Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Moment bounds for the corrector in stochastic homogenization of a percolation model

Agnes Lamacz [1], Stefan Neukamm [2], Felix Otto[3]

submitted: September 2, 2013

[1] Technische Universität Dortmund
Fakultät für Mathematik
Vogelpothsweg 87
44227 Dortmund
Germany
E-Mail: agnes.lamacz@tu-dortmund.de

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: stefan.neukamm@wias-berlin.de

[3] Max Planck Institute for Mathematics in the Sciences
Inselstr. 22
04103 Leipzig
Germany
E-Mail: felix.otto@mis.mpg.de

No. 1863
Berlin 2013

2010 *Mathematics Subject Classification.* 35B27, 60K37, 60F99, 60H25, 39A70.

Key words and phrases. stochastic homogenization, percolation, corrector equation, quantitative results.

Stefan Neukamm was partially supported by ERC-2010-AdG no.267802 AnaMultiScale.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We study the corrector equation in stochastic homogenization for a simplified Bernoulli percolation model on $\mathbb{Z}^d$, $d > 2$. The model is obtained from the classical $\{0,1\}$-Bernoulli bond percolation by conditioning all bonds parallel to the first coordinate direction to be open. As a main result we prove (in fact for a slightly more general model) that stationary correctors exist and that all finite moments of the corrector are bounded. This extends a previous result in [8], where uniformly elliptic conductances are treated, to the degenerate case. Our argument is based on estimates on the gradient of the elliptic Green's function.

1 Introduction

We consider the lattice graph $(\mathbb{Z}^d, \mathbb{B}^d)$, $d > 2$, where $\mathbb{B}^d$ denotes the set of nearest-neighbor edges. Given a stationary and ergodic probability measure $\langle\cdot\rangle$ on Ω – the space of conductance fields $\boldsymbol{a} : \mathbb{B}^d \to [0,1]$ – we study the *corrector equation* from stochastic homogenization, i.e. the elliptic difference equation

$$\nabla^*(\boldsymbol{a}(\nabla\phi + e)) = 0, \qquad x \in \mathbb{Z}^d. \tag{1}$$

Here, ∇ and ∇^* denote discrete versions of the continuum gradient and (negative) divergence, cf. Section 2, and $e \in \mathbb{R}^d$ denotes a vector of unit length, which is fixed throughout the paper. The corrector equation (1) emerges in the homogenization of discrete elliptic equations with random coefficients: For random conductances that are stationary and ergodic (with respect to the shifts $\boldsymbol{a}(\cdot) \mapsto \boldsymbol{a}(\cdot + z)$, $z \in \mathbb{Z}^d$, cf. Section 2), and under the assumption of uniform ellipticity (i.e. there exists $\lambda_0 > 0$ such that $\boldsymbol{a} \geq \lambda_0$ on $\mathbb{B}^d$ almost surely), a classical result from stochastic homogenization (e. g. see [12, 13]) shows that the effective behavior of $\nabla^*\boldsymbol{a}\nabla$ on large length scales is captured by the homogenized elliptic operator $\nabla^*\boldsymbol{a}_{\text{hom}}\nabla$ where $\boldsymbol{a}_{\text{hom}}$ is a deterministic, symmetric and positive definite $d \times d$ matrix. It is characterized by the minimization problem

$$e \cdot \boldsymbol{a}_{\text{hom}} e = \inf_{\varphi} \langle (e + \nabla\varphi) \cdot \boldsymbol{a}(e + \nabla\varphi) \rangle, \tag{2}$$

where the infimum is taken over random fields φ that are $\langle\cdot\rangle$-stationary in the sense of $\varphi(\boldsymbol{a}, x + z) = \varphi(\boldsymbol{a}(\cdot + z), x)$ for all $x, z \in \mathbb{Z}^d$ and $\langle\cdot\rangle$-almost every $\boldsymbol{a} \in \Omega$. Minimizers to (2) are called *stationary correctors* and are characterized as the stationary solutions to the corrector equation (1). Due to the lack of a Poincaré Inequality for ∇ on the infinite dimensional space of stationary random fields, the elliptic operator $\nabla^*\boldsymbol{a}\nabla$ is highly degenerate and the minimum in (2) may not be obtained in general. In fact, it is known to fail generally for $d = 2$. The only existence result of a stationary corrector (in dimensions $d > 2$) has been obtained recently in [8] by Gloria and the third author under the assumption that the $\boldsymbol{a}$'s are uniformly elliptic, and that $\langle\cdot\rangle$ satisfies a Spectral Gap Estimate, which is in particular the case for independent and identically distributed coefficients. They also show that $\langle|\phi|^p\rangle \lesssim 1$ for all $p < \infty$.

The goal of the present paper is to extend this result to the case of conductances with degenerate ellipticity. To be definite, consider the probability measure $\langle\cdot\rangle_\lambda$ constructed by the following procedure:

> Take the classical $\{0,1\}$-Bernoulli-bond percolation on $\mathbb{B}^d$ with parameter $\lambda \in (0,1]$ and declare all bonds parallel to the coordinate direction e_1 to be open. (3)

(We adapt the convention to call a bond "open" if the associated coefficient is "1", while a bond is "closed" if the associated coefficient is "0". The parameter λ denotes the probability that a bond is "open"). As for d-dimensional Bernoulli percolation, $\langle\cdot\rangle_\lambda$ describes a random graph of open bonds, which is locally disconnected with positive probability. However, as a merit of the modification any two vertices in the random graph are almost surely connected by some open path. As a main result we show that (1) admits a stationary solution, all finite moments of which are bounded:

Theorem (main result). Let $d > 2$ and $\lambda \in (0,1]$. There exits a $\langle\cdot\rangle_\lambda$-stationary function $\phi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ such that $\phi(\boldsymbol{a},\cdot)$ solves (1) for $\langle\cdot\rangle_\lambda$ almost every $\boldsymbol{a} \in \Omega$, and

$$\forall p < \infty : \qquad \langle|\phi|^p\rangle_\lambda^{\frac{1}{p}} \leq C.$$

Here C denotes a constant that only depends on p, λ and d.

The modified Bernoulli percolation model $\langle\cdot\rangle_\lambda$ fits into a slightly more general framework that we introduce in Section 2 below, cf. Lemma 1. The result above will then follow as a special case of Theorem 1 stated below.

The corrector and in particular its *gradient* play a prominent role in the derivation of invariance principles for random conductance models, cf. [10, 17], see also [3] for a recent survey in this direction. For supercritical Bernoulli percolation quenched invariance principles have been obtained in [16] for $d \geq 4$ and in [4, 14] for $d \geq 2$; see also [1, 2] for recent results on degenerate elliptic conductances. A key step in these results is to establish *sublinear growth* of the (non-stationary) corrector. In Theorem 1 we prove existence of stationary correctors and establish bounds on the finite moments of the corrector (and thus on its gradient). These estimates are stronger than the qualitative property of sublinearity. In fact, the latter can be deduced from bounds on sufficiently high moments of the gradient of the corrector.

Moment bounds for stationary correctors are a key ingredient to develop a quantitative theory for stochastic homogenization. While the property of *sublinearity* holds for general stationary and ergodic conductances (at least in the uniformly elliptic case), the estimates on the moments (and even the existence) of a stationary corrector require to quantify ergodicity. In a series of papers (see [6, 7, 8, 9]) two of the authors and Gloria developed a quantitative theory for the corrector equation (1) (and regularized versions) based on the assumption that the underlying statistics satisfies a Spectral Gap Estimate (SG) for a Glauber dynamics on the coefficient fields, as it is the case for independent and identically distributed (i. i. d. coefficients). In these works, see in particular [6, 7], moment bounds are used to establish various optimal error estimates, e. g. for approximations of $\boldsymbol{a}_{\text{hom}}$ and the homogenization error. In the present paper we extend some of these results to a model of degenerate elliptic conductances.

Let us discuss some of the difficulties that emerge due to the loss of uniform ellipticity. A crucial element of the approach in [8] is a quenched estimate on the gradient of the Green's function associated with $\nabla^*\boldsymbol{a}\nabla$, whose proof extensively uses elliptic regularity theory in the spirit of De Giorgi. The estimate reflects the optimal spatial decay of the gradient of the Green's function and the constant in the estimate only depends on $\boldsymbol{a}$ via the constant of ellipticity, say $\lambda_0 > 0$. The arguments in [8] do not extend to the degenerate elliptic case, since the pointwise (and deterministic) inequality $\nabla u \cdot \boldsymbol{a}\nabla u \geq \lambda_0|\nabla u|^2$ breaks down. To overcome this, we provide a coercivity estimate (see Lemma 4 below) that replaces the missing pointwise inequality by a weighted, integrated version. Thanks to the coercivity estimate we establish estimates on the gradient of the Green's function and develop a quantitative theory for (1) for dimensions $d > 2$ under the assumption that high moments of the chemical distance of nearest neighbors are bounded. In a work in progress we study the case of standard supercritical Bernoulli percolation based on the ideas of the present paper.

The paper is structured as follows: In Section 2 we gather basic definitions and introduce the slightly more general framework studied in this paper. We then present the main result in the general framework. Section 3.1 is devoted to the proof of the main result: we first discuss the general strategy of the proof and present several auxiliary lemmas needed for the proof of the main theorem – in particular, the coercivity estimates, see Lemmas 4 and 5, and an estimate for the gradient of the elliptic Green's function, see Proposition 1, which play a key role in our argument. The proof of the main result is given at the end of Section 3.1, while the auxiliary results are proven in Section 4.

Throughout this article, we use the following notation, see Section 2 for more details:

- d is the dimension;
- $\mathbb{Z}^d$ is the integer lattice;
- $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{Z}^d$;
- $e \in \mathbb{R}^d$, which appears in (1), denotes a vector of unit length and is fixed throughout the paper;
- $\mathbb{B}^d := \{\, \mathrm{b} = \{x, x + e_i\} \,:\, x \in \mathbb{Z}^d,\, i = 1, \ldots, d\,\}$ is the set of nearest neighbor bonds of $\mathbb{Z}^d$;
- $B_R(x_0)$ is the cube of vertices $x \in x_0 + ([-R, R] \cap \mathbb{Z})^d$;
- $Q_R(x_0)$ is the cube of bonds $\mathrm{b} = \{x, x + e_i\} \in \mathbb{B}^d$ with $x \in B_R(x_0)$ and $i \in \{1, \ldots, d\}$;
- $|A|$ denotes the number of elements in $A \subset \mathbb{Z}^d$ (resp. $A \subset \mathbb{B}^d$).

2 General framework

In the first part of this section, we introduce the general framework following the presentation of [6]: We introduce a discrete differential calculus, the random conductance model, and finally recall the standard definitions of the corrector and the modified corrector.

2.1 Lattice and discrete differential calculus

We consider the lattice graph $(\mathbb{Z}^d, \mathbb{B}^d)$, where $\mathbb{B}^d := \{\, \mathrm{b} = \{x, x + e_i\} \,:\, x \in \mathbb{Z}^d,\, i = 1, \ldots, d\,\}$ denotes the set of nearest-neighbor bonds. We write $\ell^p(\mathbb{Z}^d)$ and $\ell^p(\mathbb{B}^d)$, $1 \le p \le \infty$, for the usual spaces of p-summable (resp. bounded for $p = \infty$) functions on $\mathbb{Z}^d$ and $\mathbb{B}^d$. For $u : \mathbb{Z}^d \to \mathbb{R}$ the *discrete derivative* $\nabla u(\mathrm{b})$, $\mathrm{b} \in \mathbb{B}^d$, is defined by the expression

$$\nabla u(\mathrm{b}) := u(y_{\mathrm{b}}) - u(x_{\mathrm{b}}).$$

Here x_{b} and y_{b} denote the unique vertices with $\mathrm{b} = \{x_{\mathrm{b}}, y_{\mathrm{b}}\} \in \mathbb{B}^d$ satisfying $y_{\mathrm{b}} - x_{\mathrm{b}} \in \{e_1, \ldots, e_d\}$. We denote by ∇^* the adjoined of ∇, so that we have for $F : \mathbb{B}^d \to \mathbb{R}$

$$\nabla^* F(x) = \sum_{i=1}^{d} F(\{x - e_i, x\}) - F(\{x, x + e_i\}).$$

Furthermore, the discrete integration by parts formula reads

$$\sum_{\mathrm{b} \in \mathbb{B}^d} \nabla u(\mathrm{b}) F(\mathrm{b}) = \sum_{x \in \mathbb{Z}^d} u(x) \nabla^* F(x), \tag{4}$$

and holds whenever the sums converge.

2.2 Random conductance field

To each bond $\mathrm{b} \in \mathbb{B}^d$ a *conductance* $\boldsymbol{a}(\mathrm{b}) \in [0,1]$ is attached. Hence, a *configuration* of the lattice is described by a *conductance field* $\boldsymbol{a} \in \Omega$, where $\Omega := [0,1]^{\mathbb{B}^d}$ denotes the *configuration space.* Given $\boldsymbol{a} \in \Omega$ we define the chemical distance between vertices $x, y \in \mathbb{Z}^d$ by

$$\operatorname{dist}_{\boldsymbol{a}}(x,y) := \inf \left\{ \sum_{\mathrm{b}\in\pi} \boldsymbol{a}(\mathrm{b})^{-1} \;:\; \pi \text{ is a path from } x \text{ to } y \right\} \qquad (\text{where } \tfrac{1}{0} := +\infty).$$

We equip Ω with the product topology (induced by $[0,1] \subset \mathbb{R}$) and the usual product σ-algebra, and describe *random configurations* by means of a probability measure on Ω, called the *ensemble.* The associated expectation is denoted by $\langle\cdot\rangle$.

