Set $d(z) = (1 + \theta_2 z)^2$.

We have

$$\frac{\partial G}{\partial \theta} = \frac{1}{(1+\theta_2 z^{-1})^2} \begin{pmatrix} z^{-1} + \theta_2 z^{-2} \\ -\theta_1 z^{-2} \end{pmatrix},$$

and the information matrix is given by

$$\bar{M} = \frac{1}{2\pi\lambda_0} \int_{-\pi}^{+\pi} \frac{\partial G(e^{j\omega})}{\partial \theta} \Phi_u \left(\frac{\partial G(e^{j\omega})}{\partial \theta} \right)^* d\omega = \lambda_0^{-1} \begin{pmatrix} (1+\theta_2^2)m_{0,11} + 2\theta_2 m_{1,11} & -\theta_1\theta_2 m_{0,11} - \theta_1 m_{1,11} \\ -\theta_1\theta_2 m_{0,11} - \theta_1 m_{1,11} & \theta_1^2 m_{0,11} \end{pmatrix}.$$

Further,

$$\begin{pmatrix} G & H \end{pmatrix} = \frac{1}{(1+\theta_2 z^{-1})^2} \begin{pmatrix} \theta_1 z^{-1} + \theta_1\theta_2 z^{-2} & 1 + 2\theta_2 z^{-1} + \theta_2^2 z^{-2} \end{pmatrix},$$

and the output power is given by

$$\begin{aligned} \bar{E}y^2 &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} \begin{pmatrix} G & H \end{pmatrix} \Phi_{\chi_0} \begin{pmatrix} G & H \end{pmatrix}^* d\omega \\ &= 2\theta_1\theta_2 m_{2,21} + 2(\theta_1(1+2\theta_2^2)m_{1,21} + \theta_1^2\theta_2 m_{1,11} + \theta_1\theta_2^2 m_{1,12}) \\ &\quad +\theta_1^2(1+\theta_2^2)m_{0,11} + 2\theta_1\theta_2(2+\theta_2^2)m_{0,12} + \lambda_0 \end{aligned}$$

The generalized moments of Φ_e are given by

$$m_{0,22} = \frac{(1+\theta_2^2)\lambda_0}{(1-\theta_2^2)^3}, \qquad m_{1,22} = -\frac{2\theta_2\lambda_0}{(1-\theta_2^2)^3}, \qquad m_{2,22} = \frac{\theta_2^2(3-\theta_2^2)\lambda_0}{(1-\theta_2^2)^3}.$$

The recursion on $m_{k,21}$ reads $m_{k,21} = -(2\theta_2 m_{k-1,21} + \theta_2^2 m_{k-2,21})$ for $k > 0$, which amounts to $m_{1,21} = -2\theta_2 m_{0,12} - \theta_2^2 m_{1,12}$, $m_{2,21} = 3\theta_2^2 m_{0,12} + 2\theta_2^3 m_{1,12}$. Then the output power simplifies to

$$\bar{E}y^2 = \theta_1^2((1+\theta_2^2)m_{0,11} + 2\theta_2 m_{1,11}) + \lambda_0.$$

The output power and the information matrix contain only the moments $m_{0,11}, m_{1,11}$. As a result, the output power is fixed by the fact that the information matrix is fixed. It remains to construct a power spectrum Φ_{χ_0} that generates these two moments.

The moments $m_{2,11}, m_{k,12}$ enter only in the positivity constraint, but not in the output power and the information matrix. A possible choice for these moments is $m_{k,12} = 0$, $m_{2,11} = \frac{m_{1,11}^2}{m_{0,11}}$, with $|m_{1,11}| \leq m_{0,11}$ imposed by the positivity condition.

Then the moments m_0, m_1, m_2 are diagonal. It is not hard to see that the moments of the central extension of (m_0, m_1, m_2) are also diagonal, and $\Phi_{ue} = 0$. The corresponding experiment is hence open-loop. Moreover, the central extension of the sequence $(m_{0,11}, m_{1,11}, m_{2,11})$ equals the central extension of the sequence $(m_{0,11}, m_{1,11})$. This leads to $A(z) = \frac{m_{0,11} - m_{1,11}z}{m_{0,11}^2 - m_{1,11}^2}$,

$$\Phi_u = \Phi_r = \frac{m_{0,11}(m_{0,11}^2 - m_{1,11}^2)|1 + \theta_2 e^{j\omega}|^4}{|m_{0,11} - m_{1,11}e^{j\omega}|^2}.$$

7 Conclusions

We have provided a solution to the closed loop optimal experiment design for MIMO systems. The solution uses the so-called *partial correlation approach* in which the criterion and the constraints are expressed as a function of a finite set of generalized moments. The optimal moments are then obtained as the solution of a semi-definite program. The key difficulty of this approach, which had been a stumbling block so far, is to extend the finite set of optimal moments into an infinite set, or equivalently into a spectrum, because the spectrum must obey some constraints which are due to the closed loop setup. Thus, the classical Carathéodory-Fejer theorem cannot be used to produce a feasible extension.