Our assumptions on $\langle\cdot\rangle$ are the following:

Assumption 1.

(A1) (*Stationarity*). The *shift operators* $\Omega \ni \boldsymbol{a} \mapsto \boldsymbol{a}(\cdot + z) \in \Omega$, $z \in \mathbb{Z}^d$ preserve the measure $\langle\cdot\rangle$. (For a bond $\mathrm{b} = \{x, y\} \in \mathbb{B}^d$ and $z \in \mathbb{Z}^d$ we write $\mathrm{b} + z := \{x + z, y + z\}$ for the *shift* of b by z.)

(A2) (*Moment condition*). There exists a modulus of integrability $\Lambda : [1,\infty) \to [0,\infty)$ such that the distance of neighbors is finite on average in the sense that

$$\forall p < \infty \;:\; \max_{i=1,\ldots,d} \langle (\operatorname{dist}_{\boldsymbol{a}}(0,e_i))^p \rangle^{\frac{1}{p}} \le \Lambda(p).$$

(A3) (*Spectral Gap Estimate*). There exists a constant $\rho > 0$ such that for all $\zeta \in L^2(\Omega)$ we have

$$\left\langle (\zeta - \langle\zeta\rangle)^2 \right\rangle \le \frac{1}{\rho} \sum_{\mathrm{b}\in\mathbb{B}^d} \left\langle \left(\frac{\partial\zeta}{\partial\mathrm{b}} \right)^2 \right\rangle,$$

where $\frac{\partial\zeta}{\partial\mathrm{b}}$ denotes the *vertical derivative* as defined in Definition 1 below.

For technical reasons we need to strengthen (A2):

(A2+) We assume that

$$\forall p < \infty \;:\; \max_{i=1,\ldots,d} \langle (\operatorname{dist}_{\boldsymbol{a}^{e_i,0}}(0,e_i))^p \rangle^{\frac{1}{p}} \le \Lambda(p),$$

where $\boldsymbol{a}^{e_i,0}$ denotes the conductance field obtained by "deleting" the bond $\{0, e_i\}$ (i. e. $\boldsymbol{a}^{e_i,0}(\mathrm{b}) = \boldsymbol{a}(\mathrm{b})$ for all $\mathrm{b} \neq \{0, e_i\}$ and $\boldsymbol{a}^{e_i,0}(\{0, e_i\}) = 0$).

Let us comment on these properties. A minimal requirement needed for qualitative stochastic homogenization in the uniformly elliptic case is stationarity and ergodicity of the ensemble. The basic example for such an ensemble are i. i. d. coefficients which means that $\langle\cdot\rangle$ is a $\mathbb{B}^d$-fold product of a "single edge" probability measure on $[0,1]$. The assumption (A3) is weaker than assuming i. i. d., but stronger than ergodicity. Indeed, in [6] it is shown that any i. i. d. ensemble satisfies (A3) with constant $\rho = 1$. Moreover, it is shown that (A3) can be seen as a quantification of ergodicity. From the functional analytic point of view the spectral gap estimate is a Poincaré Inequality where the derivative is taken in vertical direction, see below. (The terminology "vertical" versus "horizontal" is motivated from viewing $\boldsymbol{a} \in \Omega$ as a "height"-function defined on the "horizontal" plane $\mathbb{B}^d$). We recall from [6] the definition of the vertical derivative:

Definition 1. For $\zeta \in L^1(\Omega)$ the *vertical derivative* w. r. t. $\mathrm{b} \in \mathbb{B}^d$ is given by

$$\frac{\partial \zeta}{\partial \mathrm{b}} := \zeta - \langle \zeta \rangle_{\mathrm{b}}\,,$$

where $\langle \zeta \rangle_{\mathrm{b}}$ denotes the conditional expectation where we condition on $\{\boldsymbol{a}(\mathrm{b}')\}_{\mathrm{b}' \neq \mathrm{b}}$. For $\zeta : \Omega \to \mathbb{R}$ sufficiently smooth we denote by $\frac{\partial \zeta}{\partial \boldsymbol{a}(\mathrm{b})}$ the classical partial derivative of ζ w. r. t. the coordinate $\boldsymbol{a}(\mathrm{b})$.

Property (A2) is a crucial assumption on the connectedness of the graph. In particular it implies that almost surely every pair of vertices can be connected by a path with finite intrinsic length. However, (A2) and (A2+) do not exclude configurations with coefficients that vanish with non-zero probability, as it is the case for $\langle \cdot \rangle_\lambda$ – the model considered in the introduction:

Lemma 1. *The modified Bernoulli percolation model $\langle \cdot \rangle_\lambda$ defined via* (3) *satisfies Assumption 1 with $\rho = 1$.*

Proof. Evidently, $\langle \cdot \rangle_\lambda$ can be written as the (infinite) product of probability measures attached to the bonds in $\mathbb{B}^d$. These "single-bond" probability measures only depend on the direction of the bond. Hence, $\langle \cdot \rangle_\lambda$ is stationary. Another consequence of the product structure is that $\langle \cdot \rangle_\lambda$ satisfies (A3) with constant $\rho = 1$ (see [6, Lemma 7] for the argument). It remains to check (A2+). By stationarity and symmetry we may assume that $e_i = e_d$. Consider the (random) set

$$\mathcal{L}(\boldsymbol{a}) := \{\, j \in \mathbb{Z} \,:\, \boldsymbol{a}^{e_d,0}(\{je_1, je_1 + e_d\}) = 1 \,\}.$$

Clearly, each $j \in \mathcal{L}(\boldsymbol{a})$ yields an open path connecting 0 and e_d, for instance the "U-shaped" path through the sites 0, je_1, $je_1 + e_d$ and e_d. Hence, $\mathrm{dist}_{\boldsymbol{a}^{e_d,0}}(0, e_d) \leq 2\mathrm{dist}(0, \mathcal{L}(\boldsymbol{a})) + 1$ almost surely, where $\mathrm{dist}(0, \mathcal{L}(\boldsymbol{a})) := \min_{j \in \mathcal{L}(\boldsymbol{a})} |j|$. Consequently, it suffices to prove that

$$\langle (2\mathrm{dist}(0, \mathcal{L}(\boldsymbol{a})) + 1)^p \rangle_\lambda^{\frac{1}{p}} < \infty$$

for any $p \geq 1$. Note that due to the definition $\boldsymbol{a}^{e_d,0}(\{0, e_d\}) = 0$ and thus $\mathrm{dist}(0, \mathcal{L}(\boldsymbol{a})) \in \mathbb{N}$. Hence,

$$\langle (2\mathrm{dist}(0, \mathcal{L}(\boldsymbol{a})) + 1)^p \rangle_\lambda = \sum_{k=1}^{\infty} (2k+1)^p \, \langle \mathbf{1}(A_k) \rangle_\lambda\,,$$

where $\mathbf{1}(A_k)$ denotes the set indicator function of $A_k := \{\, \boldsymbol{a} \,:\, \mathrm{dist}(0, \mathcal{L}(\boldsymbol{a})) = k \,\}$. Evidently, we have

$$A_k \subset A_k' := \Big\{\, \boldsymbol{a} \,:\, \boldsymbol{a}(\{je_1, je_1 + e_d\}) = 0 \text{ for all } j = 1, \ldots, k-1 \,\Big\}.$$

From $\langle \mathbf{1}(A_k') \rangle_\lambda = (1-\lambda)^{k-1}$, we deduce that

$$\langle (2\mathrm{dist}(0, \mathcal{L}(\boldsymbol{a})) + 1)^p \rangle_\lambda \leq \sum_{k=1}^{\infty} (2k+1)^p (1-\lambda)^{k-1}.$$

The sum on the right-hand side converges, since $0 < \lambda \leq 1$ by assumption. This completes the proof. □

3 Main result

We are interested in stationary solutions to the corrector equation (1). Note that we tacitly identify the vector $e \in \mathbb{R}^d$ with the translation invariant vector field $e(\mathrm{b}) := e \cdot (y_{\mathrm{b}} - x_{\mathrm{b}})$. For conciseness we write

$$\mathcal{S} := \Big\{ \varphi : \Omega \times \mathbb{Z}^d \to \mathbb{R} \,\big|\, \varphi \text{ is measurable and stationary, i. e. } \varphi(\boldsymbol{a}(\cdot + z), x) = \varphi(\boldsymbol{a}, x+z) \\ \text{for all } x, z \in \mathbb{Z}^d \text{ and } \langle\cdot\rangle\text{-almost every } \boldsymbol{a} \in \Omega \Big\}$$

for the space of *stationary random fields.* Thanks to (A1) the expectation $\langle \varphi \rangle = \langle \varphi(\cdot, x) \rangle$ of a stationary random variable does not depend on x. Therefore, $\|\varphi\|_{L^2(\Omega)} := \big\langle |\varphi|^2 \big\rangle^{\frac{1}{2}}$ defines a norm on $(\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$.

We are interested in solutions to (1) in $(\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$. Thanks to discreteness, the operator $\nabla^*(\boldsymbol{a}\nabla)$ is bounded and linear on $(\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$. However, it is degenerate-elliptic for two-reasons:

- In general the Poincaré Inequality does not hold in $(\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$.
- The conductances $\boldsymbol{a}$ may vanish with positive probability.

Therefore, following [15], we regularize the equation by adding a 0th order term and consider for $T > 0$ the modified corrector equation

$$\frac{1}{T}\phi_T(x) + \nabla^* \boldsymbol{a}(x)(\nabla \phi_T(x) + e) = 0 \qquad \text{for all } x \in \mathbb{Z}^d \text{ and } \boldsymbol{a} \in \Omega. \tag{5}$$

Thanks to the regularization, (5) admits (for all $T > 0$) a unique solution in $(\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$ as follows from Riesz' representation theorem.

Definition 2 (modified corrector)**.** The unique solution $\phi_T \in (\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$ to (5) is called the modified corrector.

We think about the modified corrector as an approximation for the stationary corrector and hope to recover a solution to (1) in the limit $T \uparrow \infty$. This is possible as soon as we have estimates on (some) moments of ϕ_T that are uniform in T — this is the main result of the paper:

Theorem 1 (Moment bounds for the modified corrector)**.** *Let $d > 2$ and $\langle\cdot\rangle$ satisfy Assumption 1 for some ρ and Λ. Let ϕ_T denote the modified corrector as defined in Definition 2. Then for all $T > 0$ and $1 \le p < \infty$ we have*

$$\langle |\phi_T|^p \rangle^{\frac{1}{p}} \lesssim 1. \tag{6}$$

Here $\lesssim$ means $\le$ up to a constant that only depends on p, Λ, ρ, and d.

Since the estimate in Theorem 1 is uniform in T we get as a corollary:

Corollary 1. *Let $d > 2$ and $\langle\cdot\rangle$ satisfy Assumption 1 for some ρ and Λ. Then the corrector equation* (1) *has a unique stationary solution $\phi \in (\mathcal{S}, \|\cdot\|_{L^2(\Omega)})$ with $\langle \phi \rangle = 0$. Moreover, we have*

$$\langle |\phi|^p \rangle^{\frac{1}{p}} \lesssim 1$$

for all $1 \le p < \infty$. Here $\lesssim$ means $\le$ up to a constant that only depends on p, Λ, ρ and d.

3.1 Auxiliary lemmas and proof of Theorem 1

The proof of Theorem 1 is inspired by the approach in [8] where uniformly elliptic conductances are treated. The starting point of our argument is the following p-version of the *Spectral Gap Estimate* (A3), which we recall from [6, Lemma 2]:

Lemma 2 (p-version of (SG))**.** *Let $\langle\cdot\rangle$ satisfy (A3) with constant $\rho > 0$. Then for $p \in \mathbb{N}$ and all $\zeta \in L^{2p}(\Omega)$ with $\langle\zeta\rangle = 0$ we have*

$$\langle \zeta^{2p} \rangle \lesssim \left\langle \left(\sum_{\mathrm{b}\in\mathbb{B}^d} \left(\frac{\partial\zeta}{\partial \mathrm{b}} \right)^2 \right)^p \right\rangle,$$

where $\lesssim$ means $\leq$ up to a constant that only depends on p, ρ and d.