Our main contribution has been to show that the so-called *central extension* is a feasible extension, which satisfies these constraints. In addition, using properties of the central extension, as well as results on the *positive matrix completion theorem*, we have shown how to construct families of parametrized optimal extensions which also obey the constraints of the optimal experiment design problem.

One of the key advantages of the solution method developed in the present paper is that it allows one to explicitly compute an optimal solution for the spectrum Φ_r of the external excitation signal and the feedback controller K. They can be computed straightforwardly from the optimal moments that result from the solution of the semi-definite program. This is a significant progress over our previous result [18] which only proved the existence of an optimal spectrum, but without an explicit computational procedure.

Appendix

In this Appendix we provide auxiliary results related to the *positive matrix completion problem*. This is the problem of completing a real symmetric matrix, only part of whose entries are specified, to a full positive semi-definite matrix. A partially specified matrix M is said to be *partial positive semi-definite* if all diagonal entries of M are specified, and every principal submatrix of M which is fully specified is positive semi-definite. A partially specified matrix M is said to be *positive semi-definite completable* if there exists a specification of the unspecified entries of M such that the resulting fully specified matrix is positive semi-definite. Clearly partial positive semi-definiteness is a necessary condition for positive semi-definite completability. There exist specification patterns for which this condition is also sufficient. These patterns have been completely described in [16] by graph-theoretic means. We shall need only a special case of such specification patterns, namely when the unspecified entries can be arranged in a rectangular block by a suitable permutation of the row and column indices of M. In this case the set of all completions has a closed-form description as an affine image of a matrix ball. This fact has been brought to our attention by Keith Glover.

The results in this Appendix, and in particular Lemma 2 and Lemma 4, are required to prove that the moment extension in Theorem 1 is an admissible extension in that it produces $T_{n+1} \succeq 0$.

Lemma 1. *[14, Theorem 16.1, p.435] A real symmetric matrix* $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ *is positive semi-definite if and only if* $C \succeq 0$, $(I - CC^\dagger)B^T = 0$, *and* $A - BC^\dagger B^T \succeq 0$. *In this case we have the factorization*

$$M = \begin{pmatrix} I & BC^\dagger \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BC^\dagger B^T & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ C^\dagger B^T & I \end{pmatrix}.$$

Here $C^\dagger$ *denotes the pseudo-inverse of* C, *and* I *denote identity matrices of appropriate size.* □

Lemma 2. *[15] Let* $M = \begin{pmatrix} A & B & * \\ B^T & C & D \\ * & D^T & E \end{pmatrix}$ *be a real partial positive semi-definite matrix, where* A, B, C, D, E *are blocks of compatible sizes. Then the matrix* $M_X = \begin{pmatrix} A & B & X \\ B^T & C & D \\ X^T & D^T & E \end{pmatrix}$ *is a positive semi-definite completion of* M *if and only if the block* X *can be written as* $X = BC^\dagger D + (A - BC^\dagger B^T)^{1/2}\Delta(E - D^T C^\dagger D)^{1/2}$*, where* Δ *is a real matrix satisfying the condition* $\sigma_{\max}(\Delta) \leq 1$*. Here* $\sigma_{\max}$ *denotes the maximal singular value and* $W^{1/2}$ *the positive semi-definite matrix square root of the positive semi-definite matrix* W*.*

Proof. Since M is partial positive semi-definite, the matrices $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ and $\begin{pmatrix} E & D^T \\ D & C \end{pmatrix}$ are positive semi-definite. Applying Lemma 1 to these matrices, we obtain that $C \succeq 0$, $(I - CC^\dagger)B^T = 0$, $(I - CC^\dagger)D = 0$, $A - BC^\dagger B^T \succeq 0$, $E - D^T C^\dagger D \succeq 0$. Applying Lemma 1 to the matrix $\left(\begin{array}{cc|c} A & X & B \\ X^T & E & D^T \\ \hline B^T & D & C \end{array}\right)$, we obtain that $M_X \succeq 0$ if and only if

$$\begin{pmatrix} A & X \\ X^T & E \end{pmatrix} - \begin{pmatrix} B \\ D^T \end{pmatrix} C^\dagger \begin{pmatrix} B^T & D \end{pmatrix} = \begin{pmatrix} A - BC^\dagger B^T & X - BC^\dagger D \\ (X - BC^\dagger D)^T & E - D^T C^\dagger D \end{pmatrix} \succeq 0.$$

The claim of the lemma now easily follows. □

The next result deals with the specific choice $\Delta = 0$.