Applied to $\zeta = \phi_T(x=0)$, this estimate yields a bound on stochastic moments of ϕ_T in terms of the vertical derivatives $\frac{\partial\phi_T(x=0)}{\partial\mathrm{b}}$, $\mathrm{b} \in \mathbb{B}^d$ (see Definition 1). Heuristically, we expect the vertical derivative $\frac{\partial\phi_T(x=0)}{\partial\mathrm{b}}$ to behave as the classical partial derivative $\frac{\partial\phi_T(x=0)}{\partial\boldsymbol{a}(\mathrm{b})}$. As we shall see, the latter admits the Green's function representation

$$\frac{\partial\phi_T(x=0)}{\partial\boldsymbol{a}(\mathrm{b})} = -\nabla G_T(\boldsymbol{a},\mathrm{b},0)(\nabla\phi_T(\mathrm{b}) + e(\mathrm{b})). \tag{7}$$

Here G_T denotes the Green's function associated with $(\frac{1}{T} + \nabla^*\boldsymbol{a}\nabla)$ and is defined as follows:

Definition 3. For $T > 0$ the Green's function $G_T : \Omega \times \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is defined as follows: For each $\boldsymbol{a} \in \Omega$ and $y \in \mathbb{Z}^d$ the function $x \mapsto G_T(\boldsymbol{a}, x, y)$ is the unique solution in $\ell^2(\mathbb{Z}^d)$ to

$$\frac{1}{T}G_T(\boldsymbol{a},\cdot,y) + \nabla^*\boldsymbol{a}\nabla G_T(\boldsymbol{a},\cdot,y) = \delta(\cdot - y). \tag{8}$$

For uniformly elliptic conductances we have $\frac{\partial\phi_T(x=0)}{\partial\mathrm{b}} \sim \frac{\partial\phi_T(x=0)}{\partial\boldsymbol{a}(\mathrm{b})}$ up to a constant that only depends on the ratio of ellipticity. In the case of degenerate ellipticity this is no longer true. However, the discrepancy between the vertical and classical partial derivative of ϕ_T can be quantified in terms of weights defined as follows: We introduce the weight function $\omega : \Omega\times\mathbb{B}^d \to [0,\infty]$ as

$$\omega(\boldsymbol{a},\mathrm{b}) := (\mathrm{dist}_{\boldsymbol{a}}(x_{\mathrm{b}}, y_{\mathrm{b}}))^{d+2} \qquad (\boldsymbol{a} \in \Omega,\ \mathrm{b} = \{x_{\mathrm{b}}, y_{\mathrm{b}}\} \in \mathbb{B}^d). \tag{9}$$

For $\mathrm{b} \in \mathbb{B}^d$ and $\boldsymbol{a} \in \Omega$ we denote by $\boldsymbol{a}^{\mathrm{b},0}$ the conductance field obtained by "deleting" the bond b (i. e. $\boldsymbol{a}^{\mathrm{b},0}(\mathrm{b}') = \boldsymbol{a}(\mathrm{b}')$ for all $\mathrm{b}' \neq \mathrm{b}$ and $\boldsymbol{a}^{\mathrm{b},0}(\mathrm{b}) = 0$), and introduce the modified weight ω_0 as

$$\omega_0(\boldsymbol{a},\mathrm{b}) := \omega(\boldsymbol{a}^{\mathrm{b},0},\mathrm{b}). \tag{10}$$

Lemma 3. *Assume that $\langle\cdot\rangle$ satisfies (A1) and (A2+). For $T > 0$ let ϕ_T denote the modified corrector. Then for all $\mathrm{b} \in \mathbb{B}^d$ we have*

$$\left|\frac{\partial\phi_T(x=0)}{\partial\mathrm{b}}\right| \lesssim \omega_0^2(\mathrm{b})\,|\nabla G_T(\mathrm{b},0)|\;|\nabla\phi_T(\mathrm{b}) + e(\mathrm{b})|\,.$$

Here $\lesssim$ means $\leq$ up to a constant that only depends on d.

To benefit from (7) (in the form of Lemma 3) we require an *estimate on the gradient of the Green's function.* As it is well known, the constant coefficient Green's function $G_T^0(x) :=$

$G_T(\boldsymbol{a}=\mathbf{1},x,0)$ (which is associated with the modified Laplacian $\frac{1}{T}+\nabla^*\nabla$) satisfies the pointwise estimate

$$\forall \mathrm{b} := \{x, x+e_i\}\ :\qquad |\nabla G_T^0(\mathrm{b})| \lesssim (1+|x|)^{1-d}\qquad \text{uniformly in } T>0. \tag{11}$$

We require an estimate that captures the same decay in x. It is known from the continuum, uniformly elliptic case, that such an estimate cannot hold pointwise in x and pointwise in $\boldsymbol{a}$. In [8, Lemma 2.9], for uniformly elliptic conductances, a spatially averaged version of (11) is established, where the averages are taken over dyadic annuli. The constant in this estimate depends on the conductances only through their contrast of ellipticity. In the degenerate elliptic case, the ellipticity contrast is infinite. In order to keep the optimal decay in x, we need to allow the constant in the estimate to depend on $\boldsymbol{a}$. For $x_0 \in \mathbb{Z}^d$, $R>1$ and $1\le q<\infty$ consider the spatial average of the weight ω (cf. (9))

$$C(\boldsymbol{a}, Q_R(x_0), q) := \left(\frac{1}{|Q_R(x_0)|}\sum_{\mathrm{b}\in Q_R(x_0)} \omega^q(\boldsymbol{a},\mathrm{b})\right)^{\frac{1}{q}}. \tag{12}$$

We shall prove the following estimate:

Proposition 1. *For $R_0>1$ and $k\in\mathbb{N}_0$ consider*

$$A_k := \begin{cases} Q_{R_0}(0) & k=0,\\ Q_{2^kR_0}(0)\setminus Q_{2^{k-1}R_0}(0) & k\ge 1.\end{cases}$$

Then for all $\frac{2d}{d+2}<p<2$ we have

$$\left(\frac{1}{|A_k|}\sum_{\mathrm{b}\in A_k}|\nabla G_T(\boldsymbol{a},\mathrm{b},0)|^p\right)^{\frac{1}{p}} \lesssim C(\boldsymbol{a})\,2^{k(1-d)},$$

where $\lesssim$ means $\le$ up to a constant that only depends on R_0, d and p, and

$$C(\boldsymbol{a}) := C^{\frac{\beta}{2}}(\boldsymbol{a}, Q_{2^{k+1}R_0}(0), \tfrac{p}{2-p}) \tag{13}$$

with $\beta := 2\frac{p^*-1}{p^*-2}+p^*$ and $p^* := \frac{dp}{d-p}$.

The precise form of the constant C in (13) is not crucial. In fact, in the random setting, when Ω is equipped with a probability measure satisfying (A1) and (A2), we may view C as a random variable with controlled finite moments:

Remark 1. *Let $\langle\cdot\rangle$ satisfy Assumption (A1). Then the spatial average introduced in* (12) *satisfies*

$$\left\langle C^q(\boldsymbol{a}, Q_R(x_0), q')\right\rangle = \left\langle \left(\frac{1}{|Q_R(x_0)|}\sum_{\mathrm{b}\in Q_R(x_0)}\omega^{q'}(\boldsymbol{a},\mathrm{b})\right)^{\frac{q}{q'}}\right\rangle \le \begin{cases} \left\langle \omega^{q'}\right\rangle^{\frac{q}{q'}} & \text{if } q'\ge q,\\ \langle \omega^q\rangle & \text{if } q'<q,\end{cases}$$

as can be seen by appealing to Jensen's inequality and stationarity. Moreover, if $\langle\cdot\rangle$ additionally fulfills (A2), then C defined in (13) *satisfies*

$$\forall m\in\mathbb{N}\,:\,\langle C^m\rangle^{\frac{1}{m}} \lesssim 1,$$

where $\lesssim$ means $\le$ up to a constant that only depends on m, p, Λ and d.

The proof of Proposition 1 relies on arguments from elliptic regularity theory, which in the uniformly elliptic case are standard. They typically involve the pointwise inequality

$$\lambda_0|\nabla u(\mathrm{b})|^2 \leq \nabla u(\mathrm{b})\,\boldsymbol{a}(\mathrm{b})\nabla u(\mathrm{b}), \qquad (\mathrm{b}\in\mathbb{B}^d), \tag{14}$$

where $\lambda_0 > 0$ denotes the constant of ellipticity. In the degenerate case, the conductances $\boldsymbol{a}$ may vanish on a non-negligible set of bonds and (14) breaks down. As a replacement we establish estimates which provide a weighted, integrated version of (14):

Lemma 4. *Let $p > d+1$. For any function $u : \mathbb{Z}^d \to \mathbb{R}$ and all $\boldsymbol{a} \in \Omega$ we have (with the convention $\frac{1}{\infty} = 0$)*

$$\sum_{\mathrm{b}\in\mathbb{B}^d} |\nabla u(\mathrm{b})|^2 \mathrm{dist}_{\boldsymbol{a}}^{-p}(x_\mathrm{b}, y_\mathrm{b}) \leq C(p,d) \sum_{\mathrm{b}\in\mathbb{B}^d} \boldsymbol{a}(\mathrm{b})|\nabla u(\mathrm{b})|^2, \tag{15}$$

where $C(p,d) := \sum_{x\in\mathbb{Z}^d}(|x|+1)^{1-p}$ and the inequality holds whenever the sums converge.

While Lemma 4 is purely deterministic, we also need the following statistically averaged version:

Lemma 5. *Let $\langle\cdot\rangle$ be stationary, cf. (A1), and $p > d+1$. Then for any stationary random field u and any bond $\mathrm{b}\in\mathbb{B}^d$ we have (with the convention $\frac{1}{\infty} = 0$)*

$$\left\langle |\nabla u(\mathrm{b})|^2 \mathrm{dist}_{\boldsymbol{a}}^{-p}(x_\mathrm{b}, y_\mathrm{b}) \right\rangle \leq C(p,d) \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1\ldots d}} \left\langle \boldsymbol{a}(\mathrm{b}')|\nabla u(\mathrm{b}')|^2 \right\rangle,$$

where $C(p,d) := \sum_{k=0}^\infty 2^{k(1-p)}|B_{2^{k+1}}(0)| < \infty$.

A last ingredient required for the proof of Theorem 1 is a *Caccioppoli inequality in probability* that yields a gain of stochastic integrability and helps to treat the $\nabla\phi_T$-term on the right-hand side in (7). In the uniformly elliptic case, i. e. when $0 < \lambda_0 \leq \boldsymbol{a} \leq 1$, the Caccioppoli inequality

$$\left\langle |\nabla\phi_T|^{2p+2} \right\rangle^{\frac{1}{2p+2}} \lesssim \left\langle \phi_T^{2p} \right\rangle^{\frac{1}{2p}\frac{p}{p+1}} \tag{16}$$

holds for any integer exponents p (see [8, Lemma 2.7]). The inequality follows from combining the elementary discrete inequality

$$|\nabla u(\mathrm{b})| = |u(y_\mathrm{b}) - u(x_\mathrm{b})| \leq |u(y_\mathrm{b})| + |u(x_\mathrm{b})|, \tag{17}$$

with the estimate

$$\left\langle \phi_T^{2p}|\nabla\phi_T|^2 \right\rangle \lesssim \frac{1}{\lambda_0} \left\langle \phi_T^{2p}|\nabla\phi_T| \right\rangle. \tag{18}$$

The latter is obtained by testing the modified corrector equation (5) with ϕ_T^{2p+1} and uses the uniform ellipticity of $\boldsymbol{a}$. In the degenerate elliptic case (18) is not true any longer. However, by appealing to Lemma 5 the following weaker version of (18) survives:

$$\left\langle |\nabla\phi_T|^{(2p+2)\theta} \right\rangle^{\frac{1}{(2p+2)\theta}} \lesssim \left\langle \phi_T^{2p} \right\rangle^{\frac{1}{2p}\frac{p}{p+1}} \tag{19}$$

for any factor $0 < \theta < 1$. Hence, we only gain an increase of integrability by exponents strictly smaller two. As a matter of fact, in the proof of our main result we only need the estimate in the following form:

Lemma 6 (Caccioppoli estimate in probability). *Let $\langle\cdot\rangle$ satisfy (A1) and (A2). Let ϕ_T denote the corrector associated with $e \in \mathbb{R}^d$, $|e| = 1$, $T > 0$. For every even integer p we have*

$$\left\langle|\nabla\phi_T|^{2p+1}\right\rangle^{\frac{1}{2p+1}} \lesssim \left\langle\phi_T^{2p}\right\rangle^{\frac{1}{2p}\frac{p}{p+1}}, \tag{20}$$

where $\lesssim$ means $\leq$ up to a constant that only depends on p, Λ and d.

Now we are ready to prove our main result:

Proof of Theorem 1. It suffices to consider exponents $p \in 2\mathbb{N}$ that are larger than a threshold only depending on d – the threshold is determined by (22) below.
Further, we only need to prove

$$\left\langle\phi_T^{2p}\right\rangle^{\frac{1}{2p}} \lesssim \max_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle|\nabla\phi_T(\mathrm{b}')|^{2p+1}\right\rangle^{\frac{1}{2p+1}} + 1. \tag{21}$$

Indeed, in combination with the Caccioppoli estimate in probability, cf. Lemma 6, estimate (21) yields $\left\langle\phi_T^{2p}\right\rangle^{\frac{1}{2p}} \lesssim \left\langle\phi_T^{2p}\right\rangle^{\frac{1}{2p}\frac{p}{p+1}} + 1$. Since $\frac{p}{p+1} < 1$ the first term can be absorbed and the desired estimate follows.
We prove (21). For reasons that will become clear at the end of the argument we fix an exponent $\frac{2d}{d+2} < q < 2$ such that

$$d(\frac{1}{q} + \frac{1}{2p} - 1) + 1 < 0. \tag{22}$$

This is always possible for $p \gg 1$ and $0 < 2 - q \ll 1$, since

$$\lim_{q\uparrow 2, p\uparrow\infty} d(\frac{1}{q} + \frac{1}{2p} - 1) = -\frac{d}{2} < -1 \qquad \text{for } d > 2.$$

Our argument for (21) starts with the p-version of the spectral gap estimate, see Lemma 2, that we combine with Lemma 3:

$$\begin{aligned}\left\langle\phi_T^{2p}\right\rangle^{\frac{1}{p}} &= \left\langle\phi_T^{2p}(x=0)\right\rangle^{\frac{1}{p}} \lesssim \left\langle\left(\sum_{\mathrm{b}\in\mathbb{B}^d}\left(\frac{\partial\phi_T(x=0)}{\partial\mathrm{b}}\right)^2\right)^p\right\rangle^{\frac{1}{p}}\\ &\lesssim \left\langle\left(\sum_{\mathrm{b}\in\mathbb{B}^d}(\nabla G_T(\mathrm{b},0))^2(\nabla\phi_T(\mathrm{b}) + e(\mathrm{b}))^2\omega_0^4(\mathrm{b})\right)^p\right\rangle^{\frac{1}{p}}.\end{aligned}$$

Now we wish to benefit from the decay estimate for ∇G_T in Proposition 1, and therefore decompose $\mathbb{B}^d$ into dyadic annuli: Let the dyadic annuli A_k, $k \in \mathbb{N}_0$ be defined as in Proposition 1 with initial radius $R_0 = 2$. Note that $\mathbb{B}^d$ can be written as the disjoint union of $A_0, A_1, A_2, \dots$. With the triangle inequality w. r. t. $\langle(\cdot)^p\rangle^{\frac{1}{p}}$ and Hölder's inequality in b-space with exponents $(\frac{p}{p-1}, p)$ we get

$$\begin{aligned}\left\langle\phi_T^{2p}\right\rangle^{\frac{1}{p}} &\lesssim \sum_{k\in\mathbb{N}_0}\left\langle\left(\sum_{\mathrm{b}\in A_k}(\nabla G_T(\mathrm{b},0))^2(\nabla\phi_T(\mathrm{b}) + e(\mathrm{b}))^2\omega_0^4(\mathrm{b})\right)^p\right\rangle^{\frac{1}{p}}\\ &\lesssim \sum_{k\in\mathbb{N}_0}\left\langle\left(\sum_{\mathrm{b}\in A_k}|\nabla G_T(\mathrm{b},0)|^{\frac{2p}{p-1}}\right)^{p-1}\left(\sum_{\mathrm{b}\in A_k}(\nabla\phi_T(\mathrm{b}) + e(\mathrm{b}))^{2p}\omega_0^{4p}(\mathrm{b})\right)\right\rangle^{\frac{1}{p}}.\end{aligned} \tag{23}$$

Because $\frac{2d}{d+2} < q < 2 < \frac{2p}{p-1}$, the discrete ℓ^q-$\ell^{\frac{2p}{p-1}}$-estimate combined with the decay estimate of Proposition 1 yields

$$\left(\sum_{\mathrm{b}\in A_k} |\nabla G_T(\mathrm{b},0)|^{\frac{2p}{p-1}}\right)^{p-1} \quad \leq \quad \left(\sum_{\mathrm{b}\in A_k} |\nabla G_T(\mathrm{b},0)|^{q}\right)^{\frac{2p}{q}} \leq C 2^{k(2p(1-(1-\frac{1}{q})d))}. \tag{24}$$

Here and below, C denotes a generic, non-negative random variable with the property that $\langle C^m\rangle \lesssim 1$ for all $m < \infty$, where $\lesssim$ means $\leq$ up to a constant that only depends on m, p, q, Λ and d. Combining (23) and (24) yields

$$\left\langle \phi_T^{2p}\right\rangle^{\frac{1}{p}} \lesssim \sum_{k\in\mathbb{N}_0} 2^{2k(1-(1-\frac{1}{q})d)} \left(\sum_{\mathrm{b}\in A_k} \left\langle C\ (\nabla\phi_T(\mathrm{b})+e(\mathrm{b}))^{2p}\ \omega_0^{4p}(\mathrm{b})\right\rangle\right)^{\frac{1}{p}}. \tag{25}$$

Next we apply a triple Hölder inequality in probability with exponents $(\theta, \theta', \theta')$, where we choose $\theta = \frac{2p+1}{2p}$ (so that $2p\theta = 2p+1$). We have

$$\left\langle C\ (\nabla\phi_T(\mathrm{b})+e(\mathrm{b}))^{2p}\ \omega_0^{4p}(\mathrm{b})\right\rangle \leq \left\langle(\nabla\phi_T(\mathrm{b})+e(\mathrm{b}))^{2p+1}\right\rangle^{\frac{2p}{2p+1}} \left\langle C^{\theta'}\right\rangle^{\frac{1}{\theta'}} \left\langle \omega_0^{4p\theta'}(\mathrm{b})\right\rangle^{\frac{1}{\theta'}}.$$

The first term is estimated by stationarity of $\nabla\phi_T$ and the assumption $|e| = 1$ as

$$\left\langle(\nabla\phi_T(\mathrm{b})+e(\mathrm{b}))^{2p+1}\right\rangle^{\frac{2p}{2p+1}} \lesssim \max_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\ldots,d}} \left\langle|\nabla\phi_T(\mathrm{b}')|^{2p+1}\right\rangle^{\frac{2p}{2p+1}} + 1.$$

For the second term we have $\left\langle C^{\theta'}\right\rangle^{\frac{1}{\theta'}} \left\langle \omega_0^{4p\theta'}(\mathrm{b})\right\rangle^{\frac{1}{\theta'}} \lesssim 1$ due to (A2+), so that we obtain

$$\left\langle C\ (\nabla\phi_T(\mathrm{b})+e(\mathrm{b}))^{2p}\ \omega_0^{4p}(\mathrm{b})\right\rangle \lesssim \max_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\ldots,d}} \left\langle|\nabla\phi_T(\mathrm{b}')|^{2p+1}\right\rangle^{\frac{2p}{2p+1}} + 1. \tag{26}$$

Combined with (25) we get

$$\begin{aligned}
\left\langle \phi_T^{2p}\right\rangle^{\frac{1}{p}} \quad &\lesssim \quad \left(\max_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\ldots,d}} \left\langle|\nabla\phi_T(\mathrm{b}')|^{2p+1}\right\rangle^{\frac{2}{2p+1}} + 1\right) \times \sum_{k\in\mathbb{N}_0} 2^{2k(1-(1-\frac{1}{q})d)} |A_k|^{\frac{1}{p}} \\
&\lesssim \quad \left(\max_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\ldots,d}} \left\langle|\nabla\phi_T(\mathrm{b}')|^{2p+1}\right\rangle^{\frac{2}{2p+1}} + 1\right).
\end{aligned}$$

In the last line we used that

$$\sum_{k\in\mathbb{N}_0} 2^{2k(1-(1-\frac{1}{q})d)} |A_k|^{\frac{1}{p}} \lesssim \sum_{k\in\mathbb{N}_0} 2^{2k(1-(1-\frac{1}{2p}-\frac{1}{q})d)} \lesssim 1,$$

which holds since the exponent is negative, cf. (22). This proves (21). □

4 Proofs of the auxiliary lemmas

4.1 Proof of Lemma 3

The argument for Lemma 3 is split into three lemmas.

Lemma 7. *Let* $\mathrm{b} \in \mathbb{B}^d$ *be fixed. For* $T > 0$ *let* ϕ_T *and* G_T *denote the modified corrector and the Green's function, respectively. Then*

$$\begin{aligned}
\frac{\partial \phi_T(x=0)}{\partial \boldsymbol{a}(\mathrm{b})} &= -\nabla G_T(\mathrm{b},0)(\nabla\phi_T(\mathrm{b})+e(\mathrm{b})), && (27)\\
\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}\frac{\partial \phi_T(x=0)}{\partial \boldsymbol{a}(\mathrm{b})} &= -2\nabla\nabla G_T(\mathrm{b},\mathrm{b})\frac{\partial \phi_T(x=0)}{\partial \boldsymbol{a}(\mathrm{b})}, && (28)\\
\frac{\partial}{\partial \boldsymbol{a}(b)}\nabla\nabla G_T(\mathrm{b},\mathrm{b}) &= -\left(\nabla\nabla G_T(\mathrm{b},\mathrm{b})\right)^2. && (29)
\end{aligned}$$

Moreover, $\nabla\nabla G_T(\mathrm{b},\mathrm{b})$ *and* $1-\boldsymbol{a}(\mathrm{b})\nabla\nabla G_T(\mathrm{b},\mathrm{b})$ *are strictly positive.*

Proof of Lemma 7. For simplicity we write ϕ and G instead of ϕ_T and G_T.

Step 1. Argument for (29).
We first claim that

$$\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}G(x,y) = -\nabla G(\mathrm{b},y)\nabla G(\mathrm{b},x), \tag{30a}$$

$$\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}\nabla G(x,\mathrm{b}) = -\nabla\nabla G(\mathrm{b},\mathrm{b})\nabla G(\mathrm{b},x). \tag{30b}$$

Indeed, since ∇ and $\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}$ commute, an application of $\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}$ to (8) yields

$$\left(\frac{1}{T}+\nabla^*\boldsymbol{a}\nabla\right)\frac{\partial G(\cdot,y)}{\partial \boldsymbol{a}(\mathrm{b})} = -\nabla^*\frac{\partial \boldsymbol{a}(\cdot)}{\partial \boldsymbol{a}(\mathrm{b})}\nabla G(\cdot,y). \tag{31}$$

We test this identity with $G(\cdot,x)$:

$$\begin{aligned}
\frac{\partial G(x,y)}{\partial \boldsymbol{a}(\mathrm{b})} &= \sum_{y'\in\mathbb{Z}^d}\frac{\partial G(y',y)}{\partial \boldsymbol{a}(\mathrm{b})}\delta(x-y') && (32)\\
&\overset{(8)}{=} \sum_{y'\in\mathbb{Z}^d}\frac{\partial G(y',y)}{\partial \boldsymbol{a}(\mathrm{b})}\left(\frac{1}{T}+\nabla^*\boldsymbol{a}\nabla\right)G(y',x)\\
&\overset{(4)}{=} \sum_{y'\in\mathbb{Z}^d}G(y',x)\left(\frac{1}{T}+\nabla^*\boldsymbol{a}\nabla\right)\frac{\partial G(y',y)}{\partial \boldsymbol{a}(\mathrm{b})}\\
&\overset{(31),(4)}{=} -\sum_{\mathrm{b}'\in\mathbb{B}^d}\frac{\partial \boldsymbol{a}(\mathrm{b}')}{\partial \boldsymbol{a}(\mathrm{b})}\nabla G(\mathrm{b}',y)\,\nabla G(\mathrm{b}',x).
\end{aligned}$$

Since $\frac{\partial \boldsymbol{a}(\mathrm{b}')}{\partial \boldsymbol{a}(\mathrm{b})}$ is equal to 1 if $\mathrm{b}'=\mathrm{b}$ and 0 else, the sum on the right-hand side reduces to $\nabla G(\mathrm{b},y)\nabla G(\mathrm{b},x)$ and we get (30a). An application of ∇ to (30a) yields (30b), and an application of ∇ to (30b) finally yields (29).

Step 2. Argument for (27) and (28).
We apply $\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}$ to the modified corrector equation (5):

$$\frac{1}{T}\frac{\partial \phi}{\partial \boldsymbol{a}(\mathrm{b})}+\nabla^*\boldsymbol{a}\nabla\frac{\partial \phi}{\partial \boldsymbol{a}(\mathrm{b})} = -\nabla^*\frac{\partial \boldsymbol{a}(\cdot)}{\partial \boldsymbol{a}(\mathrm{b})}(\nabla\phi+e(\mathrm{b})). \tag{33}$$

As in (32) testing with $G(\cdot,x)$ yields

$$\frac{\partial \phi(x)}{\partial \boldsymbol{a}(\mathrm{b})} = -(\nabla\phi(\mathrm{b})+e(\mathrm{b}))\nabla G(\mathrm{b},x), \tag{34}$$

and (27) follows. By applying $\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}$ and ∇ to (34) we obtain the two identities

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}\frac{\partial \phi(x)}{\partial \boldsymbol{a}(\mathrm{b})} &= -\frac{\partial(\nabla\phi(\mathrm{b})+e(\mathrm{b}))}{\partial \boldsymbol{a}(\mathrm{b})}\nabla G(\mathrm{b},x) - (\nabla\phi(\mathrm{b})+e(\mathrm{b}))\frac{\partial \nabla G(\mathrm{b},x)}{\partial \boldsymbol{a}(\mathrm{b})},\\
\nabla\frac{\partial \phi(\mathrm{b})}{\partial \boldsymbol{a}(\mathrm{b})} &= -(\nabla\phi(\mathrm{b})+e(\mathrm{b}))\nabla\nabla G(\mathrm{b},\mathrm{b}).
\end{aligned}$$

By combining the first with the second identity, (30b) and (34) we get

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{a}(\mathrm{b})}\frac{\partial \phi(x)}{\partial \boldsymbol{a}(\mathrm{b})} &= 2(\nabla\phi(\mathrm{b})+e(\mathrm{b}))\nabla\nabla G(\mathrm{b},\mathrm{b})\nabla G(\mathrm{b},x)\\
&= -2\frac{\partial \phi(x)}{\partial \boldsymbol{a}(\mathrm{b})}\nabla\nabla G(\mathrm{b},\mathrm{b}),
\end{aligned}$$

and thus (28).