Lemma 3. *Assume the conditions of Lemma 2, and set* $X = BC^\dagger D$*. Assume that there exists a matrix* F *of appropriate size such that* $\begin{pmatrix} C & D \end{pmatrix} F = 0$*. Then we have also* $\begin{pmatrix} B & X \end{pmatrix} F = 0$*.*

Proof. Partition $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ into subblocks of appropriate size. We have $CF_1 + DF_2 = 0$, and hence $BF_1 + XF_2 = B(F_1 + C^\dagger DF_2) = B(I - C^\dagger C)F_1 = 0$. Here the last equality follows from Lemma 1. □

Lemma 2 permits to obtain a parametrization of all positive semi-definite matrix completions not only in the case when the unspecified elements form a rectangular block in the upper right corner, but also when such a situation can be achieved by a suitable permutation of the row and column indices.

Lemma 4. *Assume the conditions of Lemma 2, but let the unknown block be partitioned as* $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$*. Let the blocks* $D = \begin{pmatrix} D_1 & D_2 \end{pmatrix}$*,* $E = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{22} \end{pmatrix}$ *be partitioned in a compatible manner.*

Then the partially specified matrix $\hat{M} = \begin{pmatrix} A & B & * & X_2 \\ B^T & C & D_1 & D_2 \\ * & D_1^T & E_{11} & E_{12} \\ X_2^T & D_2^T & E_{12}^T & E_{22} \end{pmatrix}$*, where* $X_2 = BC^\dagger D_2$*, is partial* *positive semi-definite. The general form of a positive semi-definite completion X_1 of $\hat{M}$ is given by*

$$\begin{aligned}\begin{pmatrix} B & X_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^{\dagger} \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix} + \left(A - \begin{pmatrix} B & X_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^{\dagger} \begin{pmatrix} B^T \\ X_2^T \end{pmatrix} \right)^{1/2} \hat{\Delta} \\ \times \left(E_{11} - \begin{pmatrix} D_1^T & E_{12} \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^{\dagger} \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix} \right)^{1/2},\end{aligned} \tag{28}$$

where $\hat{\Delta}$ is any real matrix of size compatible with those of A and E_{11} such that $\sigma_{\max}(\hat{\Delta}) \leq 1$.

Proof. The choice $\Delta = 0$ in Lemma 2 leads to $X_\alpha = BC^\dagger D_\alpha$, $\alpha = 1, 2$. Hence $\hat{M}$ is positive semi-definite completable. In particular, it must be partial positive semi-definite. The general form of its positive semi-definite completion X_1 follows by application of Lemma 2 to $\hat{M}$, after an appropriate permutation of rows and columns. □

Lemma 5. *Assume the conditions of Lemma 2 and Lemma 4. Completing the matrix M by $X = BC^\dagger D$, i.e., by the choice $\Delta = 0$, leads to the same result as first setting $X_2 = BC^\dagger D_2$ and then completing $\hat{M}$ by $X_1 = \begin{pmatrix} B & X_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^{\dagger} \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix}$, i.e., by the choice $\hat{\Delta} = 0$.*

Proof. We have to show that $BC^\dagger D_1 = \begin{pmatrix} B & BC^\dagger D_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^{\dagger} \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix}$. By Lemma 1 we have

$$\begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ D_2^T C^\dagger & I \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & E_{22} - D_2^T C^\dagger D_2 \end{pmatrix} \begin{pmatrix} I & C^\dagger D_2 \\ 0 & I \end{pmatrix},$$

and hence

$$\begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^{\dagger} = \begin{pmatrix} I & -C^\dagger D_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} C^\dagger & 0 \\ 0 & (E_{22} - D_2^T C^\dagger D_2)^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ -D_2^T C^\dagger & I \end{pmatrix}.$$

It follows that $\begin{pmatrix} I & C^\dagger D_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^{\dagger} = \begin{pmatrix} C^\dagger & 0 \end{pmatrix}$, which implies our claim. □

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