Step 3. Positivity of $\nabla\nabla G(\mathrm{b},\mathrm{b})$ and $1-\boldsymbol{a}(\mathrm{b})\nabla\nabla G(\mathrm{b},\mathrm{b})$.
Let $\mathrm{b}=(x_\mathrm{b},y_\mathrm{b})\in\mathbb{B}^d$ be fixed. An application of ∇ (w. r. t. the y-component) to (8) yields

$$(\frac{1}{T}+\nabla^*\boldsymbol{a}\nabla)\nabla G(\cdot,\mathrm{b}) = \delta(\cdot - y_\mathrm{b}) - \delta(\cdot - x_\mathrm{b}).$$

We test this equation with $\nabla G(\cdot,\mathrm{b})$ and get

$$\frac{1}{T}\sum_{x\in\mathbb{Z}^d}(\nabla G(x,\mathrm{b}))^2 + \sum_{\mathrm{b}'\in\mathbb{B}^d}\boldsymbol{a}(\mathrm{b}')\left(\nabla\nabla G(\mathrm{b}',\mathrm{b})\right)^2 = \nabla\nabla G(\mathrm{b},\mathrm{b}). \tag{35}$$

This identity implies that $\nabla\nabla G(\mathrm{b},\mathrm{b})$ and $1-\boldsymbol{a}(\mathrm{b})\nabla\nabla G(\mathrm{b},\mathrm{b})$ are strictly positive. Indeed, $\nabla\nabla G(\mathrm{b},\mathrm{b})$ must be strictly positive, since otherwise $\sum_{x\in\mathbb{Z}^d}|\nabla G(x,\mathrm{b})|^2=0$ and thus $G(\cdot,\mathrm{b})=0$ in contradiction to (8). The strict positivity of $1-\nabla\nabla G(\mathrm{b},\mathrm{b})$ follows from the strict positivity of $\nabla\nabla G(\mathrm{b},\mathrm{b})-\boldsymbol{a}(\mathrm{b})\left(\nabla\nabla G(\mathrm{b},\mathrm{b})\right)^2$. The latter can be seen by the following argument:

$$\begin{aligned}
&\nabla\nabla G(\mathrm{b},\mathrm{b}) - \boldsymbol{a}(\mathrm{b})\left(\nabla\nabla G(\mathrm{b},\mathrm{b})\right)^2\\
&= \left(\nabla\nabla G(\mathrm{b},\mathrm{b}) - \frac{1}{T}\sum_{x\in\mathbb{Z}^d}(\nabla G(x,\mathrm{b}))^2 - \sum_{\mathrm{b}'\in\mathbb{B}^d}\boldsymbol{a}(\mathrm{b}')\left(\nabla\nabla G(\mathrm{b}',\mathrm{b})\right)^2\right)\\
&\quad + \frac{1}{T}\sum_{x\in\mathbb{Z}^d}(\nabla G(x,\mathrm{b}))^2 + \sum_{\mathrm{b}'\neq\mathrm{b}}\boldsymbol{a}(\mathrm{b}')\left(\nabla\nabla G(\mathrm{b}',\mathrm{b})\right)^2\\
&\overset{(35)}{\geq} \frac{1}{T}\sum_{x\in\mathbb{Z}^d}(\nabla G(x,\mathrm{b}))^2 > 0.
\end{aligned}$$

□

The next lemma establishes a (quantitative) link between the vertical and classical partial derivative of ϕ_T.

Lemma 8. *Let* $\mathrm{b}\in\mathbb{B}^d$ *be fixed. For* $T>0$ *let* ϕ_T *and* G_T *denote the modified corrector and the Green's function. Then*

$$\left|\frac{\partial\phi_T(x=0)}{\partial \mathrm{b}}\right| \leq \left(1+\frac{\boldsymbol{a}(\mathrm{b})}{1-\boldsymbol{a}(\mathrm{b})\nabla\nabla G_T(\mathrm{b},\mathrm{b})}\right)\left|\frac{\partial\phi_T(x=0)}{\partial \boldsymbol{a}(\mathrm{b})}\right|. \tag{36}$$

Proof of Lemma 8. Fix $\boldsymbol{a} \in \Omega$ and $\mathrm{b} \in \mathbb{B}^d$. Set $a_0 := \boldsymbol{a}(\mathrm{b})$. We shall use the following shorthand notation

$$\varphi(a) := \frac{\partial \phi_T(\boldsymbol{a}^{\mathrm{b},a}, x=0)}{\partial \boldsymbol{a}(\mathrm{b})}, \qquad g(a) := \nabla\nabla G_T(\boldsymbol{a}^{\mathrm{b},a}, \mathrm{b}, \mathrm{b}), \qquad (a \in [0,1]), \tag{37}$$

where $\boldsymbol{a}^{\mathrm{b},a}$ denotes the coefficient field obtained from $\boldsymbol{a}$ by setting $\boldsymbol{a}^{\mathrm{b},a}(\mathrm{b}') = a$ if $\mathrm{b}' = \mathrm{b}$ and $\boldsymbol{a}^{\mathrm{b},a}(\mathrm{b}') := \boldsymbol{a}(\mathrm{b}')$ else. With that notation (28) and (29) turn into

$$\varphi' = -2g\varphi, \tag{38}$$

$$g' = -g^2. \tag{39}$$

Since we have $\left|\frac{\partial \phi_T(x=0)}{\partial \mathrm{b}}\right| \leq \int_0^1 |\varphi(a)|\, da$, it suffices to show

$$\int_0^1 |\varphi(a)|\, da \leq \left(1 + \frac{a_0}{1 - a_0 g(a_0)}\right) |\varphi(a_0)|. \tag{40}$$

The positivity of g and (38) imply that φ is either strictly positive, strictly negative or that it vanishes identically. In the latter case, the claim is trivial. In the other cases we have

$$\varphi(a) = \exp(h(a))\varphi(a_0), \qquad \text{where } h(a) := \ln \frac{\varphi(a)}{\varphi(a_0)},$$

and (40) reduces to the inequality

$$\int_0^1 \exp(h(a))\, da \leq 1 + \frac{a_0}{1 - a_0 g(a_0)}. \tag{41}$$

From (38) we learn that $h' = -2g$. Since $g > 0$, h is decreasing. Combined with the identity $h(a_0) = 0$ we get

$$h(a) \leq \begin{cases} 2\displaystyle\int_a^{a_0} g(a')\, da' & \text{for } a \in [0, a_0), \\ 0 & \text{for } a \in [a_0, 1]. \end{cases} \tag{42}$$

On the other hand, we learn from integrating (39) that $g(a') = \frac{g(a_0)}{1+(a'-a_0)g(a_0)}$. Hence, for $a < a_0$ the right-hand side in (42) turns into

$$2\int_a^{a_0} g(a')\, da' = -2\ln(1 + (a - a_0)g(a_0)),$$

which in combination with (42) yields (41). □

Lemma 3 is a direct consequence of (36), (27) and the following estimate:

Lemma 9. *Let G_T denote the Green's function. Assume that (A1) is satisfied. Then for all $T > 0$, $\boldsymbol{a} \in \Omega$ and $\mathrm{b} \in \mathbb{B}^d$ we have*

$$1 + \frac{\boldsymbol{a}(b)}{1 - \boldsymbol{a}(b)\nabla\nabla G_T(\mathrm{b}, \mathrm{b})} \lesssim \omega_0^2(\boldsymbol{a}, \mathrm{b}), \tag{43}$$

where $\lesssim$ means up to a constant that only depends on d.

Proof of Lemma 9. Step 1. Reduction to an estimate for $\boldsymbol{a}^{\mathrm{b},0}$.
We claim that

$$\frac{\boldsymbol{a}(\mathrm{b})}{1-\boldsymbol{a}(\mathrm{b})\nabla\nabla G_T(\boldsymbol{a},\mathrm{b},\mathrm{b})} \leq (1+\nabla\nabla G_T(\boldsymbol{a}^{\mathrm{b},0},\mathrm{b},\mathrm{b}))^2$$

For the argument let $\boldsymbol{a} \in \Omega$ and $\mathrm{b} \in \mathbb{B}^d$ be fixed. With the shorthand notation introduced in (37), the claim reads

$$\frac{a_0}{1-a_0 g(a_0)} \leq (1+g(0))^2. \tag{44}$$

For $a_0 = 0$ the statement is trivial. For $a_0 > 0$ consider the function

$$f(a) := \frac{1}{a}g(a) - g^2(a),$$

with help of which the left-hand side in (44) can be written as $\frac{g(a_0)}{f(a_0)}$. The function f is non-negative and decreasing, as can be seen by combining the inequality $0 < g(a) < \frac{1}{a}$ from Lemma 7 with the identity $f'(a) = g(a)(g^2(a) - \frac{1}{a^2} + g^2(a) - \frac{1}{a}g(a))$ which follows from (39). The latter also implies that $g(1) = \frac{g(0)}{1+g(0)}$ and thus $f(1) = g(1)(1-g(1)) = \frac{g(0)}{(1+g(0))^2}$. Hence,

$$\frac{a_0}{1-a_0 g(a_0)} = \frac{g(a_0)}{f(a_0)} \leq \frac{g(a_0)}{f(1)} = (1+g(0))^2 \frac{g(a_0)}{g(0)} \leq (1+g(0))^2;$$

in the last step we used in addition that $g(a_0) \leq g(0)$ which is a consequence of (39).

Step 2. Conclusion.
To complete the argument we only need to show that

$$\nabla\nabla G_T(\boldsymbol{a}^{\mathrm{b},0},\mathrm{b},\mathrm{b}) \lesssim \omega_0(\boldsymbol{a},\mathrm{b}). \tag{45}$$

For simplicity set $\boldsymbol{a}_0 := \boldsymbol{a}^{\mathrm{b},0}$. Note that $\omega_0(\boldsymbol{a},\mathrm{b}) = \omega(\boldsymbol{a}_0,\mathrm{b})$. From (35) we obtain

$$\begin{aligned}\nabla\nabla G_T(\boldsymbol{a}_0,\mathrm{b},\mathrm{b}) &\overset{(35)}{\geq} \sum_{\mathrm{b}'\in\mathbb{B}^d} \boldsymbol{a}_0(\mathrm{b}')\left(\nabla\nabla G_T(\boldsymbol{a}_0,\mathrm{b}',\mathrm{b})\right)^2 \overset{(15)}{\gtrsim} \sum_{\mathrm{b}'\in\mathbb{B}^d} \omega^{-1}(\boldsymbol{a}_0,\mathrm{b}')\left(\nabla\nabla G_T(\boldsymbol{a}_0,\mathrm{b}',\mathrm{b})\right)^2 \\ &\geq \omega^{-1}(\boldsymbol{a}_0,\mathrm{b})\left(\nabla\nabla G_T(\boldsymbol{a}_0,\mathrm{b},\mathrm{b})\right)^2.\end{aligned}$$

Dividing both sides by $\omega^{-1}(\boldsymbol{a}_0,\mathrm{b})\nabla\nabla G_T(\boldsymbol{a}_0,\mathrm{b},\mathrm{b})$ yields (45). □

4.2 Proof of Lemma 4 and Lemma 5

Proof of Lemma 4. Fix for a moment $\boldsymbol{a} \in \Omega$. For $\mathrm{b} \in \mathbb{B}^d$ with $\mathrm{dist}_{\boldsymbol{a}}(x_{\mathrm{b}}, y_{\mathrm{b}}) < \infty$, let $\pi_{\boldsymbol{a}}(\mathrm{b})$ denote a shortest open path that connects x_{b} and y_{b}, i.e.

$$\mathrm{dist}_{\boldsymbol{a}}(x_{\mathrm{b}}, y_{\mathrm{b}}) = \sum_{\mathrm{b}'\in\pi_{\boldsymbol{a}}(\mathrm{b})} \frac{1}{\boldsymbol{a}(\mathrm{b}')}.$$

Thanks to the triangle inequality and the Cauchy-Schwarz inequality we have

$$\begin{aligned}|\nabla u(\mathrm{b})| &\leq \sum_{\mathrm{b}'\in\pi(\mathrm{b})} |\nabla u(\mathrm{b}')| \leq \left(\sum_{\mathrm{b}'\in\pi_{\boldsymbol{a}}(\mathrm{b})} \frac{1}{\boldsymbol{a}(\mathrm{b}')}\right)^{\frac{1}{2}} \left(\sum_{\mathrm{b}'\in\pi_{\boldsymbol{a}}(\mathrm{b})} |\nabla u(\mathrm{b}')|^2 \boldsymbol{a}(\mathrm{b}')\right)^{\frac{1}{2}} \\ &= \mathrm{dist}_{\boldsymbol{a}}^{\frac{1}{2}}(x_{\mathrm{b}}, y_{\mathrm{b}}) \left(\sum_{\mathrm{b}'\in\pi_{\boldsymbol{a}}(\mathrm{b})} |\nabla u(\mathrm{b}')|^2 \boldsymbol{a}(\mathrm{b}')\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, using the convention $\frac{1}{\infty}=0$, we conclude that for all $\mathrm{b}\in\mathbb{B}^d$ and $\boldsymbol{a}\in\Omega$:

$$\mathrm{dist}_{\boldsymbol{a}}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2 \;\le\; \mathrm{dist}_{\boldsymbol{a}}^{1-p}(x_{\mathrm{b}},y_{\mathrm{b}})\sum_{\mathrm{b}'\in\pi_{\boldsymbol{a}}(\mathrm{b})}|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}'). \tag{46}$$

We drop the "$\boldsymbol{a}$" in the notation from now on. Summation of (46) in $\mathrm{b}\in\mathbb{B}^d$ yields

$$\begin{aligned}\sum_{\mathrm{b}\in\mathbb{B}^d}\mathrm{dist}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2 \;&\le\; \sum_{\mathrm{b}\in\mathbb{B}^d}\sum_{\mathrm{b}'\in\pi(\mathrm{b})}\mathrm{dist}^{1-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}')\\ &=\; \sum_{\mathrm{b}'\in\mathbb{B}^d}\sum_{\substack{\mathrm{b}\in\mathbb{B}^d\text{ with}\\ \pi(\mathrm{b})\ni\mathrm{b}'}}\mathrm{dist}^{1-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}').\end{aligned}$$

Since $\pi(\mathrm{b})$ is a shortest path, and because $\boldsymbol{a}\le 1$, we have $\mathrm{dist}(x_{\mathrm{b}},y_{\mathrm{b}})\ge|x_{\mathrm{b}}-x_{\mathrm{b}'}|+1$ for all $\mathrm{b},\mathrm{b}'\in\mathbb{B}^d$ with $\mathrm{b}'\in\pi(\mathrm{b})$. Combined with the previous estimate we get

$$\begin{aligned}\sum_{\mathrm{b}\in\mathbb{B}^d}\mathrm{dist}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2 \;&\le\; \sum_{\mathrm{b}'\in\mathbb{B}^d}\sum_{\substack{\mathrm{b}\in\mathbb{B}^d\text{ with}\\ \pi(\mathrm{b})\ni\mathrm{b}'}}(|x_{\mathrm{b}}-x_{\mathrm{b}'}|+1)^{1-p}|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}')\\ &\le\; C(d,p)\sum_{\mathrm{b}'\in\mathbb{B}^d}|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}').\end{aligned}$$

□

Proof of Lemma 5. Fix $\mathrm{b}\in\mathbb{B}^d$. For $L\in\mathbb{N}$ consider the indicator function

$$\chi_L(\boldsymbol{a}) := \begin{cases}1 & \text{if } L\le\mathrm{dist}_{\boldsymbol{a}}(x_{\mathrm{b}},y_{\mathrm{b}})<2L,\\ 0 & \text{else.}\end{cases} \tag{47}$$

With the convention $\frac{1}{\infty}=0$, we have

$$\sum_{k=0}^{\infty}\chi_{2^k}(\boldsymbol{a})\mathrm{dist}_{\boldsymbol{a}}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}}) = \mathrm{dist}_{\boldsymbol{a}}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}}) \tag{48}$$

for all $\boldsymbol{a}\in\Omega$. In the following we drop "$\boldsymbol{a}$" in the notation. We recall (46) in the form of

$$\chi_L\mathrm{dist}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2 \;\le\; \chi_L\mathrm{dist}^{1-p}(x_{\mathrm{b}},y_{\mathrm{b}})\sum_{\mathrm{b}'\in\pi(\mathrm{b})}|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}'). \tag{49}$$

From $\boldsymbol{a}\le 1$ and $\mathrm{dist}(x_{\mathrm{b}},y_{\mathrm{b}})<2L$ for $\chi_L\neq 0$, cf. (47), we learn that $\pi(\mathrm{b})$ is contained in the box $Q_{2L}(x_{\mathrm{b}})$. Hence, (49) turns into

$$\chi_L\mathrm{dist}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2 \overset{(47)}{\le} \chi_L L^{1-p}\sum_{\mathrm{b}'\in Q_{2L}(x_{\mathrm{b}})}|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}').$$

We take the expectation on both sides and appeal to stationarity:

$$\begin{aligned}\left\langle\chi_L\mathrm{dist}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2\right\rangle \;&\le\; L^{1-p}\sum_{\mathrm{b}'\in Q_{2L}(x_{\mathrm{b}})}\left\langle\chi_L|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}')\right\rangle\\ &\overset{\chi_L\le 1}{\le}\; L^{1-p}\sum_{x\in B_{2L}(x_{\mathrm{b}})}\sum_{\substack{\mathrm{b}'=\{x,x+e_i\}\\ i=1,\dots,d}}\left\langle|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}')\right\rangle\\ &\overset{\text{stationarity}}{\le}\; L^{1-p}|B_{2L}(0)|\sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}}\left\langle|\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}')\right\rangle.\end{aligned}$$

Using $1+d-p<0$ we get

$$\begin{aligned}\left\langle \operatorname{dist}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2\right\rangle &\overset{(48)}{=} \sum_{k=0}^{\infty}\left\langle \chi_{2^k}\operatorname{dist}^{-p}(x_{\mathrm{b}},y_{\mathrm{b}})|\nabla u(\mathrm{b})|^2\right\rangle \\ &\leq C(p,d)\sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\ldots,d}}\left\langle |\nabla u(\mathrm{b}')|^2\boldsymbol{a}(\mathrm{b}')\right\rangle .\end{aligned}$$

□

4.3 Proof of Proposition 1 – Green's function estimates

We first establish an estimate for the Green's function itself:

Lemma 10 (BMO-estimate)**.** *Let $d\geq 2$ and consider $u,f\in\ell^1(\mathbb{Z}^d)$ with*

$$\nabla^*\boldsymbol{a}\nabla u = f \qquad \textit{in } \mathbb{Z}^d. \tag{50}$$

Then for all $\frac{2d}{d+2}<p<2$, $R\geq 1$ and $x_0\in\mathbb{Z}^d$ we have

$$\sum_{x\in B_R(x_0)}|u(x)-\bar{u}|\lesssim C\,R^2\sum_{x\in\mathbb{Z}^d}|f(x)|. \tag{51}$$

Here, $\bar{u}:=\frac{1}{|B_R(x_0)|}\sum_{x\in B_R(x_0)}u(x)$ denotes the average of u on $B_R(x_0)$, $C:=C(\boldsymbol{a},Q_R(x_0),\frac{p}{2-p})$, and $\lesssim$ means $\leq$ up to a constant that only depends on d and p.

Proof of Lemma 10. W. l. o. g. we assume $\sum_{\mathbb{Z}^d}|f|=1$ and $R\in\mathbb{N}$. To shorten the notation we write B_R and Q_R for $B_R(x_0)$ and $Q_R(x_0)$, respectively. Let $M(u)$ denote a median of u on B_R, i. e.

$$|\{u\geq M(u)\}\cap B_R|,|\{u\leq M(u)\}\cap B_R|\geq\frac{1}{2}|B_R|.$$

By Jensen's inequality we have $|\bar{u}-M(u)|\leq\frac{1}{|B_R|}\sum_{B_R}|u-M(u)|$, so that it suffices to prove for $v:=u-M(u)$ the estimate

$$\sum_{B_R}|v|\lesssim C\,R^2\sum_{\mathbb{Z}^d}|f|=C\,R^2.$$

For $0\leq M<\infty$ consider the cut-off version of v

$$v_M:=\max\{\min\{v,M\},0\}.$$

Then v_M satisfies

$$\sum_{\mathbb{B}^d}\nabla v_M\,\boldsymbol{a}\nabla v_M=\sum_{\mathbb{B}^d}\nabla u\,\boldsymbol{a}\nabla v_M.$$

Since $u\in\ell^1(\mathbb{Z}^d)$ (by assumption) and $v_M\in\ell^\infty(\mathbb{Z}^d)$ (by construction), we may integrate by parts:

$$\sum_{\mathbb{B}^d}\nabla u\,\boldsymbol{a}\nabla v_M=\sum_{\mathbb{Z}^d}v_M\,\nabla^*\boldsymbol{a}\nabla u=\sum_{\mathbb{Z}^d}f v_M\leq M\sum_{\mathbb{Z}^d}|f|=M.$$

Hence,

$$\sum_{\mathbb{B}^d}\nabla v_M\,\boldsymbol{a}\nabla v_M\leq M. \tag{52}$$

Set $p^* = \frac{pd}{d-p}$ and $q^* := \frac{p^*}{p^*-1}$. By construction we have $|\{v_M = 0\} \cap B_R| = |\{v_M \leq 0\} \cap B_R| \geq \frac{1}{2}|B_R|$. Hence, the Sobolev-Poincaré inequality yields

$$\left(R^{-d}\sum_{B_R}|v_M|^{p^*}\right)^{\frac{1}{p^*}} \lesssim R\left(R^{-d}\sum_{Q_R}|\nabla v_M|^p\right)^{\frac{1}{p}}.$$

Lemma 4 combined with Hölder's inequality with exponents $(\frac{2}{2-p}, \frac{2}{p})$ yields

$$\begin{aligned}
\left(R^{-d}\sum_{Q_R}|\nabla v_M|^p\right)^{\frac{1}{p}} &= \left(R^{-d}\sum_{Q_R}\omega^{\frac{p}{2}}|\nabla v_M|^p\omega^{-\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\leq \left(R^{-d}\sum_{Q_R}\omega^{\frac{p}{2-p}}\right)^{\frac{2-p}{2p}}\left(R^{-d}\sum_{Q_R}|\nabla v_M|^2\omega^{-1}\right)^{\frac{1}{2}} \\
&\overset{\text{Lemma 4}}{\lesssim} C^{\frac{1}{2}}\left(R^{-d}\sum_{\mathbb{B}^d}\nabla v_M\,\boldsymbol{a}\nabla v_M\right)^{\frac{1}{2}},
\end{aligned} \tag{53}$$

so that

$$\left(R^{-d}\sum_{B_R}|v_M|^{p^*}\right)^{\frac{1}{p^*}} \lesssim C^{\frac{1}{2}}R\left(R^{-d}\sum_{\mathbb{B}^d}\nabla v_M\,\boldsymbol{a}\nabla v_M\right)^{\frac{1}{2}} \overset{(52)}{\lesssim} (CR^{2-d}M)^{\frac{1}{2}}. \tag{54}$$

Next we use Chebyshev's inequality in the form of

$$M\left(R^{-d}|\{\,v > M\,\} \cap B_R|\right)^{\frac{1}{p^*}} \lesssim \left(R^{-d}\sum_{B_R}|v_M|^{p^*}\right)^{\frac{1}{p^*}}.$$

With (54) we get

$$R^{-d}|\{\,v > M\,\} \cap B_R| \lesssim C^{\frac{p^*}{2}}\,R^{(2-d)\frac{p^*}{2}}M^{-\frac{p^*}{2}},$$

which upgrades by symmetry to

$$R^{-d}|\{\,|v| > M\,\} \cap B_R| \lesssim C^{\frac{p^*}{2}}\,R^{(2-d)\frac{p^*}{2}}M^{-\frac{p^*}{2}}.$$

Since $p > \frac{2d}{d+2}$ (by assumption), we have $\frac{p^*}{2} > 1$ and the "wedding cake formula" for $M := CR^{2-d}$ yields

$$\begin{aligned}
R^{-d}\sum_{B_R}|v| &= \int_0^\infty R^{-d}|\{\,|v| > M'\,\} \cap B_R|\,dM' \lesssim M + \int_M^\infty R^{-d}|\{\,|v| > M'\,\} \cap B_R|\,dM' \\
&\lesssim M + C^{\frac{p^*}{2}}R^{(2-d)\frac{p^*}{2}}M^{1-\frac{p^*}{2}} \lesssim CR^{2-d}.
\end{aligned}$$

□

A careful Caccioppoli estimate combined with the previous lemma yields:

Lemma 11. *Let $d \geq 2$, $x_0 \in \mathbb{Z}^d$ and $R \geq 1$. Consider $f \geq 0$ and u related as*

$$\nabla^* \boldsymbol{a} \nabla u = -f \qquad \text{in } B_{2R}(x_0). \tag{55}$$

Then for $\frac{2d}{d+2} < p < 2$ we have

$$\left(R^{-d} \sum_{Q_R(x_0)} |R\nabla u|^p \right)^{\frac{1}{p}} \lesssim C^{\frac{\alpha}{2}} \left(R^{-d} \sum_{B_{2R}(x_0)} |u| + \left(R^{2-d} \sum_{B_{2R}(x_0)} f u_- \right)^{\frac{1}{2}} \right), \tag{56}$$

where $u_- := \max\{-u, 0\}$ denotes the negative part of u, $C := C(\boldsymbol{a}, Q_{2R}(x_0), \frac{p}{2-p})$, $\alpha := 2\frac{p^*-1}{p^*-2}$ and $p^* := \frac{dp}{d-p}$. Here $\lesssim$ stands for $\leq$ up to a constant that only depends on p and d

Proof of Lemma 11. Step 1. Caccioppoli estimate.
We claim that for every cut-off function η that is supported in $B_{2R-1}(x_0)$ (so that in particular $\nabla\eta = 0$ outside of $Q_{2R}(x_0)$) we have

$$\left(R^{-d} \sum_{\mathbb{B}^d} |R\nabla(u\eta)|^p \right)^{\frac{1}{p}} \lesssim C^{\frac{1}{2}} \left(R^{2-d} \sum_{\mathbb{Z}^d} f u_- \eta^2 + R^{-d} \sum_{\mathrm{b} \in \mathbb{B}^d} u(x_\mathrm{b}) u(y_\mathrm{b}) |R\nabla\eta(\mathrm{b})|^2 \boldsymbol{a}(\mathrm{b}) \right)^{\frac{1}{2}}. \tag{57}$$

Indeed, we get with Lemma 4 (using an argument similar to (53)):

$$\left(R^{-d} \sum_{\mathbb{B}^d} |R\nabla(u\eta)|^p \right)^{\frac{1}{p}} = \left(R^{-d} \sum_{Q_{2R}(x_0)} |R\nabla(u\eta)|^p \right)^{\frac{1}{p}} \lesssim C^{\frac{1}{2}} \left(R^{-d} \sum_{\mathbb{B}^d} |R\nabla(u\eta)|^2 \boldsymbol{a} \right)^{\frac{1}{2}},$$

Combined with the elementary identity

$$|\nabla(u\eta)(\mathrm{b})|^2 = \nabla u(\mathrm{b}) \nabla(u\eta^2)(\mathrm{b}) + u(x_\mathrm{b}) u(y_\mathrm{b}) |\nabla\eta(\mathrm{b})|^2,$$

the equation for u, and the fact that $-fu\eta^2 \leq f u_- \eta^2$ (here we use $f \geq 0$), the claimed estimate (57) follows.

Step 2. Conclusion.
Set $\theta := \frac{\alpha-1}{\alpha}$ and note that α is defined in such a way that for the considered range of p we have

$$\frac{1}{2} = \theta \frac{1}{p^*} + (1-\theta) \qquad \text{and} \qquad 2(1-\theta) < 1. \tag{58}$$

As we shall see below in Step 3, there exists a cut-off function η with $\eta = 1$ in $B_{R+1}(x_0)$ and $\eta = 0$ outside of $B_{2R-1}(x_0)$, such that

$$\begin{aligned} \left(R^{-d} \sum_{\mathrm{b} \in \mathbb{B}^d} |u(x_\mathrm{b})||u(y_\mathrm{b})||R\nabla\eta(\mathrm{b})|^2 \right)^{\frac{1}{2}} &\lesssim \left(R^{-d} \sum_{\mathbb{Z}^d} |u\eta|^{p^*} \right)^{\frac{\theta}{p^*}} \left(R^{-d} \sum_{B_{2R}(x_0)} |u| \right)^{1-\theta} \\ &\quad + \left(R^{-d} \sum_{\mathbb{Z}^d} |u\eta|^{p^*} \right)^{\frac{1}{2p^*}} \left(R^{-d} \sum_{B_{2R}(x_0)} |u| \right)^{\frac{1}{2}}. \end{aligned} \tag{59}$$

Let us explain the right-hand side of this estimate. While the first term on the right-hand side would also appear in the continuum case (i.e. when $\mathbb{Z}^d$ is replaced by $\mathbb{R}^d$), the second term is

an error term coming from discreteness. In fact, it is of lower order: A sharp look at (63) below shows that (59) holds with the vanishing factor $R^{-\epsilon}$ (for some $\epsilon > 0$ only depending on p and d) in front of the second term on the right-hand side.
By combining this estimate with the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{Z}^d$, i.e. $\big(R^{-d}\sum_{\mathbb{Z}^d}|u\eta|^{p^*}\big)^{\frac{1}{p^*}} \lesssim \big(R^{-d}\sum_{\mathbb{B}^d}|R\nabla(u\eta)|^p\big)^{\frac{1}{p}}$, and two applications of Youngs' inequality, we find that for all $\delta > 0$ there exists a constant $C(\delta) > 0$ only depending on δ, p and d, such that

$$\begin{aligned}
&\left(CR^{-d}\sum_{\mathrm{b}\in\mathbb{B}^d}|u(x_\mathrm{b})||u(y_\mathrm{b})|(\nabla\eta(\mathrm{b}))^2\boldsymbol{a}(\mathrm{b})\right)^{\frac{1}{2}}\\
&\qquad\le \delta\left(R^{-d}\sum_{\mathbb{B}^d}|R\nabla(u\eta)|^p\right)^{\frac{1}{p}} + C(\delta)\left(C^{\frac{1}{2(1-\theta)}}R^{-d}\sum_{B_{2R}(x_0)}|u| + CR^{-d}\sum_{B_{2R}(x_0)}|u|\right)\\
&\qquad\overset{2(1-\theta)<1}{\le} \delta\left(R^{-d}\sum_{\mathbb{B}^d}|R\nabla(u\eta)|^p\right)^{\frac{1}{p}} + 2C(\delta)C^{\frac{1}{2(1-\theta)}}R^{-d}\sum_{B_{2R}(x_0)}|u|.
\end{aligned}$$

We combine this estimate with (57) and absorb the first term on the right-hand side of the previous estimate into the left-hand side of (57). Since $\nabla(\eta u) = \nabla u$ in $Q_R(x_0)$ this yields (56).

Step 3. Proof of (59).
We first construct a suitable cut-off function η for $B_{R+1}(x_0)$ in $B_{2R-1}(x_0)$. W. l. o. g. we assume that $x_0 = 0$. Recall that $\alpha = 2\frac{p^*-1}{p^*-2}$. For $t \ge 0$ set

$$\tilde{\eta}(t) := \max\{1 - 2\max\{\tfrac{t}{R+1} - 1, 0\}, 0\}^\alpha,$$

and define

$$\eta(x) := \prod_{i=1}^d \tilde{\eta}(|x_i|). \tag{60}$$

Using the relation $\alpha - 1 = \theta\alpha$, cf. (58), it is straightforward to check that η satisfies for all edges b with $|\nabla\eta(\mathrm{b})| > 0$:

$$R|\nabla\eta(\mathrm{b})| \lesssim \begin{cases}\min\{\eta^\theta(x_\mathrm{b}), \eta^\theta(y_\mathrm{b})\} & \text{if } \min\{\eta(x_\mathrm{b}), \eta(y_\mathrm{b}) > 0\},\\ R^{1-\alpha} & \text{if } \min\{\eta(x_\mathrm{b}), \eta(y_\mathrm{b})\} = 0.\end{cases} \tag{61}$$

Now we turn to (59). We split the sum into a "interior" and a "boundary" contribution:

$$\begin{aligned}
&\sum_{\mathrm{b}\in\mathbb{B}^d}|u(x_\mathrm{b})||u(y_\mathrm{b})|(\nabla\eta(\mathrm{b}))^2\\
&\qquad = \sum_{\mathrm{b}\in A_{\mathrm{int}}}|u(x_\mathrm{b})||u(y_\mathrm{b})|(\nabla\eta(\mathrm{b}))^2 + \sum_{\mathrm{b}\in A_{\mathrm{bound}}}|u(x_\mathrm{b})||u(y_\mathrm{b})|(\nabla\eta(\mathrm{b}))^2,
\end{aligned}$$

where

$$\begin{aligned}
A_{\mathrm{int}} &:= \{\,\mathrm{b} : |\nabla\eta(\mathrm{b})| > 0 \text{ and } \min\{\eta(x_\mathrm{b}), \eta(y_\mathrm{b})\} > 0\,\},\\
A_{\mathrm{bound}} &:= \{\,\mathrm{b} : |\nabla\eta(\mathrm{b})| > 0 \text{ and } \min\{\eta(x_\mathrm{b}), \eta(y_\mathrm{b})\} = 0\,\}.
\end{aligned}$$

For A_{int} we get with (61), Young's inequality, and Hölder's inequality with exponents $(p^*\frac{1}{2\theta}, \frac{1}{2(1-\theta)})$:

$$\begin{aligned} R^{-d} \sum_{\mathrm{b}\in A_{\text{int}}} |u(x_{\mathrm{b}})||u(y_{\mathrm{b}})||R\nabla\eta(\mathrm{b})|^2 &\lesssim R^{-d}\sum_{\mathbb{Z}^d} u^2\eta^{2\theta} \\ = R^{-d}\sum_{\mathbb{Z}^d}(u\eta)^{2\theta}u^{2(1-\theta)} &\le \left(R^{-d}\sum_{\mathbb{Z}^d}(u\eta)^{p^*}\right)^{\frac{2\theta}{p^*}} \left(R^{-d}\sum_{B_{2R}}|u|\right)^{2(1-\theta)}. \end{aligned} \tag{62}$$

Next we treat A_{bound}, which is an error term coming from discreteness. By the definition of A_{bound} the cut-off function η vanishes at one and only one of the two sites adjacent to $\mathrm{b} \in A_{\text{bound}}$. Given $\mathrm{b} \in A_{\text{bound}}$ we denote by $\tilde{x}_{\mathrm{b}}$ (resp. $\tilde{y}_{\mathrm{b}}$) the site adjacent to b with $\eta(\tilde{x}_{\mathrm{b}}) = 0$ (resp. $\eta(\tilde{y}_{\mathrm{b}}) \neq 0$), so that

$$R^{-d} \sum_{\mathrm{b}\in A_{\text{bound}}} |u(x_{\mathrm{b}})||u(y_{\mathrm{b}})||R\nabla\eta(\mathrm{b})|^2 = R^{1-d} \sum_{\mathrm{b}\in A_{\text{bound}}} |u(\tilde{x}_{\mathrm{b}})||u(\tilde{y}_{\mathrm{b}})|\eta(\tilde{y}_{\mathrm{b}})|R\nabla\eta(\mathrm{b})|.$$

We combine this with (61), Hölder's inequality with exponents $(p^*, q^* := \frac{p^*}{p^*-1})$, and the discrete ℓ^1-ℓ^{q^*}-estimate:

$$\begin{aligned} R^{-d} \sum_{\mathrm{b}\in A_{\text{bound}}} |u(x_{\mathrm{b}})||u(y_{\mathrm{b}})||R\nabla\eta(\mathrm{b})|^2 &\lesssim R^{2-d-\alpha} \sum_{\mathrm{b}\in A_{\text{bound}}} |u(\tilde{x}_{\mathrm{b}})||u(\tilde{y}_{\mathrm{b}})|\eta(\tilde{y}_{\mathrm{b}}) \\ \le R^{2-d-\alpha}\left(\sum_{B_{2R}}|u\eta|^{p^*}\right)^{\frac{1}{p^*}}\left(\sum_{B_{2R}}|u|^{q^*}\right)^{\frac{1}{q^*}} &\le R^{2-d-\alpha}\left(\sum_{B_{2R}}|u\eta|^{p^*}\right)^{\frac{1}{p^*}}\sum_{B_{2R}}|u| \\ = R^{\frac{d}{p^*}-2-\alpha}\left(R^{-d}\sum_{B_{2R}}|u\eta|^{p^*}\right)^{\frac{1}{p^*}}&\left(R^{-d}\sum_{B_{2R}}|u|\right). \end{aligned} \tag{63}$$

From the definition of α and p^*, and the fact that $\alpha > 2$, we deduce that the exponent $\frac{d}{p^*} - 2 - \alpha$ is negative. Together with (62) the desired estimate (59) follows. ☐

Now we are ready to prove Proposition 1. We distinguish the cases $k \geq 1$ and $k = 0$.

Proof of Proposition 1. Step 1. Argument for $k \geq 1$.
For brevity set $R := 2^{k-1}R_0$ and recall that $A_k = Q_{2R}(0) \setminus Q_R(0)$. We cover the annulus A_k by boxes $Q_{\frac{R}{2}}(x_0)$, $x_0 \in X_R \subset \mathbb{Z}^d$, such that

$$A_k \subset \bigcup_{x_0\in X_R} Q_{\frac{R}{2}}(x_0) \subset \bigcup_{x_0\in X_R} Q_R(x_0) \subset Q_{3R}(0) \setminus \{0\}. \tag{64}$$

Since the diameter of the annulus and the side length of the boxes are comparable, we may choose X_R such that its cardinality is bounded by a constant only depending on d. Since in addition we have for $x_0 \in X_R$ the inequality $C(\boldsymbol{a}, Q_R(x_0), \frac{p}{2-p}) \lesssim C(\boldsymbol{a}, Q_{3R}(0), \frac{p}{2-p})$ (thanks to the third inclusion in (64)), it suffices to prove

$$\left(R^{-d} \sum_{\mathrm{b}\in Q_{\frac{R}{2}}(x_0)} |\nabla G_T(\boldsymbol{a}, \mathrm{b}, 0)|^p\right)^{\frac{1}{p}} \lesssim C^{\frac{\beta}{2}}\, R^{1-d}, \qquad \text{where } C := C(\boldsymbol{a}, Q_R(x_0), \tfrac{p}{2-p}),$$

for each $x_0 \in X_R$ separately. We use the shorthand $G_T(x) := G_T(\boldsymbol{a}, x, 0)$ and set $\bar{G}_T := \frac{1}{|B_R(x_0)|}\sum_{x\in B_R(x_0)} G_T(x)$. In view of (8), $u(x) := G_T(x) - \bar{G}_T$ satisfies (50) with $f = \delta - \frac{1}{T}G_T$. Since

$$\sum_{\mathbb{Z}^d} |\delta - \frac{1}{T}G_T| \leq 1 + \frac{1}{T}\sum_{\mathbb{Z}^d} G_T(x) = 2, \tag{65}$$

Lemma 10 yields

$$R^{-d}\sum_{B_R(x_0)} |u| \lesssim C^{\frac{1}{2}p^*}\, R^{2-d}. \tag{66}$$

Thanks to the third inclusion in (64) we have $0 \notin B_R(x_0)$, and thus u satisfies (55) with $f = \frac{1}{T}G_T$ (with $B_{2R}(x_0)$ replaced by $B_R(x_0)$). Hence, Lemma 11 yields

$$\begin{aligned}\left(R^{p-d}\sum_{Q_{\frac{R}{2}}(x_0)} |\nabla G_T|^p\right)^{\frac{1}{p}} &= \left(R^{p-d}\sum_{Q_{\frac{R}{2}}(x_0)} |\nabla u|^p\right)^{\frac{1}{p}} \\ &\lesssim C^{\frac{1}{2}\alpha}\, R^{-d}\sum_{B_R(x_0)} |u| \;+\; C^{\frac{1}{2}\alpha}\left(R^{2-d}\sum_{B_R(x_0)} \frac{1}{T}G_T u_-\right)^{\frac{1}{2}} \\ &\overset{(66)}{\lesssim} C^{\frac{1}{2}(\alpha+p^*)}\, R^{2-d} \;+\; C^{\frac{1}{2}\alpha}\left(R^{2-d}\sum_{B_R(x_0)} \frac{1}{T}G_T u_-\right)^{\frac{1}{2}}.\end{aligned} \tag{67}$$

Regarding the second term on the right-hand side we only need to show

$$\frac{1}{T}\sum_{B_R(x_0)} G_T u_- \lesssim C^{p^*}\, R^{2-d}. \tag{68}$$

We note that $(G_T - \bar{G}_T)(G_T - \bar{G}_T)_- \leq 0$, so that

$$\frac{1}{T}\sum_{B_R(x_0)} G_T u_- \;=\; \frac{1}{T}\sum_{B_R(x_0)} (G_T - \bar{G}_T + \bar{G}_T)(G_T - \bar{G}_T)_- \leq \frac{1}{T}\bar{G}_T \sum_{B_R(x_0)} |G_T - \bar{G}_T|.$$

Combined with (66) and the inequality $\frac{1}{T}\bar{G}_T \lesssim R^{-d}\frac{1}{T}\sum_{B_R(x_0)} G_T \leq R^{-d}$, (68) follows.

Step 2. Argument for $k = 0$. Fix $\boldsymbol{a} \in \Omega$. For brevity set $G_T(x) := G_T(\boldsymbol{a}, x, 0)$ and $\bar{G}_T := \frac{1}{|B_{2R_0}(0)|}\sum_{x\in B_{2R_0}(0)} G_T(x)$. By the discrete ℓ^1-ℓ^p-estimate and the elementary inequality $|\nabla G_T(\mathrm{b})| \leq |G_T(x_\mathrm{b}) - \bar{G}_T| + |G_T(y_\mathrm{b}) - \bar{G}_T|$ we have

$$\left(\frac{1}{|Q_{R_0}(0)|}\sum_{\mathrm{b}\in Q_{R_0}(0)} |\nabla G_T(\mathrm{b})|^p\right)^{\frac{1}{p}} \lesssim \sum_{B_{2R_0}(0)} |G_T - \bar{G}_T|.$$

As in Step 1 an application of Lemma 10 yields

$$\sum_{B_{2R_0}(0)} |G_T - \bar{G}_T| \lesssim C^{\frac{p^*}{2}}(\boldsymbol{a}, Q_{2R_0}(0), \tfrac{p}{2-p})\, R_0^2.$$

Since $R_0^2 \sim R_0^{1-d}$ and because the exponent of the constant satisfies $\frac{p^*}{2} \leq \frac{\beta}{2}$, the desired estimate follows. □

4.4 Proof of Lemma 6

In order to deal with the failure of the Leibniz rule we will appeal to a number of discrete estimates, which are stated in Lemma 12 below. As already mentioned, we replace the missing uniform ellipticity of $\boldsymbol{a}$ by the coercivity estimate of Lemma 5 which makes use of the weight ω defined in (9). Morally speaking it plays the role of $\frac{1}{\lambda_0}$ in (18). In view of Assumption (A2) all moments of ω are bounded, i. e. $\langle \omega^k \rangle \lesssim 1$, where $\lesssim$ means $\leq$ up to a constant that only depends on k, p, Λ and d. We split the proof of Lemma 6 into the following two inequalities:

$$\left\langle |\nabla\phi(\mathrm{b})|^{2p+1} \right\rangle^{\frac{2p+2}{2p+1}} \lesssim \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle |\nabla(\phi^{p+1})(\mathrm{b}')|^2 \boldsymbol{a}(\mathrm{b}') \right\rangle, \tag{69}$$

$$\sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle |\nabla(\phi^{p+1})(\mathrm{b}')|^2 \boldsymbol{a}(\mathrm{b}') \right\rangle \lesssim \left\langle \phi^{2p}(x) \right\rangle. \tag{70}$$

Here and below we write ϕ instead of ϕ_T for simplicity. Note that due to stationarity the left-hand side of (69) and the right-hand side of (70) do not depend on $\mathrm{b} \in \mathbb{B}^d$ (resp. $x \in \mathbb{Z}^d$). Therefore, we suppress these arguments in the following. We start with (69). We smuggle in ω by appealing to Hölder's inequality with exponent $\frac{2p+2}{2p+1}$ and exploit that all moments of ω are bounded by Assumption (A2):

$$\left\langle |\nabla\phi|^{2p+1} \right\rangle^{\frac{2p+2}{2p+1}} \lesssim \left\langle |\nabla\phi|^{2p+2} \omega^{-1} \right\rangle.$$

We combine (17) in the form of $|\nabla\phi(\mathrm{b})|^{2p+2} \lesssim (\frac{\phi^p(x_\mathrm{b})+\phi^p(y_\mathrm{b})}{2})^2 |\nabla\phi(\mathrm{b})|^2$ (where we use that p is even) with the discrete version of the Leibniz rule $F^p \nabla F = \frac{1}{p+1}\nabla(F^{p+1})$, see (74) in Corollary 2 below:

$$\left\langle |\nabla\phi|^{2p+2} \omega^{-1} \right\rangle \lesssim \left\langle |\nabla(\phi^{p+1})|^2 \omega^{-1} \right\rangle. \tag{71}$$

Now (69) follows from the coercivity estimate of Lemma 5.
Next we prove (70). The discrete version of the Leibniz rule $|\nabla(F^{p+1})|^2 = \frac{(p+1)^2}{(2p+1)} \nabla F \nabla(F^{2p+1})$ (see Lemma 12 (ii)) yields

$$\sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle |\nabla(\phi^{p+1})(\mathrm{b}')|^2 \boldsymbol{a}(\mathrm{b}') \right\rangle \lesssim \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle \nabla\phi(\mathrm{b}') \boldsymbol{a}(\mathrm{b}') \nabla(\phi^{2p+1})(\mathrm{b}') \right\rangle.$$

By stationarity and the modified corrector equation (5) we have

$$\begin{aligned}
&\sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle \nabla\phi(\mathrm{b}') \boldsymbol{a}(\mathrm{b}') \nabla(\phi^{2p+1})(\mathrm{b}') \right\rangle = \left\langle (\nabla^* \boldsymbol{a} \nabla\phi)\ \phi^{2p+1} \right\rangle \\
&= \quad -\frac{1}{T} \left\langle \phi^{2(p+1)} \right\rangle - \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle \nabla\phi^{2p+1}(\mathrm{b}') \boldsymbol{a}(\mathrm{b}') e(\mathrm{b}') \right\rangle \\
&\leq \quad \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle |\nabla(\phi^{2p+1})(\mathrm{b}')| \boldsymbol{a}(\mathrm{b}') \right\rangle,
\end{aligned}$$

where for the last inequality we use that $\phi^{2(p+1)} \geq 0$ and $|e| = 1$. By Corollary 2 and Young's inequality we get for any $\epsilon > 0$

$$\begin{aligned}
\sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle |\nabla(\phi^{2p+1})(\mathrm{b}')| \boldsymbol{a}(\mathrm{b}')\right\rangle &\overset{(73)}{\lesssim} \epsilon \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle |\nabla\phi(\mathrm{b}')|^2 \left(\tfrac{\phi^p(x_{\mathrm{b}'})+\phi^p(y_{\mathrm{b}'})}{2}\right)^2 \boldsymbol{a}(\mathrm{b}')\right\rangle \\
&\quad + \frac{1}{\epsilon} \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle \left(\tfrac{\phi^p(x_{\mathrm{b}'})+\phi^p(y_{\mathrm{b}'})}{2}\right)^2 \right\rangle \\
&\overset{(74)}{\lesssim} \epsilon \sum_{\substack{\mathrm{b}'=\{0,e_i\}\\ i=1,\dots,d}} \left\langle |\nabla(\phi^{p+1})(\mathrm{b}')|^2 \boldsymbol{a}(\mathrm{b}')\right\rangle + \frac{1}{\epsilon}\left\langle \phi^{2p}\right\rangle .
\end{aligned}$$

Since we may choose $\epsilon > 0$ as small as we wish, the first term on the right-hand side can be absorbed into the left-hand side of (70) and the claim follows.

Acknowledgments

We thank Artem Sapozhnikov for stimulating discussions on percolation models. Stefan Neukamm was partially supported by ERC-2010-AdG no.267802 AnaMultiScale. Most of this work was done while all three authors where employed at the Max-Planck-Institute for Mathematics in the Sciences, Leipzig.

Appendix: Replacements of the Leibniz rule for the discrete derivative

Lemma 12. *Let F be a scalar function on $\mathbb{Z}^d$ and $\mathrm{b} \in \mathbb{B}^d$.*

(i) Assume that $p \in 2\mathbb{N}$. Then we have

$$|\nabla(F^{p+1})(\mathrm{b})| \sim |\nabla F(\mathrm{b})| \frac{F^p(x_\mathrm{b}) + F^p(y_\mathrm{b})}{2}.$$

(ii) For every integer p we have

$$|\nabla(F^{p+1})(\mathrm{b})|^2 \lesssim \nabla F(\mathrm{b}) \nabla(F^{2p+1})(\mathrm{b}).$$

Here $\lesssim$ (resp. $\sim$) means up to a constant that only depends on p.

Proof of Lemma 12. Let $x, y \in \mathbb{Z}^d$ denote the vertices with $\mathrm{b} = \{x, y\}$ and $y - x \in \{e_1, \dots, e_d\}$ so that $\nabla F(\mathrm{b}) = F(y) - F(x)$.

Proof of part (i). The statement " $\lesssim$ " is equivalent to [8, Equation (5.29)] and is proven there. Concerning $\gtrsim$ we appeal to [8, Equation (5.28)]. From that equation we learn that

$$\nabla(F^{p+1})(\mathrm{b})\nabla F(b) \gtrsim \frac{F^p(x_\mathrm{b}) + F^p(y_\mathrm{b})}{2} |\nabla F(\mathrm{b})|^2.$$

By dividing by $|\nabla F(\mathrm{b})|$ one immediately finds the claimed result.

Proof of part (ii). We have to distinguish two cases.
First case: $F(x), F(y) \geq 0$ or $F(x), F(y) \leq 0$. It suffices to show the statement for $F(x), F(y) \geq 0$, since then the case $F(x), F(y) \leq 0$ follows by symmetry. We have to prove that

$$(F^{p+1}(y) - F^{p+1}(x))^2 \lesssim (F(y) - F(x))(F^{2p+1}(y) - F^{2p+1}(x)).$$

By symmetry and and scale invariance, it suffices to show the elementary inequality

$$\forall f \geq 0: \quad (1 - f^{p+1})^2 \leq c(1 - f)(1 - f^{2p+1}), \tag{72}$$

where $c > 0$ only depends on p. We omit its proof for the sake of brevity.
Second case: $F(x) \leq 0, F(y) \geq 0$ or $F(x) \geq 0, F(y) \leq 0$. It suffices to show the statement for $F(x) \leq 0, F(y) \geq 0$, since then the case $F(x) \geq 0, F(y) \leq 0$ follows by symmetry. We have to prove that

$$(F^{p+1}(y) - F^{p+1}(x))^2 \lesssim (F(y) - F(x))(F^{2p+1}(y) - F^{2p+1}(x))$$

or equivalently

$$\begin{aligned} &F^{2(p+1)}(y) + F^{2(p+1)}(x) - 2F^{p+1}(y)F^{p+1}(x) \\ &\qquad \lesssim F^{2p+2}(y) + F^{2p+2}(x) - F(x)F^{2p+1}(y) - F(y)F^{2p+1}(x). \end{aligned}$$

Note that since $2p+1$ is an odd integer, the last two terms on the right hand side of the above inequality are positive. Hence, it suffices to prove that

$$F^{2(p+1)}(y) + F^{2(p+1)}(x) - 2F^{p+1}(y)F^{p+1}(x) \lesssim F^{2p+2}(y) + F^{2p+2}(x),$$

which follows due to $-2F^{p+1}(y)F^{p+1}(x) \leq F^{2p+2}(y) + F^{2p+2}(x)$. □

In the course of proving our main result we will use the discrete Leibniz rule, (i) in the above lemma, in the following form.

Corollary 2. *For every scalar function F, every bond* b *and every even integer p we have*

$$|\nabla(F^{2p+1})(\mathrm{b})| \lesssim |\nabla F(\mathrm{b})| \left(\frac{F^p(x_\mathrm{b}) + F^p(y_\mathrm{b})}{2} \right)^2, \tag{73}$$

$$|\nabla F(\mathrm{b})|^2 \left(\frac{F^p(x_\mathrm{b}) + F^p(y_\mathrm{b})}{2} \right)^2 \lesssim |\nabla(F^{p+1})(\mathrm{b})|^2. \tag{74}$$

References

[1] S. Andres, M. T. Barlow, J. D. Deuschel, and B. M. Hambly. Invariance principle for the random conductance model. *Probab. Theory Rel. Fields*, to appear.

[2] S. Andres, J. D. Deuschel, and M. Slowik. Invariance principle for the random conductance model in a degenerate ergodic environment. preprint.

[3] M. Biskup. Recent progress on the Random Conductance Model. *Probability Surveys* 8:294373, 2011.

[4] N. Berger, M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Relat. Fields* 137(1-2):83–120, 2007.

[5] G. D. Birkhoff. Proof of the ergodic theorem. *Proc Natl Acad Sci USA* 17(12):656–660, 1931.

[6] A. Gloria, S. Neukamm and F. Otto. Quantifiation of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. MPI-Preprint 91/2013.

[7] A. Gloria, S. Neukamm and F. Otto. An optimal quantitative two-scale expansion in stochastic homogenization of discrete linear elliptic equations. MPI-Preprint 41/2013.

[8] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.*, 39(3), 779–856, 2011.

[9] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.*, 22(1):1–28, 2012.

[10] C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functional of reversible Markov processes and applications to simple exclusion. *Commun. Math. Phys.*, 104:1–19, 1986.

[11] S.M. Kozlov. The averaging of random operators. *Mat. Sb. (N.S.)*, 109(151)(2):188–202, 327, 1979.

[12] S.M. Kozlov. Averaging of difference schemes. *Math. USSR Sbornik*, 57(2):351–369, 1987.

[13] R. Künnemann. The diffusion limit for reversible jump processes on $\mathbb{Z}^d$ with ergodic random bond conductivities. *Commun. Math. Phys.*, 90:27–68, 1983.

[14] P. Mathieu, A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 463(2085):2287–2307, 2007.

[15] G.C. Papanicolaou and S.R.S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.

[16] V. Sidoravicius, Vladas, A.-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Relat. Fields*, 129(2):219–244, 2004.

[17] A.-S. Sznitman, O. Zeitouni. An Invariance principle for Isotropic Diffusions in Random Environment. *Invent. Math*, 164:455–567, 2006.